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A GENERALIZED BIVARIATE LIFETIME DISTRIBUTION BASED ON PARALLEL-SERIES STRUCTURES

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In this paper, a generalized bivariate lifetime distribution is introduced. This new model is constructed based on a dependent model consisting of two parallel-series systems which have a random number of parallel subsystems with fixed components connected in series. The probability that one system fails before the other one is measured by using competing risks. Using the extreme-value copulas, the dependence structure of the proposed model is studied. Kendall's tau, Spearman's rho and tail dependences are investigated for some special cases. Simulation results are given to examine the effectiveness of the proposed model.

Keywords: copula, extreme-value copula, dependence measures, distortion, competing risks

Classification: 60E05, 62N05, 62H20

1. INTRODUCTION

Constructing flexible families of lifetime distributions is interesting for researchers who work in distribution theory, applied probability and reliability theory. In the literature several methods have been presented to construct such models. Marshall and Olkin [18] introduced a method to expand the family of bivariate distributions. Their model was constructed by considering the component-wise maximum (minimum) of N independent and identical bivariate random vectors, when N has a geometric distribution. Kundu and Gupta [15] studied this model when the bivariate random vectors have a bivariate Weibull distribution. Furthermore, Zhang et al. [29] used this method to construct a new class of dependent models involving geometric distribution. Roozegar and Nadarajah [26] used a similar method for the component-wise maximum (minimum) of the first component and the component-wise minimum (maximum) for the second component of N independent and identical bivariate random vectors by taking N as a power series random variable. Another method was introduced by Durrleman et al. [8] who apply a distortion function to a bivariate distribution. Genest and Rivest [9] explored how the Kendall distribution is affected by this transformation. Some other references for this model are Morillas [21], Crane and Hoek [2] and Dolati et al. [5]. In addition, the dependence properties of the new model was investigated by Durante et al. [7]. Recently, Popović and Genç [24] introduced bivariate Student-t distribution of the Marshall–Olkin

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type and studied the distribution properties and moments of minimum and maximum of a bivariate random variable from bivariate Marshall–Olkin Student-t distribution.

In these decades, modeling dependence is one of the most popular topics in applied probability which is developing by copula theory. Because copulas are useful tools for characterizing the dependence between the lifetime distributions. Let X and Y be continuous random variables with the joint distribution (survival) function $F(\cdot,\cdot)$ ($\bar{F}(\cdot,\cdot)$) and marginal cumulative distribution functions (cdf) F_1 and F_2, respectively. According to Sklar [28] the joint distribution function of the pair (X, Y) can be characterized by the relation

$$F(x,y) = P(X \le x, Y \le y) = D(F_1(x), F_2(y)), \quad (x,y) \in \mathbb{R}^2,$$

and

$$\bar{F}(x,y) = P(X > x, Y > y) = \hat{D}(\bar{F}_1(x), \bar{F}_2(y)), \quad (x,y) \in \mathbb{R}^2,$$

where $D(\cdot,\cdot)$ and $\hat{D}(\cdot,\cdot)$ are unique copula and survival copula, respectively, that expresses the dependence between X and Y. See for example, Nelsen [22] for more details on copula theory.

Since in some cases the lifetime distributions are of great importance in extreme events and in such situations the corresponding dependent models are related to the extreme value copulas. So, it would be required to know the properties of these copulas. It is said that $D(\cdot,\cdot)$ is an extreme value copula if it satisfies $D(u,v) = D^t(u^{1/t}, v^{1/t})$ for $t > 0$. For more details, we refer the reader to Pikhands [23], Kotz and Nadarajah [13] and Goudendorf and Segers [11]. Also, a useful way of formulating properties of a dependent model is to know the form of its stochastic dependence. In the literature, there are several notions of dependence, and measures of association and concordance. We recall briefly some of them which will be used in the sequel:

(i) It is said that X and Y are positively quadrant dependent (denoted by $PQD(X,Y)$) if and only if $D(u,v) \ge uv$ for all $u, v \in (0,1)$.

(ii) The random variable Y is said to be left tail decreasing (denoted by $LTD(Y \mid X)$) when $D(u,v)/u$ is non-increasing in u for all $v \in (0,1)$.

(iii) The random vector (X,Y) is said to be totally positive of order two (TP_2), if its joint pdf $f(x,y)$ is TP_2 that is, $f(x_1,y_1)f(x_2,y_2) \ge f(x_1,y_2)f(x_2,y_1)$, for all $x_1 \le x_2$ and $y_1 \le y_2$.

(v) The random vector (X,Y) is said to be right corner set increasing (denoted by $RCSI(X,Y)$) if $P(X > x, Y > y \mid X > x', Y > y')$ is non-decreasing in x' and in y' for all x and y.

See for example, Nelsen [22], for dependence concepts. It is known that the Kendall's tau τ and Spearman's rho ρ measure the strength of association between random variables. For the pair (X,Y) they are given by

$$\tau = 4\int_0^1 \int_0^1 D(u,v)\,\mathrm{d}D(u,v) - 1$$

$$= 1 - 4 \int_0^1 \int_0^1 \frac{\partial}{\partial u} D(u, v) \frac{\partial}{\partial v} D(u, v)\, \mathrm{d}u\mathrm{d}v, \tag{1}$$

and

$$\rho = 12 \int_0^1 \int_0^1 D(u, v)\, \mathrm{d}u\mathrm{d}v - 3$$
$$= 12 \int_0^1 \int_0^1 uv\, \mathrm{d}D(u, v) - 3, \tag{2}$$

respectively. Tail dependences are used to measure the dependence of extreme values. The upper and the lower tail dependences are defined as $\lambda_L = \lim_{u \to 0^+} \frac{D(u,u)}{u}$ and $\lambda_U = 2 - \lim_{u \to 1^-} \frac{1 - D(u,u)}{1-u}$, respectively. For further information about the structure and measure of dependence, we refer the reader to Joe [12] and Nelsen [22].

Asgharzadeh et al. [1] constructed a family of continuous lifetime distributions based on $\min\{X_1, \ldots, X_N\}$ where X_i's are iid from a continuous distribution function G and N is a zero-truncated Poission–Lindley random variable independent of the X_i's. Goldoust et al. [10] considered a system with parallel subsystems consisting of a random number of series components and modeled the lifetime of the system by compounding the exponential, geometric and power series distributions. In this paper, by extending the model proposed in Goldoust et al. [10], we intend to construct a new bivariate lifetime distribution based on the two dependent parallel-series systems with random number of sub-systems so that the number of components in each subsystem are fixed. It may be noted that some reliability properties of the parallel (series) systems that have statistical dependent lifetimes were studied by Li and Li [16].

The rest of this paper are organized as follows. The general form of the proposed model is discussed in Section 2. In Section 3, the probability that one system fails before the other one is stated. The dependence structure of the model is studied in special cases in Section 4. Simulation studies are provided in Section 5.

2. MODEL DESCRIPTION

It is known that a system is a collection of components that are arranged for a specific purpose. In systems with parallel structure at least one of the components have to succeed for the system to succeed and in systems with series structure the failure of any component results in the failure of the system. A parallel-series system is composed of a fixed number of series sub-systems connected in parallel. There have been several models proposed to explain parallel-series systems. We are going to present a model for two dependent systems with parallel-series sub-systems. First, let us fix the assumptions and notations.

Assumptions: Let $Z_{i,j}^{(1)}$ and $Z_{i,j}^{(2)}$ be the random lifetimes of the jth component of the ith sub-system corresponding to system (I) and system (II), respectively, for $j = 1, \ldots, m$, $i = 1, \ldots, N$, where N is a positive integer-valued random variable independent of $(Z_{i,j}^{(1)}, Z_{i,j}^{(2)})$. Suppose that $X_i = \min\{Z_{i,1}^{(1)}, \ldots, Z_{i,m}^{(1)}\}$ and $Y_i = \min\{Z_{i,1}^{(2)}, \ldots, Z_{i,m}^{(2)}\}$ for $i = 1, \ldots, N$ are the random lifetimes of the ith sub-system of system (I) and system

(II), respectively. Also, let us take $T_1 = \max\{X_1, \dots, X_N\}$ and $T_2 = \max\{Y_1, \dots, Y_N\}$ as the random lifetimes of system (I) and system (II), respectively. Moreover, we suppose that $Z_{i,j}^{(1)}$ for $i = 1, \dots, N$ and $j = 1, \dots, m$ are iid with common marginal cdf F_1 and $Z_{l,j}^{(2)}$ for $l = 1, \dots, N$ and $j = 1, \dots, m$ are iid with common marginal cdf F_2. In addition the random vector $(Z_{i,j}^{(1)}, Z_{i,j}^{(2)})$ has a bivariate cdf $F(\cdot,\cdot)$. Furthermore, let $g(\cdot)$ be the probability generating function (pgf) of N.

Then, by assumptions the cdf of (T_1, T_2) is given by

$$\begin{aligned} H_{T_1,T_2}(x,y) &= P(T_1 \leq x, T_2 \leq y) \\ &= \sum_{n=1}^{\infty} P(T_1 \leq x, T_2 \leq y \mid N = n) P(N = n) \\ &= \sum_{n=1}^{\infty} P(\max\{X_1, \dots, X_N\} \leq x, \max\{Y_1, \dots, Y_N\} \leq y \mid N = n) P(N = n). \end{aligned}$$

Since by assumption the random vectors $(Z_{i,j}^{(1)}, Z_{i,j}^{(2)})$ for $j = 1, \dots, m$ and $i = 1 \dots, N$ are independent of N, then the random vectors (X_i, Y_i), for $i = 1, \dots, N$, are independent of N, too. Consequently, $H_{T_1,T_2}(\cdot,\cdot)$ can be reexpressed as

$$\begin{aligned} H_{T_1,T_2}(x,y) &= \sum_{n=1}^{\infty} P(X_1 \leq x, \dots, X_n \leq x, Y_1 \leq y, \dots, Y_n \leq y) P(N = n) \\ &= \sum_{n=1}^{\infty} \prod_{i=1}^{n} P(X_i \leq x, Y_i \leq y) P(N = n) \\ &= \sum_{n=1}^{\infty} \prod_{i=1}^{n} P(\min\{Z_{i1}^{(1)}, \dots, Z_{im}^{(1)}\} \leq x, \min\{Z_{i1}^{(2)}, \dots, Z_{im}^{(2)}\} \leq y) P(N = n). \end{aligned} \tag{3}$$

Also, by assumption $Z_{i,j}^{(1)}$ for $i = 1, \dots, N$ and $j = 1, \dots, m$ are iid, and $Z_{l,j}^{(2)}$ for $l = 1, \dots, N$ and $j = 1, \dots, m$ are iid. Accordingly, we have

$$\prod_{i=1}^{n} P(\min\{Z_{i1}^{(1)}, \dots, Z_{im}^{(1)}\} \leq x, \min\{Z_{i1}^{(2)}, \dots, Z_{im}^{(2)}\} \leq y) = [1 - \bar{F}_1^{\,m}(x) - \bar{F}_2^{\,m}(y) + \bar{F}^m(x,y)]^n.$$

So, by substituting in (3), we have an expression for the cdf of (T_1, T_2) in terms of the pgf of N as follows

$$H_{T_1,T_2}(x,y) = g(1 - \bar{F}_1^{\,m}(x) - \bar{F}_2^{\,m}(y) + \bar{F}^m(x,y)), \tag{4}$$

where $\bar{F}(\cdot,\cdot)$ is the corresponding survival function of $(Z_{i,j}^{(1)}, Z_{i,j}^{(2)})$. Also, the marginal cdf of T_1 and T_2 in terms of the pgf of N are given by $H_1(x) = g(1 - \bar{F}_1^{\,m}(x))$ and $H_2(x) = g(1 - \bar{F}_2^{\,m}(x))$, respectively. It is clear that this new bivariate distribution function, namely $H_{T_1,T_2}(\cdot,\cdot)$, is exchangeable if $F(\cdot,\cdot)$ is exchangeable and $F_1(\cdot) = F_2(\cdot)$.

Durante [6] proposed a method for constructing non-exchangeable bivariate distribution functions.

To fix the concept, let us first consider N as a discrete random variable having a power series distribution (truncated at zero) with the following probability mass function

$$P(N=n)=\frac{a_n\theta^n}{A(\theta)},\quad n=1,2,...,\quad \theta>0,\ a_n>0, \tag{5}$$

where $A(\theta)=\sum_{n=1}^{\infty}a_n\theta^n<\infty$. Then, it is easy to show that the pgf of N is $g(z)=\frac{A(\theta z)}{A(\theta)}$. So, in this case by (4), $H_{T_1,T_2}(x,y)$ can be written as

$$H_{T_1,T_2}(x,y)=A(\theta[1-\bar{F_1}^m(x)-\bar{F_2}^m(y)+\bar{F}^m(x,y)])/A(\theta). \tag{6}$$

It should be mentioned that for the special case $m=1$, the expression in (6) reduced to Equation (2.3) in Roozegar and Jafari [25], where the authors called it as bivariate generalized linear failure rate-power series model.

Now, let us consider N as a random variable with two values 1 and 2 with probability $1-\theta$ and θ, respectively, where $0\le\theta\le 1$. Then, the pgf of N is $g(t)=t(1-\theta(1-t))$, $0\le t\le 1$. If $F(x,y)=F_1(x)F_2(y),\forall x,y\in\mathbb{R}$, then from (4) we have

$$H_{T_1,T_2}(x,y)=(1-\bar{F_1}^m(x))(1-\bar{F_2}^m(y))[1-\theta(1-(1-\bar{F_1}^m(x))(1-\bar{F_2}^m(y)))], \tag{7}$$

and

$$\bar{H}_{T_1,T_2}(x,y)=\bar{F_1}^m(x)\bar{F_2}^m(y)[1-\theta(1-(2-\bar{F_1}^m(x))(2-\bar{F_2}^m(y)))]. \tag{8}$$

It should be noted that (7) is a member of the model proposed by Mirhosseini et al. [20]. Moreover, the distorted distribution $\tilde{F_i}(x)=g(F_i(x))=F_i(x)(1-\theta\bar{F_i}(x))$, $i=1,2$ and a bivariate distribution mixtured by $\tilde{F_i}(x)$ was introduced by Mirhosseini et al. [19].

By simple calculations, from (8) we obtain

$$\frac{\partial^2}{\partial x\partial y}\ln\bar{H}_{T_1,T_2}(x,y)=\frac{m^2\theta(1-\theta)f_1(x)f_2(y)\bar{F_1}^{m-1}(x)\bar{F_2}^{m-1}(y)}{\left(1-\theta+\theta(2-\bar{F_1}^m(x))(2-\bar{F_2}^m(y))\right)^2}\ge 0.$$

This proves that $\bar{H}_{T_1,T_2}(\cdot,\cdot)$ is TP_2, so we have the following result.

Proposition 2.1. Suppose that the random vector (T_1,T_2) is modeled as (8), then $RCSI(T_1,T_2)$.

Now, let us take $\bar{F_i}(x)=e^{-\alpha_i x}$, $x>0$ for $i=1,2$, then by (7) and (8), we have

$$H_{T_1,T_2}(x,y)=(1-e^{-m\alpha_1 x})(1-e^{-m\alpha_2 y})[1-\theta(1-(1-e^{-m\alpha_1 x})(1-e^{-m\alpha_2 y}))],x>0,y>0$$

and

$$\bar{H}_{T_1,T_2}(x,y)=e^{-m(\alpha_1 x+\alpha_2 y)}[1-\theta(1-(2-e^{-m\alpha_1 x})(2-e^{-m\alpha_2 y}))],x>0,y>0. \tag{9}$$

By exploiting the definition of the conditional concepts and some mathematical computations, from (9) we obtain the next result.

Proposition 2.2. Let (T_1, T_2) be modeled as (9). Then for $i, j = 1, 2$, $i \neq j$,

i) $P(T_j > y \mid T_i > x) = \dfrac{1 - \theta + \theta(2 - e^{-\beta_i x})(2 - e^{-\beta_j y})}{1 + \theta - \theta e^{-\beta_i x}} e^{-\beta_j y}$, $x \geq 0$, $y \geq 0$,

ii) $E(T_j \mid T_i > x) = \dfrac{2 + 4\theta - 3\theta e^{-\beta_i x}}{2\beta_j(1 + \theta - 2\theta e^{-\beta_i x})}$, $x \geq 0$,

iii) $E(T_j \mid T_i = x) = \dfrac{1 + 2\theta - 3\theta e^{-\beta_i x}}{\beta_j(1 + \theta - 2\theta e^{-\beta_i x})}$, $x \geq 0$,

where $\beta_i = m\alpha_i$ and $\beta_j = m\alpha_j$ for $i, j = 1, 2$ and $i \neq j$.

Figure 1 displays the attitude of the conditional tail expectation $E(T_2 \mid T_1 > x)$ and the regression function $E(T_2 \mid T_1 = x)$ in Proposition 2.2.

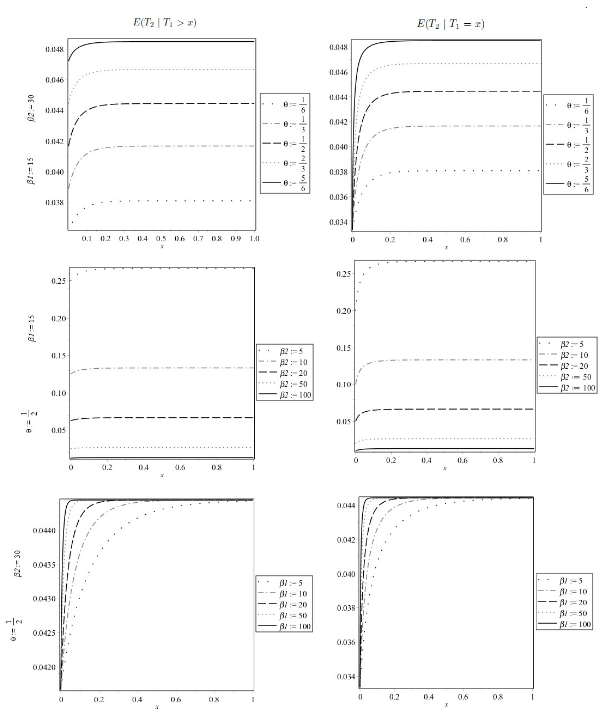

Fig. 1. Plots of $E(T_2 \mid T_2 > x)$ and $E(T_2 \mid T_1 = x)$ in Proposition 2.2.

From Figure 1, we observe that in the first row by considering $\beta_1 = 15$ and $\beta_2 = 30$, the conditional concepts increase by increasing θ and in the second row by fixing $\theta = 1/2$ and $\beta_1 = 15$, both conditional expectations decrease by increasing β_2. Also, in the last row, it is observed that for $\theta = 1/2$ and $\beta_2 = 30$, these conditional concepts increase as β_1 increases.

3. COMPETING RISKS MEASURES

It is known that competing risks are events that prevent an event of interest from occurring, or modify the chance of its occurrence. We refer the reader to Crowder [3], Shih and Emura [27] and Lu [17] for more details on competing risks models. In this section, we deal with measures related to sub-distribution functions which plays an important role in competing risks models. Let T_1 and T_2 be two dependent failure times (latent failure times), also $T = \min\{T_1, T_2\}$ be the first occurring failure time with the corresponding failure cause $C = 1$ if $T_1 \le T_2$ and $C = 2$ if $T_1 > T_2$. Then, the sub-distribution function of the failure cause i, $(i = 1, 2)$ is given by

$$\begin{aligned} H(i,t) &= P(C = i, T \le t) \\ &= \int_0^t h(i,z)\,\mathrm{d}z, \end{aligned} \tag{10}$$

where $h(i,t) = -\dfrac{\partial}{\partial x_i}\bar{H}_{T_1,T_2}(x_i,x_j)\mid_{x_i=x_j=t}$ for $i,j = 1,2$ $(i \ne j)$ are the sub-density functions and $\bar{H}_{T_1,T_2}(x,y) = 1 - H_1(x) - H_2(y) + H_{T_1,T_2}(x,y)$, where $H_1(\cdot)$ and $H_2(\cdot)$ are the marginal cdfs of T_1 and T_2, respectively.

The corresponding sub-distribution functions of failure causes 1 $(C = 1)$ and 2 $(C = 2)$ of model (4) are expressed in the next proposition.

Proposition 3.1. Suppose that (T_1, T_2) have the bivariate distribution function as given in (4), then

$$\begin{aligned} H(1,t) = g(1 - \bar{F_1}^m(t)) - \int_0^t &\Big(mf_1(x)\bar{F_1}^{m-1}(x) + m\frac{\partial}{\partial x}\bar{F}(x,y)\bar{F}^{m-1}(x,y)\Big) \\ &\times g'(1 - \bar{F_1}^m(x) - \bar{F_2}^m(y) + \bar{F}^m(x,y)) \mid_{x=y=z}\ \mathrm{d}z \end{aligned} \tag{11}$$

and

$$\begin{aligned} H(2,t) = g(1 - \bar{F_2}^m(t)) - \int_0^t &\Big(mf_2(y)\bar{F_2}^{m-1}(y) + m\frac{\partial}{\partial y}\bar{F}(x,y)\bar{F}^{m-1}(x,y)\Big) \\ &\times g'(1 - \bar{F_1}^m(x) - \bar{F_2}^m(y) + \bar{F}^m(x,y)) \mid_{x=y=z}\ \mathrm{d}z, \end{aligned} \tag{12}$$

where $g'(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}g(t)$.

Proof. By (4), we have the following expression for the sub-density function of failure cause $C = 1$

$$\begin{aligned} h(1,z) = mf_1(x)\bar{F_1}^{m-1}(x)g'(1 - \bar{F_1}^m(x)) - \Big(&mf_1(x)\bar{F_1}^{m-1}(x) + m\frac{\partial}{\partial x}\bar{F}(x,y)\bar{F}^{m-1}(x,y)\Big) \\ \times g'(1 - \bar{F_1}^m(x) - \bar{F_2}^m(y) + \bar{F}^m(x,y)) \mid_{x=y=z}. \end{aligned}$$

Similarly, the sub-density function of failure cause $C = 2$ is given by

$$\begin{aligned} h(2,z) = mf_2(y)\bar{F_2}^{m-1}(y)g'(1 - \bar{F_2}^m(y)) - \Big(&mf_2(y)\bar{F_2}^{m-1}(y) + m\frac{\partial}{\partial y}\bar{F}(x,y)\bar{F}^{m-1}(x,y)\Big) \\ \times g'(1 - \bar{F_1}^m(x) - \bar{F_2}^m(y) + \bar{F}^m(x,y)) \mid_{x=y=z}. \end{aligned}$$

The expressions in (11) and (12) are readily obtained upon substituting the above equations in (10). □

It should be mentioned that $P(T_2 < T_1) = \lim_{t\to\infty} H(2,t)$, so by putting $t \to \infty$ in (12) the probability that system (II) failed before system (I) is given by

$$P(T_2 < T_1) = 1 - \int_0^\infty (mf_2(y)\bar{F_2}^{m-1}(y) + m\frac{\partial \bar{F}(x,y)}{\partial y}\bar{F}^{m-1}(x,y)) \times g'(1 - \bar{F_1}^m(x) - \bar{F_2}^m(y) + \bar{F}^m(x,y)) \,|_{x=y=z}\, \mathrm{d}z. \tag{13}$$

If $\bar{F}(x,y) = \bar{F_1}(x)\bar{F_2}(y)$, then

$$P(T_2 < T_1) = 1 - \int_0^\infty mf_2(z)\bar{F_2}^{m-1}(z)(1 - \bar{F_1}^m(z))g'((1 - \bar{F_1}^m(z))(1 - \bar{F_2}^m(z)))\,\mathrm{d}z.$$

It should be noted that the expression $P(T_2 < T_1) = \lim_{t\to\infty} H(2,t)$ coincides with the stress-strength reliability measure. See, Kotz et al. [14] for a comprehensive treatment of stress-strength models up to 2003.

Example 3.2. Let $g(t) = \dfrac{\theta t}{1-(1-\theta)t}$, i. e., the random variable N has a geometric distribution, then from (13) we have

$$P(T_2 < T_1) = 1 - \int_0^\infty \frac{\theta m f_2(z)\bar{F_2}^{m-1}(z)(1 - \bar{F_1}^m(z))}{(1-(1-\theta)(1-\bar{F_1}^m(z))(1-\bar{F_2}^m(z)))^2}\,\mathrm{d}z.$$

Moreover, if $\bar{F_1}(z) = \bar{F_2}(z) = e^{-z}$ then

$$P(T_2 < T_1) = 1 - \int_0^\infty \frac{m\theta e^{-mz}(1-e^{-mz})}{(1-(1-\theta)(1-e^{-mz}))^2}\,\mathrm{d}z. \tag{14}$$

By setting $u = 1 - e^{-mz}$ in (14) the formula simplifies as

$$\begin{aligned} P(T_2 < T_1) &= 1 - \int_0^1 \frac{\theta u}{(1-(1-\theta)u)^2}\,\mathrm{d}u \\ &= \frac{\theta}{(1-\theta)^2}(\theta - 1 - \ln(\theta)), \end{aligned} \tag{15}$$

which depends on θ. It can be easily shown that $P(T_2 < T_1)$ in (15) is an increasing function of θ and $\lim_{\theta\to1} P(T_2 < T_1) = \frac{1}{2}$.

Example 3.3. Let $g(t) = \dfrac{e^{\theta t}-1}{e^{\theta}-1}$, i. e., N has a truncated Poisson distribution, then from (13) we have

$$P(T_2 < T_1) = 1 - \frac{\theta}{e^\theta - 1}\int_0^\infty mf_2(z)\bar{F_2}^{m-1}(z)(1 - \bar{F_1}^m(z))e^{\theta(1-\bar{F_1}^m(z))(1-\bar{F_2}^m(z))}\,\mathrm{d}z. \tag{16}$$

Furthermore, substituting $\bar{F}_1(z) = \bar{F}_2(z) = e^{-z}$ implies

$$P(T_2 < T_1) = 1 - \frac{\theta}{e^\theta - 1} \int_0^\infty m e^{-z} e^{-(m-1)z} (1 - e^{-mz}) e^{\theta(1-e^{-mz})^2} \, \mathrm{d}z.$$

By setting $v = (1 - e^{-mz})^2$ in (16) the simplified formula is

$$P(T_2 < T_1) = 1 - \frac{\theta}{e^\theta - 1} \int_0^1 e^{\theta v} \, \mathrm{d}v = 1/2.$$

Example 3.4. Let $g(t) = t - \theta(1-t)t$, i. e., N is a random variable with two values 1 and 2 with probabilities $1-\theta$ and θ, respectively, when $0 \le \theta \le 1$. Then

$$P(T_2 < T_1) = 1 - \int_0^{+\infty} m f_2(z) \bar{F}_2^{m-1}(z)(1 - \bar{F}_1^{m}(z))[1 - \theta + 2\theta(1 - \bar{F}_1^{m}(z))(1 - \bar{F}_2^{m}(z))] \, \mathrm{d}z.$$

In addition, if $\bar{F}_i(z) = e^{-\alpha_i z}$ for $i = 1, 2$ then

$$\begin{aligned} P(T_2 < T_1) &= 1 - \int_0^{+\infty} m\alpha_2 e^{-m\alpha_2 z}(1 - e^{-m\alpha_1 z})[1 - \theta + 2\theta(1 - e^{-m\alpha_1 z})(1 - e^{-m\alpha_2 z})] \, \mathrm{d}z \\ &= \frac{(4\theta+1)\alpha_2}{\alpha_1 + \alpha_2} - \frac{4\theta\alpha_2}{2\alpha_2 + \alpha_1} - \frac{2\theta\alpha_2}{2\alpha_1 + \alpha_2}. \end{aligned} \tag{17}$$

If $\alpha_1 = \alpha_2 = 1$, then (17) simplifies to $P(T_2 < T_1) = \frac{1}{2}$.

4. RESULTS BASED ON EXTREME-VALUE COPULAS

In this section, the corresponding copula function of (4) will be represented. The dependence structure of the proposed model will be investigated in detail for an extreme-value copula. Let $\hat{D}(\cdot,\cdot)$ be the survival copula corresponding to $\bar{F}(\cdot,\cdot)$ and let $C_g(D)(\cdot,\cdot)$ be the corresponding copula of $H_{T_1,T_2}(\cdot,\cdot)$ in (4), then by using Sklar's Theorem and some algebraic calculations, we can write

$$\begin{aligned} C_g(D)(u,v) &= H_{T_1,T_2}(H_1^{-1}(u), H_2^{-1}(v)) \\ &= g(g^{-1}(u) + g^{-1}(v) - 1 + \hat{D}^m((1 - g^{-1}(u))^{1/m}, (1 - g^{-1}(v))^{1/m}). \end{aligned} \tag{18}$$

In the special case, if $\hat{D}(\cdot,\cdot)$ is an extreme-value copula, then (18) simplifies to

$$C_g(D)(u,v) = g(D(g^{-1}(u), g^{-1}(v))). \tag{19}$$

If $g(\cdot)$ satisfies the relation $g^m(t^{1/m}) = g(t)$ for some $m \ge 1$, then the model (19) preserves dependence structure of extreme-value copulas.

4.1. Dependence structure

In modeling dependence, it is important to know the dependence structure of the model which shows the attitude of two random variables relative to each other. The next proposition provides conditions that the model (19) is PQD and LTD.

Proposition 4.1. Let (T_1, T_2) be a random vector distributed as (19). Then

i) $PQD(T_1, T_2)$ if and only if $g(D(t_1, t_2)) \geq g(t_1)g(t_2); \forall t_1, t_2 \in [0, 1]$,

ii) $LTD(T_2 \mid T_1)$ if and only if $\dfrac{g(D(t_1, t_2))}{g(t_1)}$ is decreasing in t_1.

Example 4.2. Let $g(\cdot)$ be as in Example 3.2. By applying Proposition 4.1 we have

i) $PQD(T_1, T_2)$ if and only if $D(t_1, t_2) \geq \dfrac{\theta t_1 t_2}{\theta + (1-\theta)(1-t_1)(1-t_2)}$,

ii) $LTD(T_2 \mid T_1)$ if and only if $D_1(t_1, t_2) \leq \dfrac{1-(1-\theta)D(t_1, t_2)}{t_1 - (1-\theta)t_1^2} D(t_1, t_2)$,

where $D_1(t_1, t_2) = \dfrac{\partial}{\partial t_1} D(t_1, t_2)$.

Example 4.3. Let $g(\cdot)$ be as in Example 3.3. Then by exploiting Proposition 4.1 we have

i) $PQD(T_1, T_2)$ if and only if $D(t_1, t_2) \geq \dfrac{1}{\theta} \ln\left(1 + \dfrac{(e^{\theta t_1} - 1)(e^{\theta t_2} - 1)}{e^{\theta} - 1}\right)$,

ii) $LTD(T_2 \mid T_1)$ if and only if $D_1(t_1, t_2) \leq \dfrac{e^{\theta t_1}(e^{\theta D(t_1, t_2)} - 1)}{e^{\theta D(t_1, t_2)}(e^{\theta t_1} - 1)}$.

The following proposition expresses conditions such that the random vector (T_1, T_2) is TP_2.

Proposition 4.4. The copula $C_g(D)(\cdot, \cdot)$ defined by (19) is TP_2 if $D(\cdot, \cdot)$ is a TP_2 copula.

Proof. Since $g(\cdot)$ is one-to-one, it is an isomorphism. Moreover, $g(e^t)$ is log-convex on $(-\infty, 0] \to [0, 1]$ so, by Theorem 3.1 of Durante et al. [7] the proof is completed. □

It may be noted that the model (19) preserves dependence structure of $M(u, v) = \min\{u, v\}$.

Proposition 4.5. Suppose that $D(u, v) = M(u, v)$, then $C_M(u, v) = M(u, v)$.

Proof. The proof is straightforward since $g(\cdot)$ is an increasing function, we take

$$C_M(u, v) = g(M(g^{-1}(u), g^{-1}(v))) = M(u, v).$$

□

4.2. Measures of dependence

It is known that measures of dependence quantify the strength of dependence between two random variables. Kendall's tau (τ) and Spearman's rho (ρ) are the most popular dependence measures which are free from the marginal distributions. Shih and Emura [27] studied the properties of some common bivariate dependence measures. The next proposition represents a formula for Kendall's tau and Spearman's rho of the proposed model.

Proposition 4.6. Let (T_1, T_2) be a random vector distributed as (19). Then

i) If $D(\cdot,\cdot)$ and $g(\cdot)$ are twice differentiable, then

$$\tau_{C_g(D)} = 1 - 4\int_0^1\int_0^1 (g'(D(x,y)))^2 D_1(x,y)D_2(x,y)\,\mathrm{d}x\mathrm{d}y, \tag{20}$$

where $D_2(x,y) = \dfrac{\partial}{\partial y}D(x,y)$.

ii) If $g(\cdot)$ is differentiable, then

$$\rho_{C_g(D)} = 12\int_0^1\int_0^1 g'(x)g'(y)g(D(x,y))\,\mathrm{d}x\mathrm{d}y - 3. \tag{21}$$

Proof.

i) By using formula (1), we have

$$\begin{aligned}\tau_{C_g(D)} &= 1 - 4\int_0^1\int_0^1 \frac{\partial}{\partial u}g(D(g^{-1}(u),g^{-1}(v)))\frac{\partial}{\partial v}g(D(g^{-1}(u),g^{-1}(v)))\,\mathrm{d}u\mathrm{d}v\\ &= 1 - 4\int_0^1\int_0^1 \frac{\partial g(D(g^{-1}(u),g^{-1}(v)))}{\partial g^{-1}(u)}\frac{\partial g(D(g^{-1}(u),g^{-1}(v)))}{\partial g^{-1}(v)}\,\mathrm{d}g^{-1}(u)\,\mathrm{d}g^{-1}(v).\end{aligned}$$

Setting $x = g^{-1}(u)$ and $y = g^{-1}(v)$ then

$$\begin{aligned}\tau_{C_g(D)} &= 1 - 4\int_0^1\int_0^1 \frac{\partial}{\partial x}g(D(x,y))\frac{\partial}{\partial y}g(D(x,y))\,\mathrm{d}x\mathrm{d}y\\ &= 1 - 4\int_0^1\int_0^1 (g'(D(x,y)))^2 D_1(x,y)D_2(x,y)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

ii) Using (2), the Spearman's rho of model (19) is

$$\rho_{C_g(D)} = 12\int_0^1\int_0^1 g(D(g^{-1}(u),g^{-1}(v)))\,\mathrm{d}u\mathrm{d}v - 3. \tag{22}$$

Take $x = g^{-1}(u)$ and $y = g^{-1}(v)$ in (22), we have

$$\rho_{C_g(D)} = 12\int_0^1\int_0^1 g'(x)g'(y)g(D(x,y))\,\mathrm{d}x\mathrm{d}y - 3.$$

□

For given $D(\cdot,\cdot)$ and $g(\cdot)$ by (20) and (21), we obtained exact expressions for τ and ρ, these are presented in the subsequent sections.

4.3. Dependence orders

Stochastic orders are useful tools for comparing the strength of dependence between the considered random variables (or two random vectors). Let $D^*(\cdot,\cdot)$ and $D^{**}(\cdot,\cdot)$ be two copulas, it is said that $D^{**}(\cdot,\cdot)$ is more concordant than $D^*(\cdot,\cdot)$ (denoted by $D^* \prec_c D^{**}$) if

$$D^*(u,v) \leq D^{**}(u,v), \quad \forall u,v \in [0,1].$$

Moreover, the function $f : [0,1] \to [0,1]$ is said to be Supra-D if

$$D(f(x), f(y)) \leq f(D(x,y)), \quad \forall x,y \in [0,1].$$

Dolati and Ućbeda-Flores [5] proposed a method to construct a new copula based on the choice of pairs of order statistics of the marginal distributions and studied several dependence properties including concordant order. The following proposition expresses the concordance order between two distortion copulas, the result follows by Theorem 3.14 in Morillas [21].

Proposition 4.7. Let $g_1(\cdot)$ and $g_2(\cdot)$ be two distortion functions and assume that $C_g(D)$ satisfies (19), then $C_{g_1}(D) \prec_c C_{g_2}(D)$ if and only if $g_1^{-1} \circ g_2$ is Supra-D.

Since by assumption $g(\cdot)$ is an increasing function and it is known that the order is preserved by concordance functions. Consequently, we have the following proposition.

Proposition 4.8. Let $D_1(\cdot,\cdot)$ and $D_2(\cdot,\cdot)$ be two copulas and assume that $C_g(D)$ satisfies in Equation (19). If $D_1 \prec_c D_2$ then

i) $C_g(D_1) \prec_c C_g(D_2)$,

ii) $\tau_{C_g(D_1)} \leq \tau_{C_g(D_2)}$,

iii) $\rho_{C_g(D_1)} \leq \rho_{C_g(D_2)}$.

4.4. Sub-families

In this subsection, the dependence properties of model (19) are investigated when N has geometric and truncated Poisson distributions.

Case I (N has a geometric distribution): In this case, by Example 3.2, we get $g^{-1}(t) = \dfrac{t}{\theta + (1-\theta)t}$, so from (19), $C_g(D)(u,v)$ is given by

$$C_g(D)(u,v) = \frac{\theta D\left(\dfrac{u}{\theta + u(1-\theta)}, \dfrac{v}{\theta + v(1-\theta)}\right)}{1-(1-\theta)D\left(\dfrac{u}{\theta + u(1-\theta)}, \dfrac{v}{\theta + v(1-\theta)}\right)}. \tag{23}$$

By substituting the functions $g(t) = \frac{\theta t}{1-(1-\theta)t}$ and $g'(t) = \frac{\theta}{(1-(1-\theta)t)^2}$ in equations (20) and (21), we obtain the Kendall's tau and Spearman's rho of model (23) which are stated in the next proposition.

Proposition 4.9. Let $D(\cdot,\cdot)$ be a copula which is twice differentiable, then the Kendall's tau and Spearman's rho of model (23) are given by

$$\tau_{C_g(D)} = 1 - 4\theta^2 \int_0^1 \int_0^1 \Big(\frac{D_1(x,y)}{(1-(1-\theta)D(x,y))^2}\Big)\Big(\frac{D_2(x,y)}{(1-(1-\theta)D(x,y))^2}\Big)\,\mathrm{d}x\mathrm{d}y$$

and

$$\rho_{C_g(D)} = 12\theta^3 \int_0^1 \int_0^1 \frac{D(x,y)}{(1-(1-\theta)x)^2(1-(1-\theta)y)^2(1-(1-\theta)D(x,y))}\,\mathrm{d}x\mathrm{d}y - 3,$$

respectively.

The upper (lower) tail dependence measures the probability of having large (small) values. We obtained them for model (23) which are stated in the next proposition.

Proposition 4.10. If $\lambda_L(D)$ and $\lambda_U(D)$ exist and both are finite, then $\lambda_L(C_g(D)) = \lambda_L(D)$ and $\lambda_U(C_g(D)) = \lambda_U(D)$.

Proof. Since $g(t) = \frac{\theta t}{1-(1-\theta)t}$ is an isomorphism and also

$$\lim_{t\to 0^+} \frac{g(t)}{t^\alpha} = \lim_{t\to 0^+} \frac{\theta}{t^{\alpha-1}(1-(1-\theta)t)} = \begin{cases} 0, & 0<\alpha<1,\\ \theta, & \alpha=1,\\ \infty, & \alpha>1, \end{cases} \tag{24}$$

and

$$\lim_{t\to 1^-} \frac{1-g(t)}{(1-t)^\alpha} = \lim_{t\to 1^-} \frac{1}{(1-t)^\alpha(1-(1-\theta)t)} = \begin{cases} 0, & 0<\alpha<1,\\ \frac{1}{\theta}, & \alpha=1,\\ \infty, & \alpha>1. \end{cases} \tag{25}$$

Thus, by exploiting (24), (25) and Theorem 4.2 of Durante et al. [7], we arrive at $\lambda_L(C_g(D)) = \lambda_L(D)$. Also, from Theorem 4.3 of Durante et al. [7] we conclude that $\lambda_U(C_g(D)) = \lambda_U(D)$. □

Example 4.11. If $D(u,v) = \Pi(u,v)$, then we have the following results.

i) For all $0<\theta<1$; $C_g(\Pi)(u,v) = \dfrac{uv}{1-(1-\theta)(1-u)(1-v)}$. This copula is the well known $AMH(1-\theta)$, that is TP_2.

ii) Since $\lambda_L(\Pi) = \lambda_U(\Pi) = 0$, hence Proposition 4.10 implies that

$$\lambda_L(C_g(\Pi)) = \lambda_U(C_g(\Pi)) = 0.$$

iii) By Nelsen [22], Kendall's tau and Spearman's rho are given by

$$\tau_{C_g(\Pi)} = -\frac{2}{3\theta^2}[(1-\theta)^2\ln(1-\theta)+1] - 1$$

and

$$\rho_{C_g(\Pi)} = -\frac{3}{\theta^2}[14(1-\theta)dilog(1-\theta)+8(1-\theta)\ln(1-\theta)-8dilog(1-\theta)+\theta^2-16\theta+14],$$

where $dilog(x) = \int_1^x \frac{\ln(t)}{1-t}\,\mathrm{d}t$.

Example 4.12. Let $G(\cdot,\cdot)$ be the Gumbel survival copula with parameter $\alpha \geq 1$, then

i) For $0 < \theta < 1$,

$$C_g(G)(u,v) = \frac{\theta(w(u,v) + p(u)p(v)\exp\{-[q(u)+q(v)]^{1/\alpha}\})}{p(u)p(v)[1-(1-\theta)\exp\{-[q(u)+q(v)]^{1/\alpha}\}] - (1-\theta)w(u,v)},$$

where $p(u) = \theta + u(1-\theta)$, $q(u) = \left(-\ln(\frac{(1-\theta)u}{\theta+(1-\theta)u})\right)^\alpha$ and $w(u,v) = (uv - \theta^2(1-u)(1-v))$.

ii) Since $G(\cdot,\cdot)$ is TP_2 by using Proposition 4.4, $C_g(G)(\cdot,\cdot)$ is TP_2.

iii) It is easy to show that $\lambda_L(G) = 0$ and $\lambda_U(G) = 2 - 2^{1/\theta}$. So, Proposition 4.10 implies that

$$\lambda_L(C_g(G)) = 0 \quad \text{and} \quad \lambda_U(C_g(G)) = 2 - 2^{1/\theta}.$$

iv) Since $\Pi \prec_c G \prec_c M$, so by Proposition 4.8, the following inequalities are deduced

$$\tau_{C_g(\Pi)} \leq \tau_{C_g(G)} \leq \tau_{C_g(M)} \quad \text{and} \quad \rho_{C_g(\Pi)} \leq \rho_{C_g(G)} \leq \rho_{C_g(M)}.$$

As it is shown in Example 4.11, $C_g(\Pi)(u,v) = AMH(1-\theta)$. Hence,

$$\tau_{(AMH(1-\theta))} \leq \tau_{C_g(G)} \leq 1, \quad \text{and} \quad \rho_{(AMH(1-\theta))} \leq \rho_{C_g(G)} \leq 1.$$

Case II (N has a truncated Poisson distribution): In this case by Example 3.3, we get $g^{-1}(t) = \dfrac{1}{\theta}\ln(1+t(e^\theta-1))$. So, we can write

$$C_g(D)(u,v) = \frac{1}{e^\theta - 1}\{\exp(\theta D\left(\frac{\ln(1+u(e^\theta-1))}{\theta}, \frac{\ln(1+v(e^\theta-1))}{\theta}\right) - 1\}. \tag{26}$$

The Kendall's tau and Spearman's rho of model (26) are given in the next result.

Proposition 4.13. Let $D(\cdot,\cdot)$ be a twice differentiable copula, then Kendall's tau and Spearman's rho of model (26) are given by

$$\tau_{C_g(D)} = 1 - \frac{4\theta^2}{(e^\theta-1)^2}\int_0^1\int_0^1 D_1(x,y)D_2(x,y)e^{2\theta D(x,y)}\,\mathrm{d}x\mathrm{d}y \tag{27}$$

and

$$\rho_{C_g(D)} = \frac{12\theta^2}{(e^\theta-1)^3}\int_0^1\int_0^1 (e^{\theta D(x,y)}-1)e^{\theta(x+y)}\,\mathrm{d}x\mathrm{d}y - 3,$$

respectively.

The next result provides the upper and lower tail dependence of model (26).

Proposition 4.14. If $\lambda_L(D)$ and $\lambda_U(D)$ exist, then $\lambda_L(C_g(D)) = \lambda_L(D)$ and $\lambda_U(C_g(D)) = \lambda_U(D)$.

Proof. The function $g(t) = \frac{e^{\theta t}-1}{e^{\theta}-1}$ is an isomorphism and also

$$\lim_{t\to 0^+} \frac{g(t)}{t^\alpha} = \lim_{t\to 0^+} \frac{e^{\theta t}-1}{t^\alpha(e^\theta-1)} = \begin{cases} 0, & 0<\alpha<1, \\ \dfrac{\theta}{e^\theta-1}, & \alpha=1, \\ \infty, & \alpha>1, \end{cases} \tag{28}$$

and

$$\lim_{t\to 1^-} \frac{1-g(t)}{(1-t)^\alpha} = \lim_{t\to 1^-} \frac{(e^\theta-e^{\theta t})}{(e^\theta-1)(1-t)^\alpha} = \begin{cases} 0, & 0<\alpha<1, \\ \dfrac{\theta e^\theta}{e^\theta-1}, & \alpha=1, \\ \infty, & \alpha>1. \end{cases} \tag{29}$$

So, by (28), (29) and Theorems 4.2-4.3 of Durante et al. (2010) the proof is completed. □

Example 4.15. If $D(u,v) = \Pi(u,v)$, then we have the following results.

i) For all $\theta > 0$; $C_g(\Pi) = \dfrac{1}{e^\theta-1}\left\{exp(\dfrac{\ln(1+(e^\theta-1)u)\ln(1+(e^\theta-1)v)}{\theta}) - 1\right\}$.
Since $\Pi(\cdot,\cdot)$ is TP_2 referring to Proposition 4.4, $C_g(D)(\cdot,\cdot)$ is TP_2.

ii) By using Proposition 4.14 we obtain

$$\lambda_L(C_g(\Pi)) = \lambda_L(\Pi) = 0 \quad \text{and} \quad \lambda_U(C_g(\Pi)) = \lambda_U(\Pi) = 0.$$

iii) From the equation (27), we have

$$\begin{aligned} \tau_{C_g(\Pi)} &= 1 - \frac{4\theta^2}{(e^\theta-1)^2}\int_0^1\int_0^1 xye^{2\theta xy}\,\mathrm{d}x\mathrm{d}y \\ &= 1 - \frac{2\theta}{(e^\theta-1)^2}\left\{\int_0^1 e^{2\theta y}dy - \frac{1}{2\theta}\int_0^1 (\frac{e^{2\theta y}-1}{y})\,\mathrm{d}y\right\} \\ &= 1 - \frac{1}{(e^\theta-1)^2}\left\{e^{2\theta}-1-\int_0^1 \frac{e^{2\theta y}-1}{y}\,\mathrm{d}y\right\}. \end{aligned} \tag{30}$$

By replacing $e^{\theta t} = \sum_{k=0}^\infty \frac{(\theta t)^k}{k!}$ in (30) we get

$$\begin{aligned} \tau_{C_g(\Pi)} &= 1 - \frac{1}{(e^\theta-1)^2}\left\{e^{2\theta}-1-\sum_{k=1}^\infty \frac{(2\theta)^k}{k!}\int_0^1 y^{k-1}\,\mathrm{d}y\right\} \\ &= 1 - \frac{1}{(e^\theta-1)^2}\left\{e^{2\theta}-1-\sum_{k=1}^\infty \frac{(2\theta)^k}{k(k!)}\right\}. \end{aligned}$$

Moreover, by applying Proposition 4.9, we have

$$\begin{aligned}\rho_{C_g(\Pi)} &= \frac{12\theta^2}{(e^\theta-1)^3}\int_0^1\int_0^1 (e^{\theta xy}-1)e^{\theta(x+y)}\,\mathrm{d}x\mathrm{d}y - 3\\ &= \frac{12\theta^2}{(e^\theta-1)^3}\int_0^1\int_0^1 (e^{\theta xy+\theta x+\theta y}-e^{\theta x+\theta y})\,\mathrm{d}x\mathrm{d}y - 3\\ &= \frac{12\theta}{(e^\theta-1)^3}\left\{\int_0^1 \frac{e^{2\theta y+1}-e^{\theta y}}{1+y}dy - \int_0^1 (e^{\theta y+\theta}-e^{\theta y})\,\mathrm{d}y\right\} - 3.\end{aligned}$$

Using $e^{\theta t} = \sum_{k=0}^{\infty}\frac{(\theta t)^k}{k!}$, the Spearman's rho of $C_g(\Pi)$ is given by,

$$\rho_{C_g(\Pi)} = \frac{12\theta}{(e^\theta-1)^3}\left\{1-\frac{(e^\theta-1)^2}{\theta}+\sum_{k=2}^{\infty}\frac{1}{k!}\int_0^1\frac{(2\theta y+\theta)^k-(\theta y)^k}{1+y}\right\} - 3.$$

Example 4.16. Let $G(\cdot,\cdot)$ be the Gumbel survival copula with parameter $\alpha \geq 1$, then the following statements hold.

i) $C_g(G) = \frac{1}{e^\theta-1}\{\exp\{r(u)+r(v)-\theta+\theta\exp(-((-\ln(1-r(u)))^\alpha$
$+(-\ln(1-r(v)))^\alpha)^{1/\alpha})-1\}$,
where $r(u) = \ln(1+u(e^\theta-1))$ and $\theta > 0$.

ii) According to Proposition 4.4, $C_g(G)$ is TP_2.

iii) By Proposition 4.14,

$$\lambda_L(C_g(G)) = \lambda_L(G) = 0 \quad \text{and} \quad \lambda_U(C_g(G)) = \lambda_L(G) = 2-2^{1/\theta}.$$

iv) Since $\Pi \prec_c G \prec_c M$, thus by Proposition 4.8 we have,

$$\tau_{C_g(\Pi)} \leq \tau_{C_g(G)} \leq 1, \quad \text{and} \quad \rho_{C_g(\Pi)} \leq \rho_{C_g(G)} \leq 1.$$

5. SIMULATION RESULTS

It is known that one of the effective ways for evaluating the behaviour of dependent models is to simulate random data from their corresponding copula. Zhang et al. [29] presented an algorithm to generate random data from $U = \max\{X_{1,1},\ldots,X_{1,N}\}$ and $V = \max\{X_{2,1},\ldots,X_{2,N}\}$ where $(X_{1,i},X_{2,i}), i = 1,\ldots,N$ are iid random variables and N follows a geometric distribution and is independent of $(X_{1,1},X_{2,1})$. These assumptions hold for our proposed model, so we can use the algorithm proposed by Zhang et al. [29] to generate data from Examples 4.12 – 4.16.

Algorithm 1. Let $D(\cdot,\cdot)$ be the appropriate copula of the proposed model, then we can follow the following steps for generating data:

1. For given θ, generate a random variable from a discrete distribution with parameter θ, say N.

2. For $i = 1 \dots, N$ and $j = 1 \dots, m$ generate random vectors $(Z_{i,j}^{(1)}, Z_{i,j}^{(2)})$ from their corresponding copula $D(\cdot, \cdot)$ with known parameter α.

3. For $i = 1 \dots, N$ calculate $X_i = \max\{Z_{i,1}^{(1)}, \dots, Z_{i,m}^{(1)}\}$ and $Y_i = \max\{Z_{i,1}^{(2)}, \dots, Z_{i,m}^{(2)}\}$.

4. Calculate $T_1 = \min\{X_1, \dots, X_N\}$ and $T_2 = \min\{Y_1, \dots, Y_N\}$.

5. For given $g(\cdot)$, compute $H_1(T_1) = g(1 - T_1^m)$ and $H_2(T_2) = g(1 - T_2^m)$.

6. Set $U = H_1(T_1)$, $V = H_2(T_2)$,

then (U, V) is a random vector generated from $C_g(D)$.

Table 1 displays Kendall's tau and Spearman's rho of Example 4.12 based on a sample of size 100 when $m = 5$, $\alpha = 8$ and 10^4 data has been replicated from Example 4.12 by using Algorithm 1. The results in Table 1 indicate that, $\frac{\rho}{\tau} < \frac{3}{2}$ and $\rho \geq \tau \geq 0$. Also, from Table 1, it is observed that by increasing θ both the Kendall's tau and Spearman's rho increase.

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
τ	0.803	0.830	0.844	0.851	0.858	0.863	0.866	0.870	0.870	0.875
ρ	0.940	0.954	0.961	0.964	0.967	0.969	0.971	0.972	0.972	0.974
$\frac{\rho}{\tau}$	1.170	1.149	1.139	1.133	1.127	1.124	1.121	1.118	1.118	1.114

Tab. 1. The simulation results of Example 4.12.

Table 2 illustrates the values of Kendall's tau and Spearman's rho of Example 4.15 based on a sample of size 100, $m = 5$, when 10^4 data which has been simulated from Example 4.15 by using Algorithm 1. From Table 2, we have $\frac{\rho}{\tau} < \frac{3}{2}$ and $\rho \geq \tau \geq 0$. It is also observed that, from Table 2, both the Kendall's tau and Spearman's rho first increase and then decrease as θ increases.

θ	1	2	3	4	5	6	7	8	9	10	50	100
τ	0.026	0.052	0.067	0.077	0.079	0.077	0.071	0.064	0.057	0.052	0.012	0.004
ρ	0.039	0.077	0.100	0.115	0.118	0.115	0.106	0.096	0.086	0.078	0.017	0.007
$\frac{\rho}{\tau}$	1.498	1.497	1.498	1.496	1.496	1.496	1.495	1.496	1.495	1.496	1.496	1.496

Tab. 2. The simulation results of Example 4.15.

Table 3 displays Kendall's tau and Spearman's rho of Example 4.16 for a sample of size 100 when $m = 5$, $\alpha = 8$ and 10^4 data which has been replicated from Example 4.16 by using Algorithm 1. As we can see $\frac{\rho}{\tau} < \frac{3}{2}$ and $\rho \geq \tau \geq 0$. Furthermore by increasing θ the Kendall's tau and Spearman's rho decrease.

θ	1	5	10	50	100
τ	0.734	0.724	0.695	0.635	0.616
ρ	0.899	0.890	0.868	0.816	0.800
$\frac{\rho}{\tau}$	1.224	1.229	1.249	1.286	1.297

Tab. 3. The simulation results of Example 4.16.

6. CONCLUSIONS

In this paper, we have tackled a new dependent model which has been constructed based on the structure of two parallel-series systems. Some reliability properties of the proposed model such as conditional tail expectation has been calculated. Furthermore, by considering extreme-value copulas as their base copula their dependence properties such as their structure, Kendall's tau, Spearman's rho and tail dependences have been investigated in special cases. It should be noted that the proposed model can be extended to the case of a general number of systems.

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