Mathematica Bohemica

Renukadevi S. Dyavanal; Rajalaxmi V. Desai

Uniqueness of q-shift difference polynomials of meromorphic functions sharing a small function

Mathematica Bohemica, Vol. 145 (2020), No. 3, 241–253

Persistent URL: http://dml.cz/dmlcz/148347

UNIQUENESS OF q-SHIFT DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION

Renukadevi S. Dyavanal, Rajalaxmi V. Desai, Dharwad

Received August 1, 2018. Published online July 10, 2019.
Communicated by Grigore Sălăgean

Abstract. We investigate the uniqueness of a q-shift difference polynomial of meromorphic functions sharing a small function which extend the results of N. V. Thin (2017) to q-difference operators.

Keywords: Nevanlinna theory; meromorphic function; q-shift difference polynomial; uniqueness

MSC 2010: 30D35

1. Introduction and results

In this paper, a meromorphic function always means it is meromorphic in the complex plane $\mathbb{C}$. We use the standard notation in Nevanlinna's value distribution theory (see, e.g. [9], [14], [15]). We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite logarithmic measure. We write $\varrho(f)$ for order of $f(z)$.

The following definitions we use while proving our results.

Definition 1.1. Let a be a finite complex number, and k a positive integer. We denote by $N_{(k}(r,a,f)$ the counting function for zeros of $f-a$ with multiplicities at least k, and by $\overline{N}_{(k}(r,a,f)$ the one for which multiplicity is not counted. Similarly, we denote by $N_{k)}(r,a,f)$ the counting function for zeros of $f-a$ with multiplicities

The first author is supported by UGC-SAP-DRS-III Programme with Ref. No. F.510/3/DRS-III/2016(SAP-I) Dated: 29th February 2016. And the Second author was supported by UGC-UPE fellowship, Department of Mathematics, Karnatak University, Dharwad. Ref. No. KU/Sch/UGC-UPE/2013-14/1144.

DOI: 10.21136/MB.2019.0093-18

at most k, and by $\overline{N}_{(k}(r,a,f)$ the one for which multiplicity is not counted. Then

$$N_k(r,a,f) = \overline{N}_{(1}(r,a,f) + \overline{N}_{(2}(r,a,f) + \ldots + \overline{N}_{(k}(r,a,f).$$

Definition 1.2. Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane $\mathbb{C}$. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM, and if we do not consider the multiplicity, then we say that $f(z)$ and $g(z)$ share the value a IM, where a is a complex number.

Recently, people have raised great interest in difference analogues of Nevanlinna's theory and many articles have focused on value distribution and uniqueness of difference polynomials of entire or meromorphic functions (see for example [1]–[8]).

In 2015, Zhao and Zhang (see [17]) proved the following results

Theorem 1.A. *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order and let n, k be positive integers. If $n > 2k+5$, then $(f^n f(qz+c))^{(k)}$ and $(g^n g(qz+c))^{(k)}$ share z or 1 CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem 1.B. *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order and let n, k be positive integers. If $n > 5k+11$, then $(f^n f(qz+c))^{(k)}$ and $(g^n g(qz+c))^{(k)}$ share z or 1 IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

In 2017, Thin proved the following theorems for meromorphic functions (see [12]).

Theorem 1.C. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (entire) functions of zero order, q and c be complex constants, $q \neq 0$, k be a positive integer. Let $a(z) \not\equiv 0$ be a meromorphic (entire) small function and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n (\neq 0)$ and m be the distinct zeros of $P(z)$. If $n > 2m(k+1)+2k+6$ (respectively, $n > 2m(k+1)+4$) and $(P(f)f(qz+c))^{(k)}$ and $(P(g)g(qz+c))^{(k)}$ share $a(z)$, ∞ CM, then one of the following two results holds:*

(1) *$f = tg$ for a constant t with $t^d = 1$, where $d = LCM\{\lambda_j\colon\ j = 0,1,\ldots,n\}$, and*

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(2) *f and g satisfy the algebraic equation $R(f,g) = 0$, where*

$$R(\omega_1,\omega_2) = P(\omega_1)\omega_1(qz+c) - P(\omega_2)\omega_2(qz+c).$$

Theorem 1.D. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order, q and c be complex constants, $q \neq 0$, k be a positive integer. Let $a(z) \not\equiv 0$ be a meromorphic (entire) small function and let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n (\neq 0)$ and m be the distinct zeros of $P(z)$. If $n > 2m(k+2) + 3m(k+1) + 8k + 21$ and $(P(f)f(qz+c))^{(k)}$ and $(P(g)g(qz+c))^{(k)}$ share $a(z)$ IM, then one of the following three results holds:*

(1) $(P(f)f(qz+c))^{(k)}(P(g)g(qz+c))^{(k)} \equiv a^2$,

(2) *$f = tg$ for a constant t with $t^d = 1$, where $d = LCM\{\lambda_j \colon j = 0, 1, \ldots, n\}$, and*

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(3) *f and g satisfy the algebraic equation $R(f, g) = 0$, where*

$$R(\omega_1, \omega_2) = P(\omega_1)\omega_1(qz+c) - P(\omega_2)\omega_2(qz+c).$$

In this paper, we extend Theorem 1.C and Theorem 1.D to the q-difference operator $\Delta_q f = f(qz+c) - f(z)$ and prove the following theorems.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (entire) functions of zero order, such that $f(qz+c) - f(z) \not\equiv 0$ and $g(qz+c) - g(z) \not\equiv 0$, where q and c are nonzero complex constants, k, n, m are positive integers. Let $a(z)$ $(\not\equiv 0)$ be a small function of $f(z)$ and $g(z)$. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n (\neq 0)$ and m the distinct zeros of $P(z)$. If $n > 2mk + 2m + 2k + 7$ (respectively, $n > 2mk + 2m + 5$) and $(P(f)(f(qz+c) - f(z)))^{(k)}$ and $(P(g)(g(qz+c) - g(z)))^{(k)}$ share $a(z)$, ∞ CM, then one of the following two cases holds:*

(1) *$f \equiv tg$ for a constant t with $t^d = 1$, where $d = LCM\{\lambda_j \colon j = 0, 1, \ldots, n\}$, and*

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(2) *f and g satisfy the algebraic equation $R(f, g) = 0$, where*

$$R(\omega_1, \omega_2) = P(\omega_1)(\omega_1(qz+c) - \omega_1(z)) - P(\omega_2)(\omega_2(qz+c) - \omega_2(z)).$$

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order such that $f(qz+c)-f(z)\not\equiv 0$ and $g(qz+c)-g(z)\not\equiv 0$, where q and c are nonzero complex constants and k, n, m are positive integers. Let $a(z)$ ($\not\equiv 0$) be a small function of $f(z)$ and $g(z)$, let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ be a nonconstant polynomial with constant coefficients $a_0,a_1,\ldots,a_{n-1},a_n(\neq 0)$ and m the distinct zeros of $P(z)$. If $n>5mk+7m+8k+25$ and $(P(f)(f(qz+c)-f(z)))^{(k)}$ and $(P(g)(g(qz+c)-g(z)))^{(k)}$ share $a(z)$ IM, then one of the following three cases holds:*

(1) $(P(f)(f(qz+c)-f(z)))^{(k)}(P(g)(g(qz+c)-g(z)))^{(k)}\equiv a^2$,

(2) $f\equiv tg$ *for a constant t with $t^d=1$, where $d=LCM\{\lambda_j\colon\ j=0,1,\ldots,n\}$, and*

$$\lambda_j=\begin{cases} j+1 & \text{if } a_j\neq 0,\\ n+1 & \text{if } a_j=0,\end{cases}$$

(3) *f and g satisfy the algebraic equation $R(f,g)=0$, where*

$$R(\omega_1,\omega_2)=P(\omega_1)(\omega_1(qz+c)-\omega_1(z))-P(\omega_2)(\omega_2(qz+c)-\omega_2(z)).$$

As a particular case of the above theorems, we deduce the following corollaries.

Corollary 1.1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order such that $f(qz+c)-f(z)\not\equiv 0$ and $g(qz+c)-g(z)\not\equiv 0$, where q and c are nonzero complex constants. Let k, n be positive integers, $a(z)$ ($\not\equiv 0$) be a small function of $f(z)$ and $g(z)$, and α a complex constant. If $n>4k+9$ and $((f-\alpha)^n(f(qz+c)-f(z)))^{(k)}$ and $((g-\alpha)^n(g(qz+c)-g(z)))^{(k)}$ share $a(z)$, ∞ CM, then one of the following two cases holds:*

(1) $f\equiv tg$ *for a constant t with $t^{n+1}=1$,*

(2) *f and g satisfy the algebraic equation $R(f,g)=0$, where*

$$R(\omega_1,\omega_2)=(\omega_1-\alpha)^n(\omega_1(qz+c)-\omega_1(z))-(\omega_2-\alpha)^n(\omega_2(qz+c)-\omega_2(z)).$$

Corollary 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order such that $f(qz+c)-f(z)\not\equiv 0$ and $g(qz+c)-g(z)\not\equiv 0$, where q and c are nonzero complex constants. Let k, n be positive integers, $a(z)$ ($\not\equiv 0$) be a small function of $f(z)$ and $g(z)$, and α a complex constant. If $n>13k+32$ and $((f-\alpha)^n(f(qz+c)-f(z)))^{(k)}$ and $((g-\alpha)^n(g(qz+c)-g(z)))^{(k)}$ share $a(z)$ IM, then one of the following three cases holds:*

(1) $((f-\alpha)^n(f(qz+c)-f(z)))^{(k)}((g-\alpha)^n(g(qz+c)-g(z)))^{(k)}\equiv a^2$,

(2) $f\equiv tg$ *for a constant t with $t^{n+1}=1$,*

(3) *f and g satisfy the algebraic equation $R(f,g)=0$, where*

$$R(\omega_1,\omega_2)=(\omega_1-\alpha)^n(\omega_1(qz+c)-\omega_1(z))-(\omega_2-\alpha)^n(\omega_2(qz+c)-\omega_2(z)).$$

2. Some preliminary results

To prove our theorems we require the following lemmas.

Lemma 2.1 ([14]). *Let $f(z)$ be a nonconstant meromorphic function, then*

$$T(r,P_n(f))=nT(r,f)+S(r,f).$$

Lemma 2.2 ([13]). *Let $f(z)$ be a nonconstant meromorphic function of zero order and let q, η be two nonzero complex constants. Then on a set of lower logarithmic density 1, we have*

$$T(r,f(qz+\eta))=T(r,f)+S(r,f).$$

Lemma 2.3 ([13]). *Let $f(z)$ be a nonconstant meromorphic function of zero order and let q, η be two nonzero complex constants. Then on a set of lower logarithmic density 1, we have*

$$N(r,f(qz+\eta))=N(r,f)+S(r,f),$$
$$N\Big(r,\frac{1}{f(qz+\eta)}\Big)=N\Big(r,\frac{1}{f}\Big)+S(r,f).$$

Lemma 2.4 ([11], Theorem 2.1). *Let $f(z)$ be a nonconstant zero order meromorphic function and let q be a nonzero complex number. Then on a set of logarithmic density 1, we have*

$$m\Big(r,\frac{f(qz+\eta)}{f(z)}\Big)=S(r,f).$$

Lemma 2.5 ([16]). *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, and let $a(z)(\not\equiv 0,\infty)$ be a small function of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a(z)$ IM, then one of the following three cases holds:*

(1) $T(r,f)\leqslant N_2(r,1/f)+N_2(r,f)+N_2(r,1/g)+N_2(r,g)+2(\overline{N}(r,1/f)+\overline{N}(r,f))+(\overline{N}(r,1/g)+\overline{N}(r,g))+S(r,f)+S(r,g)$, *and a similar inequality holds for* $T(r,g)$,
(2) $fg\equiv 1$,
(3) $f\equiv g$.

Lemma 2.6 ([10], Lemma 2.11). *Let $f(z)$ be a nonconstant meromorphic function, and let p, k be positive integers. Then*

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f),$$
$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant N_{p+k}\Big(r, \frac{1}{f}\Big) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of zero order and let $F = P(f)(f(qz+c) - f(z))$, where n is a positive integer. Then*

$$(n-1)T(r, f) + S(r, f) \leqslant T(r, F).$$

P r o o f. From First fundamental theorem, Lemma 2.1 and Lemma 2.4, we obtain

$$\begin{aligned}(n+1)T(r, f) &= T(r, f(z)P(f)) + S(r, f) \leqslant T\Big(r, \frac{f(z)F}{f(qz+c) - f(z)}\Big) + S(r, f)\\ &\leqslant T(r, F) + T\Big(r, \frac{f(qz+c) - f(z)}{f(z)}\Big) + S(r, f)\\ &\leqslant T(r, F) + T\Big(r, \frac{f(qz+c)}{f(z)}\Big) + S(r, f)\\ &\leqslant T(r, F) + m\Big(r, \frac{f(qz+c)}{f(z)}\Big) + N\Big(r, \frac{f(qz+c)}{f(z)}\Big) + S(r, f)\\ &\leqslant T(r, F) + 2T(r, f) + S(r, f).\end{aligned}$$

Therefore $(n-1)T(r, f) + S(r, f) \leqslant T(r, F)$ on a set of logarithmic density 1. □

Lemma 2.8. *Let $f(z)$ be a transcendental entire function of zero order and $P(f) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0$. Let $F = P(f)(f(qz+c) - f(z))$, where n is a positive integer. Then*

$$nT(r, f) + S(r, f) \leqslant T(r, F)$$

P r o o f. From First fundamental theorem, Lemma 2.1 and Lemma 2.4, we obtain

$$(n+1)T(r, f) = T(r, f(z)P(f)) + S(r, f) \leqslant T\Big(r, \frac{f(z)F}{f(qz+c) - f(z)}\Big) + S(r, f)$$

$$\begin{aligned}
&\leqslant T(r,F)+T\Big(r,\frac{f(qz+c)-f(z)}{f(z)}\Big)+S(r,f)\\
&\leqslant T(r,F)+T\Big(r,\frac{f(qz+c)}{f(z)}\Big)+S(r,f)\\
&\leqslant T(r,F)+m\Big(r,\frac{f(qz+c)}{f(z)}\Big)+N\Big(r,\frac{f(qz+c)}{f(z)}\Big)+S(r,f)\\
&\leqslant T(r,F)+T(r,f)+S(r,f).
\end{aligned}$$

Therefore $nT(r,f)+S(r,f)\leqslant T(r,F)$ on a set of logarithmic density 1. □

3. Proof of theorems

Proof of Theorem 1.1. Let $F(z)=P(f)(f(qz+c)-f(z))$ and $F^{(k)}(z)=(P(f)(f(qz+c)-f(z)))^{(k)}$ and $G(z)=P(g)(g(qz+c)-g(z))$ and $G^{(k)}(z)=(P(g)(g(qz+c)-g(z)))^{(k)}$. Since $F^{(k)}$ and $G^{(k)}$ share $a(z)$, ∞ CM, there exists a nonzero constant β such that

$$\frac{(P(f)(f(qz+c)-f(z)))^{(k)}/a(z)-1}{(P(g)(g(qz+c)-g(z)))^{(k)}/a(z)-1}=\beta, \tag{3.1}$$

and we get

$$(P(f)(f(qz+c)-f(z)))^{(k)}-a(z)(1-\beta)=\beta(P(g)(g(qz+c)-g(z)))^{(k)}.$$

Now, we will prove that $\beta=1$. Let, on the contrary, $\beta\neq1$. Using the Second fundamental theorem and by Lemma 2.6 , we get

$$\begin{aligned}
T(r,F^{(k)})&\leqslant \overline{N}(r,F^{(k)})+\overline{N}\Big(r,\frac{1}{F^{(k)}}\Big)+\overline{N}\Big(r,\frac{1}{F^{(k)}-a(z)(1-\beta)}\Big)+S(r,f)\\
&\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F^{(k)}}\right)+\overline{N}\Big(r,\frac{1}{G^{(k)}}\Big)+S(r,f)\\
&\leqslant \overline{N}(r,F)+T(r,F^{(k)})-T(r,F)+N_{k+1}\Big(r,\frac{1}{F}\Big)+k\overline{N}(r,G)\\
&\quad+N_{k+1}\Big(r,\frac{1}{G}\Big)+S(r,f)+S(r,g),
\end{aligned}$$

which implies

$$\begin{aligned}
T(r,F) &\leqslant \overline{N}(r,F) + N_{k+1}\Big(r,\frac{1}{F}\Big) + k\overline{N}(r,G) + N_{k+1}\Big(r,\frac{1}{G}\Big) + S(r,f) + S(r,g)\\
&\leqslant \overline{N}(r,f) + \overline{N}(r,f(qz+c)) + N_{k+1}\Big(r,\frac{1}{P(f)}\Big) + N\Big(r,\frac{1}{f(qz+c)-f(z)}\Big)\\
&\quad + k\overline{N}(r,g) + k\overline{N}(r,g(qz+c)) + N_{k+1}\Big(r,\frac{1}{P(g)}\Big)\\
&\quad + N\Big(r,\frac{1}{g(qz+c)-g(z)}\Big) + S(r,f) + S(r,g)\\
&\leqslant T(r,f) + T(r,f) + m(k+1)T(r,f) + 2T(r,f) + m(k+1)T(r,g)\\
&\quad + 2T(r,g) + kT(r,g) + kT(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Combining this with Lemma 2.7, we obtain

$$(n-1)T(r,f) \leqslant (mk+m+4)T(r,f)+(mk+m+2k+2)T(r,g)+S(r,f)+S(r,g). \tag{3.2}$$

Similarly

$$(n-1)T(r,g) \leqslant (mk+m+4)T(r,g)+(mk+m+2k+2)T(r,f)+S(r,f)+S(r,g). \tag{3.3}$$

From (3.2)and (3.3), we get

$$(n-1)(T(r,f)+T(r,g)) \leqslant (2mk+2m+2k+6)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which implies $(n-1-2mk-2m-2k-6)(T(r,f)+T(r,g)) \leqslant S(r,f)+S(r,g)$, that is $(n-2mk-2m-2k-7)(T(r,f)+T(r,g)) \leqslant S(r,f)+S(r,g)$. This is a contradiction to $n > 2mk+2m+2k+7$. Thus, we get $\beta = 1$. Hence from (3.1), we have

$$(P(f)(f(qz+c)-f(z)))^{(k)} = (P(g)(g(qz+c)-g(z)))^{(k)},$$

and we get

$$P(f)(f(qz+c)-f(z)) = P(g)(g(qz+c)-g(z)) + r(z), \tag{3.4}$$

where $r(z)$ is a polynomial of degree at most $k-1$. Suppose $r(z) \not\equiv 0$, then we get

$$\frac{P(f)(f(qz+c)-f(z))}{r(z)} = \frac{P(f)(g(qz+c)-g(z))}{r(z)} + 1.$$

Therefore, from Lemma 2.7 and the Second fundamental theorem, we have

$$\begin{aligned}(n-1)T(r,f) &\leqslant T\Big(r, \frac{P(f)(f(qz+c)-f(z))}{r(z)}\Big) + S(r,f)\\ &\leqslant \overline{N}\Big(r, \frac{P(f)(f(qz+c)-f(z))}{r(z)}\Big)\\ &\quad + \overline{N}\Big(r, \frac{r(z)}{P(f)(f(qz+c)-f(z))}\Big)\\ &\quad + \overline{N}\Big(r, \frac{r(z)}{P(g)(g(qz+c)-g(z))}\Big) + S(r,f)\\ &\leqslant \overline{N}(r, f(z)) + \overline{N}(r, f(qz+c)) + \overline{N}\Big(r, \frac{1}{P(f)}\Big)\\ &\quad + \overline{N}\Big(r, \frac{1}{f(qz+c)-f(z)}\Big) + \overline{N}\Big(r, \frac{1}{P(g)}\Big)\\ &\quad + \overline{N}\Big(r, \frac{1}{g(qz+c)-g(z)}\Big) + S(r,f),\end{aligned}$$

$$(n-1)T(r,f) \leqslant (m+2)(T(r,f)+T(r,g)) + 2T(r,f) + S(r,f). \tag{3.5}$$

Similarly, we have

$$(n-1)T(r,g) \leqslant (m+2)(T(r,f)+T(r,g)) + 2T(r,g) + S(r,g). \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\begin{aligned}(n-1)(T(r,f)+T(r,g)) \leqslant 2(m+2)(T(r,f)+T(r,g)) + 2(T(r,f)+T(r,g))\\ + S(r,f) + S(r,g),\end{aligned}$$

$$(n-1-2m-4-2)(T(r,f)+T(r,g)) \leqslant S(r,f)+S(r,g),$$

that is

$$(n-2m-7)(T(r,f)+T(r,g)) \leqslant S(r,f)+S(r,g).$$

This is a contradiction to $n > 2mk+2m+2k+7 > 2m+7$. Therefore $r(z) \equiv 0$. Hence (3.4) becomes

$$P(f)(f(qz+c)-f(z)) = P(g)(g(qz+c)-g(z)). \tag{3.7}$$

That is $(a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0)(f(qz+c)-f(z)) = (a_ng^n+a_{n-1}g^{n-1}+\ldots+a_1g+a_0)(g(qz+c)-g(z))$, which implies $(a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0)\times f(qz+c)-(a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0)f(z) = (a_ng^n+a_{n-1}g^{n-1}+\ldots+a_1g+a_0)\times g(qz+c)-(a_ng^n+a_{n-1}g^{n-1}+\ldots+a_1g+a_0)g(z)$. Let $h = f/g$, we consider the following cases.

Case 1. If $h(z)$ is a constant, then substituting $f = gh$ into (3.7), we have $(a_nh^ng^n + a_{n-1}h^{n-1}g^{n-1} + \ldots + a_1hg + a_0)hg(qz + c) - (a_ng^n + a_{n-1}g^{n-1} + \ldots + a_1g + a_0)g(qz + c) - ((a_nh^ng^n + a_{n-1}h^{n-1}g^{n-1} + \ldots + a_1hg + a_0)hg(z) - (a_ng^n + a_{n-1}g^{n-1} + \ldots + a_1g + a_0)g(z)) = 0$, which implies $a_ng^ng(qz + c)(h^{n+1} - 1) + a_{n-1}g^{n-1}g(qz + c)(h^n - 1) + \ldots + a_1g^1g(qz + c)(h^2 - 1) + a_0g(qz + c)(h - 1) - (a_ng^{n+1}(h^{n+1} - 1) + a_{n-1}g^n(h^n - 1) + \ldots + a_1g^2(h^2 - 1) + a_0g(h - 1)) = 0$. Therefore $a_ng^n(g(qz + c) - g(z))(h^{n+1} - 1) + a_{n-1}g^{n-1}(g(qz + c) - g(z))(h^n - 1) + \ldots + a_1g(g(qz + c) - g(z))(h^2 - 1) + a_0(g(qz + c) - g(z))(h - 1) = 0$. This implies $h^d = 1$, where $d = LCM\{\lambda_j\colon\ j = 0, 1, \ldots, n\}$, and

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0. \end{cases}$$

Thus, $f \equiv tg$, where t is a constant with $t^d = 1$, where $d = LCM\{\lambda_j\colon\ j = 0, 1, \ldots, n\}$, and

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0. \end{cases}$$

Case 2. Suppose $h(z)$ is not a constant, then f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = P(\omega_1)(\omega_1(qz + c) - \omega_1(z)) - P(\omega_2)(\omega_2(qz + c) - \omega_2(z)).$$

Note that, when $f(z)$ and $g(z)$ are transcendental entire functions, we have $N(r, F) = 0$ and $N(r, G) = 0$. By computing similarly to the case of meromorphic functions, we easily obtain the conclusion of Theorem 1.1 with $n > 2mk + 2m + 5$ $\square$

Proof of Theorem 1.2. Let $F(z) = P(f)(f(qz + c) - f(z))$ and $G(z) = P(f)(g(qz+c) - g(z))$. We see that $F^{(k)}$ and $G^{(k)}$ share $a(z)$ IM. If (1) of Lemma 2.5 holds, then using Lemma 2.7, we obtain

$$\begin{aligned} T(r, F^{(k)}) &\leqslant N_2\Big(r, \frac{1}{F^{(k)}}\Big) + N_2(r, F^{(k)}) + N_2\Big(r, \frac{1}{G^{(k)}}\Big) + N_2(r, G^{(k)}) \\ &\quad + 2\Big(\overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}(r, F^{(k)})\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) \\ &\quad + \overline{N}(r, G^{(k)}) + S(r, F) + S(r, G) \\ &\leqslant N_2(r, F^{(k)}) + T(r, F^{(k)}) - T(r, F) + N_{k+2}\Big(r, \frac{1}{F}\Big) + N_{k+2}\Big(r, \frac{1}{G}\Big) \\ &\quad + k\overline{N}(r, G) + N_2(r, G^{(k)}) + 2\Big(N_{k+1}\Big(r, \frac{1}{F}\Big) + k\overline{N}(r, F) + \overline{N}(r, F^{(k)})\Big) \\ &\quad + N_{k+1}\Big(r, \frac{1}{G}\Big) + k\overline{N}(r, G) + \overline{N}(r, G^{(k)}) + S(r, f) + S(r, g), \end{aligned}$$

which implies

$$T(r,F) \leqslant N_2(r,F) + N_{k+2}\Big(r,\frac{1}{F}\Big) + N_{k+2}\Big(r,\frac{1}{G}\Big) + k\overline{N}(r,G)$$
$$+ N_2(r,G) + 2\Big(N_{k+1}\Big(r,\frac{1}{F}\Big) + k\overline{N}(r,F) + \overline{N}(r,F)\Big)$$
$$+ N_{k+1}\Big(r,\frac{1}{G}\Big) + k\overline{N}(r,G) + \overline{N}(r,G) + S(r,f) + S(r,g).$$

Therefore

$$\begin{aligned}T(r,F) \leqslant{}& (2k+4)\overline{N}(r,F) + N_{k+2}\Big(r,\frac{1}{F}\Big) + 2N_{k+1}\Big(r,\frac{1}{F}\Big)\\ &+ (2k+3)\overline{N}(r,G) + N_{k+2}\Big(r,\frac{1}{G}\Big) + N_{k+1}\Big(r,\frac{1}{G}\Big)\\ &+ S(r,F) + S(r,G).\end{aligned} \tag{3.8}$$

Similarly

$$\begin{aligned}T(r,G) \leqslant{}& (2k+4)\overline{N}(r,G) + N_{k+2}\Big(r,\frac{1}{G}\Big) + 2N_{k+1}\Big(r,\frac{1}{G}\Big)\\ &+ (2k+3)\overline{N}(r,F) + N_{k+2}\Big(r,\frac{1}{F}\Big) + N_{k+1}\Big(r,\frac{1}{F}\Big)\\ &+ S(r,F) + S(r,G).\end{aligned} \tag{3.9}$$

We have

$$\overline{N}(r,F) \leqslant \overline{N}(r,f) + \overline{N}(r,f(qz+c)-f(z)) \leqslant 2T(r,f) + S(r,f), \tag{3.10}$$

$$\begin{aligned}N_{k+2}\Big(r,\frac{1}{F}\Big) &\leqslant N_{k+2}\Big(r,\frac{1}{P(f)}\Big) + N\Big(r,\frac{1}{f(qz+c)-f(z)}\Big)\\ &\leqslant (m(k+2)+2)T(r,f) + S(r,f),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}N_{k+1}\Big(r,\frac{1}{F}\Big) &\leqslant N_{k+1}\Big(r,\frac{1}{P(f)}\Big) + N\Big(r,\frac{1}{f(qz+c)-f(z)}\Big)\\ &\leqslant (m(k+1)+2)T(r,f) + S(r,f).\end{aligned} \tag{3.12}$$

Similarly,

$$\overline{N}(r,G) \leqslant 2T(r,g) + S(r,g), \tag{3.13}$$

$$N_{k+2}\Big(r,\frac{1}{G}\Big) \leqslant (m(k+2)+2)T(r,g) + S(r,g), \tag{3.14}$$

and

$$N_{k+1}\Big(r,\frac{1}{G}\Big) \leqslant (m(k+1)+2)T(r,g)+S(r,g). \tag{3.15}$$

Substituting (3.10)–(3.12) in (3.8), we get

$$\begin{aligned} T(r,F) \leqslant{}& 2(2k+4)T(r,f)+(m(k+2)+2)T(r,f)+2(m(k+1)+2)T(r,f)\\ &+2(2k+3)T(r,g)+(m(k+2)+2)T(r,g)+(m(k+1)+2)T(r,g)\\ &+S(r,f)+S(r,g),\end{aligned}$$

$$\begin{aligned}(n-1)T(r,f) \leqslant{}& (3mk+4m+4k+14)T(r,f)\\ &+(2mk+3m+4k+10)T(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{3.16}$$

Similarly, substituting (3.13)–(3.15) in (3.9), we get

$$\begin{aligned}(n-1)T(r,g) \leqslant{}& (3mk+4m+4k+14)T(r,g)\\ &+(2mk+3m+4k+10)T(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{3.17}$$

From (3.16) and (3.17), we get

$$\begin{aligned}(n-1)(T(r,f)+T(r,g)) \leqslant{}& (5mk+7m+8k+24)(T(r,f)+T(r,g))\\ &+S(r,f)+S(r,g),\end{aligned} \tag{3.18}$$

$$(n-1-5mk-7m-8k-24)(T(r,f)+T(r,g)) \leqslant S(r,f)+S(r,g),$$

and

$$(n-5mk-7m-8k-25)(T(r,f)+T(r,g)) \leqslant S(r,f)+S(r,g),$$

which is a contradiction to $n > 5mk+7m+8k+25$. Thus, by Lemma 2.5 we have either $F^{(k)}G^{(k)} \equiv a^2(z)$ or $F^{(k)} = G^{(k)}$.

Case 1. Suppose $F^{(k)}G^{(k)} \equiv a^2(z)$ i.e.,

$$(P(f)(f(qz+c)-f(z)))^{(k)}(P(g)(g(qz+c)-g(z)))^{(k)} \equiv a^2(z).$$

This is one of the conclusion of Theorem 1.2.

Case 2. Now, we consider $F^{(k)} = G^{(k)}$. By an argument as in Theorem 1.1, we obtain that f and g satisfy one of the following two statements:

(1) $f \equiv tg$ for a constant t with $t^d = 1$, where $d = LCM\{\lambda_j\colon\ j = 0,1,\dots,n\}$, and

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0,\\ n+1 & \text{if } a_j = 0,\end{cases}$$

(2) f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(\omega_1,\omega_2) = P(\omega_1)(\omega_1(qz+c)-\omega_1(z)) - P(\omega_2)(\omega_2(qz+c)-\omega_2(z)).$$

$\square$

4. Open Question

Question 4.1. Are the conditions on n in Theorem 1.1 and Theorem 1.2 sharp?

Acknowledgement. The authors are grateful to the referees for their suggestions towards the improvement of the paper.

References

[1] *R. S. Dyavanal, R. V. Desai*: Uniqueness of q-difference and differential polynomials of entire functions. Math. Sci. Int. Research J. *4* (2015), 267–271.
[2] *R. S. Dyavanal, R. V. Desai*: Uniqueness of q-shift difference and differential polynomials of entire functions. Far East J. Appl. Math. *91* (2015), 189–202. zbl doi
[3] *R. S. Dyavanal, R. V. Desai*: Uniqueness of product of difference polynomials of meromorphic functions sharing fixed point with finite weight. Glob. J. Pure Appl. Math. *14* (2018), 331–346.
[4] *R. S. Dyavanal, A. M. Hattikal*: Weighted sharing of difference-differential polynomials of entire functions. Math. Sci. Int. Research J. *4* (2015), 276–280.
[5] *R. S. Dyavanal, A. M. Hattikal*: Weighted sharing of uniqueness of difference polynomials of meromorphic functions. Far East J. Math. Sci. (FJMS) *98* (2015), 293–313. zbl doi
[6] *R. S. Dyavanal, A. M. Hattikal*: On the uniqueness of product of difference polynomials of meromorphic functions. Konuralp J. Math. *4* (2016), 42–55. zbl MR
[7] *R. S. Dyavanal, A. M. Hattikal*: Unicity theorems on difference polynomials of meromorphic functions sharing one value. Int. J. Pure Appl. Math. Sci. *9* (2016), 89–97.
[8] *R. S. Dyavanal, A. M. Hattikal*: Uniqueness of q-shift difference polynomials of meromorphic functions sharing a small function. New Trends Math. Sci. *5* (2017), 250–263. doi
[9] *W. K. Hayman*: Meromorphic Functions. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964. zbl MR
[10] *W.-C. Lin, H.-X. Yi*: Uniqueness theorems for meromorphic function. Indian J. Pure Appl. Math. *35* (2004), 121–132. zbl MR
[11] *K. Liu, X.-G. Qi*: Meromorphic solutions of q-shift difference equations. Ann. Pol. Math. *101* (2011), 215–225. zbl MR doi
[12] *N. V. Thin*: Uniqueness of meromorphic functions and Q-differential polynomials sharing small functions. Bull. Iran. Math. Soc. *43* (2017), 629–647. zbl MR
[13] *J. Xu, X. Zhang*: The zeros of q-shift difference polynomials of meromorphic functions. Adv. Difference Equ. *2012* (2012), Paper No. 200, 10 pages. zbl MR doi
[14] *C. C. Yang, H. X. Yi*: Uniqueness Theory of Meromorphic Functions. Mathematics and Its Applications 557. Kluwer Academic Publishers, Dordrecht, 2003. zbl MR
[15] *L. Yang*: Value Distribution Theory. Springer, Berlin; Science Press, Beijing, 1993. zbl MR doi
[16] *X. Zhang*: Value sharing of meromorphic functions and some questions of Dyavanal. Front. Math. China *7* (2012), 161–176. zbl MR doi
[17] *Q. Zhao, J. Zhang*: Zeros and shared one value of q-shift difference polynomials. J. Contemp. Math. Anal., Armen. Acad. Sci. *50* (2015), 63–69. zbl MR doi

Authors' address: *Renukadevi S. Dyavanal*, *Rajalaxmi V. Desai*, Department of Mathematics, Karnatak University, Dharwad-580003, Karnataka, India, e-mail: `renukadyavanal@gmail.com`, `desairajlakshmi@gmail.com`.