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THE TORSION THEORY AND THE MELKERSSON CONDITION

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Abstract. We consider a generalization of the notion of torsion theory, which is associated with a Serre subcategory over a commutative Noetherian ring. In 2008 Aghapournahr and Melkersson investigated the question of when local cohomology modules belong to a Serre subcategory of the module category. In their study, the notion of Melkersson condition was defined as a suitable condition in local cohomology theory. One of our purposes in this paper is to show how naturally the concept of Melkersson condition appears in the context of torsion theories.

Keywords: Melkersson condition; Serre subcategory; torsion theory

MSC 2010: 13C60, 13D30

1. Introduction

Let R be a commutative Noetherian ring. We denote by R-Mod the category of R-modules and let $\mathcal{S}$ be a Serre subcategory of R-Mod. Dickson in [3] introduced the notion of torsion theory (or torsion pair) in an abelian category. After this, torsion theory has been investigated as a significant tool in abelian categories, triangulated categories, and stable categories, for example see [2], [5], [6], [7]. It is a well-known fact that a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of R-Mod is a hereditary torsion theory if and only if the torsion class $\mathcal{T}$ is a Serre subcategory that is closed under taking arbitrary direct sums. In this case the torsion class $\mathcal{T}$ can be represented as the subcategory $\mathcal{T}_W = \{M \in R\text{-Mod}\colon \Gamma_W(M) = M\}$ and the torsion-free class $\mathcal{F}$ as the subcategory $\mathcal{F}_W = \{M \in R\text{-Mod}\colon \Gamma_W(M) = 0\}$ for a specialization closed subset W of $\operatorname{Spec}(R)$. By virtue of Gabriel's classification theorem in [4], we can give a bijective correspondence between hereditary torsion theories in R-Mod and Serre subcategories of R-Mod that are closed under taking arbitrary direct sums.

With regard to the study of Serre subcategory, our next purpose is to investigate Serre subcategories with a weaker condition than the condition that they are

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closed under taking arbitrary direct sums. We will focus on Serre subcategories that are closed under taking injective hulls, and more generally, those which satisfy the following condition:

$$(\mathrm{C}_I) \qquad \text{If } \Gamma_I(M) = M \text{ and } (0 :_M I) \text{ is in } \mathcal{S}, \text{ then } M \text{ is in } \mathcal{S}$$

for an ideal I of R and an R-module M. The above condition was introduced as a suitable condition in local cohomology theory by Aghapournahr and Melkersson, see [1], and it came to be called the *Melkersson condition.* It is known that a Serre subcategory that is closed under taking injective hulls satisfies the Melkersson condition with respect to all ideals. Furthermore, the converse implication does not hold in general, see [9]. However, it is a well-known fact that a Serre subcategory is closed under taking injective hulls if it is closed under taking arbitrary direct sums, and thus it satisfies the Melkersson condition. Therefore, if we treat the torsion theory under more general assumptions, there is a possibility that the Melkersson condition on a Serre subcategory occurs naturally in the context of torsion theories.

The aims of this paper are to generalize the notion of the torsion theory and to show that the notion of Melkersson condition naturally arises in studying this generalization. For an ordinary torsion theory, all homomorphisms from a module in the torsion class to a module in the torsion-free class are zero. In other words, these homomorphisms must go through the zero object in the zero subcategory. Our idea will be based on rethinking this condition. Namely, the above homomorphisms will be allowed to pass through objects in a Serre subcategory. With this in mind, we define a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of R-Mod to be a *torsion theory connected by* $\mathcal{S}$ in R-Mod if it satisfies the following conditions:

(1) The module $f(T)$ is in $\mathcal{S}$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$ and $f \in \mathrm{Hom}_R(T, F)$.
(2) If $f(M) \in \mathcal{S}$ for all $F \in \mathcal{F}$ and $f \in \mathrm{Hom}_R(M, F)$, then $M \in \mathcal{T}$.
(3) If $f(T) \in \mathcal{S}$ for all $T \in \mathcal{T}$ and $f \in \mathrm{Hom}_R(T, M)$, then $M \in \mathcal{F}$.

Moreover, if $\mathcal{T}$ is closed under taking submodules then $(\mathcal{T}, \mathcal{F})$ is called *hereditary.* A torsion theory connected by the zero subcategory is an ordinary torsion theory. (Note that the extension subcategories $\mathcal{C} * \mathcal{S}$ and $\mathcal{S} * \mathcal{C}$ for a subcategory $\mathcal{C}$ of R-Mod are in general not equal. Furthermore, we shall see that $\mathcal{T}$ is not necessarily closed under taking extensions.) In addition, we define $\mathcal{T}(W, \mathcal{S}) = \{M \in R\text{-Mod}\colon M/\Gamma_W(M) \in \mathcal{S}\} = \mathcal{T}_W * \mathcal{S}$ as a generalization of $\mathcal{T}_W$ for a specialization closed subset W of $\mathrm{Spec}(R)$. However, there are two possible ways to generalize a torsion-free class as follows: One is defined by $\mathcal{FG}(W, \mathcal{S}) = \{M \in R\text{-Mod}\colon \Gamma_W(M) \in \mathcal{S}\} = \mathcal{S} * \mathcal{F}_W$ associated with the section functor as a generalization of $\mathcal{F}_W$ and the other is defined by $\mathcal{FH}(I, \mathcal{S}) = \{M \in R\text{-Mod}\colon \mathrm{Hom}_R(R/I, M) \in \mathcal{S}\}$ associated with the Hom functor as a generalization of $\mathcal{F}_{V(I)}$ for an ideal I of R. It is natural to ask whether $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ and $(\mathcal{T}(V(I),\mathcal{S}), \mathcal{FH}(I,\mathcal{S}))$ are hereditary torsion theories connected by $\mathcal{S}$ or not. Our first result is given by the following:

Theorem 1.1. *The pair $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ is a hereditary torsion theory connected by the Serre subcategory $\mathcal{S}$ for a specialization closed subset W of* $\mathrm{Spec}(R)$.

In general, however, the generalized torsion-free class $\mathcal{FG}(V(I),\mathcal{S})$ does not coincide with $\mathcal{FH}(I,\mathcal{S})$. Moreover, the pair $(\mathcal{T}(V(I),\mathcal{S}), \mathcal{FH}(I,\mathcal{S}))$ is not necessarily a hereditary torsion theory connected by $\mathcal{S}$. Therefore, one of our purposes in this paper is to discuss the question of when the pair $(\mathcal{T}(V(I),\mathcal{S}), \mathcal{FH}(I,\mathcal{S}))$ is a hereditary torsion theory connected by $\mathcal{S}$. By using the notion of Melkersson condition, our main result is stated as follows:

Theorem 1.2. *Let I be an ideal of R and $\mathcal{S}$ a Serre subcategory. Then the following conditions are equivalent:*
(1) *The pair $(\mathcal{T}(V(I),\mathcal{S}), \mathcal{FH}(I,\mathcal{S}))$ is a hereditary torsion theory connected by $\mathcal{S}$.*
(2) *One has $\mathcal{T}(V(I),\mathcal{S}) \cap \mathcal{FH}(I,\mathcal{S}) = \mathcal{S}$.*
(3) *One has $\mathcal{FH}(I,\mathcal{S}) = \mathcal{FG}(V(I),\mathcal{S})$.*
(4) *$\mathcal{S}$ satisfies the Melkersson condition* (C_I).

The organization of this paper is as follows. In Section 2, we recall definitions and basic properties of several subcategories, the Melkersson condition, and the notion of torsion theory. In Section 3, we give the notion of a torsion theory connected by a Serre subcategory and a sufficient condition to be such a generalized torsion theory. After proving Theorem 1.1 in Section 4, we show Theorem 1.2 in Section 5. In Section 6, we construct an example of a hereditary torsion theory connected by a Serre subcategory but which is not an ordinary torsion theory.

2. Preliminaries

Throughout this paper, all rings are commutative Noetherian and all modules are unitary. For a ring R, we suppose that all full subcategories $\mathcal{X}$ of the R-module category R-Mod are closed under isomorphisms.

In this section, we recall notions and results for several subcategories and the torsion theory. First of all, we recall the definitions of a Serre subcategory and the Melkersson condition on a Serre subcategory of R-Mod. The Melkersson condition was introduced by Aghapournahr and Melkersson in [1], Definition 2.1.

Definition 2.1.
(1) A full subcategory $\mathcal{S}$ of R-Mod is called a *Serre subcategory* if $\mathcal{S}$ is closed under taking submodules, quotient modules and extensions.

(2) A Serre subcategory $\mathcal{S}$ of R-Mod is said to satisfy the *Melkersson condition* (C_I) *with respect to an ideal* I of R if it satisfies the following condition:

(C_I) If $\Gamma_I(M) = M$ and $(0 :_M I)$ is in $\mathcal{S}$ for an R-module M, then M is in $\mathcal{S}$.

In the rest of this paper, a Serre subcategory of R-Mod will be referred to as a Serre subcategory simply.

Next, let us recall the notion of a subcategory consisting of extension modules in two given subcategories of R-Mod.

Definition 2.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of R-Mod. We denote by $\mathcal{X} * \mathcal{Y}$ the subcategory consisting of those R-modules M that there exists a short exact sequence $0 \to X \to M \to Y \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

The following lemma will be used later. We denote by $E_R(M)$ the injective hull of an R-module M.

Lemma 2.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories which are closed under taking submodules. Then the following hold:*

(1) *The subcategory $\mathcal{X} * \mathcal{Y}$ is closed under taking submodules.*

(2) *If $\mathcal{X}$ and $\mathcal{Y}$ are closed under taking injective hulls, then the subcategory $\mathcal{X} * \mathcal{Y}$ is closed under taking injective hulls.*

Proof. (1) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. We suppose $M \in \mathcal{X} * \mathcal{Y}$ and shall show $L \in \mathcal{X} * \mathcal{Y}$. It follows from the definition of $\mathcal{X} * \mathcal{Y}$ that there exists a short exact sequence $0 \to X \xrightarrow{\varphi} M \to Y \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then our assertion is proved by the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \cap L & \longrightarrow & X & \longrightarrow & \dfrac{X}{X \cap L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \overline{\varphi}} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{L}{X \cap L} & \longrightarrow & Y & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

of R-modules with exact rows and columns, where $\overline{\varphi}$ is a natural homomorphism induced by φ and $N' = \mathrm{Coker}(\overline{\varphi})$.

(2) For an R-module M in $\mathcal{X} * \mathcal{Y}$, there exists a short exact sequence $0 \to X \to M \to Y \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & E_R(X) & \longrightarrow & E_R(X) \oplus E_R(Y) & \longrightarrow & E_R(Y) & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows and vertical injective homomorphisms. (For $m \in M$, we define $\eta(m) = (\mu(m), \tau \circ \psi(m))$ where $\mu\colon M \to E_R(X)$ is a homomorphism induced by the injectivity of $E_R(X)$ such that $\sigma = \mu \circ \varphi$.) Since $\mathcal{X}$ and $\mathcal{Y}$ are closed under taking injective hulls, the module $E_R(X) \oplus E_R(Y)$ is in $\mathcal{X} * \mathcal{Y}$. We note that $E_R(M)$ is a direct summand of $E_R(X) \oplus E_R(Y)$. Therefore, we see that $E_R(M) \in \mathcal{X} * \mathcal{Y}$ because $\mathcal{X} * \mathcal{Y}$ is closed under taking submodules by (1). Consequently, the subcategory $\mathcal{X} * \mathcal{Y}$ is closed under taking injective hulls. $\square$

The purpose of this paper is to generalize the notion and results of the torsion theory in R-Mod. Regarding the following or more detailed facts about the torsion theory, we will refer to Dickson, see [3], Lambek, see [5], and Stenström, see [6], [7].

Definition 2.4. A *torsion theory* in R-Mod is a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of R-Mod satisfying the following conditions:

(1) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
(2) If $\mathrm{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$.
(3) If $\mathrm{Hom}_R(T, M) = 0$ for all $T \in \mathcal{T}$, then $M \in \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ is a *torsion-free class*. In addition, if $\mathcal{T}$ is closed under taking submodules, then a torsion theory $(\mathcal{T}, \mathcal{F})$ is called a *hereditary* torsion theory.

We recall that a subset W of $\mathrm{Spec}(R)$ is said to be *specialization closed* if it has the following property: if $\mathfrak{p}$ is a prime ideal in W and $\mathfrak{q}$ is a prime ideal containing $\mathfrak{p}$, then $\mathfrak{q}$ is also in W.

Remark 2.5. The following are well-known facts concerning torsion theories, see [3], [5], [6], [7].

(1) A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of R-Mod is a torsion theory if and only if it satisfies the following conditions:
    (a) One has $\mathcal{T} \cap \mathcal{F} = \{0\}$.

(b) $\mathcal{T}$ is closed under taking quotient modules.
(c) $\mathcal{F}$ is closed under taking submodules.
(d) One has $\mathcal{T} * \mathcal{F} = R\text{-Mod}$.

(2) Let $\mathcal{T}$ and $\mathcal{F}$ be subcategories of R-Mod.
(a) $\mathcal{T}$ is closed under taking quotient modules, extensions, and arbitrary direct sums if and only if $\mathcal{T}$ is a torsion class for some torsion theory.
(b) $\mathcal{F}$ is closed under taking submodules, extensions, and direct products if and only if $\mathcal{F}$ is a torsion-free class for some torsion theory.

(3) A torsion theory $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under taking injective hulls.

(4) There is a bijective correspondence between
(a) hereditary torsion theories in R-Mod;
(b) left exact radical functors on R-Mod;
(c) section functors with support in a specialization closed subset of $\mathrm{Spec}(R)$ on R-Mod.

We denote by $(\mathcal{T}_W, \mathcal{F}_W)$ the hereditary torsion theory corresponding to a specialization closed subset W of $\mathrm{Spec}(R)$, that is,

$$\mathcal{T}_W = \{M \in R\text{-Mod}\colon\ \Gamma_W(M) = M\} \text{ and } \mathcal{F}_W = \{M \in R\text{-Mod}\colon\ \Gamma_W(M) = 0\}.$$

Note that we have

$$\mathcal{F}_{V(I)} = \{M \in R\text{-Mod}\colon\ \Gamma_I(M) = 0\} = \{M \in R\text{-Mod}\colon\ \mathrm{Hom}_R(R/I, M) = 0\}$$

for an ideal I of R, where $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R)\colon\ I \subseteqq \mathfrak{p}\}$.

3. A torsion theory connected by a Serre subcategory

The aim of this section is to generalize the notion of the torsion theory. We start by giving the definition of the torsion theory connected by the Serre subcategory in R-Mod.

Definition 3.1. A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of R-Mod is called a *torsion theory connected by a Serre subcategory* $\mathcal{S}$ in R-Mod if it satisfies the following conditions:

(TT1) The module $f(T)$ is in $\mathcal{S}$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$, and $f \in \mathrm{Hom}_R(T, F)$.
(TT2) If $f(M)$ is in $\mathcal{S}$ for all $F \in \mathcal{F}$ and $f \in \mathrm{Hom}_R(M, F)$, then M is in $\mathcal{T}$.
(TT3) If $f(T)$ is in $\mathcal{S}$ for all $T \in \mathcal{T}$ and $f \in \mathrm{Hom}_R(T, M)$, then M is in $\mathcal{F}$.

In addition, if $\mathcal{T}$ is closed under taking submodules, then $(\mathcal{T}, \mathcal{F})$ is called a *hereditary torsion theory connected by* $\mathcal{S}$.

Remark 3.2. A torsion theory connected by the zero subcategory is an ordinary torsion theory. (The *zero subcategory* means the subcategory consisting of the zero module.)

Any subcategory generates a torsion theory connected by a Serre subcategory.

Example 3.3. Let $\mathcal{C}$ be a subcategory of R-Mod and $\mathcal{S}$ a Serre subcategory. We set

$$\mathcal{F}_{\mathcal{C}}(\mathcal{S}) = \{F \in R\text{-Mod}\colon\ f(C) \in \mathcal{S}\ \forall\, C \in \mathcal{C} \text{ and } f \in \operatorname{Hom}_R(C, F)\} \text{ and,}$$
$$\mathcal{T}_{\mathcal{C}}(\mathcal{S}) = \{T \in R\text{-Mod}\colon\ f(T) \in \mathcal{S}\ \forall\, F \in \mathcal{F}_{\mathcal{C}}(\mathcal{S}) \text{ and } f \in \operatorname{Hom}_R(T, F)\}.$$

Then it is clear that the pair $(\mathcal{T}_{\mathcal{C}}(\mathcal{S}), \mathcal{F}_{\mathcal{C}}(\mathcal{S}))$ satisfies the conditions (TT1) and (TT2). We suppose that an R-module M satisfies $f(T) \in \mathcal{S}$ for all $T \in \mathcal{T}_{\mathcal{C}}(\mathcal{S})$ and $f \in \operatorname{Hom}_R(T, M)$. We note that $\mathcal{C}$ is contained in $\mathcal{T}_{\mathcal{C}}(\mathcal{S})$ by virtue of the definition of $\mathcal{F}_{\mathcal{C}}(\mathcal{S})$. Therefore, the above assumption can be applied to R-modules in $\mathcal{C}$. Namely, we have $f(C) \in \mathcal{S}$ for all $C \in \mathcal{C}$ and $f \in \operatorname{Hom}_R(C, M)$. By the definition of $\mathcal{F}_{\mathcal{C}}(\mathcal{S})$, we obtain $M \in \mathcal{F}_{\mathcal{C}}(\mathcal{S})$. Therefore, the condition (TT3) is satisfied. Consequently, the pair $(\mathcal{T}_{\mathcal{C}}(\mathcal{S}), \mathcal{F}_{\mathcal{C}}(\mathcal{S}))$ is a torsion theory connected by $\mathcal{S}$.

The name of torsion theory connected by a Serre subcategory comes from the following property, corresponding to Remark 2.5, case (1) (a).

Proposition 3.4. *Let $(\mathcal{T}, \mathcal{F})$ be a pair of subcategories of R-Mod and $\mathcal{S}$ a Serre subcategory. Then the following assertions hold.*

(1) *The condition (TT1) implies $\mathcal{T} \cap \mathcal{F} \subseteqq \mathcal{S}$.*
(2) *The condition (TT2) implies $\mathcal{T} \supseteqq \mathcal{S}$.*
(3) *The condition (TT3) implies $\mathcal{F} \supseteqq \mathcal{S}$.*

In particular, a torsion theory $(\mathcal{T}, \mathcal{F})$ connected by $\mathcal{S}$ satisfies $\mathcal{T} \cap \mathcal{F} = \mathcal{S}$.

Proof. The last part is clear if we can prove that the assertions (1)–(3) hold.

(1) We suppose that M is in $\mathcal{T} \cap \mathcal{F}$. Then, the identity map id_M on M can be regarded as a homomorphism from $M \in \mathcal{T}$ to $M \in \mathcal{F}$. By the condition (TT1), we see that $M = \mathrm{id}_M(M)$ is in $\mathcal{S}$.

(2) We take an R-module M in $\mathcal{S}$. For all $F \in \mathcal{F}$ and $f \in \operatorname{Hom}_R(M, F)$, one has $f(M) \in \mathcal{S}$ because $\mathcal{S}$ is closed under taking quotient modules. The condition (TT2) implies $M \in \mathcal{T}$.

(3) Let M be an R-module in $\mathcal{S}$. Then we have $f(T) \subseteqq M$ for all $T \in \mathcal{T}$ and $f \in \operatorname{Hom}_R(T, M)$. Since $\mathcal{S}$ is closed under taking submodules, we have $f(T) \in \mathcal{S}$. Consequently, the condition (TT3) yields $M \in \mathcal{F}$. □

Remark 2.5, cases (1) (b) and (1) (c) correspond to the following proposition.

Proposition 3.5. *Let $(\mathcal{T}, \mathcal{F})$ be a pair of subcategories of R-*Mod *and $\mathcal{S}$ a Serre subcategory. Then the following assertions hold.*

(1) *If $(\mathcal{T}, \mathcal{F})$ satisfies the conditions* (TT1) *and* (TT2), *then $\mathcal{T}$ is closed under taking quotient modules.*

(2) *If $(\mathcal{T}, \mathcal{F})$ satisfies the conditions* (TT1) *and* (TT3), *then $\mathcal{F}$ is closed under taking submodules.*

In particular, a torsion theory $(\mathcal{T}, \mathcal{F})$ connected by $\mathcal{S}$ satisfies the conclusion in (1) *and* (2).

P r o o f. (1) Let T be an R-module in $\mathcal{T}$ and N a quotient module of T with a surjective homomorphism $\varphi \in \operatorname{Hom}_R(T, N)$. By the condition (TT2), it is sufficient to show that $f(N)$ is in $\mathcal{S}$ for all $F \in \mathcal{F}$ and $f \in \operatorname{Hom}_R(N, F)$. We consider the homomorphism $f \circ \varphi \in \operatorname{Hom}_R(T, F)$. Then we see that $f(N) = f \circ \varphi(T) \in \mathcal{S}$ by the condition (TT1). Consequently, $\mathcal{T}$ is closed under taking quotient modules.

(2) Let F be an R-module in $\mathcal{F}$ and L a submodule of F with an injective homomorphism $\psi \in \operatorname{Hom}_R(L, F)$. We shall show that $f(T) \in \mathcal{S}$ for all $T \in \mathcal{T}$ and $f \in \operatorname{Hom}_R(T, L)$. Since $\psi \circ f \in \operatorname{Hom}_R(T, F)$, we have $f(T) = \psi \circ f(T) \in \mathcal{S}$ by the condition (TT1). Therefore, the condition (TT3) implies $L \in \mathcal{F}$. Consequently, $\mathcal{F}$ is closed under taking submodules. □

Next, we give a sufficient condition to be closed under taking extensions for $\mathcal{T}$ and $\mathcal{F}$.

Proposition 3.6. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory connected by a Serre subcategory $\mathcal{S}$. Then the following hold.*

(1) *If $\mathcal{T}$ is closed under taking submodules, then $\mathcal{F}$ is closed under taking extensions.*

(2) *If $\mathcal{F}$ is closed under taking quotient modules, then $\mathcal{T}$ is closed under taking extensions.*

P r o o f. (1) Let $0 \to F_1 \to M \to F_2 \to 0$ be a short exact sequence of R-modules with $F_1, F_2 \in \mathcal{F}$. If we can show $f(T) \in \mathcal{S}$ for all $T \in \mathcal{T}$ and $f \in \operatorname{Hom}_R(T, M)$, then one has $M \in \mathcal{F}$ by the condition (TT3). We consider a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f^{-1}(F_1) & \longrightarrow & T & \longrightarrow & T/f^{-1}(F_1) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f|_{f^{-1}(F_1)}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \overline{f}} & & \\
0 & \longrightarrow & f(T) \cap F_1 & \longrightarrow & f(T) & \longrightarrow & f(T)/(f(T) \cap F_1) & \longrightarrow & 0
\end{array}$$

of R-modules with exact rows and vertical surjective homomorphisms, where $\overline{f}$ is a natural homomorphism induced by f. Since $\mathcal{T}$ is closed under taking submodules by our assumption and quotient modules by Proposition 3.5, case (1), the

R-modules $f^{-1}(F_1)$ and $T/f^{-1}(F_1)$ are in $\mathcal{T}$. (We note that the condition (TT2) was used here.) However, we have $f(T) \cap F_1 \subseteqq F_1$ and $f(T)/(f(T) \cap F_1) \subseteqq M/F_1 \cong F_2$. Since $\mathcal{F}$ is closed under taking submodules by Proposition 3.5, case (2), the R-modules $f(T) \cap F_1$ and $f(T)/(f(T) \cap F_1)$ are in $\mathcal{F}$. Therefore, we see that $f|_{f^{-1}(F_1)}$ and $\overline{f}$ are homomorphisms from an R-module in $\mathcal{T}$ to an R-module in $\mathcal{F}$. The condition (TT1) implies that $f(T) \cap F_1$ and $f(T)/(f(T) \cap F_1)$ are in $\mathcal{S}$. Consequently, the R-module $f(T)$ is also in $\mathcal{S}$ because $\mathcal{S}$ is closed under taking extensions.

(2) Let $0 \to T_1 \to M \to T_2 \to 0$ be a short exact sequence of R-modules with $T_1, T_2 \in \mathcal{T}$. If we can prove that $f(M)$ is in $\mathcal{S}$ for all $F \in \mathcal{F}$ and $f \in \mathrm{Hom}_R(M, F)$, then one has $M \in \mathcal{T}$ by the condition (TT2). We consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1 & \longrightarrow & M & \longrightarrow & T_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f|_{T_1}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & f(T_1) & \longrightarrow & f(M) & \longrightarrow & f(T)/f(T_1) & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows and vertical surjective homomorphisms where $\overline{f}$ is a natural homomorphism induced by f. Since $f(T_1) \subseteqq f(M) \subseteqq F$ and $\mathcal{F}$ is closed under taking submodules by Proposition 3.5, case (2), the R-modules $f(T_1)$ and $f(M)$ are in $\mathcal{F}$. (Here, we used the condition (TT3).) Furthermore, $\mathcal{F}$ is closed under taking quotient modules by our assumption, the R-module $f(M)/f(T_1)$ is also in $\mathcal{F}$. Therefore, we see that $f|_{T_1}$ and $\overline{f}$ are homomorphisms from an R-module in $\mathcal{T}$ to an R-module in $\mathcal{F}$. The condition (TT1) implies that $f(T_1)$ and $f(M)/f(T_1)$ are in $\mathcal{S}$. Consequently, the R-module $f(M)$ is also in $\mathcal{S}$ because $\mathcal{S}$ is closed under taking extensions. □

Here, we will give a sufficient condition to be a torsion theory connected by a Serre subcategory. If we take the zero subcategory as a Serre subcategory, the proposition below gives a necessary and sufficient condition to be an ordinary torsion theory by combining Remark 2.5, cases (1) and (2). However, this proposition does not give a necessary condition to be a generalized torsion theory. Indeed, we will show the existence of a torsion theory $(\mathcal{T}, \mathcal{F})$ connected by a Serre subcategory that is not closed under taking extensions for $\mathcal{T}$, see Example 6.2.

Proposition 3.7. *Let $\mathcal{S}$ be a Serre subcategory. For a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of R-*Mod*, we suppose that the following statements hold:*

(1) *One has $\mathcal{T} \cap \mathcal{F} = \mathcal{S}$.*
(2) *$\mathcal{T}$ is closed under taking quotient modules and extensions.*
(3) *$\mathcal{F}$ is closed under taking submodules and extensions.*
(4) *One has $\mathcal{T} * \mathcal{F} = R$-*Mod*.*

Then $(\mathcal{T}, \mathcal{F})$ is a torsion theory connected by $\mathcal{S}$.

P r o o f. First of all, we shall see that the condition (TT1) is satisfied by the assumptions (1), that $\mathcal{T}$ is closed under taking quotient modules, and that $\mathcal{F}$ is closed under taking submodules. Let us take an R-module T in $\mathcal{T}$, an R-module F in $\mathcal{F}$, and $f \in \mathrm{Hom}_R(T, F)$. Then, since $\mathcal{T}$ is closed under taking quotient modules, we have $f(T) \in \mathcal{T}$. However, since $\mathcal{F}$ is closed under taking submodules, one has $f(T) \in \mathcal{F}$. Therefore, we see that $f(T) \in \mathcal{T} \cap \mathcal{F} = \mathcal{S}$ by (1).

Next, we shall show that $(\mathcal{T}, \mathcal{F})$ satisfies the condition (TT2) by the assumptions (1), that $\mathcal{T}$ is closed under taking extensions, and (4). Let M be an R-module such that $f(M) \in \mathcal{S}$ for all $F \in \mathcal{F}$ and $f \in \mathrm{Hom}_R(M, F)$. By (4), there exists a short exact sequence

$$0 \to T \to M \overset{\varphi}{\to} F' \to 0$$

of R-modules with $T \in \mathcal{T}$ and $F' \in \mathcal{F}$. Since φ is a homomorphism from M to an R-module in $\mathcal{F}$, we can apply our assumption to φ. Then, we obtain $\varphi(M) \in \mathcal{S} = \mathcal{T} \cap \mathcal{F} \subsetneqq \mathcal{T}$ by (1). Since $\mathcal{T}$ is closed under taking extensions, the above short exact sequence implies $M \in \mathcal{T}$.

Finally, we shall show that the condition (TT3) holds for $(\mathcal{T}, \mathcal{F})$ by the assumptions (1), that $\mathcal{F}$ is closed under taking extensions, and (4). Let M be an R-module such that $f(T) \in \mathcal{S}$ for all $T \in \mathcal{T}$ and $f \in \mathrm{Hom}_R(T, M)$. By (4), there exists a short exact sequence

$$0 \to T' \overset{\psi}{\to} M \to F \to 0$$

of R-modules with $T' \in \mathcal{T}$ and $F \in \mathcal{F}$. We apply our assumption to $\psi \in \mathrm{Hom}_R(T', M)$. Then one has $\psi(T') \in \mathcal{S} = \mathcal{T} \cap \mathcal{F} \subsetneqq \mathcal{F}$ by (1). Since $\mathcal{F}$ is closed under taking extensions, the above short exact sequence yields $M \in \mathcal{F}$. $\square$

4. A generalized torsion theory associated with the section functor

To present examples of hereditary torsion theories connected by a Serre subcategory, we try to generalize the torsion class $\mathcal{T}_W$ and the torsion-free class $\mathcal{F}_W$ associated with a specialization closed subset W of $\mathrm{Spec}(R)$. In particular, there exist two possible ways to generalize the torsion-free class: one is associated with the section functor and the other is associated with the Hom functor. After investigating torsion-free classes associated with the section functor in this section, we will study torsion-free classes associated with the Hom functor in the next section.

First of all, we introduce the notions of generalized torsion class and generalized torsion-free class associated with the section functor. The main purpose of this section is to prove that a pair of these classes is always a torsion theory connected by a Serre subcategory.

Definition 4.1. If W is a specialization closed subset of $\operatorname{Spec}(R)$ and $\mathcal{S}$ is a Serre subcategory, we denote by

$$\mathcal{T}(W,\mathcal{S}) = \{M \in R\text{-Mod}\colon\ M/\Gamma_W(M) \in \mathcal{S}\}$$

the *torsion class connected by* $\mathcal{S}$ *for* W, and by

$$\mathcal{FG}(W,\mathcal{S}) = \{M \in R\text{-Mod}\colon\ \Gamma_W(M) \in \mathcal{S}\}$$

the Γ-*torsion-free class connected by* $\mathcal{S}$ *for* W. In particular, for an ideal I of R, we will simply denote $\mathcal{T}(V(I),\mathcal{S})$ and $\mathcal{FG}(V(I),\mathcal{S})$ by $\mathcal{T}(I,\mathcal{S})$ and $\mathcal{FG}(I,\mathcal{S})$, respectively.

Remark 4.2. Let $\mathcal{S}$ be a Serre subcategory.

(1) It is easy to see that if $\mathcal{S} = \{0\}$ then the pair $(\mathcal{T}(W,\{0\}),\mathcal{FG}(W,\{0\}))$ coincides with the ordinary torsion theory $(\mathcal{T}_W,\mathcal{F}_W)$ corresponding to the specialization closed subset W of $\operatorname{Spec}(R)$.

(2) We have $\mathcal{T}(I,\mathcal{S}) = \mathcal{T}(\sqrt{I},\mathcal{S})$ and $\mathcal{FG}(I,\mathcal{S}) = \mathcal{FG}(\sqrt{I},\mathcal{S})$ for an ideal I of R.

Here, we will observe the relationship between $\mathcal{T}(W,\mathcal{S})$ and $\mathcal{T}_W$ for a specialization closed subset W of $\operatorname{Spec}(R)$ and a Serre subcategory $\mathcal{S}$.

Proposition 4.3. *Let W be a specialization closed subset of* $\operatorname{Spec}(R)$ *and $\mathcal{S}$ a Serre subcategory. Then the following hold.*

(1) *$\mathcal{T}(W,\mathcal{S})$ is closed under taking submodules and quotient modules.*

(2) *One has $\mathcal{T}(W,\mathcal{S}) = \mathcal{T}_W * \mathcal{S} \subseteqq \mathcal{S} * \mathcal{T}_W$.*

(3) *$\mathcal{T}(W,\mathcal{S})$ is closed under taking extensions if and only if one has $\mathcal{T}(W,\mathcal{S}) = \mathcal{T}_W * \mathcal{S} = \mathcal{S} * \mathcal{T}_W$. In this case, $\mathcal{T}(W,\mathcal{S})$ is a Serre subcategory.*

(4) *If $\mathcal{S}$ is closed under taking injective hulls, then one has $\mathcal{T}(W,\mathcal{S}) = \mathcal{T}_W * \mathcal{S} = \mathcal{S} * \mathcal{T}_W$.*

Proof. (1) It is directly proved from the definition of $\mathcal{T}(W,\mathcal{S})$.

(2) Let M be an R-module in $\mathcal{T}(W,\mathcal{S})$. Then we have a short exact sequence

$$0 \to \Gamma_W(M) \to M \to M/\Gamma_W(M) \to 0$$

of R-modules with $\Gamma_W(M) \in \mathcal{T}_W$ and $M/\Gamma_W(M) \in \mathcal{S}$. Therefore, M is in $\mathcal{T}_W * \mathcal{S}$.

Conversely, let us take an R-module $M \in \mathcal{T}_W * \mathcal{S}$. Then there exists a short exact sequence $0 \to L \to M \xrightarrow{\varphi} S \to 0$ of R-modules with $L \in \mathcal{T}_W$ and $S \in \mathcal{S}$. Applying the left exact functor $\Gamma_W(-)$ to this sequence, we obtain the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_W(L) & \longrightarrow & \Gamma_W(M) & \xrightarrow{\varphi'} & \operatorname{Im}\varphi' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \xrightarrow{\varphi} & S & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M/\Gamma_W(M) & \xrightarrow{\overline{\varphi}} & S/\operatorname{Im}\varphi' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

of R-modules where $\varphi' = \varphi|_{\Gamma_W(M)}$ and $\overline{\varphi}$ is a natural homomorphism induced by φ. The snake lemma implies that $\overline{\varphi}$ is an isomorphism. Since $\mathcal{S}$ is closed under taking quotient modules, we see that $M/\Gamma_W(M) \cong S/\operatorname{Im}\varphi' \in \mathcal{S}$. Consequently, M is in $\mathcal{T}(W, \mathcal{S})$.

Finally, since $\mathcal{T}_W$ is a Serre subcategory that is closed under taking injective hulls, the subcategory $\mathcal{S} * \mathcal{T}_W$ is Serre by [8], Corollary 3.5. In particular, this subcategory is closed under taking extensions. Therefore, we have $\mathcal{T}_W * \mathcal{S} \subseteqq \mathcal{S} * \mathcal{T}_W$ because $\mathcal{T}_W$ and $\mathcal{S}$ are contained in $\mathcal{S} * \mathcal{T}_W$.

(3) It follows from the assertions (1), (2), and [8], Theorem 3.2.

(4) We suppose that $\mathcal{S}$ is closed under taking injective hulls. Then the subcategory $\mathcal{T}(W, \mathcal{S}) = \mathcal{T}_W * \mathcal{S}$ is Serre by [8], Corollary 3.5. Therefore, the assertion (3) implies $\mathcal{T}(W, \mathcal{S}) = \mathcal{T}_W * \mathcal{S} = \mathcal{S} * \mathcal{T}_W$. □

Remark 4.4. Let W be a specialization closed subset of $\operatorname{Spec}(R)$ and $\mathcal{S}$ a Serre subcategory. By virtue of Proposition 4.3, case (2), for any R-module M in $\mathcal{T}_W * \mathcal{S}$, we can see that there is a short exact sequence

$$0 \to \Gamma_W(M) \to M \to M/\Gamma_W(M) \to 0$$

of R-modules with $\Gamma_W(M) \in \mathcal{T}_W$ and $M/\Gamma_W(M) \in \mathcal{S}$.

Now, we will see that a torsion class connected by a Serre subcategory for a specialization closed subset of $\operatorname{Spec}(R)$ is not necessarily closed under taking extensions. We denote by $\mathcal{S}_{\mathrm{fg}}$ the subcategory consisting of finitely generated (fg) modules.

Example 4.5. We suppose that R is a 1-dimensional Gorenstein local ring with maximal ideal $\mathfrak{m}$. Then, let us prove that $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$ is not closed under taking extensions. We consider a minimal injective resolution of R:

$$0 \to R \to \bigoplus_{\substack{\mathfrak{p}\in \operatorname{Spec}(R),\\ \operatorname{ht}\mathfrak{p}=0}} E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{m}) \to 0.$$

Since $R/\Gamma_{\mathfrak{m}}(R)$ and $E_R(R/\mathfrak{m})/\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{m})) = 0$ are finitely generated R-modules, the modules R and $E_R(R/\mathfrak{m})$ are in $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$. Here, we denote by E the middle term $\bigoplus_{\operatorname{ht}\mathfrak{p}=0} E_R(R/\mathfrak{p})$ in the above resolution. Then one has $E/\Gamma_{\mathfrak{m}}(E) = E$ because we have $\operatorname{Ass}_R(E) \cap V(\mathfrak{m}) = \{\mathfrak{p} \in \operatorname{Spec}(R)\colon \operatorname{ht}\mathfrak{p} = 0\} \cap V(\mathfrak{m}) = \emptyset$. Since R is a non-Artinian local ring, there exists no nonzero finitely generated injective R-module. This fact yields $E/\Gamma_{\mathfrak{m}}(E) \notin \mathcal{S}_{\mathrm{fg}}$, whence one has $E \notin \mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$. Consequently, $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$ is not closed under taking extensions. Moreover, we note that the above argument also gives an inclusion relation

$$\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) = \mathcal{T}_{V(\mathfrak{m})} * \mathcal{S}_{\mathrm{fg}} \subsetneqq \mathcal{S}_{\mathrm{fg}} * \mathcal{T}_{V(\mathfrak{m})}$$

because E is obviously in $\mathcal{S}_{\mathrm{fg}} * \mathcal{T}_{V(\mathfrak{m})}$.

However, the following torsion class connected by a Serre subcategory is closed under taking extensions.

Example 4.6. Let V, W be specialization closed subsets of $\operatorname{Spec}(R)$ and $\mathcal{S}$ a Serre subcategory. Then we have

$$\mathcal{T}(V, \mathcal{S} * \mathcal{T}_{V\cup W}) \supseteqq \mathcal{S} * \mathcal{T}_{V\cup W} \supseteqq \mathcal{T}(V, \mathcal{S} * \mathcal{T}_W).$$

Therefore, if $V \subseteqq W$, then $\mathcal{T}(V, \mathcal{S} * \mathcal{T}_W) = \mathcal{S} * \mathcal{T}_W$ is a Serre subcategory. In particular, the subcategory $\mathcal{T}(\mathfrak{p}, \mathcal{S} * \mathcal{T}_W)$ is Serre for $\mathfrak{p} \in W$.

Let us prove that our assertion holds. It is clear that $\mathcal{T}_V$, $\mathcal{T}_W$ and $\mathcal{T}_{V\cup W}$ are Serre subcategories that are closed under taking injective hulls. Therefore, the subcategories $\mathcal{S} * \mathcal{T}_{V\cup W}$, $\mathcal{S} * \mathcal{T}_W$, and $(\mathcal{S} * \mathcal{T}_W) * \mathcal{T}_V$ are Serre by [8], Corollary 3.5. Then we have

$$\begin{aligned}\mathcal{T}(V, \mathcal{S} * \mathcal{T}_{V\cup W}) &= \mathcal{T}_V * (\mathcal{S} * \mathcal{T}_{V\cup W}) \supseteqq \mathcal{S} * \mathcal{T}_{V\cup W} \overset{\text{(a)}}{=} \mathcal{S} * (\mathcal{T}_W * \mathcal{T}_V)\\ &\overset{\text{(b)}}{=} (\mathcal{S} * \mathcal{T}_W) * \mathcal{T}_V \overset{\text{(c)}}{\supseteqq} \mathcal{T}_V * (\mathcal{S} * \mathcal{T}_W) = \mathcal{T}(V, \mathcal{S} * \mathcal{T}_W).\end{aligned}$$

The equality (a) is given by [8], Lemma 2.2, case (1). Next, the equality (b) is given by the push out diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & L & \to & M & \to & T_V & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_W & \to & Y & \to & T_V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

of R-modules with $L \in \mathcal{S} * \mathcal{T}_W$, $S \in \mathcal{S}$, $T_W \in \mathcal{T}_W$, and $T_V \in \mathcal{T}_V$, and the pull back diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & S' & \to & Z & \to & T'_W & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & S' & \to & M & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T'_V & = & T'_V & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

of R-modules with $N \in \mathcal{T}_W * \mathcal{T}_V$, $T'_W \in \mathcal{T}_W$, $T'_V \in \mathcal{T}_V$, and $S' \in \mathcal{S}$. Finally, since $(\mathcal{S} * \mathcal{T}_W) * \mathcal{T}_V$ is closed under taking extensions, we see that the relation (c) holds.

Next, we observe the relationship between $\mathcal{FG}(W, \mathcal{S})$ and $\mathcal{F}_W$ for a specialization closed subset W of $\mathrm{Spec}(R)$ and a Serre subcategory $\mathcal{S}$.

Proposition 4.7. *Let W be a specialization closed subset of* $\mathrm{Spec}(R)$ *and $\mathcal{S}$ a Serre subcategory. Then the following hold.*

(1) *$\mathcal{FG}(W, \mathcal{S})$ is closed under taking submodules and extensions.*

(2) *One has $\mathcal{FG}(W, \mathcal{S}) = \mathcal{S} * \mathcal{F}_W \supseteqq \mathcal{F}_W * \mathcal{S}$.*

(3) *If $\mathcal{S}$ is closed under taking injective hulls, then one has $\mathcal{FG}(W, \mathcal{S}) = \mathcal{S} * \mathcal{F}_W = \mathcal{F}_W * \mathcal{S}$.*

P r o o f. (1) For a short exact sequence $0 \to L \to M \to N \to 0$ of R-modules, there exists an exact sequence $0 \to \Gamma_W(L) \to \Gamma_W(M) \to \Gamma_W(N)$. Thus, our assertion obviously holds due to the definition of $\mathcal{FG}(W, \mathcal{S})$ because $\mathcal{S}$ is closed under taking submodules and extensions.

(2) Let M be an R-module in $\mathcal{FG}(W, \mathcal{S})$. Then we have a short exact sequence

$$0 \to \Gamma_W(M) \to M \to M/\Gamma_W(M) \to 0$$

of R-modules with $\Gamma_W(M) \in \mathcal{S}$ and $M/\Gamma_W(M) \in \mathcal{F}_W$. Therefore, M is in $\mathcal{S} * \mathcal{F}_W$.

Conversely, let us take an R-module $M \in \mathcal{S} * \mathcal{F}_W$. Then there exists a short exact sequence $0 \to S \to M \to F \to 0$ of R-modules with $S \in \mathcal{S}$ and $F \in \mathcal{F}_W$. We apply the left exact functor $\Gamma_W(-)$ to this exact sequence. Since $\Gamma_W(F) = 0$ and $\mathcal{S}$ is closed under taking submodules, one has $\Gamma_W(M) = \Gamma_W(S) \in \mathcal{S}$. This yields $M \in \mathcal{FG}(W, \mathcal{S})$.

Finally, we shall see that $\mathcal{FG}(W, \mathcal{S}) \supseteqq \mathcal{F}_W * \mathcal{S}$. Let M be an R-module in $\mathcal{F}_W * \mathcal{S}$. Then there exists a short exact sequence

$$0 \to F \to M \to S \to 0$$

of R-modules with $F \in \mathcal{F}_W$ and $S \in \mathcal{S}$. Applying the left exact functor $\Gamma_W(-)$ to the above short exact sequence, we obtain an exact sequence $0 \to \Gamma_W(M) \to \Gamma_W(S)$. Since $\mathcal{S}$ is closed under taking submodules, we see that $\Gamma_W(M) \in \mathcal{S}$. Consequently, M is in $\mathcal{FG}(W, \mathcal{S})$.

(3) We only have to show $\mathcal{S} * \mathcal{F}_W \subseteqq \mathcal{F}_W * \mathcal{S}$. Let us take an R-module M in $\mathcal{S} * \mathcal{F}_W$. Then there exists a short exact sequence

$$0 \to S \to M \to F \to 0$$

of R-modules with $S \in \mathcal{S}$ and $F \in \mathcal{F}_W$. Here, we consider a push out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & M & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \Vert & & \\ 0 & \longrightarrow & E_R(S) & \longrightarrow & X & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows and vertical injective homomorphisms. The second row splits, and thus we have an injective homomorphism $M \to F \oplus E_R(S)$. Since $\mathcal{S}$ is closed under taking injective hulls, we have $F \oplus E_R(S) \in \mathcal{F}_W * \mathcal{S}$. Furthermore, Lemma 2.3, case (1) implies that $\mathcal{F}_W * \mathcal{S}$ is closed under taking submodules. Consequently, we obtain $M \in \mathcal{F}_W * \mathcal{S}$. □

Remark 4.8. Let W be a specialization closed subset of $\mathrm{Spec}(R)$ and $\mathcal{S}$ a Serre subcategory. By virtue of Proposition 4.7, case (2), we can see that an R-module M in $\mathcal{S} * \mathcal{F}_W$ has a short exact sequence

$$0 \to \Gamma_W(M) \to M \to M/\Gamma_W(M) \to 0$$

of R-modules with $\Gamma_W(M) \in \mathcal{S}$ and $M/\Gamma_W(M) \in \mathcal{F}_W$.

Here, we investigate the extension and the intersection of a torsion class and a Γ-torsion free class connected by a Serre subcategory.

Lemma 4.9. *Let W be a specialization closed subset of* $\mathrm{Spec}(R)$ *and $\mathcal{S}$ a Serre subcategory. Then the following hold.*
(1) *One has $\mathcal{T}(W,\mathcal{S}) * \mathcal{FG}(W,\mathcal{S}) = R$-Mod.*
(2) *One has $\mathcal{T}(W,\mathcal{S}) \cap \mathcal{FG}(W,\mathcal{S}) = \mathcal{S}$.*

Proof. (1) We note that $\mathcal{T}(W,\mathcal{S}) \supseteqq \mathcal{T}_W$ and $\mathcal{FG}(W,\mathcal{S}) \supseteqq \mathcal{F}_W$. Therefore, one has

$$R\text{-Mod} \supseteqq \mathcal{T}(W,\mathcal{S}) * \mathcal{FG}(W,\mathcal{S}) \supseteqq \mathcal{T}_W * \mathcal{F}_W = R\text{-Mod}.$$

(2) It is clear that $\mathcal{T}(W,\mathcal{S}) \supseteqq \mathcal{S}$ and $\mathcal{FG}(W,\mathcal{S}) \supseteqq \mathcal{S}$. Therefore, $\mathcal{T}(W,\mathcal{S}) \cap \mathcal{FG}(W,\mathcal{S}) \supseteqq \mathcal{S}$ holds. Conversely, let M be an R-module in $\mathcal{T}(W,\mathcal{S}) \cap \mathcal{FG}(W,\mathcal{S})$. Since $M \in \mathcal{T}(W,\mathcal{S})$, there exists a short exact sequence $0 \to \Gamma_W(M) \to M \to M/\Gamma_W(M) \to 0$ of R-modules with $M/\Gamma_W(M) \in \mathcal{S}$. However, since $M \in \mathcal{FG}(W,\mathcal{S})$, we have $\Gamma_W(M) \in \mathcal{S}$. Consequently, the above short exact sequence implies $M \in \mathcal{S}$ because $\mathcal{S}$ is closed under taking extensions. □

Now, we can give the main result of this section.

Theorem 4.10. *The pair $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ is a hereditary torsion theory connected by the Serre subcategory $\mathcal{S}$ for a specialization closed subset W of* $\mathrm{Spec}(R)$.

Proof. It must be checked that the pair $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ satisfies the conditions (TT1), (TT2), and (TT3) in Definition 3.1.

In the first and third parts of the proof for Proposition 3.7, we have already seen that the following assertions hold for a pair $(\mathcal{T},\mathcal{F})$ of subcategories of R-Mod in general:

(a) We suppose that one has $\mathcal{T} \cap \mathcal{F} = \mathcal{S}$, $\mathcal{T}$ is closed under taking quotient modules, and $\mathcal{F}$ is closed under taking submodules. Then $(\mathcal{T},\mathcal{F})$ satisfies the condition (TT1).
(b) We suppose that one has $\mathcal{T} \cap \mathcal{F} = \mathcal{S}$, $\mathcal{F}$ is closed under taking extensions, and one has $\mathcal{T} * \mathcal{F} = R$-Mod. Then $(\mathcal{T},\mathcal{F})$ satisfies the condition (TT3).

The pair $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ satisfies the conditions in (a) by Proposition 4.3, case (1), Proposition 4.7, case (1), and Lemma 4.9, case (2). However, it also satisfies the conditions in (b) by Proposition 4.7, case (1) and Lemma 4.9. Consequently, we can conclude that the pair $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ satisfies the conditions (TT1) and (TT3). It remains to be checked that the condition (TT2) is satisfied. Let us take an R-module M such that $f(M) \in \mathcal{S}$ for all $F \in \mathcal{FG}(W,\mathcal{S})$ and $f \in \operatorname{Hom}_R(M,F)$. We consider the short exact sequence $0 \to \Gamma_W(M) \to M \xrightarrow{\varphi} M/\Gamma_W(M) \to 0$ of R-modules. Since $M/\Gamma_W(M) \in \mathcal{F}_W \subsetneqq \mathcal{FG}(W,\mathcal{S})$, we see that φ is a homomorphism from M to an R-module in $\mathcal{FG}(W,\mathcal{S})$. Therefore, we can apply the above assumption for M to φ. Then, one has $M/\Gamma_W(M) = \varphi(M) \in \mathcal{S}$. Consequently, we see that $M \in \mathcal{T}(W,\mathcal{S})$, and thus the pair $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ satisfies the condition (TT2).

Finally, the subcategory $\mathcal{T}(W,\mathcal{S})$ is closed under taking submodules by Proposition 4.3, case (1). Therefore, the pair $(\mathcal{T}(W,\mathcal{S}), \mathcal{FG}(W,\mathcal{S}))$ is a hereditary torsion theory connected by $\mathcal{S}$. □

5. A generalized torsion-free class associated with the Hom functor and the relationship with the Melkersson condition

We start by giving a generalization of torsion-free class associated with the Hom functor.

Definition 5.1. If I is an ideal of R and $\mathcal{S}$ is a Serre subcategory, we denote

$$\mathcal{FH}(I,\mathcal{S}) = \{M \in R\text{-Mod}\colon\ \operatorname{Hom}_R(R/I,M) \in \mathcal{S}\}.$$

If $\mathcal{FH}(I,\mathcal{S})$ is a torsion-free class for some torsion theory connected by $\mathcal{S}$, then $\mathcal{FH}(I,\mathcal{S})$ is called the *Hom-torsion-free class connected by $\mathcal{S}$ for I.*

Remark 5.2. Let I be an ideal of R and $\mathcal{S}$ a Serre subcategory.

(1) It is easy to see that if $\mathcal{S} = \{0\}$ then the pair $(\mathcal{T}(I,\{0\}), \mathcal{FH}(I,\{0\}))$ coincides with the ordinary torsion theory $(\mathcal{T}_{V(I)}, \mathcal{F}_{V(I)})$ corresponding to the closed subset $V(I)$ of $\operatorname{Spec}(R)$.

(2) It is possible to consider the subcategory $\mathcal{TH}(I,\mathcal{S}) = \{M \in R\text{-Mod}\colon\ M/(0 :_M I) \in \mathcal{S}\}$ associated with the Hom functor. In general, however, we see that $\mathcal{TH}(I,\{0\})$ does not coincide with $\mathcal{T}_{V(I)}$. To see this, let us take the zero subcategory as $\mathcal{S}$ over a local ring R with maximal ideal $\mathfrak{m}$. Then the module $E_R(R/\mathfrak{m})$ is in $\mathcal{T}_{V(\mathfrak{m})}$. However, if we suppose that $E_R(R/\mathfrak{m})$ is in $\mathcal{TH}(\mathfrak{m},\{0\})$, then we have $E_R(R/\mathfrak{m})/(0 :_{E_R(R/\mathfrak{m})} \mathfrak{m}) = E_R(R/\mathfrak{m})/(R/\mathfrak{m}) \in \{0\}$. Namely, one has $E_R(R/\mathfrak{m}) = R/\mathfrak{m}$. This equality means that R must be an Artinian ring.

(3) It is clear that $\mathcal{T}(I,\mathcal{S}) \supseteqq \mathcal{T}_{V(I)}$ and $\mathcal{FH}(I,\mathcal{S}) \supseteqq \mathcal{F}_{V(I)}$. Therefore, we have $\mathcal{T}(I,\mathcal{S}) * \mathcal{FH}(I,\mathcal{S}) = \mathcal{T}_{V(I)} * \mathcal{F}_{V(I)} = R\text{-Mod}$.

The main purpose of this section is to investigate the problem of whether the pair $(\mathcal{T}(I,\mathcal{S}), \mathcal{FH}(I,\mathcal{S}))$ is a hereditary torsion theory connected by a Serre subcategory $\mathcal{S}$ or not for an ideal I of R. However, we shall see that this problem has a negative answer. Therefore, we will try to give necessary and sufficient conditions for the pair $(\mathcal{T}(I,\mathcal{S}), \mathcal{FH}(I,\mathcal{S}))$ to be the hereditary torsion theory connected by $\mathcal{S}$. One of these conditions will be given by using the notion of Melkersson condition.

First of all, we observe the relationship between $\mathcal{FH}(I,\mathcal{S})$ and a Γ-torsion-free class $\mathcal{FG}(I,\mathcal{S})$ connected by the Serre subcategory $\mathcal{S}$ for an ideal I of R. We note that these subcategories may not be Serre subcategories because $\mathcal{FH}(I,\{0\}) = \mathcal{FG}(I,\{0\}) = \mathcal{F}_{V(I)}$ is not necessarily closed under taking quotient modules.

Proposition 5.3. *Let I be an ideal of R and $\mathcal{S}$ a Serre subcategory. Then the following hold.*

(1) *$\mathcal{FH}(I,\mathcal{S})$ is closed under taking submodules and extensions.*

(2) *One has $\mathcal{FH}(I,\mathcal{S}) = \mathcal{FH}(I^n,\mathcal{S}) = \mathcal{FH}(\sqrt{I},\mathcal{S})$ for each positive integer n.*

(3) *One has $\mathcal{FH}(I,\mathcal{S}) \supseteqq \mathcal{FG}(I,\mathcal{S}) = \mathcal{S} * \mathcal{F}_{V(I)} \supseteqq \mathcal{F}_{V(I)} * \mathcal{S}$.*

Proof. (1) For a short exact sequence $0 \to L \to M \to N \to 0$ of R-modules, there exists an exact sequence

$$0 \to \operatorname{Hom}_R(R/I, L) \to \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, N).$$

Thus, our assertion is proved from the definition of $\mathcal{FH}(I,\mathcal{S})$ because $\mathcal{S}$ is closed under taking submodules and extensions.

(2) We take an R-module M and a positive integer n. Since we have

$$\operatorname{Hom}_R(R/\sqrt{I}, M) \subseteqq \operatorname{Hom}_R(R/I, M) \subseteqq \operatorname{Hom}_R(R/I^n, M)$$

and $\mathcal{S}$ is closed under taking submodules, one has

$$\mathcal{FH}(\sqrt{I},\mathcal{S}) \supseteqq \mathcal{FH}(I,\mathcal{S}) \supseteqq \mathcal{FH}(I^n,\mathcal{S}).$$

Next, we will show $\mathcal{FH}(\sqrt{I},\mathcal{S}) \subseteqq \mathcal{FH}(I^n,\mathcal{S})$. Let us take an R-module M in $\mathcal{FH}(\sqrt{I},\mathcal{S})$. There exists a positive integer s such that $(\sqrt{I})^s \subseteqq I$. Then, we have

$$\operatorname{Hom}_R(R/I^n, M) \subseteqq \operatorname{Hom}_R(R/(\sqrt{I})^{ns}, M).$$

Thus, if we can show $\operatorname{Hom}_R(R/(\sqrt{I})^{ns}, M) \in \mathcal{S}$, then one has $M \in \mathcal{FH}(I^n,\mathcal{S})$ because $\mathcal{S}$ is closed under taking submodules.

We denote $\sqrt{I}$ by J. Since J/J^2 is a finitely generated R/J-module, there exists a surjective homomorphism φ from $\bigoplus^t R/J$ to J/J^2 for some positive integer t. Then, we have

$$\mathrm{Hom}_R(J/J^2, M) \subseteq \mathrm{Hom}_R\Big(\bigoplus^t R/J, M\Big) \cong \bigoplus^t \mathrm{Hom}_R(R/J, M).$$

Thus $\mathrm{Hom}_R(J/J^2, M)$ is in $\mathcal{S}$ because $\mathcal{S}$ is closed under taking submodules and extensions. The short exact sequence $0 \to J/J^2 \to R/J^2 \to R/J \to 0$ provides an exact sequence

$$0 \to \mathrm{Hom}_R(R/J, M) \to \mathrm{Hom}_R(R/J^2, M) \to \mathrm{Hom}_R(J/J^2, M).$$

Consequently, since M is in $\mathcal{FH}(J, \mathcal{S})$, we see that $\mathrm{Hom}_R(R/J^2, M) \in \mathcal{S}$. By repeating the same argument as above, we can prove that $\mathrm{Hom}_R(R/J^{ns}, M)$ is in $\mathcal{S}$.

(3) By virtue of Proposition 4.7, case (2), it remains to show $\mathcal{FH}(I, \mathcal{S}) \supseteq \mathcal{FG}(I, \mathcal{S})$. Let us take an R-module M in $\mathcal{FG}(I, \mathcal{S})$. Then, we have

$$\mathrm{Hom}_R(R/I, M) \cong (0 :_M I) \subseteq \Gamma_I(M) \in \mathcal{S}.$$

Therefore, since $\mathcal{S}$ is closed under taking submodules, we obtain $M \in \mathcal{FH}(I, \mathcal{S})$. □

In general, $\mathcal{FH}(I, \mathcal{S})$ does not coincide with $\mathcal{FG}(I, \mathcal{S})$ for a Serre subcategory $\mathcal{S}$ and an ideal I of R.

Example 5.4. Let R be a non-Artinian local ring with maximal ideal $\mathfrak{m}$.

(1) Since $E_R(R/\mathfrak{m})/\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{m})) = E_R(R/\mathfrak{m})/E_R(R/\mathfrak{m}) = 0 \in \mathcal{S}_{\mathrm{fg}}$, one has $E_R(R/\mathfrak{m}) \in \mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$.

(2) We have $\mathrm{Hom}_R(R/\mathfrak{m}, E_R(R/\mathfrak{m})) \cong R/\mathfrak{m} \in \mathcal{S}_{\mathrm{fg}}$. Thus, we obtain $E_R(R/\mathfrak{m}) \in \mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$.

(3) One has $\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{m})) = E_R(R/\mathfrak{m}) \notin \mathcal{S}_{\mathrm{fg}}$ because there exists no nonzero finitely generated injective module over a non-Artinian local ring. Therefore, we see that $E_R(R/\mathfrak{m}) \notin \mathcal{FG}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$.

Consequently, we see that $\mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) \supsetneqq \mathcal{FG}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$ by Proposition 5.3, case (3). Moreover, we have $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) \cap \mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) \supsetneqq \mathcal{S}_{\mathrm{fg}}$. It follows from Theorem 4.10 that the pair $(\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}), \mathcal{FG}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}))$ is a hereditary torsion theory connected by $\mathcal{S}_{\mathrm{fg}}$. However, Proposition 3.4 yields the pair $(\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}), \mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}))$ is not a hereditary torsion theory connected by $\mathcal{S}_{\mathrm{fg}}$.

Now, the following natural questions arise: For an ideal I of R and a Serre subcategory $\mathcal{S}$:

(1) When is the pair $(\mathcal{T}(I, \mathcal{S}), \mathcal{FH}(I, \mathcal{S}))$ a hereditary torsion theory connected by $\mathcal{S}$?

(2) What kind of Serre subcategory $\mathcal{S}$ does the equality $\mathcal{T}(I,\mathcal{S}) \cap \mathcal{FH}(I,\mathcal{S}) = \mathcal{S}$ hold?
(3) When does $\mathcal{FH}(I,\mathcal{S})$ coincide with $\mathcal{FG}(I,\mathcal{S})$?

The following theorem gives answers to these questions, which is the main result of this paper.

Theorem 5.5. *Let I be an ideal of R and $\mathcal{S}$ a Serre subcategory. Then the following conditions are equivalent:*
(1) *A pair $(\mathcal{T}(I,\mathcal{S}), \mathcal{FH}(I,\mathcal{S}))$ is a hereditary torsion theory connected by $\mathcal{S}$.*
(2) *One has $\mathcal{T}(I,\mathcal{S}) \cap \mathcal{FH}(I,\mathcal{S}) = \mathcal{S}$.*
(3) *One has $\mathcal{FH}(I,\mathcal{S}) = \mathcal{FG}(I,\mathcal{S})$.*
(4) *$\mathcal{S}$ satisfies the Melkersson condition (C_I).*

Proof. We will show the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1) hold.

(1) $\Rightarrow$ (2): It follows from Proposition 3.4.

(2) $\Rightarrow$ (4): We suppose that $\Gamma_I(M) = M$ and $(0 :_M I)$ is in $\mathcal{S}$ for an R-module M. Since $M/\Gamma_I(M) = 0 \in \mathcal{S}$, we have $M \in \mathcal{T}(I,\mathcal{S})$. Moreover, we have $\mathrm{Hom}_R(R/I, M) \cong (0 :_M I) \in \mathcal{S}$. This means $M \in \mathcal{FH}(I,\mathcal{S})$. Consequently, we obtain $M \in \mathcal{T}(I,\mathcal{S}) \cap \mathcal{FH}(I,\mathcal{S}) = \mathcal{S}$, and thus $\mathcal{S}$ satisfies the Melkersson condition (C_I).

(4) $\Rightarrow$ (3): We have already seen that $\mathcal{FH}(I,\mathcal{S}) \supseteq \mathcal{FG}(I,\mathcal{S})$ in Proposition 5.3. We suppose that $\mathcal{S}$ satisfies the Melkersson condition (C_I) and an R-module M is in $\mathcal{FH}(I,\mathcal{S})$. We apply the left exact functor $\mathrm{Hom}_R(R/I, -)$ to a short exact sequence $0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$ of R-modules. Then we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(R/I, \Gamma_I(M)) \to \mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/I, M/\Gamma_I(M)).$$

Since $\mathrm{Hom}_R(R/I, M/\Gamma_I(M)) = 0$ and $M \in \mathcal{FH}(I,\mathcal{S})$, one has

$$\mathrm{Hom}_R(R/I, \Gamma_I(M)) \cong \mathrm{Hom}_R(R/I, M) \in \mathcal{S}.$$

Furthermore, it is clear that $\Gamma_I(\Gamma_I(M)) = \Gamma_I(M)$. Thus, we see that $\Gamma_I(M) \in \mathcal{S}$ by the Melkersson condition (C_I) for $\mathcal{S}$. Consequently, we have $M \in \mathcal{FG}(I,\mathcal{S})$.

(3) $\Rightarrow$ (1): By Theorem 4.10, we can conclude that the pair

$$(\mathcal{T}(I,\mathcal{S}), \mathcal{FH}(I,\mathcal{S})) = (\mathcal{T}(I,\mathcal{S}), \mathcal{FG}(I,\mathcal{S}))$$

is a hereditary torsion theory connected by $\mathcal{S}$. $\square$

The following corollary is the $\mathcal{FH}(I,\mathcal{S})$-version for Proposition 4.7, case (3).

Corollary 5.6. *Let I be an ideal of R. If a Serre subcategory $\mathcal{S}$ is closed under taking injective hulls, then one has*

$$\mathcal{FH}(I,\mathcal{S}) = \mathcal{FG}(I,\mathcal{S}) = \mathcal{S} * \mathcal{F}_{V(I)} = \mathcal{F}_{V(I)} * \mathcal{S}.$$

Proof. A Serre subcategory that is closed under taking injective hulls satisfies Melkersson condition with respect to all ideals of R. Therefore, our assertion is proved by Proposition 4.7, case (3) and the implication (4) $\Rightarrow$ (3) in Theorem 5.5. $\square$

6. Examples of generalized torsion theories

In this section, we give examples of hereditary torsion theories connected by a Serre subcategory.

The first example is a torsion class connected by a Serre subcategory that is not closed under taking arbitrary direct sums. Therefore, the following generalized hereditary torsion theory is not an ordinary one. Let us denote by $\mathcal{S}_{\mathrm{Artin}}$ and $\mathcal{S}_{\mathrm{FA}}$ the Serre subcategory consisting of the Artinian modules and the Serre subcategory $\mathcal{S}_{\mathrm{fg}} * \mathcal{S}_{\mathrm{Artin}}$ consisting of the Minimax modules, respectively.

Example 6.1. We suppose that R is a 1-dimensional semi-local ring with at least two minimal prime ideals and $\mathfrak{p}$ is a minimal prime ideal of R. Then the following assertions (1)–(4) hold.

(1) The Serre subcategory $\mathcal{S}_{\mathrm{FA}}$ is closed under taking injective hulls. In particular, it satisfies the Melkersson condition $(\mathrm{C}_{\mathfrak{p}})$.

(2) One has $\mathcal{T}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}}) = \mathcal{T}_{V(\mathfrak{p})} * \mathcal{S}_{\mathrm{FA}} = \mathcal{S}_{\mathrm{FA}} * \mathcal{T}_{V(\mathfrak{p})}$. Furthermore, this is a Serre subcategory which is closed under taking injective hulls but not closed under taking arbitrary direct sums. Therefore, $\mathcal{T}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}})$ is not a torsion class for any torsion theory in the ordinary sense.

(3) One has $\mathcal{FH}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}}) = \mathcal{FG}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}}) = \mathcal{S}_{\mathrm{FA}} * \mathcal{F}_{V(\mathfrak{p})} = \mathcal{F}_{V(\mathfrak{p})} * \mathcal{S}_{\mathrm{FA}}$ and this is closed under taking injective hulls.

(4) The pair $(\mathcal{T}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}}), \mathcal{FH}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}}))$ is a hereditary torsion theory connected by $\mathcal{S}_{\mathrm{FA}}$ with the inclusion relations

$$\begin{array}{ccccc} & & R\text{-Mod} & & \\ & \subsetneq & & \supsetneq & \\ \mathcal{T}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}}) & & & & \mathcal{FH}(\mathfrak{p},\mathcal{S}_{\mathrm{FA}}) \\ & \supsetneq & & \subsetneq & \\ & & \mathcal{S}_{\mathrm{FA}} & & \end{array}$$

of subcategories such that $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) * \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = R\text{-Mod}$ and $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) \cap \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = \mathcal{S}_{\mathrm{FA}}$.

Let us prove that the above assertions hold.

(1): By virtue of [8], Corollary 3.3, the subcategory $\mathcal{S}_{\mathrm{FA}}$ is a Serre subcategory. Since R is a 1-dimensional semi-local ring, we see that $\mathcal{S}_{\mathrm{FA}}$ is closed under taking injective hulls by [9], Theorem 3.5. Moreover, if a Serre subcategory is closed under taking injective hulls, then it satisfies the Melkersson conditions with respect to all ideals of R. Therefore, the subcategory $\mathcal{S}_{\mathrm{FA}}$ satisfies the Melkersson condition $(\mathrm{C}_{\mathfrak{p}})$.

(2): We have $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = \mathcal{T}_{V(\mathfrak{p})} * \mathcal{S}_{\mathrm{FA}} = \mathcal{S}_{\mathrm{FA}} * \mathcal{T}_{V(\mathfrak{p})}$ by the assertion (1) and Proposition 4.3, case (4). Moreover, $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$ is a Serre subcategory that is closed under taking injective hulls by Lemma 2.3, case (2), Proposition 4.3, cases (1) and (3).

Next, we will prove that $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$ is not closed under taking arbitrary direct sums. By our assumption, we can take a minimal prime ideal $\mathfrak{q}$ with $\mathfrak{q} \neq \mathfrak{p}$. The R-module $R/\mathfrak{q}$ is in $\mathcal{S}_{\mathrm{fg}} \subsetneqq \mathcal{S}_{\mathrm{FA}} \subsetneqq \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$. Since $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$ is closed under taking injective hulls, we have $E_R(R/\mathfrak{q}) \in \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$.

Here, we denote an infinite direct sum of copies of $E_R(R/\mathfrak{q})$ by $E(\mathfrak{q})$, and shall see $E(\mathfrak{q}) \notin \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$. We assume $E(\mathfrak{q}) \in \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = \mathcal{T}_{V(\mathfrak{p})} * \mathcal{S}_{\mathrm{FA}}$. Then there exists a short exact sequence

$$0 \to T \to E(\mathfrak{q}) \to S \to 0$$

of R-modules with $T \in \mathcal{T}_{V(\mathfrak{p})}$ and $S \in \mathcal{S}_{\mathrm{FA}}$. This sequence implies $\mathrm{Ass}_R(T) \subsetneqq \mathrm{Ass}_R(E(\mathfrak{q})) = \{\mathfrak{q}\}$. However, since T is in $\mathcal{T}_{V(\mathfrak{p})}$, one has $\mathrm{Ass}_R(T) \subsetneqq V(\mathfrak{p})$. Therefore, we can see that $\mathrm{Ass}_R(T) = \emptyset$ because the minimal prime $\mathfrak{q}$ cannot belong to $V(\mathfrak{p})$. This equality means $T = 0$, and thus the above short exact sequence yields $E(\mathfrak{q}) \cong S \in \mathcal{S}_{\mathrm{FA}}$. Then, there exists a short exact sequence

$$0 \to X \to E(\mathfrak{q}) \to Y \to 0$$

of R-modules with $X \in \mathcal{S}_{\mathrm{fg}}$ and $Y \in \mathcal{S}_{\mathrm{Artin}}$. We note that $E(\mathfrak{q})$ is a direct summand of $E_R(X) \oplus E_R(Y)$, and $E_R(X) \oplus E_R(Y)$ is a finite direct sum of indecomposable injective R-modules. However, this is not possible because $E(\mathfrak{q})$ is an infinite direct sum of indecomposable injective R-modules. Consequently, the contradiction is derived, and hence we see that $E(\mathfrak{q}) \notin \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$.

In addition, we note that an ordinary torsion class must be closed under taking arbitrary direct sums. (See [7], Ch. VI, Proposition 2.1.) Hence, the above argument shows that for $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$ it is not possible to be a torsion class for some ordinary torsion theory.

(3): It follows from our assertion (1), Lemma 2.3, case (2), and Corollary 5.6.

(4): We have already seen that $\mathcal{S}_{\mathrm{FA}}$ satisfies the Melkersson condition ($\mathrm{C}_{\mathfrak{p}}$) in the assertion (1). Therefore, Theorem 5.5 implies that the pair

$$(\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}), \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}))$$

is a hereditary torsion theory connected by $\mathcal{S}_{\mathrm{FA}}$ and that one has $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) \cap \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = \mathcal{S}_{\mathrm{FA}}$. Furthermore, we have $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) * \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = R$-Mod by Remark 5.2, case (3).

Next, we shall see that the inclusion relations hold. It is easy to see that we have $\mathcal{S}_{\mathrm{FA}} \subseteqq \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) \subseteqq R$-Mod. Moreover, $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$ is strictly contained in R-Mod because we have already seen that $E(\mathfrak{q}) \notin \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$. If we denote an infinite direct sum of copies of $E_R(R/\mathfrak{p})$ by $E(\mathfrak{p})$, then one has $E(\mathfrak{p}) \in \mathcal{T}_{V(\mathfrak{p})} \subseteqq \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$. However, we can deduce $E(\mathfrak{p}) \notin \mathcal{S}_{\mathrm{FA}}$ for the same reason when discussing $E(\mathfrak{q}) \notin \mathcal{S}_{\mathrm{FA}}$ in the proof of assertion (2).

Finally, it remains to prove $\mathcal{S}_{\mathrm{FA}} \subsetneqq \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) \subsetneqq R$-Mod. It is easy to see that the inclusion relations hold. We assume $\mathcal{S}_{\mathrm{FA}} = \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$. Since $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$ is closed under taking extensions, one has

$$\begin{aligned} R\text{-Mod} &= \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) * \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) * \mathcal{S}_{\mathrm{FA}} \\ &\subseteqq \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) * \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}). \end{aligned}$$

However, this conclusion contradicts $\mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) \subsetneqq R$-Mod. On the other hand, we assume $\mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = R$-Mod. Then the following equalities hold:

$$\mathcal{S}_{\mathrm{FA}} = \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) \cap \mathcal{FH}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}) = \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}}).$$

These equalities imply a contradiction because we have $\mathcal{S}_{\mathrm{FA}} \subsetneqq \mathcal{T}(\mathfrak{p}, \mathcal{S}_{\mathrm{FA}})$.

The second example states that a torsion class connected by a Serre subcategory is not necessarily closed under taking extensions.

Example 6.2. Let R be a 1-dimensional Gorenstein local ring with maximal ideal $\mathfrak{m}$.

(1) The subcategory $\mathcal{S}_{\mathrm{fg}}$ is a Serre subcategory which does not satisfy the Melkersson condition ($\mathrm{C}_{\mathfrak{m}}$).

(2) One has $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) = \mathcal{T}_{V(\mathfrak{m})} * \mathcal{S}_{\mathrm{fg}} \subsetneqq \mathcal{S}_{\mathrm{fg}} * \mathcal{T}_{V(\mathfrak{m})}$. Moreover, $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$ is closed under taking submodules and quotient modules. However, this subcategory is not closed under taking extensions or injective hulls.

(3) One has $\mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) \supsetneqq \mathcal{FG}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) = \mathcal{S} * \mathcal{F}_{V(\mathfrak{m})}$. These two subcategories are closed under taking submodules and extensions.

(4) The pair $(\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}), \mathcal{FG}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}))$ is a hereditary torsion theory connected by $\mathcal{S}_{\mathrm{fg}}$ with $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) * \mathcal{FG}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) = R\text{-Mod}$ and $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) \cap \mathcal{FG}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) = \mathcal{S}_{\mathrm{fg}}$.

(5) The pair $(\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}), \mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}))$ is not a hereditary torsion theory connected by $\mathcal{S}_{\mathrm{fg}}$ with $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) * \mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) = R\text{-Mod}$ and $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) \cap \mathcal{FH}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}}) \supsetneqq \mathcal{S}_{\mathrm{fg}}$.

Let us prove that the above assertions hold.

(1): We have $\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{m})) = E_R(R/\mathfrak{m})$ and $(0 :_{E_R(R/\mathfrak{m})} \mathfrak{m}) \cong R/\mathfrak{m} \in \mathcal{S}_{\mathrm{fg}}$. However, since R is a non-Artinian local ring, one has $E_R(R/\mathfrak{m}) \notin \mathcal{S}_{\mathrm{fg}}$.

(2): It follows from Proposition 4.3 and Example 4.5. In particular, we have already seen that R is in $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$ but $E_R(R)$ is not in $\mathcal{T}(\mathfrak{m}, \mathcal{S}_{\mathrm{fg}})$ in Example 4.5.

(3)–(5): Our assertions have been already seen in Proposition 4.7, Theorem 4.10, Proposition 5.3, Example 5.4, and Theorem 5.5.

The third example is a trivial case. However, this example states that $\mathcal{T}(I, \mathcal{S}) \subsetneqq \mathcal{FG}(I, \mathcal{S}) \subseteqq \mathcal{FH}(I, \mathcal{S})$ or $\mathcal{T}(I, \mathcal{S}) \supsetneqq \mathcal{FH}(I, \mathcal{S}) \supseteqq \mathcal{FG}(I, \mathcal{S})$ may occur for an ideal I of R and a Serre subcategory $\mathcal{S}$.

Example 6.3. Let I be an ideal of R and W a specialization closed subset of $\mathrm{Spec}(R)$. Then the following assertions (1)–(4) hold.

(1) The subcategory $\mathcal{T}_W$ is a Serre subcategory that is closed under taking injective hulls. In particular, $\mathcal{T}_W$ satisfies the Melkersson condition (C_I).

(2) One has $\mathcal{T}(I, \mathcal{T}_W) = \mathcal{T}_{V(I)} * \mathcal{T}_W = \mathcal{T}_W * \mathcal{T}_{V(I)} = \mathcal{T}_{V(I) \cup W}$. (As for the last equality, also see [8], Lemma 2.2, case (1).) This subcategory is a Serre subcategory that is closed under taking arbitrary direct sums.

(3) One has $\mathcal{FH}(I, \mathcal{T}_W) = \mathcal{FG}(I, \mathcal{T}_W) = \mathcal{F}_{V(I)} * \mathcal{T}_W = \mathcal{T}_W * \mathcal{F}_{V(I)} = \{M \in R\text{-Mod} \colon \Gamma_W(\Gamma_I(M)) = \Gamma_I(M)\}$. (The last equality is derived from the definition of $\mathcal{FG}(I, \mathcal{T}_W)$.) This subcategory is closed under taking submodules, extensions, and injective hulls.

(4) The pair $(\mathcal{T}(I, \mathcal{T}_W), \mathcal{FH}(I, \mathcal{T}_W))$ is a hereditary torsion theory connected by $\mathcal{T}_W$ with $\mathcal{T}(I, \mathcal{T}_W) * \mathcal{FH}(I, \mathcal{T}_W) = R\text{-Mod}$ and $\mathcal{T}(I, \mathcal{T}_W) \cap \mathcal{FH}(I, \mathcal{T}_W) = \mathcal{T}_W$.

The above assertions are proved by the same arguments as in Example 6.1. In particular, if we take $V(I) \subsetneqq \mathrm{Spec}(R)$ as W, then we have

$$\mathcal{T}(I, \mathcal{T}_{V(I)}) = \mathcal{T}_{V(I)} \subsetneqq \mathcal{FH}(I, \mathcal{T}_{V(I)}) = \mathcal{FG}(I, \mathcal{T}_{V(I)}) = R\text{-Mod}.$$

Moreover, if we suppose $W \subsetneqq \mathrm{Spec}(R)$, then one has

$$\mathcal{T}((0), \mathcal{T}_W) = \mathcal{T}_{V((0))} = R\text{-Mod} \supsetneqq \mathcal{FH}((0), \mathcal{T}_W) = \mathcal{FG}((0), \mathcal{T}_W) = \mathcal{T}_W.$$

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References

[1] *M. Aghapournahr, L. Melkersson*: Local cohomology and Serre subcategories. J. Algebra *320* (2008), 1275–1287. zbl MR doi

[2] *A. Beligiannis, I. Reiten*: Homological and homotopical aspects of torsion theories. Mem. Am. Math. Soc. *883* (2007), 207 pages. zbl MR doi

[3] *S. E. Dickson*: A torsion theory for Abelian categories. Trans. Am. Math. Soc. *121* (1966), 223–235. zbl MR doi

[4] *P. Gabriel*: Des catégories abéliennes. Bull. Soc. Math. Fr. *90* (1962), 323–448. (In French.) zbl MR doi

[5] *J. Lambek*: Torsion Theories, Additive Semantics, and Rings of Quotients. Lecture Notes in Mathematics 177, Springer, Berlin, 1971. zbl MR doi

[6] *B. Stenström*: Rings and Modules of Quotients. Lecture Notes in Mathematics 237, Springer, Berlin, 1971. zbl MR doi

[7] *B. Stenström*: Rings of Quotients. An Introduction to Methods of Ring Theory. Die Grundlehren der Mathematischen Wissenschaften 217, Springer, Berlin, 1975. zbl MR doi

[8] *T. Yoshizawa*: Subcategories of extension modules by Serre subcategories. Proc. Am. Math. Soc. *140* (2012), 2293–2305. zbl MR doi

[9] *T. Yoshizawa*: On the closedness of taking injective hulls of several Serre subcategories. Commun. Algebra *45* (2017), 4846–4854. zbl MR doi

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