# Ramalingam Udhayakumar; Intan Muchtadi-Alamsyah; Chelliah Selvaraj Some results on $G_C$ -flat dimension of modules

Commentationes Mathematicae Universitatis Carolinae, Vol. 60 (2019), No. 2, 187–198

Persistent URL: http://dml.cz/dmlcz/147820

# Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2019

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

# Some results on $G_C$ -flat dimension of modules

Ramalingam Udhayakumar, Intan Muchtadi-Alamsyah, Chelliah Selvaraj

Abstract. In this paper, we study some properties of  $G_C$ -flat R-modules, where C is a semidualizing module over a commutative ring R and we investigate the relation between the  $G_C$ -yoke with the C-yoke of a module as well as the relation between the  $G_C$ -flat resolution and the flat resolution of a module over GF-closed rings. We also obtain a criterion for computing the  $G_C$ -flat dimension of modules.

 $\mathit{Keywords:}\ GF\-closed$ ring;  $G_C\-flat$ module;  $G_C\-flat$  dimension; semidualizing module

Classification: 18G20, 18G25

## 1. Introduction

In basic homological algebra, projective, injective and flat modules play an important and fundamental role. Homological properties of the Gorenstein projective, injective and flat modules have been studied by many authors, some references are [2], [3], [5], [8], [15]. The study of semidualizing modules over commutative Noetherian rings was initiated independently (with different names) by H.-B. Foxby in [6], E.S. Golod in [7], and W.V. Vasconcelos in [16]. Over a commutative Noetherian ring, E.S. Holm and P. Jørgensen in [9] introduced the C-Gorenstein projective, C-Gorenstein injective and C-Gorenstein flat modules using semidualizing modules and their associated projective, injective and flat classes which are also called  $G_C$ -projective,  $G_C$ -injective and  $G_C$ -flat module, respectively. D. White introduced in [17] the  $G_C$ -projective modules and gave a functorial description of the  $G_C$ -projective dimension of modules with respect to a semidualizing module C over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [17]. Being motivated from [17], in this paper, we give equivalent conditions for  $G_C$ -flat dimension of modules with respect to a semidualizing module C.

This paper is organized as follows. In Section 2, we recall some notions and definitions which will be needed in the later sections. In Section 3, we establish the relation between the  $G_C$ -yoke with the C-yoke of a module as well as the relation between the  $G_C$ -flat resolution and the flat resolution of a module over a GF-closed ring.

DOI 10.14712/1213-7243.2019.007

In Section 4, we get some properties of  $G_C$ -flat dimension of modules. In particular, as an application of the results obtained in Section 3, we get a criterion for computing such a dimension. Let R be a GF-closed ring and let M be an R-module and  $n \ge 0$ . We prove that the  $G_C$ -flat dimension of M is at most n if and only if for every nonnegative integer t such that  $0 \le t \le n$ , there exists an exact sequence of R-modules  $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$  such that  $X_t$ is  $G_C$ -flat and  $X_i$  are flat for  $i \ne t$ .

## 2. Preliminaries

Throughout this paper, R is a commutative ring with identity and all modules are unitary modules. Let M be an R-module. We denote  $\operatorname{Add}_R M$  (or  $\operatorname{Prod}_R M$ ) the subclass of R-modules consisting of all modules isomorphic to direct summands of direct sums (direct products, respectively) of copies of M. At the beginning of this section, we recall some notions from [10], [17].

**Definition 2.1** ([17]). A degreewise finite projective (or free) resolution of an R-module M is a *projective (or free) resolution* P of M such that each  $P_i$  is finitely generated projective (free, respectively).

**Remark 2.2.** Note that M admits a degreewise finite projective resolution if and only if it admits a degreewise finite free resolution. However, it is possible for a module to admit a bounded degreewise finite projective resolution but not to admit a bounded degreewise finite free resolution. For example, if  $R = k_1 \oplus k_2$ , where  $k_1$  and  $k_2$  are fields, then  $M = k_1 \oplus 0$  is a projective *R*-module, but it does not admit a bounded free resolution.

**Definition 2.3** ([17]). An *R*-module *C* is *semidualizing* if it satisfies the following conditions:

- (1) C admits a degreewise finite projective resolution;
- (2) the natural homothety morphism  $R \to \operatorname{Hom}_R(C, C)$  is an isomorphism; and
- (3)  $\operatorname{Ext}_{R}^{i}(C, C) = 0$  for any  $i \ge 1$ .

**Remark 2.4.** A free R-module of rank one is semidualizing. If R is Noetherian and admits a dualizing module D, then D is a semidualizing.

**Definition 2.5** ([10]). Let C be a semidualizing module for a ring R. An R-module is C-projective if it has the form  $C \otimes_R P$  for some projective module P. An R-module is called C-injective if it has the form  $\operatorname{Hom}_R(C, I)$  for some injective module I. Set

 $\mathcal{P}_C(R) = \{ C \otimes_R P \colon P \text{ is } R \text{-projective} \},\$ 

and

$$\mathcal{I}_C(R) = \{ \operatorname{Hom}_R(C, I) \colon I \text{ is } R \text{-injective} \}.$$

**Definition 2.6** ([10]). An *R*-module is called *C*-flat if it has the form  $C \otimes_R F$  for some flat module *F*. Set  $\mathcal{F}_C(R) = \{C \otimes_R F : F \text{ is } R\text{-flat}\}.$ 

**Definition 2.7.** Let R be a ring and let  $\mathfrak{X}$  be a class of R-modules.

- A class X is closed under extensions if for every short exact sequence of *R*-modules 0 → A → B → C → 0, the conditions A and C are in X imply B is in X.
- (2) A class X is closed under kernels of epimorphisms if for every short exact sequence of *R*-modules 0 → A → B → C → 0, the conditions B and C are in X imply A is in X.
- (3) A class X is projectively resolving if it contains all projective modules and it is closed under both extensions and kernels of epimorphisms, i.e., for every short exact sequence of *R*-modules 0 → A → B → C → 0 with C ∈ X, the conditions A ∈ X and B ∈ X are equivalent.

**Definition 2.8** ([5]). An *R*-module M is said to be *Gorenstein flat*, if there exists an exact sequence of flat *R*-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that  $M \cong \text{Im}(F_0 \to F^0)$  and such that  $B \otimes_R -$  leaves the sequence exact whenever B is an injective R-module.

**Definition 2.9** ([1]). Let R be a ring. We call R *GF-closed* if the class of Gorenstein flat R-modules is closed under extensions.

### **3.** $G_C$ -flat modules

We start with the following definition.

**Definition 3.1** ([9]). A complete  $\mathcal{FF}_C$ -resolution is a  $\mathcal{I}_C(R) \otimes_R$ -exact sequence:

(1) 
$$\mathcal{X}: \dots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \dots$$

in *R*-Mod with all  $F_i$  and  $F^i$  flat. An *R*-module *M* is called  $G_C$ -flat if there exists a complete  $\mathcal{FF}_C$ -resolution as in (1) with  $M = \operatorname{Coker}(F_1 \to F_0)$ . Set  $\mathcal{GF}_C(R)$  to be the class of  $G_C$ -flat *R*-modules.

It is trivial that in case C = R, the  $G_C$ -flat modules are just the usual Gorenstein flat modules.

Using the definition, we immediately get the following results.

**Proposition 3.2.** If  $(F_i)_{i \in I}$  is a family of  $G_C$ -flat *R*-modules, then  $\bigoplus F_i$  is  $G_C$ -flat.

**Proposition 3.3.** An *R*-module M is  $G_C$ -flat if and only if

 $\operatorname{Tor}_{>1}^{R}(\operatorname{Hom}_{R}(C, I), M) = 0$ 

and M admits a  $\mathcal{F}_C$ -resolution Y with  $\operatorname{Hom}_R(C, I) \otimes_R Y$  exact for any injective I.

**Proposition 3.4.** Let R be a commutative Noetherian ring and F a flat R-module. If M is an  $G_C$ -flat R-module, then  $M \otimes_R F$  is a  $G_C$ -flat R-module.

**PROOF:** There is an exact sequence

$$\cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

with  $F_i$  and  $F^i$  flat and  $M = \operatorname{Coker}(F_1 \to F_0)$ . Then the sequence

$$\cdots \to F_1 \otimes F \to F_0 \otimes F \to C \otimes_R F^0 \otimes F \to C \otimes_R F^1 \otimes F \to \cdots$$

is exact with  $F_i \otimes F$ ,  $F^i \otimes F$  flat by [12, Proposition 2.11]. Let I be any injective R-module and  $\mathcal{F} = \text{Hom}(C, I)$ . Then

$$\operatorname{Tor}_{1}^{R}(M \otimes_{R} F, \operatorname{Hom}(C, I)) = H_{i}((M \otimes_{R} F) \otimes \mathcal{F})$$
$$\cong H_{i}(M \otimes_{R} (F \otimes \mathcal{F}))$$
$$\cong \operatorname{Tor}_{1}^{R}(M, F \otimes_{R} \operatorname{Hom}(C, I)) = 0$$

by [13, page 258, 9.20] for all  $i \ge 1$ , since  $F \otimes_R \text{Hom}(C, I) \cong \text{Hom}(C, F \otimes_R I)$  is a *C*-injective module by [4, Theorem 3.2.16] and [10, (1.10)]. Hence  $M \otimes_R F$  is a  $G_C$ -flat *R*-module.

The following result is due to [14, Proposition 5.3].

**Proposition 3.5.** Let C be a semidualizing R-module. Then the class  $\mathcal{GF}_C(R)$  is closed under kernels of epimorphisms and extensions.

**Proposition 3.6.** Let C be a semidualizing R-module. If F is flat R-module, then F and  $C \otimes_R F$  are  $G_C$ -flat. Thus, every R-module admits a  $G_C$ -flat resolution.

PROOF: Follows from [9, Example 2.8 (a) and (c)] and since the class of  $G_C$ -flat modules contains the class of flat modules, every *R*-module admits a  $G_C$ -flat resolution.

**Theorem 3.7.** Let C be a semidualizing module, then the class  $\mathcal{GF}_C(R)$  of  $G_C$ -flat R-modules is projectively resolving and closed under direct summands.

PROOF: Using the dual of Theorem 2.8 in [17] and together with the [14, Lemma 5.2], we see that  $\mathcal{GF}_C(R)$  is projectively resolving. Now, comparing Proposition 3.5 with Proposition 1.4 in [8], we get  $\mathcal{GF}_C(R)$  is closed under direct summands.

**Proposition 3.8.** Let R be a GF-closed ring. Then every cokernel in a complete  $\mathcal{FF}_C$ -resolution is  $G_C$ -flat.

**PROOF:** Follows from Proposition 3.3, Theorem 3.7 and [14, Lemma 5.4].

**Lemma 3.9.** Let R be a GF-closed ring and let M be  $G_C$ -flat R-module. Then there exists  $\mathcal{I}_C(R)\otimes$ -exact sequences of R-modules:

and

$$0 \to M \to G \to N \to 0$$

$$0 \to K \to F \to M \to 0$$

with  $N, K G_C$ -flat, G, F flat.

**PROOF:** By the definition of  $G_C$ -flat modules and Proposition 3.8.

The following result plays a crucial role in this section and it follows from [11, Proposition 2.2].

**Lemma 3.10.** Let R be a GF-closed ring and suppose that

(2) 
$$0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0$$

is an exact sequence of R-modules with  $G_0, G_1, G_C$ -flat. Then we have the following exact sequences:

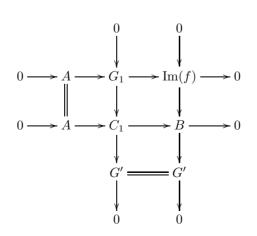
(3) 
$$0 \to A \to C_1 \to G \to M \to 0,$$

and

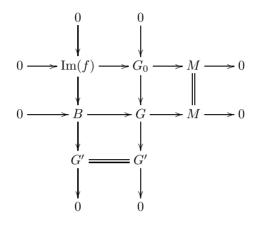
$$(4) 0 \to A \to H \to F \to M \to 0$$

with  $C_1$ , F flat, and G, H  $G_C$ -flat.

PROOF: Since  $G_1$  is  $G_C$ -flat, there exists a short exact sequence  $0 \to G_1 \to C_1 \to G' \to 0$  with  $C_1$  is flat and  $G' \to C_C$ -flat by Lemma 3.9. Then we have the following pushout diagram:

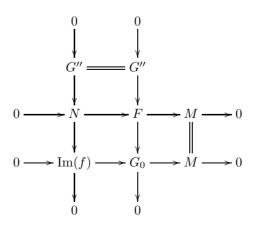


Consider the following pushout diagram:

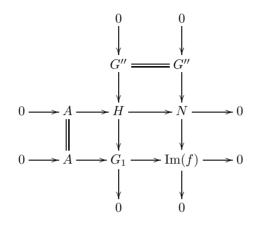


Since  $G_0$  and G' are  $G_C$ -flat, G is also  $G_C$ -flat by Theorem 3.7. Connecting the middle rows in the above two diagrams, we get the first desired exact sequence (3).

Since  $G_0$  is  $G_C$ -flat, there exists an exact sequence  $0 \to G'' \to F \to G_0 \to 0$ with F flat and  $G'' G_C$ -flat by Lemma 3.9. Then we have the following pullback diagram:



Consider the following pullback diagram:



Since  $G_1$  and G'' are  $G_C$ -flat, H is also  $G_C$ -flat by Theorem 3.7. Connecting the middle rows in the above two diagrams, we get the second desired exact sequence (4).

**Definition 3.11.** Let n be a positive integer. An R-module A is called an C-yoke module (of M) if there exists an exact sequence of R-modules

$$0 \to A \to F_{n-1} \to \dots \to F_1 \to F_0 \to M \to 0$$

with all  $F_i$  C-flat.

**Definition 3.12.** Let *n* be a positive integer, a module *A* is called an  $G_C$ -yoke module (of *M*) if there exists an exact sequence of *R*-modules

$$0 \to A \to G_{n-1} \to \dots \to G_1 \to G_0 \to M \to 0$$

with all  $G_i$   $G_C$ -flat.

The following result establishes the relation between the  $G_C$ -yoke with the C-yoke of a module as well as the relation between the  $G_C$ -flat resolution and the flat resolution of a module.

**Lemma 3.13.** Let R be a GF-closed ring and let  $n \ge 1$  and

(5) 
$$0 \to A \to G_{n-1} \to \dots \to G_1 \to G_0 \to M \to 0$$

be an exact sequence of R-modules with all  $G_i$   $G_C$ -flat. Then we have the following:

(i) There exists exact sequences of *R*-modules:

(6)  $0 \to A \to C_{n-1} \to \dots \to C_1 \to C_0 \to N \to 0$ 

and

 $0 \to M \to N \to G \to 0$ 

with all  $C_i$  flat and  $G G_C$ -flat.

(ii) There exist exact sequences of *R*-modules

(7) 
$$0 \to B \to F_{n-1} \to \dots \to F_1 \to F_0 \to M \to$$

and

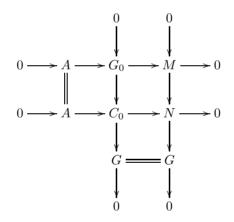
 $0 \to H \to B \to A \to 0$ 

0

with all  $F_i$  flat and H  $G_C$ -flat.

**PROOF:** We proceed by induction on n.

(i) When n = 1, we have an exact sequence of R-modules  $0 \to A \to G_0 \to M \to 0$ . Since we have a  $\mathcal{I}_C(R) \otimes_R$ -exact sequence of R-modules  $0 \to G_0 \to C_0 \to G \to 0$  with  $C_0$  is flat and  $G \ G_C$ -flat by Lemma 3.9, we have the following pushout diagram:



The middle row and the last column in the above diagram are the desired two exact sequences.

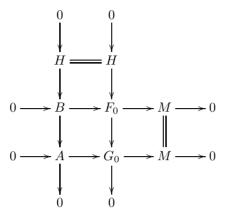
Now assume that  $n \geq 2$  and we have an exact sequence of R-modules  $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$  with all  $G_i \ G_C$ -flat. Put  $K = \operatorname{Coker}(G_{n-1} \to G_{n-2})$ . By Lemma 3.10, we get an exact sequence of R-modules

(8) 
$$0 \to A \to C_{n-1} \to G'_{n-2} \to K \to 0$$

with  $C_{n-1}$  flat and  $G'_{n-2}$   $G_C$ -flat. Put  $A' = \text{Im}(C_{n-1} \to G'_{n-2})$ . Then, we get an exact sequence of *R*-modules  $0 \to A' \to G'_{n-2} \to G_{n-3} \to \cdots \to G_1 \to G_0 \to M \to 0$ . So, by the induction hypothesis, we get the assertion.

(ii) When n = 1, we have an exact sequence of *R*-modules  $0 \to A \to G_0 \to M \to 0$ . Since we have a  $\mathcal{I}_C(R) \otimes_R$ -exact sequence of *R*-modules  $0 \to H \to F_0 \to G_0 \to 0$  with  $F_0$  flat and  $H \ G_C$ -flat by Lemma 3.9, we have the following pushout

diagram:



The middle row and the first column in the above diagram are the desired two exact sequences.

Now assume that  $n \geq 2$  and we have an exact sequence of R-modules  $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$  with all  $G_i \ G_C$ -flat. Put  $K = \text{Ker}(G_1 \to G_0)$ . By Lemma 3.10, we get an exact sequence of R-modules

(9) 
$$0 \to K \to G'_1 \to F_0 \to M \to 0$$

with  $F_0$  flat and  $G'_1 G_C$ -flat. Put  $M' = \text{Im}(G'_1 \to P_0)$ . Then we get an exact sequence of *R*-modules  $0 \to A \to G_{n-1} \to \cdots \to G_2 \to G'_1 \to G_0 \to M \to 0$ . So, by the induction hypothesis, we get the assertion.

#### 4. $G_C$ -flat dimensions of modules

The class of  $G_C$ -flat modules can be used to define the  $G_C$ -flat dimension.

**Definition 4.1.** For an *R*-module *M*, the  $G_C$ -flat dimension of *M*, denoted by  $G_C - fd_R(M)$ , is defined as  $\inf\{n: \text{ there exists an exact sequence of } R$ -modules  $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$  with all  $G_i \ G_C$ -flat $\}$ . We have  $G_C - fd_R(M) \ge 0$ , and we set  $G_C - fd_R(M) = \infty$  if no such integer exists.

We start with the following standard result.

**Lemma 4.2.** Let  $0 \to L \to M \to N \to 0$  be an exact sequence of *R*-modules.

- (i)  $G_C fd_R(N) \le \max\{G_C fd_R(M), G_C fd_R(L) + 1\}$ , and the equality holds if  $G_C fd_R(M) \ne G_C fd_R(L)$ .
- (ii)  $G_C fd_R(L) \le \max\{G_C fd_R(M), G_C fd_R(N) 1\}$ , and the equality holds if  $G_C fd_R(M) \ne G_C fd_R(N)$ .
- (iii)  $G_C fd_R(M) \leq \max\{G_C fd_R(L), G_C fd_R(N)\}$ , and the equality holds if  $G_C fd_R(N) \neq G_C fd_R(L) + 1$ .

**PROOF:** It is easy.

The proof of the following theorem is similar to [8, Theorem 3.15].

**Theorem 4.3.** Assume that R is GF-closed and C is a semidualizing module. If any two of the modules M, M' or M'' in a short exact sequence  $0 \to M' \to$  $M \to M''$  have finite  $G_C$ -flat dimension, then so has the third.

Next result is a  $G_C$ -flat version of the corresponding result about flat dimension of modules.

**Proposition 4.4.** Let  $0 \to L \to M \to N \to 0$  be an exact sequence of R-modules. If  $L \neq 0$  and N is  $G_C$ -flat, then  $G_C - fd_R(L) = G_C - fd_R(M)$ .

**PROOF:** It follows by Lemma 4.2 (3).

We give a criterion for computing the  $G_C$ -flat dimension of modules as follows. It generalizes [8, Theorem 3.14].

**Theorem 4.5.** Let R be a GF-closed ring. The following statements are equivalent for any *R*-module *M* and  $n \ge 0$ .

- (i)  $G_C fd_R(M) \leq n$ .
- (ii) For every nonnegative integer t such that  $0 \le t \le n$ , there exists an exact sequence of R-modules  $0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to X_0 \to$  $M \to 0$  such that  $X_t$  is  $G_C$ -flat and  $X_i$  are flat for  $i \neq t$ .

PROOF: (ii)  $\Rightarrow$  (i). It is trivial.

(i)  $\Rightarrow$  (ii). We proceed by induction on n. Suppose  $G_C - f d_R(M) \leq 1$ . Then there exists an exact sequence of R-modules  $0 \to G_1 \to G_0 \to M \to 0$  with  $G_0$ and  $G_1$   $G_C$ -flat. By Lemma 3.10 with A = 0, we get the exact sequences of *R*-modules  $0 \to C_1 \to G'_0 \to M \to 0$  and  $0 \to G'_1 \to F_0 \to M \to 0$  with  $C_1$  and  $F_0$  flat, and  $G'_0, G'_1$   $G_C$ -flat.

Now suppose  $G_C - fd_R(M) = n \ge 2$ . Then there exists an exact sequence of R-modules  $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$  with  $G_i \ G_C$ -flat for any  $0 \leq i \leq n$ . Set  $A = \operatorname{Coker}(G_3 \to G_2)$ . By applying Lemma 3.10 to the exact sequence  $0 \to A \to G_1 \to G_0 \to M \to 0$ , we get an exact sequence of *R*-modules  $0 \to G_n \to \cdots \to G_2 \to G'_1 \to F_0 \to M \to 0$  with  $G'_1 G_C$ -flat and  $F_0$  flat. Set  $N = \operatorname{Coker}(G_2 \to G'_1)$ . Then we have  $G_C - fd_R(N) \leq n-1$ . By the induction hypothesis, there exists an exact sequence of R-modules

$$0 \to X_n \to \dots \to X_t \to \dots \to X_1 \to F_0 \to M \to 0$$

such that  $F_0$  is flat and  $X_t$  is  $G_C$ -flat and  $X_i$  are flat for  $i \neq t$  and  $1 \leq t \leq n$ .

Now we need only to prove (ii) for t = 0. Set  $B = \operatorname{Coker}(G_2 \to G_1)$ . By the induction hypothesis, we get an exact sequence of R-modules  $0 \rightarrow X_n \rightarrow$  $\cdots \to X_3 \to X_2 \to G'_1 \to B \to 0$  with  $G'_1 G_C$ -flat and  $X_i$  being flat for any  $2 \leq i \leq n$ . Set  $D = \operatorname{Coker}(X_3 \to X_2)$ . Then by applying Lemma 3.10 to the exact sequence  $0 \to D \to G'_1 \to G_0 \to M \to 0$ , we get the exact sequence of

196

*R*-modules  $0 \to D \to C_1 \to G'_0 \to M \to 0$  with  $C_1$  flat and  $G'_0 G_C$ -flat. Thus we obtain the desired exact sequence of *R*-modules

$$0 \to X_n \to \dots \to X_2 \to X_1 \to G'_0 \to M \to 0$$

with all  $X_i$  flat and  $G'_0$   $G_C$ -flat.

#### References

- Bennis D., Rings over which the class of Gorenstein flat modules is closed under extentions, Comm. Algebra 37 (2009), no. 3, 855–868.
- [2] Christensen L. W., Gorenstein Dimensions, Lecture Notes in Mathematics, 1747, Springer, Berlin, 2000.
- [3] Christensen L. W., Frankild A., Holm H., On Gorenstein projective, injective and flat dimensions a functorial description with applications, J. Algebra 302 (2006), no. 1, 231–279.
- [4] Enochs E., Jenda O. M. G., *Relative Homological Algebra*, De Gruyter Expositions in Mathematics, 30, Walter de Gruyter, Berlin, 2000.
- [5] Enochs E., Jenda O. M. G., Torrecillas B., Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan 10 (1993), no. 1, 1–9.
- [6] Foxby H.-B., Gorenstein modules and related modules, Math. Scand. 31 (1972), 267–284.
- [7] Golod E. S., G-dimension and generalized perfect ideals, Algebraic geometry and its applications, Trudy Mat. Inst. Steklov. 165 (1984), 62–66 (Russian).
- [8] Holm H., Gorenstein homological dimensions, J. Pure Appl. Algebra 189 (2004), no. 1–3, 167–193.
- [9] Holm H., Jørgensen P., Semidualizing modules and related Gorenstein homological dimensions, J. Pure. Appl. Algebra 205 (2006), no. 2, 423–445.
- [10] Holm H., White D., Foxby equivalence over associative rings, J. Math. Kyoto Univ. 47 (2007), no. 4, 781–808.
- [11] Huang C., Huang Z., Gorenstein syzygy modules J. Algebra **324** (2010), no. 12, 3408–3419.
- [12] Liu Z., Yang X., Gorenstein projective, injective and flat modules, J. Aust. Math. Soc. 87 (2009), no. 3, 395–407.
- [13] Rotman J. J., An Introduction to Homological Algebra, Pure and Applied Mathematics, 85, Academic Press, New York, 1979.
- [14] Sather-Wagstaff S., Sharif T., White D., AB-contexts and stability for Gorenstein flat modules with respect to semidualizing modules, Algebr. Represent. Theory 14 (2011), no. 3, 403–428.
- [15] Selvaraj C., Udhayakumar R., Umamaheswaran A., Gorenstein n-flat modules and their covers, Asian-Eur. J. Math. 7 (2014), no. 3, 1450051, 13 pages.
- [16] Vasconcelos W.V., Divisor Theory in Module Categories, North-Holland Mathematics Studies, 14, North-Holland Publishing Co., Amsterdam, American Elsevier Publishing Co., New York, 1974.
- [17] White D., Gorenstein projective dimension with respect to a semidualizing module, J. Commut. Algebra 2 (2010), no. 1, 111–137.

R. Udhayakumar:

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore – 632014, Tamil Nadu, India

E-mail:udhayaram\_v@yahoo.co.in

I. Muchtadi-Alamsyah:

Algebra Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha 10, Bandung 40132, Indonesia

*E-mail:* ntan@math.itb.ac.id

C. Selvaraj:

Department of Mathematics, Periyar University, Periyar Palkalai Nagar, Salem – 636 011, Tamil Nadu, India

*E-mail:* selvavlr@yahoo.com

(Received March 16, 2018, revised December 29, 2018)