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BETTI NUMBERS OF SOME CIRCULANT GRAPHS

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Abstract. Let $o(n)$ be the greatest odd integer less than or equal to n. In this paper we provide explicit formulae to compute $\mathbb{N}$-graded Betti numbers of the circulant graphs $C_{2n}(1,2,3,5,\ldots,o(n))$. We do this by showing that this graph is the product (or join) of the cycle C_n by itself, and computing Betti numbers of $C_n * C_n$. We also discuss whether such a graph (more generally, $G * H$) is well-covered, Cohen-Macaulay, sequentially Cohen-Macaulay, Buchsbaum, or S_2.

Keywords: Betti number; Castelnuovo-Mumford regularity; projective dimension; circulant graph

MSC 2010: 13D02, 05C75

1. Introduction

Let G be a finite simple undirected graph over the vertex set $V(G)=\{x_1,\ldots,x_n\}$. Let $R=k[x_1,\ldots,x_n]$ denote the polynomial ring in n variables over the field k. The *edge ideal* of G is the ideal $I(G)$ of R generated by those square-free quadratic monomials x_ix_j such that $\{x_i,x_j\}$ is an edge of G. A subset C of $V(G)$ is called a *vertex cover* of G if $C\cap e\neq\emptyset$ for any $e\in E(G)$. A vertex cover C of G is called *minimal* if there is no proper subset of C which is a vertex cover. The minimum number of vertices in a minimal vertex cover of G is called the *covering number* of G and is denoted by $\alpha(G)$. A graph G is said to be *well-covered* if the cardinality of each minimal vertex cover equals $\alpha(G)$.

Given an integer $n\geqslant 1$ and a generating set $S\subseteq\{1,2\ldots,\lfloor\frac12 n\rfloor\}$, the *circulant graph* $C_n(S)$ is the graph with the vertex set $V=\{0,1,\ldots,n-1\}$ whose edge set is

$$E=\{\{i,j\}\colon\ |i-j|\in S \text{ or } n-|i-j|\in S\}.$$

For $S=\{a_1,\ldots,a_t\}$, we abuse the notation and use $C_n(a_1,\ldots,a_t)$ to denote $C_n(S)$. Circulant graphs belong to the family of Cayley graphs and may be considered a generalization of cycles because $C_n = C_n(1)$. In recent years, there have been a flurry of work identifying circulant graphs which are also well-covered (see e.g. [2], [3], [4], [10], [17]). A well-covered graph has the property that its independence complex is pure, and a pure complex can have some extra combinatorial (e.g., vertex decomposable and shellable) or topological (e.g., Cohen-Macaulay and Buchsbaum) structure, i.e., for a pure simplicial complex the following hierarchy is known

$$\text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen-Macaulay} \Rightarrow \text{Buchsbaum}.$$

Recently, Vander Meulen, Van Tuyl, and Watt in [21] characterized Cohen-Macaulay (vertex decomposable, shellable, or Buchsbaum) circulant graphs of the form $C_n(1, 2, \ldots, d)$ and Cohen-Macaulay cubic circulant graphs. Earl, Vander Meulen, and Van Tuyl in [6] determined whether circulant graphs of the form $C_n(d + 1, d + 2, \ldots, \lfloor \frac{1}{2}n \rfloor)$, $C_n(1, \ldots, \hat{i}, \ldots, \lfloor \frac{1}{2}n \rfloor)$, and one-paired circulants have these structures. Also Vander Meulen and Van Tuyl in [20] investigated whether the independence complex of the lexicographical product of two graphs is either vertex decomposable or shellable. They also constructed an infinite family of graphs in which the independence complex of each graph is shellable, but not vertex decomposable. Moreover, in [16] it was investigated what circulant graphs in the above mentioned families satisfy Serre's condition S_2.

For any finite simple graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ over a disjoint set of vertices (i.e., $V(G) \cap V(H) = \emptyset$), the graph theoretical *product* of G by H (in some literatures called the *join* of G and H), denoted by $G * H$, is the graph over the vertex set $V(G) \cup V(H)$ whose edge set is

$$E(G * H) = E(G) \cup E(H) \cup \{\{x, y\} \colon x \in V(G) \text{ and } y \in V(H)\}.$$

Let $o(n)$ be the greatest odd integer less than or equal to n, and let $O(n)$ be the set of all nonnegative odd integers less than or equal to n (or $o(n)$), i.e., $O(n) = \{1, 3, 5, \ldots, o(n)\}$. The aim of this paper is to study the algebraic properties of the circulant graph $C_{2n}(O(n) \cup \{2\}) = C_{2n}(1, 2, 3, 5, \ldots, o(n))$. We do this by first showing that $C_{2n}(O(n) \cup \{2\})$ is isomorphic to the product (or join) of the cycle C_n by itself. More precisely, we show:

Lemma 1.1. *Let $n, m \geqslant 3$ be integers. Then $C_n * C_m$ is a circulant graph if and only if $n = m$. In addition, if this is the case, then $C_n * C_n \simeq C_{2n}(O(n) \cup \{2\})$.*

This description enables us to investigate whether such a graph (more generally, $G * H$) is well-covered, Cohen-Macaulay, sequentially Cohen-Macaulay, Buchsbaum, or S_2.

Proposition 1.2. *Let G and H be two simple graphs with disjoint vertex sets. Also let $n, m \geqslant 3$ be integers. Then*

(i) *$G * H$ is well-covered if and only if both G and H are well-covered with $\alpha(G) + |V(H)| = \alpha(H) + |V(G)|$.*

(ii) *$G * G$ is well-covered if and only if G is well-covered.*

(iii) *$G * H$ is Cohen-Macaulay (or S_2) if and only if both G and H are complete graphs.*

(iv) *$G * H$ is sequentially Cohen-Macaulay if and only if G is complete and H is sequentially Cohen-Macaulay.*

(v) *$G * H$ is Buchsbaum if and only if both G and H are Buchsbaum graphs.*

(vi) *$C_n * C_m$ is well-covered if and only if $n = 4$ and $m = 5$, or $n = m \in \{3, 4, 5, 7\}$.*

(vii) *$C_n * C_m$ is Cohen-Macaulay (or S_2) if and only if $n = m = 3$.*

(viii) *$C_n * C_m$ is sequentially Cohen-Macaulay if and only if $n = 3$ and $m \in \{3, 5\}$.*

(ix) *$C_n * C_m$ is Buchsbaum if and only if $n, m \in \{3, 4, 5, 7\}$.*

As a special case we have:

Corollary 1.3. *$C_{2n}(O(n) \cup \{2\})$ is well-covered (equivalently Buchsbaum) if and only if $n \in \{3, 4, 5, 7\}$. Also it is Cohen-Macaulay (eqivalently S_2 or sequentially Cohen-Macaulay) if and only if $n = 3$.*

Our next topic in this paper is about the Betti numbers of $C_{2n}(O(n) \cup \{2\})$. Betti numbers belong to the most important invariants of a minimal graded free resolution of a graded module, so that Betti numbers of some families of graphs have been characterized. Among the papers that have studied Betti numbers of edge ideals, we mention [8], [9], [11], [12], [13], [18], [23], [24]. In particular, Jacques in his thesis [11] provided explicit formulae to compute Betti numbers of some families of graphs such as lines, cycles, complete, and complete multipartite graphs.

We use $\beta_{i,j}^k(G)$ to denote the graded Betti numbers of $k[x_1, \ldots, x_n]/I(G)$, where $V(G) = \{x_1, \ldots, x_n\}$. The Betti numbers are in general going to be dependent on (the characteristic of) the field k. However, for many of the families of graphs (lines, cycles, complete, and complete multipartite graphs, for example) they are independent of the choice of k (see [11]). When this is the case we will simply write $\beta_{i,j}(G)$.

Using the formula for Betti numbers of the product of graphs (see [15], Corollary 3.4 or [23], Lemma 5.4) together with the Betti numbers of cycles described in [11], Sections 7.4 and 7.5 we will provide explicit formulae for computing $\mathbb{N}$-graded Betti numbers of the circulant graph $C_{2n}(O(n) \cup \{2\})$. Indeed, in view of [15], Proposition 3.12 one has $\operatorname{reg}(C_{2n}(O(n) \cup \{2\})) = \operatorname{reg}(C_n)$. Hence we will prove the following.

Theorem 1.4. *Let $n \geqslant 5$ be an integer. Then the $\mathbb{N}$-graded Betti numbers of $C_{2n}(O(n) \cup \{2\})$ are independent of (the characteristic of) the chosen field k and*

(i) $\beta_{i,i+1} = 2n\binom{n+1}{i-1} + \binom{2n}{i+1} - 2\binom{n}{i+1}$.

(ii) *If* $2 \leqslant r < \operatorname{reg}(C_n) = \lfloor \frac{n+1}{3} \rfloor$, *then* $\beta_{i,i+r} = \frac{2n}{n-2r}\binom{n-2r}{r}\binom{n+r}{i-r}$.

(iii) *If* $r = \operatorname{reg}(C_n) = \lfloor \frac{n+1}{3} \rfloor \geqslant 2$, *then* $\beta_{i,i+r}$ *may be expressed as* $2k\binom{n}{i-p} + \frac{2n}{n-2r}\binom{n-2r}{r}\{\binom{n+r}{i-r} - \binom{n}{i-p-1}\binom{r}{p-r+1} - \binom{n}{i-p}\binom{r}{p-r}\}$, *where* $p = \operatorname{pd}(C_n) = \lfloor \frac{2n+1}{3} \rfloor$ *and*

$$k = \beta_{p,p+r}(C_n) = \begin{cases} 1 & \text{if } n \equiv_3 1, 2, \\ 2 & \text{if } n \equiv_3 0. \end{cases}$$

Recall that in the case where $n = 3, 4$ one has $\operatorname{reg}(C_{2n}(O(n) \cup \{2\})) = 1$, i.e., $C_{2n}(O(n) \cup \{2\})$ has linear resolution, and its Betti numbers may be calculated directly because $C_6(1, 2, 3) = K_6$, the complete graph over 6 vertices, and the independence complex of $C_8(1, 2, 3)$ consists of four disjoint 1-faces, or use a computer algebra system such as CoCoA [1] or Macaulay2 [7].

2. Basic setup

In this section we include the standard terminology and the basic facts which we will use throughout the paper.

A simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V, with the property that if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set). An element F of Δ is called a *face* of Δ, and maximal faces of Δ (with respect to inclusion) are called *facets* of Δ. The *dimension* of a face $F \in \Delta$ is defined by $\dim F = |F| - 1$, and the dimension of Δ is defined by $\dim \Delta = \max\{\dim F \colon F \in \Delta\}$. A simplicial complex is called *pure* if all its facets have the same cardinality. Δ is called *Cohen-Macaulay* (or *Buchsbaum, $S_2, \ldots$*) over a field k if its Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay (Buchsbaum, $S_2, \ldots$), and is called Cohen-Macaulay (Buchsbaum, $S_2, \ldots$) if it has the same property over any field k. For a face F of Δ, the *link* of F is the simplicial complex

$$\operatorname{link}_\Delta(F) = \{G \in \Delta \colon G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}.$$

By Reisner's criterion (see e.g. [5], Theorem 5.3.9), a simplicial complex Δ is Cohen-Macaulay over a field k, if and only if $\widetilde{H}_i(\operatorname{link}_\Delta(F); k) = 0$ for all $F \in \Delta$ and $i < \dim \operatorname{link}_\Delta(F)$.

Let Δ be a simplicial complex. The *pure i-th skeleton* of Δ is the pure subcomplex $\Delta^{[i]}$ of Δ whose facets are the faces F of Δ with $\dim F = i$. We say that a simplicial complex Δ is sequentially Cohen-Macaulay if $\Delta^{[i]}$ is Cohen-Macaulay for all i.

The *independence complex* of a graph G, denoted by Δ_G (or $\operatorname{Ind}(G)$), is the simplicial complex whose faces correspond to independent (or stable) sets of G, where a subset F of $V(G)$ is called an independent set if no subset of F with cardinality two belongs to $E(G)$. Since the complement of a vertex cover is an independent set, it follows that a graph G is well-covered if and only if Δ_G is a pure simplicial complex. A graph G is called Cohen-Macaulay (Buchsbaum, $S_2,\ldots$) if the independence complex Δ_G is Cohen-Macaulay (Buchsbaum, $S_2,\ldots$).

Let $G=(V,E)$ and $G'=(V',E')$ be two graphs. We say that G and G' are isomorphic and write $G\simeq G'$ if there exists a bijection $\varphi\colon V\to V'$ with $\{x,y\}\in E$ if and only if $\{\varphi(x),\varphi(y)\}\in E'$ for all $x,y\in V$.

Let M be an arbitrary graded R-module, and let

$$0\to\bigoplus_j R(-j)^{\beta_{t,j}(M)}\to\bigoplus_j R(-j)^{\beta_{t-1,j}(M)}\to\ldots\to\bigoplus_j R(-j)^{\beta_{0,j}(M)}\to M\to 0$$

be a minimal graded free resolution of M over R, where $R(-j)$ is a graded free R-module whose nth graded component is given by R_{n-j}. The number $\beta_{i,j}(M)$ is called the *ijth graded Betti number* of M and equals the number of generators of degree j in the ith syzygy module. The *(Castelnuovo-Mumford) regularity* of M, denoted by $\operatorname{reg}(M)$, is defined by

$$\operatorname{reg}(M)=\max\{j-i\colon\ \beta_{i,j}(M)\neq 0\}.$$

Recall that the *projective dimension* of an R-module M, denoted by $\operatorname{pd}(M)$, is the length of the minimal free resolution of M, i.e.,

$$\operatorname{pd}(M)=\max\{i\colon\ \beta_{i,j}(M)\neq 0 \text{ for some } j\}.$$

3. Product of cycles

In this section we will present some results concerning basic algebraic properties of the product of cycles. We begin with the next lemma which explains whether the product of two cycle graphs is circulant and what is its generating set.

Lemma 3.1. *Let $n,m\geqslant 3$ be integers. Then C_n*C_m is a circulant graph if and only if $n=m$. In addition, if this is the case, then $C_n*C_n\simeq C_{2n}(O(n)\cup\{2\})$.*

Proof. First notice that if $n\neq m$, then C_n*C_m is not a regular graph (i.e., all vertices do not have the same degree) and so it is not circulant. Now we show that

$C_n * C_n \simeq C_{2n}(O(n) \cup \{2\})$. We label the vertex sets of the cycles as $A = \{x_1, \ldots, x_n\}$ and $B = \{y_1, \ldots, y_n\}$, and the vertex set of the circulant graph $C_{2n}(O(n) \cup \{2\})$ as $V = \{0, 1, \ldots, 2n-1\}$. Define $\varphi\colon V \to V' = A \cup B$ by

$$\varphi(i) = \begin{cases} x_k & \text{if } i = 2k, \\ y_k & \text{if } i = 2k+1. \end{cases}$$

It is straightforward to check that $\{i, j\} \in E(C_{2n}(O(n) \cup \{2\}))$ if and only if $\{\varphi(i), \varphi(j)\} \in E(C_n * C_n)$. $\square$

A simplicial complex Δ satisfies *Serre's condition* S_2 if $\operatorname{link}_\Delta(F)$ is connected for every face $F \in \Delta$ with $\dim \operatorname{link}_\Delta(F) \geqslant 1$ (see [19]). Also we say that Δ is *Buchsbaum* over k if $\operatorname{link}_\Delta(x)$ is Cohen-Macaulay over k for all $x \in V(\Delta)$. Since $\operatorname{link}_{\Delta_G}(x) = \Delta_{G \setminus N_G[x]}$, a graph G is Buchsbaum if $G \setminus N_G[x]$ is Cohen-Macaulay for all $x \in V(G)$.

In the next result we characterize when $G * H$ (in particular, $C_n * C_m$) is well-covered, Cohen-Macaulay, sequentially Cohen-Macaulay, Buchsbaum, or S_2.

Proposition 3.2. *Let G and H be two simple graphs with disjoint vertex sets. Also let $n, m \geqslant 3$ be integers. Then*

(i) *$G * H$ is well-covered if and only if both G and H are well-covered with $\alpha(G) + |V(H)| = \alpha(H) + |V(G)|$.*
(ii) *$G * G$ is well-covered if and only if G is well-covered.*
(iii) *$G * H$ is Cohen-Macaulay (or S_2) if and only if both G and H are complete graphs.*
(iv) *$G * H$ is sequentially Cohen-Macaulay if and only if G is complete and H is sequentially Cohen-Macaulay.*
(v) *$G * H$ is Buchsbaum if and only if both G and H are Buchsbaum graphs.*
(vi) *$C_n * C_m$ is well-covered if and only if $n = 4$ and $m = 5$, or $n = m \in \{3, 4, 5, 7\}$.*
(viii) *$C_n * C_m$ is Cohen-Macaulay (or S_2) if and only if $n = m = 3$.*
(viii) *$C_n * C_m$ is sequentially Cohen-Macaulay if and only if $n = 3$ and $m \in \{3, 5\}$.*
(ix) *$C_n * C_m$ is Buchsbaum if and only if $n, m \in \{3, 4, 5, 7\}$.*

Proof. (i) follows from the fact that the minimal vertex covers of $G * H$ are of the following forms:

(1) $A \cup V(H)$, where A is a minimal vertex cover of G,
(2) $B \cup V(G)$, where B is a minimal vertex cover of H.

(ii) is clear by (i).

(iii) It is easy to see that $\Delta_{G*H} = \Delta_G \cup \Delta_H$ is the disjoint union of complexes. The Cohen-Macaulay case follows from the fact that a Cohen-Macaulay complex of positive dimension is connected (see e.g. [22], Proposition 5.2.3 and Theorem 5.3.5), and the S_2 case quickly follows from the definition.

(iv) First assume $G * H$ is sequentially Cohen-Macaulay. If neither G nor H are complete, then $\Delta^{[1]}_{G*H}$ is 1-dimensional disconnected and so it is not Cohen-Macaulay, a contradiction. Hence G or H must be complete. Assume G is complete. Since $\Delta^{[i]}_{G*H} = \Delta^{[i]}_{H}$ for all $i > 0$, we conclude that H is sequentially Cohen-Macaulay. The converse is clear from $\Delta^{[i]}_{G*H} = \Delta^{[i]}_{H}$ for all $i > 0$.

(v) Follows from the fact that if $x \in V(G)$, then $G * H \setminus N_{G*H}[x] = G \setminus N_G[x]$.

(vi) Follows from (i) together with the fact that C_n is well-covered if and only if $n \in \{3, 4, 5, 7\}$.

(vii) Immediately follows from (iii).

(viii) Follows from (iv) together with the fact that C_n is sequentially Cohen-Macaulay if and only if $n \in \{3, 5\}$.

(ix) Follows from (v) together with the fact that C_n is Buchsbaum if and only if $n \in \{3, 4, 5, 7\}$. □

As an immediate consequence we get the following corollary.

Corollary 3.3. *$C_{2n}(O(n) \cup \{2\})$ is well-covered (equivalently Buchsbaum) if and only if $n \in \{3, 4, 5, 7\}$. Also it is Cohen-Macaulay (eqivalently S_2 or sequentially Cohen-Macaulay) if and only if $n = 3$.*

4. Linear strand

In this section we provide explicit formulae for computing the Betti numbers in linear strand of the circulant graph $C_{2n}(O(n) \cup \{2\})$. To do this, in view of Lemma 3.1, we make use of two ingredients: the first is the formula for Betti numbers of the product of graphs given in [15], Corollary 3.4 or [23], Lemma 5.4, and the second is the formula for Betti numbers of cycles described in [11], Sections 7.4 and 7.5. We first recall them.

Lemma 4.1 ([15], Corollary 3.4 or [23], Lemma 5.4)**.** *Let G and H be two simple graphs with disjoint vertex sets having m and n vertices, respectively. Then the $\mathbb{N}$-graded Betti numbers $\beta_{i,d}(G * H)$ may be expressed as*

$$\sum_{j=0}^{d-2} \left\{ \binom{n}{j} \beta_{i-j,d-j}(G) + \binom{m}{j} \beta_{i-j,d-j}(H) \right\} \quad \text{if } d \neq i+1,$$

$$\sum_{j=0}^{d-2} \left\{ \binom{n}{j} \beta_{i-j,d-j}(G) + \binom{m}{j} \beta_{i-j,d-j}(H) \right\} + \sum_{j=1}^{d-1} \binom{m}{j} \binom{n}{d-j} \quad \text{if } d = i+1.$$

Theorem 4.2 ([11], Theorem 7.6.28). *The nonzero $\mathbb{N}$-graded Betti numbers of C_n are all of degree less than or equal to n and for $2i + j < n$,*

$$\beta_{i+j,2i+j}(C_n) = \binom{i}{j}\beta_{i,2i}(C_n) = \frac{n}{n-2i}\binom{i}{j}\binom{n-2i}{i},$$

and

$$\begin{aligned} \beta_{2m+1,n}(C_n) &= 1 \quad \text{if } n = 3m+1, \\ \beta_{2m+1,n}(C_n) &= 1 \quad \text{if } n = 3m+2, \\ \beta_{2m,n}(C_n) &= 2 \quad \text{if } n = 3m. \end{aligned}$$

We also need the following assertions in the proof of our main results. Recall that by $\mathrm{pd}(G)$ and $\mathrm{reg}(G)$ we refer to the projective dimension and (Castelnuovo-Mumford) regularity of the quotient ring $R/I(G)$ respectively, i.e., $\mathrm{pd}(G) = \mathrm{pd}(R/I(G))$ and $\mathrm{reg}(G) = \mathrm{reg}(R/I(G))$.

Proposition 4.3 ([15], Proposition 3.12). *Let G and H be two simple graphs with disjoint vertex sets. Then*

$$\mathrm{reg}(G * H) = \max\{\mathrm{reg}(G), \mathrm{reg}(H)\}.$$

Corollary 4.4. *The following statements hold:*

(i) *For $t + s < n$ and $t \geqslant s$,*

$$\beta_{t,t+s}(C_n) = \frac{n}{n-2s}\binom{s}{t-s}\binom{n-2s}{s}.$$

(ii) $\mathrm{pd}(C_n) = \lfloor \frac{2n+1}{3} \rfloor$.

(iii) $\beta_{t,n}(C_n) \neq 0$ *if and only if* $t = \mathrm{pd}(C_n)$.

(iv) *For $t + s = n$,*

$$\beta_{t,t+s}(C_n) = \begin{cases} 1 & \text{if } n \equiv_3 1, 2, \\ 2 & \text{if } n \equiv_3 0, \end{cases}$$

where $t = \mathrm{pd}(C_n)$.

(v) $\mathrm{reg}(C_n) = \lfloor \frac{n+1}{3} \rfloor$.

Proof. (i) Immediately follows from Theorem 4.2, and (ii) is the content of [11], Corollary 7.6.30.

(iii) First assume $\beta_{t,n}(C_n) \neq 0$. If $n = 3m$, then by Theorem 4.2 one has $t = 2m = \mathrm{pd}(C_n)$. The other cases are similar. Conversely, assume $t = \mathrm{pd}(C_n)$. If $n = 3m+2$, then $t = 2m+1$. Again by Theorem 4.2, $\beta_{t,n}(C_n) = 1$. The other cases are similar.

(iv) Follows from (ii), (iii), and Theorem 4.2.

(v) We just give the proof in the case when $n = 3m$ (a complete proof may be found in [14], Theorem 3.2). First observe that any $\mathbb{N}$-graded Betti number can be written in the form $\beta_{l,d} = \beta_{i+j,2i+j}$ where $i = d - l$ and $j = 2l - d$. Now we have

$$\begin{aligned}
\operatorname{reg}(C_n) &= \max\{d - l\colon \beta_{l,d}(C_n) \neq 0\} \\
&= \max\{2i + j - (i + j)\colon \beta_{i+j,2i+j}(C_n) \neq 0\} \\
&= \max\{\{i\colon \beta_{i+j,2i+j}(C_n) \neq 0,\ 2i + j < n\},\ n - \operatorname{pd}(C_n)\} \\
&= \max\{\{i\colon \beta_{i+j,2i+j}(C_n) \neq 0,\ 2i + j < n\},\ m\}.
\end{aligned}$$

Now we show that for all i with $\beta_{i+j,2i+j}(C_n) \neq 0$ and $2i + j < n$ we have $i \leqslant m$. Suppose on the contrary that $i > m$. It follows from Theorem 4.2 that $\beta_{i,2i}(C_n) \neq 0$. So by [11], Theorem 3.3.5, C_n must have an induced subgraph which consists of i disjoint edges. This implies that C_n contains at least $3m + 2$ vertices, which is impossible. Therefore $i \leqslant m$ and so $\operatorname{reg}(C_n) = m = \lfloor \frac{n+1}{3} \rfloor$. $\square$

Remark 4.5. It follows from Corollary 4.4 that $\operatorname{pd}(C_n) + \operatorname{reg}(C_n) = n$. Hence if $\beta_{t,t+s}(C_n) \neq 0$ and $t + s = n$, then $s = \operatorname{reg}(C_n)$.

Remark 4.6. Throughout this section we assume $\operatorname{reg}(C_n) > 1$ and so we may assume $n \geqslant 5$. Recall that in the case when $\operatorname{reg}(C_n) = 1$ one has $n = 3, 4$. So by Proposition 4.3 we have $\operatorname{reg}(C_{2n}(O(n) \cup \{2\})) = 1$, i.e., $C_{2n}(O(n) \cup \{2\})$ has linear resolution, and its Betti numbers may be calculated directly because $C_6(1, 2, 3) = K_6$, the complete graph over 6 vertices, and the independence complex of $C_8(1, 2, 3)$ consists of four disjoint 1-faces, and so their Betti numbers are known, or we use a computer algebra system such as CoCoA [1] or Macaulay2 [7].

We finally need the following lemma. The reason for assuming $n \geqslant 5$ is that there are two exceptions in this lemma which are $\beta_{2,3}(C_3) = 2$ and $\beta_{3,4}(C_4) = 1$.

Lemma 4.7. *Let $n \geqslant 5$ be an integer. Then*

$$\beta_{i,i+1}(C_n) = \begin{cases} n & \text{if } i = 1, 2, \\ 0 & \text{if } i > 2. \end{cases}$$

Proof. If $i = 1, 2$, then the result follows from Corollary 4.4 (i). Now let $i > 2$. If $i + 1 < n$, then $i - 1 > 1$ and hence by Corollary 4.4 (i) one has

$$\beta_{i,i+1}(C_n) = \frac{n}{n-2} \binom{1}{i-1} \binom{n-2}{1} = 0.$$

Also in the case when $i + 1 \geqslant n$ we have $i \geqslant n - 1 > \operatorname{pd}(C_n)$. So $\beta_{i,i+1}(C_n) = 0$. $\square$

Now we are ready to compute Betti numbers in linear strand of $C_{2n}(O(n) \cup \{2\})$.

Proposition 4.8. *Let $n \geqslant 5$ be an integer. Then*

$$\beta_{i,i+1}(C_{2n}(O(n)\cup\{2\}) = 2n\binom{n+1}{i-1} + \binom{2n}{i+1} - 2\binom{n}{i+1}.$$

Proof. By Lemma 3.1 one has $C_{2n}(O(n)\cup\{2\}) \simeq C_n * C_n$. Now Lemmas 4.1 and 4.7 yield

$$\begin{aligned}\beta_{i,i+1}(C_n * C_n) &= 2\sum_{j=0}^{i-1}\binom{n}{j}\beta_{i-j,i+1-j}(C_n) + \sum_{j=1}^{i}\binom{n}{j}\binom{n}{i+1-j}\\ &= 2\sum_{j=i-2}^{i-1}\binom{n}{j}\beta_{i-j,i+1-j}(C_n) + \sum_{j=1}^{i}\binom{n}{j}\binom{n}{i+1-j}\\ &= 2\left\{\binom{n}{i-1}\beta_{1,2}(C_n) + \binom{n}{i-2}\beta_{2,3}(C_n)\right\} + \sum_{j=1}^{i}\binom{n}{j}\binom{n}{i+1-j}\\ &= 2n\left\{\binom{n}{i-1} + \binom{n}{i-2}\right\} + \sum_{j=0}^{i+1}\binom{n}{j}\binom{n}{i+1-j} - 2\binom{n}{i+1}.\end{aligned}$$

To complete the proof, it is enough to apply the identities $\binom{n}{t} + \binom{n}{t-1} = \binom{n+1}{t}$ and $\sum\limits_{j=0}^{t}\binom{n}{j}\binom{m}{t-j} = \binom{n+m}{t}$. $\square$

5. Nonlinear Betti numbers

In this section we compute Betti numbers $\beta_{i,i+r}(C_{2n}(O(n)\cup\{2\}))$ where $2 \leqslant r \leqslant \operatorname{reg}(C_n)$. Recall that by Proposition 4.3 we have $\operatorname{reg}(C_{2n}(O(n)\cup\{2\})) = \operatorname{reg}(C_n)$. This implies that $\beta_{i,i+r}(C_{2n}(O(n)\cup\{2\})) = 0$ for all $r > \operatorname{reg}(C_n)$. We split the argument into two cases. The first is that $2 \leqslant r < \operatorname{reg}(C_n)$, and the second is for $r = \operatorname{reg}(C_n)$.

Theorem 5.1. *Let $n \geqslant 5$ and $2 \leqslant r < \operatorname{reg}(C_n)$. Then*

$$\beta_{i,i+r}(C_{2n}(O(n)\cup\{2\})) = \frac{2n}{n-2r}\binom{n-2r}{r}\binom{n+r}{i-r}.$$

Proof. By Lemma 4.1 one has

$$\beta_{i,i+r}(C_n * C_n) = 2\sum_{j=0}^{i+r-2}\binom{n}{j}\beta_{i-j,i+r-j}(C_n).$$

Now assume $\beta_{i-j,i+r-j}(C_n) \neq 0$. This yields that $i-j \leqslant \mathrm{pd}(C_n)$. On the other hand, $r < \mathrm{reg}(C_n)$ together with Remark 4.5 imply that $i+r-j < i-j+\mathrm{reg}(C_n) = i-j+n-\mathrm{pd}(C_n) \leqslant n$. So $i+r-j < n$ and hence using Corollary 4.4.(i) we get

$$\beta_{i-j,i+r-j}(C_n) = \frac{n}{n-2r}\binom{r}{i-j-r}\binom{n-2r}{r}.$$

Now we have

$$\begin{aligned}\beta_{i,i+r}(C_n * C_n) &= 2\sum_{j=0}^{i+r-2}\binom{n}{j}\frac{n}{n-2r}\binom{r}{i-j-r}\binom{n-2r}{r}\\ &= 2\frac{n}{n-2r}\binom{n-2r}{r}\sum_{j=0}^{i+r-2}\binom{n}{j}\binom{r}{i-j-r}.\end{aligned}$$

Note that for $j > i-r$ one has $\binom{r}{i-j-r}$=0. Hence

$$\sum_{j=0}^{i+r-2}\binom{n}{j}\binom{r}{i-j-r} = \sum_{j=0}^{i-r}\binom{n}{j}\binom{r}{i-r-j} = \binom{n+r}{i-r}.$$

This completes the proof. $\square$

To calculate Betti numbers $\beta_{i,i+r}(C_n * C_n)$ in the case when $r = \mathrm{reg}(C_n)$, we need the following two lemmas.

Lemma 5.2. *Let $p = \mathrm{pd}(C_n)$ and $r = \mathrm{reg}(C_n)$. Then*

$$\begin{aligned}\sum_{j=0}^{i-p}\binom{n}{j}\binom{r}{i-r-j} &= \begin{cases} 0 & \text{if } n \equiv_3 1,\\ \binom{n}{i-p} & \text{if } n \equiv_3 0,\\ r\binom{n}{i-p} + \binom{n}{i-p-1} & \text{if } n \equiv_3 2\end{cases}\\ &= \binom{n}{i-p-1}\binom{r}{p-r+1} + \binom{n}{i-p}\binom{r}{p-r}\end{aligned}$$

for all $i \geqslant p$.

Proof. First assume $n \equiv_3 1$ and suppose that $n = 3\alpha+1$ for some $\alpha \in \mathbb{N}$. It follows that $p = 2\alpha+1$ and $r = \alpha$. Assume $i \geqslant 2\alpha+1$ is fixed. Then for all $0 \leqslant j \leqslant i-p = i-(2\alpha+1)$ one has $i-j > 2\alpha$. This implies that $i-\alpha-j > \alpha$ and hence $\binom{r}{i-r-j} = \binom{\alpha}{i-\alpha-j} = 0$. Therefore $\sum\limits_{j=0}^{i-p}\binom{n}{j}\binom{r}{i-r-j} = 0$.

Now assume $n \equiv_3 0$ and suppose $n = 3\alpha$ for some $\alpha \in \mathbb{N}$. It follows that $p = 2\alpha$ and $r = \alpha$. Assume $i \geqslant 2\alpha$ is fixed. Then for all $0 \leqslant j < i-p = i-2\alpha$ one has

$i-\alpha-j>\alpha$ and hence $\binom{r}{i-r-j}=\binom{\alpha}{i-\alpha-j}=0$. Also for $j=i-p=i-2\alpha$ we have $i-r-j=\alpha$ and so $\binom{r}{i-r-j}=1$. Therefore $\sum\limits_{j=0}^{i-p}\binom{n}{j}\binom{r}{i-r-j}=\binom{n}{i-p}$.

Finally assume $n\equiv_3 2$ and suppose $n=3\alpha+2$ for some $\alpha\in\mathbb{N}$. It follows that $p=2\alpha+1$ and $r=\alpha+1$. Assume $i\geqslant 2\alpha+1$ is fixed. Then for all $0\leqslant j\leqslant i-p-2=i-(2\alpha+1)-2$ one has $i-j>2(\alpha+1)$ and so $i-r-j=i-(\alpha+1)-j>r=\alpha+1$. Hence $\binom{r}{i-r-j}=0$. On the other hand, for $j=i-p-1$ we have $i-r-j=\alpha+1$ and so $\binom{r}{i-r-j}=1$, and for $j=i-p$ we have $i-r-j=\alpha$ and so $\binom{r}{i-r-j}=\binom{\alpha+1}{\alpha}=\alpha+1=r$. Therefore $\sum\limits_{j=0}^{i-p}\binom{n}{j}\binom{r}{i-r-j}=r\binom{n}{i-p}+\binom{n}{i-p-1}$.

The second equality immediately follows from the above arguments in all cases. $\square$

Lemma 5.3. *Let $p=\operatorname{pd}(C_n)$ and $r=\operatorname{reg}(C_n)$. Then*

$$\sum_{j=i-p+1}^{i-r}\binom{n}{j}\binom{r}{i-r-j}=\begin{cases}\dbinom{n+r}{i-r} & \text{if } n\equiv_3 1,\\[2mm] \dbinom{n+r}{i-r}-\dbinom{n}{i-p} & \text{if } n\equiv_3 0,\\[2mm] \dbinom{n+r}{i-r}-\left\{r\dbinom{n}{i-p}+\dbinom{n}{i-p-1}\right\} & \text{if } n\equiv_3 2\end{cases}$$
$$=\binom{n+r}{i-r}-\left\{\binom{n}{i-p-1}\binom{r}{p-r+1}+\binom{n}{i-p}\binom{r}{p-r}\right\}$$

for all $i\geqslant p$.

P r o o f. It is enough to note that

$$\sum_{j=i-p+1}^{i-r}\binom{n}{j}\binom{r}{i-r-j}=\sum_{j=0}^{i-r}\binom{n}{j}\binom{r}{i-r-j}-\sum_{j=0}^{i-p}\binom{n}{j}\binom{r}{i-r-j}$$

and that $\sum\limits_{j=0}^{i-r}\binom{n}{j}\binom{r}{i-r-j}=\binom{n+r}{i-r}$, and apply Lemma 5.2. $\square$

Theorem 5.4. *Let $n\geqslant 5$ be an integer and $r=\operatorname{reg}(C_n)\geqslant 2$. Then the $\mathbb{N}$-graded Betti number $\beta_{i,i+r}(C_{2n}(O(n)\cup\{2\}))$ may be expressed as*

$$2k\binom{n}{i-p}+\frac{2n}{n-2r}\binom{n-2r}{r}$$
$$\times\left\{\binom{n+r}{i-r}-\binom{n}{i-p-1}\binom{r}{p-r+1}-\binom{n}{i-p}\binom{r}{p-r}\right\},$$

where $p = \mathrm{pd}(C_n)$ *and*

$$k = \beta_{p,p+r}(C_n) = \begin{cases} 1 & \text{if } n \equiv_3 1, 2, \\ 2 & \text{if } n \equiv_3 0. \end{cases}$$

Proof. If $i < p$, then $i + r < p + r = n$ and so $i + r - j < n$. In this case, an argument similar to that in the proof of Theorem 5.1 shows that

$$\beta_{i,i+r}(C_n * C_n) = \frac{2n}{n-2r}\binom{n-2r}{r}\binom{n+r}{i-r}.$$

Since $\binom{n}{i-p-1} = \binom{n}{i-p} = 0$, we have done this case.

Now assume $i \geqslant p = \mathrm{pd}(C_n)$. We write

$$\begin{aligned} \beta_{i,i+r}(C_n * C_n) &= 2\sum_{j=0}^{i+r-2}\binom{n}{j}\beta_{i-j,i+r-j}(C_n) \\ &= 2\sum_{j=0}^{i-p-1}\binom{n}{j}\beta_{i-j,i+r-j}(C_n) + 2\binom{n}{i-p}\beta_{p,p+r}(C_n) \\ &\quad + 2\sum_{j=i-p+1}^{i+r-2}\binom{n}{j}\beta_{i-j,i+r-j}(C_n). \end{aligned}$$

Note that if $j \leqslant i - p - 1$, then $i - j > p$ and so $\beta_{i-j,i+r-j}(C_n) = 0$. On the other hand, if $j = i - p + t$ where $t \geqslant 1$, then $i + r - j = p + r - t = n - t < n$ and so

$$\beta_{i-j,i+r-j}(C_n) = \frac{n}{n-2r}\binom{n-2r}{r}\binom{r}{i-j-r}.$$

Thus we get

$$\beta_{i,i+r}(C_n * C_n) = 2\left\{k\binom{n}{i-p} + \sum_{j=i-p+1}^{i+r-2}\binom{n}{j}\frac{n}{n-2r}\binom{n-2r}{r}\binom{r}{i-j-r}\right\}.$$

Note that since $i \geqslant p$ and $r \geqslant 2$ we have $i - p + 1 \leqslant i - r < i + r - 2$. On the other hand, $\binom{r}{i-j-r} = 0$ for all $j > i - r$. This together with Lemma 5.3 yields that

$$\begin{aligned} \beta_{i,i+r}(C_n * C_n) &= 2k\binom{n}{i-p} + 2\frac{n}{n-2r}\binom{n-2r}{r}\sum_{j=i-p+1}^{i-r}\binom{n}{j}\binom{r}{i-j-r} \\ &= 2k\binom{n}{i-p} + \frac{2n}{n-2r}\binom{n-2r}{r} \\ &\quad \times \left\{\binom{n+r}{i-r} - \binom{n}{i-p-1}\binom{r}{p-r+1} - \binom{n}{i-p}\binom{r}{p-r}\right\}. \end{aligned}$$

□

Using computer algebra systems CoCoA [1] and Macaulay2 [7], a standard PC was unable to compute the minimal graded free resolution of $C_{2n}(O(n)\cup\{2\})$ for $n\geqslant 8$, but the above formulae will do this quickly as presented in the following example for $n=8$.

Example 5.5. Let $n=8$. Then $k=1$, $r=3$, and $p=5$. Applying the above formulae will result in the following Betti Table of $C_{16}(1,2,3,5,7)$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	80	1236	2256	5600	9968	13440	14212	12016	8152	4384	1820	1120	120	16	1
2:	.	.	24	240	1080	2880	5040	6048	5040	2880	1080	240	24	.	.	.
3:	.	.	.	.	.	2	16	56	112	140	112	56	16	2	.	.
Tot:	1	80	1260	2496	6680	12850	18496	20316	17168	11172	5576	2116	1160	122	16	1

Table. Betti Table of $C_{16}(1,2,3,5,7)$.

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