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SOME REMARKS ON DESCRIPTIVE CHARACTERIZATIONS OF THE STRONG MCSHANE INTEGRAL

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Abstract. We present the full descriptive characterizations of the strong McShane integral (or the variational McShane integral) of a Banach space valued function $f\colon W \to X$ defined on a non-degenerate closed subinterval W of $\mathbb{R}^m$ in terms of strong absolute continuity or, equivalently, in terms of McShane variational measure $V_{\mathcal{M}}F$ generated by the primitive $F\colon \mathcal{I}_W \to X$ of f, where $\mathcal{I}_W$ is the family of all closed non-degenerate subintervals of W.

Keywords: strong McShane integral; McShane variational measure; Banach space, m-dimensional Euclidean space; compact non-degenerate m-dimensional interval

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1. Introduction and preliminaries

In the monograph [21] of Štefan Schwabik and Ye Guoju, a full characterization of the strong McShane integral of Banach-space valued functions defined on a compact non-degenerate subinterval of $\mathbb{R}$ is given, see Theorem 7.4.14. There is also a full descriptive characterization of the variational McShane integral in [12], Theorem 2.5.

In [13], Yeong gives some full characterizations of the strong McShane integral of Banach-space valued functions defined on a compact non-degenerate subinterval of $\mathbb{R}^m$.

In this paper, we present the full descriptive characterizations of the strong McShane integral of a Banach space valued function $f\colon W \to X$ defined on a non-degenerate closed subinterval W of $\mathbb{R}^m$ in terms of strong absolute continuity or, equivalently, in terms of McShane variational measure $V_{\mathcal{M}}F$ generated by the primitive $F\colon \mathcal{I}_W \to X$ of f, see Theorem 2.8.

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Throughout this paper, X denotes a real Banach space with the dual X^* and W denotes a compact non-degenerate subinterval of the m-dimensional Euclidean space $\mathbb{R}^m$. The Euclidean space $\mathbb{R}^m$ is equipped with the maximum norm. $B_m(t,r)$ is the open ball in $\mathbb{R}^m$ with center t and radius $r>0$. We denote by $\mathscr{B}(\mathbb{R}^m)$ the Borel σ-algebra on $\mathbb{R}^m$ and by $\mathcal{L}(\mathbb{R}^m)$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$. We put

$$\mathcal{L}(W)=\{W\cap E\colon\ E\in\mathcal{L}(\mathbb{R}^m)\}\quad\text{and}\quad\mathscr{B}(W)=\{W\cap B\colon\ B\in\mathscr{B}(\mathbb{R}^m)\}.$$

The Lebesgue measure on $\mathcal{L}(W)$ is denoted by λ and the Lebesgue measure of a Lebesgue measurable set $E\in\mathcal{L}(W)$ is denoted by $|E|$. The phrase "at almost all" always referes to λ.

If μ is a measure on $\mathcal{L}(W)$, then by $\mu\ll\lambda$ we mean that $|E|=0$ implies $\mu(E)=0$. A vector measure $\nu\colon\ \mathcal{L}(W)\to X$ is said to be a *countable additive vector measure* if ν is countable additive in the norm topology of X. A countable additive vector measure ν is said to be λ-continuous if $|E|=0$ implies $\nu(E)=0$. The *variation* of a countable additive vector measure ν is denoted by $|\nu|$; ν is said to be of *bounded variation* on W if $|\nu|(W)<\infty$.

Let $\alpha=(a_1,\dots,a_m)$ and $\beta=(b_1,\dots,b_m)$ with $-\infty<a_j<b_j<\infty$ for $j=1,\dots,m$. The set $[\alpha,\beta]=\prod\limits_{j=1}^{m}[a_j,b_j]$ is called a *closed non-degenerate interval* in $\mathbb{R}^m$, while $[\alpha,\beta)=\prod\limits_{j=1}^{m}[a_j,b_j)$ is said to be a *half-closed interval* (*or brick*) in $\mathbb{R}^m$. By $\mathscr{B}_r(\mathbb{R}^m)$, the family of all bricks in $\mathbb{R}^m$ is denoted. In particular, if $b_1-a_1=\ldots=b_m-a_m$, then $I=[\alpha,\beta]$ is called a *cube* and we set $l_I=b_1-a_1$. In this case, $|I|=(l_I)^m$. We denote by $\mathcal{I}_W$ the family of all closed non-degenerate subintervals of W.

Two intervals $I,J\in\mathcal{I}_W$ are said to be *non-overlapping* if $I^\circ\cap J^\circ=\emptyset$, where I° denotes the *interior* of I. A function $F\colon\ \mathcal{I}_W\to X$ is said to be an *additive interval function* if for each pair of non-overlapping intervals $I,J\in\mathcal{I}_W$ with $I\cup J\in\mathcal{I}_W$, we have

$$F(I\cup J)=F(I)+F(J).$$

Definition 1.1. An additive interval function $F\colon\ \mathcal{I}_W\to X$ is said to be *strongly absolutely continuous* (sAC) on W if for each $\varepsilon>0$ there exists $\eta>0$ such that for each finite collection $\{I_1,\dots,I_p\}$ of pairwise non-overlapping subintervals in $\mathcal{I}_W$ we have

$$\sum_{i=1}^{p}|I_i|<\eta\Rightarrow\sum_{i=1}^{p}\|F(I_i)\|<\varepsilon.$$

Replacing the last inequality with $\Bigl\|\sum\limits_{i=1}^{p}F(I_i)\Bigr\|<\varepsilon$, we obtain the notion of *absolute continuity* (AC) on W.

Definition 1.2. A finite collection $\{I_1, \ldots, I_p\}$ of pairwise non-overlapping intervals in $\mathcal{I}_W$ is said to be a *division* of W if $\bigcup_{i=1}^{p} I_i = W$. $\mathscr{D}_W$ denotes the family of all divisions of interval W. The *total variation* $V_F(W)$ of an additive interval function $F\colon \mathcal{I}_W \to X$ on W is defined as

$$V_F(W) = \sup\Bigl\{ \sum_{J\in\mathscr{D}} \|F(J)\| \colon \mathscr{D} \in \mathscr{D}_W \Bigr\}.$$

If $V_F(W) < \infty$, then F is said to be of *bounded variation on* W.

The following lemma has been proven in [14], Lemma 10.3.7 for the real valued functions, but the proof works also for Banach-space valued functions after trivial changes.

Lemma 1.3. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. If F is sAC on W, then F is of bounded variation on W.*

Definition 1.4. Assume that a point $t \in W$ and a function $F\colon \mathcal{I}_W \to X$ are given. We set

$$\mathcal{I}_W(t) = \{I \in \mathcal{I}_W \colon\ t \in I,\ I \text{ is a cube}\}.$$

We say that F is *cubic derivable* at t if there exists a vector $F'_{\rm c}(t) \in X$ such that

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{F(I)}{|I|} = F'_{\rm c}(t).$$

$F'_{\rm c}(t)$ is said to be the *cubic derivative* of F at t. The *cubic derivative* of F at t is a generalization of the derivative $F'(t)$ defined in [13], Definition 3.2.

A function $f\colon \mathbb{R}^m \to \mathbb{R}$ is called *locally integrable* if f is Borel measurable function and

$$\int_K |f(s)|\,\mathrm{d}\lambda < \infty \text{ for every bounded measurable set } K \in \mathscr{B}(\mathbb{R}^m).$$

The following theorem is the Lebesgue Differentiation Theorem, c.f. Theorem 3.21 in [7].

Theorem 1.5. *If a function $f\colon \mathbb{R}^m \to \mathbb{R}$ is locally integrable, then there exists $Z \in \mathscr{B}(\mathbb{R}^m)$ with $|Z| = 0$ such that*

$$\lim_{r\to 0} \frac{1}{|E_r|} \int_{E_r} |f(s) - f(t)|\,\mathrm{d}\lambda(s) = 0 \quad \text{for all } t \in \mathbb{R}^m \setminus Z$$

whenever $(E_r)_{r>0}$ is a family that shrinks nicely to t.

A family $(E_r)_{r>0}$ of Borel subsets of $\mathbb{R}^m$ is said to *shrink nicely* to $t \in \mathbb{R}^m$ if

▷ $E_r \subset B_m(t,r)$ for each r,

▷ there is a constant $\alpha > 0$, independent of r, such that $|E_r| > \alpha |B_m(t,r)|$,

c.f. [7], page 98.

A pair (t, I) of a point $t \in W$ and an interval $I \in \mathcal{I}_W$ is called an *$\mathcal{M}$-tagged interval* in W, t is the tag of I. A finite collection $\{(t_i, I_i)\colon\ i = 1, \dots, p\}$ of $\mathcal{M}$-tagged intervals in W is called an *$\mathcal{M}$-partition* in W if $\{I_i\colon\ i = 1, \dots, p\}$ is a collection of pairwise non-overlapping intervals in $\mathcal{I}_W$. Given $Z \subset W$, a positive function $\delta\colon\ Z \to (0, \infty)$ is called a *gauge* on Z. We say that an $\mathcal{M}$-partition $\pi = \{(t_i, I_i)\colon\ i = 1, \dots, p\}$ in W is

▷ a partition of W if $\bigcup\limits_{i=1}^{p} I_i = W$;

▷ Z-tagged if $\{t_1, \dots, t_p\} \subset Z$;

▷ δ-fine if for each $(t, I) \in \pi$ we have $I \subset B_m(t, \delta(t))$.

Definition 1.6. A function $f\colon\ W \to X$ is said to be *McShane integrable* on W if there is a vector $x_f \in X$ such that for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\Bigg\| \sum_{(t,I)\in\pi} f(t)|I| - x_f \Bigg\| < \varepsilon.$$

In this case, the vector x_f is said to be the *McShane integral* of f on W and we set $x_f = (M)\int_W f\,\mathrm{d}\lambda$. The function f is said to be *McShane integrable* over a subset $A \subset W$ if the function $f \cdot \mathbb{1}_A\colon\ W \to X$ is McShane integrable on W, where $\mathbb{1}_A$ is the characteristic function of the set A. The McShane integral of f over A will be denoted by $(M)\int_A f\,\mathrm{d}\lambda$. If $f\colon\ W \to X$ is McShane integrable on W, then by Theorem 4.1.6 in [21] the function f is McShane integrable on each $E \in \mathcal{L}(W)$.

Definition 1.7. The function $f\colon\ W \to X$ is said to be *variationally McShane integrable* (*or strongly McShane integrable*) on W if there exists an additive interval function $F\colon\ \mathcal{I}_W \to X$ such that for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\sum_{(t,I)\in\pi} \|f(t)|I| - F(I)\| < \varepsilon.$$

The function F is said to be the primitive of f. Clearly, if f is variationally McShane integrable with the primitive F, then f is McShane integrable, and by Proposition 3.6.16 in [21] we also have

$$F(I) = (M)\int_I f\,\mathrm{d}\lambda \quad \text{for every } I \in \mathcal{I}_W.$$

For more information about the McShane integral we refer to [21], [25], [5], [8], [9]–[11], [16], [15], [26] and [1].

Definition 1.8. Given an additive interval function $F\colon \mathcal{I}_W \to X$, a subset $Z \subset W$ and a gauge δ on Z, we define

$$V_{\mathcal{M}}F(Z,\delta) = \sup\Bigl\{\sum_{(t,I)\in\pi} \|F(I)\|\colon\ \pi \text{ is a } Z\text{-tagged } \delta\text{-fine } \mathcal{M}\text{-partition in } W\Bigr\}.$$

Then we set

$$V_{\mathcal{M}}F(Z) = \inf\{V_{\mathcal{M}}F(Z,\delta)\colon\ \delta \text{ is a gauge on } Z\}.$$

The set function $V_{\mathcal{M}}F$ is said to be the *McShane variational measure generated by* F.

The set function $V_{\mathcal{M}}F$ is a Borel metric outer measure on W, see [4] or [23]. The McShane variational measure have been used extensively for studying the primitives (indefinite integrals) of real functions. See e.g. the paper [4] of Di Piazza, the book [14] of Lee Tuo-Yeong, [20] of Pfeffer for relations to integration and the fundamental general work [24] of Thomson. The following lemma has been proven by Di Piazza in [4], Proposition 1 (there she considers real valued functions, but the proof works also for vector valued functions, after trivial changes).

Lemma 1.9. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. Then the following statements are equivalent:*

(i) *F is sAC on W;*
(ii) $V_{\mathcal{M}}F \ll \lambda$.

A function $f\colon W \to X$ is said to be *weakly measurable* if for each $x^* \in X^*$ the real function $x^* \circ f$ is Lebesgue measurable; f is said to be *measurable* if there is a sequence $f_n\colon W \to X$ of simple measurable functions such that

$$\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0 \quad \text{at almost all } t \in W.$$

The function $f\colon W \to X$ is said to be *Bochner integrable* on W if f is measurable and there exists a sequence (f_n) of simple measurable functions such that

$$\lim_{n\to\infty} \int_W \|f(t) - f_n(t)\| \,\mathrm{d}\lambda = 0.$$

In this case, $(B)\int_E f\,\mathrm{d}\lambda$ is defined for each Lebesgue measurable set $E \in \mathcal{L}(W)$ as

$$(B)\int_E f\,\mathrm{d}\lambda = \lim_{n\to\infty} (B)\int_E f_n\,\mathrm{d}\lambda,$$

where $(B)\int_E f_n\,\mathrm{d}\lambda$ is defined in the usual way.

The function $f\colon W \to X$ is said to be *Pettis integrable* on W if $x^* \circ f$ is Lebesgue integrable on W for each $x^* \in X^*$ and for every Lebesgue measurable set $E \in \mathcal{L}(W)$ there is a vector $\nu(E) \in X$ such that

$$x^*(\nu(E)) = \int_E (x^* \circ f)\,\mathrm{d}\lambda \quad \text{for all } x^* \in X^*.$$

The vector $\nu(E)$ is then called the *Pettis integral* of f over E and we set $\nu(E) = (P)\int_E f\,\mathrm{d}\lambda$. We refer to [3], [17]–[19], [22] and [2] for Pettis integral.

2. The main result

The main result is Theorem 2.8. Let us start with some auxiliary lemmas.

Lemma 2.1. *If a function $f\colon W \to \mathbb{R}$ is Lebesgue integrable on W, then*

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|}\int_I |f(s)-f(t)|\,\mathrm{d}\lambda(s) = 0 \quad \textit{for almost all } t \in W.$$

Consequently,

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|}\int_I f(s)\,\mathrm{d}\lambda(s) = f(t) \quad \textit{for almost all } t \in W. \tag{2.1}$$

Proof. Since f is Lebesgue integrable on W, there exists a Borel measurable function $h\colon W \to \mathbb{R}$ such that it is Lebesgue integrable on W and $h(t) = f(t)$ for almost all $t \in W$. Consider a function $g\colon \mathbb{R}^m \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} h(t) & \text{if } t \in W,\\ 0 & \text{if } t \in \mathbb{R}^m \setminus W.\end{cases}$$

Since g is locally integrable, by Theorem 1.5 there exists $Z \in \mathscr{B}(\mathbb{R}^m)$ with $|Z| = 0$ such that

$$\lim_{r\to 0} \frac{1}{|E_r|}\int_{E_r} |g(s)-g(t)|\,\mathrm{d}\lambda(s) = 0 \quad \text{for all } t \in \mathbb{R}^m \setminus Z,$$

whenever $(E_r)_{r>0}$ is a family that shrinks nicely to t.

Fix an arbitrary $t \in W \setminus Z$. For each real positive number $r > 0$ we can choose an arbitrary cube $I_r \in \mathcal{I}_W(t)$ such that $r = l(I_r)$. Note that

$$I_r \subset B(t,r) \quad \text{and} \quad |I_r| = r^m > \frac{1}{2^{m+1}}|B_m(t,r)|$$

whenever $r > 0$. Thus, the family $(I_r)_{r>0}$ shrinks nicely to t. Therefore

$$\lim_{r\to 0} \frac{1}{|I_r|} \int_{I_r} |g(s) - g(t)| \, \mathrm{d}\lambda(s) = 0,$$

and since t and $(I_r)_{r>0}$ are arbitrary, it follows that

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|} \int_I |g(s) - g(t)| \, \mathrm{d}\lambda(s) = 0 \quad \text{for all } t \in W \setminus Z.$$

Hence,

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|} \int_I |h(s) - h(t)| \, \mathrm{d}\lambda(s) = 0 \quad \text{for all } t \in W \setminus Z.$$

Further, since $h(t) = f(t)$ for almost all $t \in W$, it follows that

$$\lim_{\substack{I\in\mathcal{I}_W(t)\\ |I|\to 0}} \frac{1}{|I|} \int_I |f(s) - f(t)| \, \mathrm{d}\lambda(s) = 0 \quad \text{for almost all } t \in W.$$

The last result together with

$$\left| \frac{1}{|I|} \int_I f(s) \, \mathrm{d}\lambda(s) - f(t) \right| \leqslant \frac{1}{|I|} \int_I |f(s) - f(t)| \, \mathrm{d}\lambda(s)$$

yields (2.1), and this ends the proof. □

As in [6], page 156, define a function $\varrho\colon \mathcal{L}(W) \times \mathcal{L}(W) \to [0, \infty)$ by

$$\varrho(U, V) = |U \Delta V| \quad \text{for each } (U, V) \in \mathcal{L}(W) \times \mathcal{L}(W).$$

It is not difficult to check that ϱ is a semimetric in $\mathcal{L}(W)$, i.e. ϱ satisfies the following conditions:

▷ $\varrho(U, U) = 0$,
▷ $\varrho(U, V) = \varrho(V, U)$,
▷ $\varrho(U, V) \leqslant \varrho(U, H) + \varrho(H, V)$,

whenever $U, V, H \in \mathcal{L}(W)$.

Lemma 2.2. *If $\nu\colon \mathcal{L}(W) \to X$ is a countably additive λ-continuous vector measure, then*

$$\nu(\mathcal{I}_W) = \{\nu(I)\colon I \in \mathcal{I}_W\}$$

is a separable set in X.

P r o o f. We denote by $\mathcal{Q}_W$ the family of all intervals in $\mathcal{I}_W$ with vertices having rational coordinates. It is easy to see that

$$\mathcal{I}_W \subset \overline{\mathcal{Q}}_W^{\varrho}, \tag{2.2}$$

where $\overline{\mathcal{Q}}_W^{\varrho}$ is the closure of $\mathcal{Q}_W$ in the semimetric space $(\mathcal{L}(W), \varrho)$. We are going to show that

$$\nu(\mathcal{I}_W) \subset \overline{\nu(\mathcal{Q}_W)}^{\|\cdot\|}, \tag{2.3}$$

where

$$\nu(\mathcal{Q}_W) = \{\nu(I)\colon\ I \in \mathcal{Q}_W\}$$

and $\overline{\nu(\mathcal{Q}_W)}^{\|\cdot\|}$ is the closure of $\nu(\mathcal{Q}_W)$ in the Banach space X. To see this, let $\nu(I) \in \nu(\mathcal{I}_W)$. Then by (2.2), there exists a sequence $(I_k) \subset \mathcal{Q}_W$ such that

$$\lim_{k\to\infty} (|I \setminus I_k| + |I_k \setminus I|) = \lim_{k\to\infty} \varrho(I_k, I) = 0$$

and therefore by Theorem I.2.1 in [3], we obtain

$$\lim_{k\to\infty} \nu(I \setminus I_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} \nu(I_k \setminus I) = 0. \tag{2.4}$$

Since

$$I = (I \setminus I_k) \cup (I \cap I_k) \quad \text{and} \quad I_k = (I_k \setminus I) \cup (I \cap I_k),$$

it follows that

$$\|\nu(I) - \nu(I_k)\| = \|\nu(I \setminus I_k) - \nu(I_k \setminus I)\| \leqslant \|\nu(I \setminus I_k)\| + \|\nu(I_k \setminus I)\|.$$

The last result together with (2.4) yields that

$$\lim_{k\to\infty} \|\nu(I) - \nu(I_k)\| = 0.$$

This means that (2.3) holds, and this ends the proof. □

The next lemma is proved by using Caratheodory-Hahn-Kluvanek Extension theorem, see Theorem I.5.2 in [3]. We recall that a collection $\mathcal{E}$ of subsets of W is said to be an elementary family if

- ▷ $\emptyset \in \mathcal{E}$,
- ▷ if $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$,
- ▷ if $E \in \mathcal{E}$, then $E^{\mathrm{c}} = W \setminus E$ is a finite disjoint union of members of $\mathcal{E}$,

c.f. [7], page 23.

Lemma 2.3. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. If F is AC on W, then there exists a unique countably additive λ-continuous vector measure $F_{\mathcal{L}}\colon \mathcal{L}(W) \to X$ such that*

$$F(I) = F_{\mathcal{L}}(I) \quad \textit{for all } I \in \mathcal{I}_W.$$

Moreover, if F is sAC on W, then $F_{\mathcal{L}}$ is of bounded variation on W.

Proof. We set

$$\mathscr{B}_r(W) = \{W \cap B_r\colon\ B_r \in \mathscr{B}_r(\mathbb{R}^m)\}.$$

It is easy to see that $\mathcal{E} = \mathscr{B}_r(W) \cup \{\emptyset\}$ is an elementary family. Therefore, by Proposition 1.7 in [7], it follows that the collection $\mathscr{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra. Since

$$\mathscr{B}(W) = \sigma(\mathscr{A}),$$

where $\sigma(\mathscr{A})$ is the σ-algebra generated by $\mathscr{A}$, and since the closure of $\mathscr{A}$ with respect to ϱ is a σ-algebra, it follows that $\mathscr{A}$ is a dense subset of $\mathscr{B}(W)$ with respect to ϱ.

Assume that an arbitrary nonempty set $A \in \mathscr{A}$ is given. If $\{I_1, \ldots, I_p\}$ and $\{J_1, \ldots, J_q\}$ are finite collections of pairwise disjoint bricks in $\mathscr{B}_r(W)$ such that

$$A = I_1 \cup \ldots \cup I_p = J_1 \cup \ldots \cup J_q,$$

then

$$B = \{I_i \cap J_j\colon\ I_i \cap J_j \neq \emptyset,\ i = 1, \ldots, p,\ j = 1, \ldots, q\}$$

is a finite collection of pairwise disjoint bricks in $\mathscr{B}_r(W)$ and $A = \bigcup\limits_{I \in B} I$. Then, since F is additive and any two representations of A as a finite disjoint union of bricks have a common refinement, the sum

$$F(\overline{I}_1) + \ldots + F(\overline{I}_p)$$

is independent of the particular family $\{I_1, \ldots, I_p\}$ of pairwise disjoint bricks whose union is A, where $\overline{I}_i$ is the closure of I_i. Thus, we can define vector $F_{\mathscr{A}}(A)$ by equation

$$F_{\mathscr{A}}(A) = F(\overline{I}_1) + \ldots + F(\overline{I}_p).$$

In particular, we define $F_{\mathscr{A}}(\emptyset) = 0$.

From the fact that F is AC it follows that

$$\lim_{\substack{(A \in \mathscr{A}) \\ |A| \to 0}} F_{\mathscr{A}}(A) = 0.$$

Hence, $F_{\mathscr{A}}$ is a strongly additive and countably additive vector measure on $\mathscr{A}$. Therefore by Caratheodory-Hahn-Kluvanek Extension theorem, Theorem I.5.2 in [3], $F_{\mathscr{A}}$ has a unique countable additive λ-continuous extension $F_{\mathscr{B}}\colon \mathscr{B}(W) \to X$, and since

$$F_{\mathscr{B}}(B') - F_{\mathscr{B}}(B'') = F_{\mathscr{B}}(B' \setminus B'') - F_{\mathscr{A}}(B'' \setminus B'), \quad B', B'' \in \mathscr{B}(W),$$

it follows that $F_{\mathscr{B}}$ is uniformly continuous on $\mathscr{B}(W)$ with respect to ϱ.

Since $F_{\mathscr{B}}$ is a countably additive λ-continuous vector measure on $\mathscr{B}(W)$, it has a unique countable additive λ-continuous extension $F_{\mathcal{L}}\colon \mathcal{L}(W) \to X$.

We now assume that F is sAC on W. It is enough to show that $F_{\mathscr{B}}$ is of bounded variation on W. To see this, let us consider a finite collection $\{B_i\colon i = 1, 2, \ldots, p\}$ of pairwise disjoint members of $\mathscr{B}(W)$. Since $F_{\mathscr{B}}$ is uniformly continuous with respect to ϱ on $\mathscr{B}(W)$, given $0 < \varepsilon < 1$ there exists $\delta > 0$ such that for each $B, B' \in \mathscr{B}(W)$ we have

$$\varrho(B, B') = |B \Delta B'| < \delta \Rightarrow \|F_{\mathscr{B}}(B) - F_{\mathscr{B}}(B')\| < \frac{\varepsilon}{2p^2}.$$

Since $\mathscr{A}$ is dense in $\mathscr{B}(W)$ with respect to ϱ, for each B_i there exists an $A_i \in \mathscr{A}$ such that

$$\varrho(B_i, A_i) = |B_i \Delta A_i| < \frac{\delta}{2},$$

and since

$$(A_i \cap A_j) \setminus B_i \subset A_i \Delta B_i, \quad (A_i \cap A_j) \setminus B_j \subset A_j \Delta B_j$$

and

$$A_i \cap A_j \subset ((A_i \cap A_j) \setminus B_i) \cup ((A_i \cap A_j) \setminus B_j),$$

it follows that

$$\varrho((A_i \cap A_j), \emptyset) = |A_i \cap A_j| < \delta, \quad i \neq j.$$

Therefore, if we set

$$C_1 = A_1, \quad C_2 = A_2 \setminus A_1, \quad \ldots, \quad C_p = A_p \setminus \bigcup_{k=1}^{p-1} A_k,$$

then

$$\begin{aligned}
\sum_{i=1}^{p} \|F_{\mathscr{B}}(B_i)\| &\leqslant \sum_{i=1}^{p} \|F_{\mathscr{B}}(B_i) - F_{\mathscr{B}}(A_i)\| + \sum_{i=1}^{p} \|F_{\mathscr{B}}(A_i)\| < \sum_{i=1}^{p} \|F_{\mathscr{A}}(A_i)\| + \frac{\varepsilon}{2} \\
&\leqslant \sum_{i=1}^{p} \|F_{\mathscr{A}}(C_i)\| + \sum_{\substack{i \neq j \\ i,j}} \|F_{\mathscr{A}}(A_i \cap A_j)\| + \frac{\varepsilon}{2} \\
&< V_F(W) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = V_F(W) + \varepsilon < V_F(W) + 1.
\end{aligned}$$

Since F is sAC on W, the last result together with Lemma 1.3 yields

$$|F_{\mathscr{B}}|(W) \leqslant V_F(W) + 1 < \infty.$$

Thus, $F_{\mathscr{B}}$ is of bounded variation on W, and this ends the proof. □

The next lemma gives full descriptive characterizations of Lebesgue integral.

Lemma 2.4. *Let $F\colon \mathcal{I}_W \to \mathbb{R}$ be an additive interval function and let $f\colon W \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(i) *F is AC on W;*
(ii) *F is sAC on W;*
(iii) *$V_{\mathcal{M}} \ll \lambda$;*
(iv) *F is AC on W, $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$;*
(v) *F is sAC on W, $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$;*
(vi) *$V_{\mathcal{M}} \ll \lambda$, $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$;*
(vii) *f is Lebesgue integrable on W with the primitive F, i.e.*

$$F(I) = \int_I f \, \mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

P r o o f. Since F is a real valued function, it is easy to see that if F is AC on W, then F is sAC on W. Therefore (i) ⇔ (ii) and (iv) ⇔ (v). By virtue of Lemma 1.9 it follows that (ii) ⇔ (iii) and (v) ⇔ (vi).

(ii) ⇒ (vii): Assume that F is sAC on W. Then by Lemma 2.3 there exists a unique countably additive λ-continuous vector measure $F_{\mathcal{L}}\colon \mathcal{L}(W) \to \mathbb{R}$ of bounded variation on W such that $F_{\mathcal{L}}(I) = F(I)$ for all $I \in \mathcal{I}_W$. Therefore, by Lebesgue-Radon-Nikodym theorem, see Theorem 3.8 in [7], there exists a Lebesgue integrable function $f\colon W \to \mathbb{R}$ such that $F_{\mathcal{L}}(E) = \int_E f \, \mathrm{d}\lambda$ for all $E \in \mathcal{L}(W)$. In particular, we have $F(I) = \int_I f \, \mathrm{d}\lambda$ for all $I \in \mathcal{I}_W$.

(vii) ⇒ (iv): Assume that (vii) holds. Then by Corollary 3.6 in [7], F is AC on W. Also, since $F(I) = \int_I f \, \mathrm{d}\lambda$ for all $I \in \mathcal{I}_W$, by Lemma 2.1 it follows that $F'_{\rm c}(t)$ exists and $F'_{\rm c}(t) = f(t)$ for almost all $t \in W$.

Clearly, (iv) ⇒ (i), and this ends the proof. □

We now show full descriptive characterizations of Pettis integral.

Lemma 2.5. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function and let $f\colon W \to X$ be a function. Then the following statements are equivalent:*

(i) *f is Pettis integrable on W with the primitive F, i.e.*

$$F(I) = (P) \int_I f \, \mathrm{d}\lambda \quad \textit{for all } I \in \mathcal{I}_W;$$

(ii) *F is AC on W and for each $x^* \in X^*$, $(x^* \circ F)'_{\mathrm{c}}(t)$ exists and*

$$(x^* \circ F)'_{\mathrm{c}}(t) = (x^* \circ f)(t) \quad \textit{for almost all } t \in W$$

(the exceptional set may vary with x^*).

Proof. (i) $\Rightarrow$ (ii): Assume that (i) holds. Then each $(x^* \circ f)$ is Lebesgue integrable on W with the primitive $(x^* \circ F)$. Therefore for each $x^* \in X^*$, by Lemma 2.4, $(x^* \circ F)'_{\mathrm{c}}(t)$ exists and $(x^* \circ F)'_{\mathrm{c}}(t) = (x^* \circ f)(t)$ for almost all $t \in W$.

Since f is Pettis integrable on W, by Theorem II.3.5 in [3], the vector measure $\nu\colon \mathcal{L}(W) \to X$ defined as

$$\nu(E) = (P) \int_E f \, \mathrm{d}\lambda \quad \text{for all } E \in \mathcal{L}(W)$$

is a countably additive λ-continuous vector measure on $\mathcal{L}(W)$, and since λ is a finite measure on $\mathcal{L}(W)$, we obtain by Theorem I.2.1 in [3] that F is AC.

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Then by Lemma 2.4, each $(x^* \circ f)$ is Lebesgue integrable on W with the primitive $(x^* \circ F)$, i.e.

$$(x^* \circ F)(I) = \int_I (x^* \circ f) \, \mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

Since F is AC on W, by Lemma 2.3 there exists a unique countably additive λ-continuous vector measure $\nu\colon \mathcal{L}(W) \to X$ such that $F(I) = \nu(I)$ for all $I \in \mathcal{I}_W$. It follows that for each $x^* \in X^*$ we have

$$x^*(\nu(I)) = \int_I (x^* \circ f) \, \mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

It is easy to see that the family

$$\mathcal{C} = \left\{ B \in \mathscr{B}(W)\colon \forall\, x^* \in X^*, \ \left[x^*(\nu(B)) = \int_B (x^* \circ f) \, \mathrm{d}\lambda \right] \right\}$$

is a σ-algebra such that

$$\mathcal{I}_W \subset \mathcal{C} \subset \mathscr{B}(W),$$

and since $\mathscr{B}(W)=\sigma(\mathcal{I}_W)$, it follows that $\mathcal{C}=\mathscr{B}(W)$. Thus, for each $B\in\mathscr{B}(W)$ we have

$$x^*(\nu(B))=\int_B (x^*\circ f)\,\mathrm{d}\lambda \quad \text{for all } x^*\in X^*.$$

Hence, since ν is λ-continuous, for each $E\in\mathcal{L}(W)$ we have

$$x^*(\nu(E))=\int_E (x^*\circ f)\,\mathrm{d}\lambda \quad \text{for all } x^*\in X^*.$$

This means that f is Pettis integrable on W, and this ends the proof. □

By Theorem 3.5 in [13] it follows that if $V_{\mathcal{M}}F\ll\lambda$, $F'(t)$ exists and $F'(t)=f(t)$ for almost all $t\in W$, then $f\colon W\to X$ is variationally McShane integrable on W with the primitive $F\colon \mathcal{I}_W\to X$. Since $F'_{\mathrm{c}}(t)$ is a generalization of $F'(t)$, we need to prove the following theorem.

Theorem 2.6. *Let $F\colon \mathcal{I}_W\to X$ be an additive interval function and let $f\colon W\to X$ be a function. Assume that F is sAC on W, $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t)=f(t)$ for almost all $t\in W$. Then f is variationally McShane integrable function with the primitive F, i.e.*

$$F(I)=(M)\int_I f\,\mathrm{d}\lambda \quad \textit{for all } I\in\mathcal{I}_W.$$

Proof. By hypothesis, for all $x^*\in X^*$ we have $(x^*\circ F)'_{\mathrm{c}}(t)$ exists and

$$(x^*\circ F)'_{\mathrm{c}}(t)=(x^*\circ f)(t) \quad \text{for almost all } t\in W.$$

Therefore, by Lemma 2.5, f is Pettis integrable on W with the primitive F. Hence, by Theorem II.3.5 in [3], the vector measure $\nu\colon \mathcal{L}(W)\to X$ defined by

$$\nu(E)=(P)\int_E f\,\mathrm{d}\lambda \quad \text{for all } E\in\mathcal{L}(W)$$

is a countably additive λ-continuous vector measure. Since F is sAC on W and since

$$\nu(I)=F(I) \quad \text{for all } I\in\mathcal{I}_W,$$

we obtain by Lemma 2.3 that ν is of bounded variation.

We obtain by Lemma 2.2 that $Y_0=\{F(I)\colon I\in\mathcal{I}_W\}$ is a separable subset of X. If Y is the closed linear subspace spanned by Y_0, then Y is also a separable subset of X. Since $F(I)/|I|\in Y$ for all $I\in\mathcal{I}_W(t)$, we obtain that $f(t)\in Y$ for almost all $t\in W$. Hence, f is λ-essentially separably valued. Since f is Pettis integrable on W, we have also that f is weakly measurable. Therefore by Theorem II.1.2 in [3], the function f is measurable. Hence, by Remark 4.1 in [18] it follows that

$$|\nu|(E) = \int_E \|f(t)\| \, \mathrm{d}\lambda \quad \text{for each } E \in \mathcal{L}(W),$$

and since ν is of bounded variation, the function $\|f(\cdot)\|$ is Lebesgue integrable on W. Further, by Theorem II.2.2 in [3], function f is Bochner integrable on W. Since the Bochner and Pettis integrals coincide whenever they coexist, we have $F(I) = (B)\int_I f \, \mathrm{d}\lambda$ for all $I \in \mathcal{I}_W$. Thus, function f is Bochner integrable and therefore by Theorem 5.1.4 in [21], f is variationally McShane integrable on W with the primitive F, and this ends the proof. □

According to Theorem 3.1 in [13], if $F\colon \mathcal{I}_W \to X$ is the primitive of a variationally McShane integrable function $f\colon W \to X$, then $V_{\mathcal{M}}F \ll \lambda$. Therefore, to prove (i) ⇒ (ii) in Theorem 2.8, it is enough to prove that if F is the primitive of a variationally McShane integrable function f, then $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$.

Theorem 2.7. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function. Assume that a function $f\colon W \to X$ is variationally McShane integrable on W with the primitive F, i.e.*

$$F(I) = (M)\int_I f \, \mathrm{d}\lambda \quad \textit{for all } I \in \mathcal{I}_W.$$

Then $F'_{\mathrm{c}}(t)$ exists and $F'_{\mathrm{c}}(t) = f(t)$ for almost all $t \in W$.

Proof. By Theorem 5.1.4 in [21], f is Bochner integrable on W and

$$F(I) = (B)\int_I f \, \mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W.$$

Since f is measurable, we assume without loss of generality that f is separably valued. Then there exists a countable set

$$Y = \{x_k \in X\colon \ k \in \mathbb{N}\}$$

such that Y is a dense subset of $f(W)$. By virtue of Theorem II.2.2 in [3], $\|f(\cdot) - x_k\|$ is Lebesgue integrable on W. Hence, by Lemma 2.1 there exists a subset $Z_k \subset W$ with $|Z_k| = 0$ such that for all $t \in W \setminus Z_k$ we have

$$\lim_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|} \int_I \|f(s) - x_k\| \, \mathrm{d}\lambda(s) = \|f(t) - x_k\|.$$

Fix an arbitrary $t \in W \setminus Z$, where $Z = \bigcup\limits_{k=1}^{\infty} Z_k$. Since

$$\frac{1}{|I|}\int_I \|f(s) - f(t)\| \,\mathrm{d}\lambda(s) \leqslant \frac{1}{|I|}\int_I \|f(s) - x_k\| \,\mathrm{d}\lambda(s) + \|x_k - f(t)\|,$$

we obtain

$$\limsup_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}\int_I \|f(s) - f(t)\| \,\mathrm{d}\lambda(s) \leqslant 2\|x_k - f(t)\| \quad \text{for all } k \in \mathbb{N}.$$

The last inequality together with the fact that Y is a dense subset of $f(W)$ yields

$$\limsup_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}\int_I \|f(s) - f(t)\| \,\mathrm{d}\lambda(s) = 0$$

and therefore

$$\lim_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}\int_I \|f(s) - f(t)\| \,\mathrm{d}\lambda(s) = 0.$$

The last result together with

$$\left\| \frac{1}{|I|}(B)\int_I f(s)\,\mathrm{d}\lambda(s) - f(t) \right\| \leqslant \frac{1}{|I|}\int_I \|f(s) - f(t)\| \,\mathrm{d}\lambda(s)$$

yields

$$\lim_{\substack{I \in \mathcal{I}_W(t) \\ |I| \to 0}} \frac{1}{|I|}(B)\int_I f(s)\,\mathrm{d}\lambda(s) = f(t).$$

Since t is arbitrary, the last equality holds at all $t \in W \setminus Z$. Thus, $F_{\mathrm{c}}'(t)$ exists and $F_{\mathrm{c}}'(t) = f(t)$ for almost all $t \in W$, and this ends the proof. □

We are now ready to present the main result.

Theorem 2.8. *Let $F\colon \mathcal{I}_W \to X$ be an additive interval function and let $f\colon W \to X$ be a function. Then the following statements are equivalent:*

(i) *f is variationally McShane integrable on W with the primitive F, i.e.*

$$F(I) = (M)\int_I f\,\mathrm{d}\lambda \quad \text{for all } I \in \mathcal{I}_W;$$

(ii) *F is sAC on W, $F_{\mathrm{c}}'(t)$ exists and $F_{\mathrm{c}}'(t) = f(t)$ for almost all $t \in W$;*

(iii) *$V_{\mathcal{M}}F \ll \lambda$, $F_{\mathrm{c}}'(t)$ exists and $F_{\mathrm{c}}'(t) = f(t)$ for almost all $t \in W$.*

P r o o f. By virtue of Lemma 1.9, we obtain immediately that (ii) $\Leftrightarrow$ (iii). By Theorem 2.6 it follows that (ii) $\Rightarrow$ (i). Theorem 2.7 together with Theorem 3.1 in [13] yields that (i) $\Rightarrow$ (iii), and this ends the proof. □

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