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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING VALUE SHARING OF NONLINEAR DIFFERENTIAL MONOMIALS

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Abstract. With the idea of normal family we study the uniqueness of meromorphic functions f and g when $f^n(f^{(k)})^m - p$ and $g^n(g^{(k)})^m - p$ share two values, where p is any nonzero polynomial. The result of this paper significantly improves and generalizes the result due to A. Banerjee and S. Majumder (2018).

Keywords: uniqueness; meromorphic function; small function; nonlinear differential polynomial; normal family

MSC 2010: 30D35, 30D30

1. Introduction, definitions and results

In this paper, by meromorphic functions we mean meromorphic functions in the whole complex plane $\mathbb{C}$. We adopt the standard notations of value distribution theory (see [11]). Let $T(r) = \max\{T(r, f), T(r, g)\}$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r, a) = S(r, f)$. We use the symbol $\varrho(f)$ to denote the order of f.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to both $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if the zeros of $f(z) - a(z)$ and $g(z) - a(z)$ have the same locations and same multiplicities and we say that $f(z)$ and $g(z)$ share $a(z)$ IM (ignoring multiplicities) if the zeros of $f(z) - a(z)$ and $g(z) - a(z)$ have the same locations but different multiplicities.

For the sake of simplicity, we use the notion $(m)^*$ defined by $(m)^* = m - 1$ when m is a positive integer and $(m)^* = [m]$ when m is not integer but positive rational.

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Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leqslant M$ for all $z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on compact subsets of D (see [20]).

The following theorem well known in value distribution theory was posed by Hayman and settled by several authors almost at the same time (see [4]–[7]).

Theorem A. *Let f be a transcendental meromorphic function, $n \in \mathbb{N}$. Then $f^n f' = 1$ has infinitely many solutions.*

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua in [9], Yang and Hua in [24] obtained the following result.

Theorem B. *Let f and g be two non-constant entire (meromorphic) functions, $n \in \mathbb{N}$ such that $n \geqslant 6$ ($n \geqslant 11$). If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 \mathrm{e}^{cz}$, $g(z) = c_2 \mathrm{e}^{-cz}$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

We say that a finite value z_0 is called a fixed point of f if $f(z_0) = z_0$. Considering the uniqueness question of entire or meromorphic functions having fixed points, Fang and Qiu in [10] obtained the following result.

Theorem C. *Let f and g be two non-constant meromorphic (entire) functions, $n \in \mathbb{N}$ such that $n \geqslant 11$ ($n \geqslant 6$). If $f^n(z) f'(z) - z$ and $g^n(z) g'(z) - z$ share 0 CM, then either $f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

Gradually the research work in the above directions gained pace and today it has become one of the most prominent branches of uniqueness theory. During the last couple of years a large amount of research papers have been published by different authors (see [5]–[10], [17]–[21], [24], [28], [30], [31]).

We recall the following result obtained by Xu, Yi and Zhang, see [21].

Theorem D. *Let f be a transcendental meromorphic function, $k \in \mathbb{N}$, $n \in \mathbb{N}\setminus\{1\}$. Then $f^n f^{(k)}$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Recently, Cao and Zhang in [5] proved the following result.

Theorem E. *Let f and g be two non-constant meromorphic functions whose zeros are of multiplicities at least $k+1$, where $k \in \mathbb{N}$ such that $1 \leqslant k \leqslant 5$ and $n \in \mathbb{N}$ such that $n \geqslant 10$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM, f and g share ∞ IM, then one of the following two conclusions holds:*

(i) *$f \equiv tg$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$;*

(ii) *$f(z) = c_1 \mathrm{e}^{cz}$, $g(z) = c_2 \mathrm{e}^{-cz}$, where $c, c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $(-1)^k (c_3 c_4)^{n+1} \times d^{2k} = 1$.*

Regarding Theorem E the following questions are inevitable.

Question 1. Can the lower bound of n in Theorem E be further reduced?

Question 2. Can the condition "Let f and g be two non-constant meromorphic functions whose zeros are of multiplicities at least $k+1$, $k \in \mathbb{N}$" in Theorem E be further weakened?

Question 3. Does Theorem E hold for $k \geqslant 6$?

We now explain the notation of weighted sharing as introduced in [13], [14].

Definition 1 ([13], [14])**.** Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. If $a(z)$ is a small function with respect to $f(z)$ and $g(z)$, we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight l when $f(z) - a(z)$ and $g(z) - a(z)$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$, respectively.

Keeping in mind the above questions, in 2018 Banerjee and Majumder obtained the following result (see [3]).

Theorem F. *Let f, g be two transcendental meromorphic functions whose zeros are of multiplicities at least k, where $k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that*

$$n > \Big(\frac{k^2 + 4k + 4}{k}\Big)^*.$$

Let p be a nonzero polynomial such that either $\deg(p) \leqslant n-1$ *or zeros of p are of multiplicities at most $n-1$. If $f^n f^{(k)} - p$ and $g^n g^{(k)} - p$ share $(0, k_1)$, where $k_1 = ((k+2)/(n-k)) + 3$, and f, g share ∞ IM and $f^{(k)}$, $g^{(k)}$ share 0 CM, then $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

Regarding Theorem F, it is natural to ask the following questions which are the motive of the present paper.

Question 4. Can one remove the condition "$\deg(p) \leqslant n-1$ or zeros of p be of multiplicities at most $n-1$" in Theorem F?

Question 5. What happens when "$f^n(f^{(k)})^m - p$ and $g^n(g^{(k)})^m - p$" share the value 0 CM, where p is a nonzero polynomial in Theorem F?

Question 6. Can the lower bound of n be further reduced in Theorem F?

2. Main result

In this paper, taking the possible answers of the above questions into background we obtain the following result which significantly improves and generalizes Theorem F.

Theorem 1. *Let f, g be two transcendental meromorphic functions having zeros of multiplicities at least k, where $k \in \mathbb{N}$ and let $m, n, k_1 \in \mathbb{N}$ such that*

$$n \geqslant \frac{k^2 + 2mk + 6}{k}.$$

Let p be a nonzero polynomial. If $f^n(f^{(k)})^m - p$ and $g^n(g^{(k)})^m - p$ share $(0, k_1)$, where $k_1 = ((3 + (k-1)m)/(n + m + (m-2)k - 1)) + 3$, and f, g share ∞ IM and $f^{(k)}$, $g^{(k)}$ share 0 CM, then $f \equiv tg$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+m} = 1$.

We now explain some definitions and notations which are used in the paper.

Definition 2 ([17])**.** Let $p \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid \geqslant p)$ ($\overline{N}(r, a; f \mid \geqslant p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.
(ii) $N(r, a; f \mid \leqslant p)$ ($\overline{N}(r, a; f \mid \leqslant p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 3. We denote by $\overline{N}(r, a; f \mid = k)$ the reduced counting function of those a-points of f whose multiplicities are exactly k, where $k \in \mathbb{N} \setminus \{1\}$.

Definition 4 ([26]). For $a \in \mathbb{C}\cup\{\infty\}$ and $p \in \mathbb{N}$ we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geqslant 2)+\ldots+\overline{N}(r,a;f\mid\geqslant p)$. Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 5 ([1]). Let f and g be two non-constant meromorphic functions such that f and g share 1 IM. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p=q\geqslant 2$; each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g), N_E^{1)}(r,1;g), \overline{N}_E^{(2}(r,1;g)$.

Definition 6 ([14]). Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

3. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth, we shall denote by H and V the following two functions:

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big) \tag{3.1}$$

and

$$V=\Big(\frac{F'}{F-1}-\frac{F'}{F}\Big)-\Big(\frac{G'}{G-1}-\frac{G'}{G}\Big)=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}. \tag{3.2}$$

Lemma 1 ([29]). *Let f be a non-constant meromorphic function and $k,p\in\mathbb{N}$. Then*

$$N_p(r,0;f^{(k)})\leqslant N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2 ([16]). *If $N(r,0;f^{(k)}\,|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}\,|f\neq 0)\leqslant k\overline{N}(r,\infty;f)+N(r,0;f\,|<k)+k\overline{N}(r,0;f\,|\geqslant k)+S(r,f).$$

Lemma 3 ([11])**.** *Suppose that f is a non-constant meromorphic function, $k \in \mathbb{N} \setminus \{1\}$. If*

$$N(r, \infty, f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S\Big(r, \frac{f'}{f}\Big),$$

then $f(z) = \mathrm{e}^{az+b}$, where $a, b \in \mathbb{C}$, $a \neq 0$.

Lemma 4 ([23])**.** *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n \in \mathbb{C}$ $(a_n \neq 0)$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 5 ([15])**.** *Let f be a transcendental meromorphic function and α $(\alpha \not\equiv 0$, $\alpha \not\equiv \infty)$ be a small function of f. Then $\psi = \alpha(f)^n (f^{(k)})^p$ is non-constant, where $k \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{N}$.*

Lemma 6 ([25])**.** *Let f_j, $j = 1, 2, 3$ be meromorphic and f_1 be non-constant. Suppose that*

$$\sum_{j=1}^{3} f_j \equiv 1$$

and

$$\sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) < (\lambda + o(1)) T_1(r),$$

as $r \to \infty$, $r \in I$, where I is a set of $r \in (0, \infty)$ with infinite linear measure, $\lambda < 1$ and $T_1(r) = \max_{1 \leqslant j \leqslant 3} T(r, f_j)$. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 7 ([25], Theorem 1.24)**.** *Let f be a non-constant meromorphic function and let $k \in \mathbb{N}$. Suppose that $f^{(k)} \not\equiv 0$. Then $N(r, 0; f^{(k)}) \leqslant N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f)$.*

Lemma 8. *Let f, g be two transcendental meromorphic functions, whose zeros are of multiplicities at least k, where $k \in \mathbb{N}$ and $F = f^n (f^{(k)})^m / p$, $G = g^n (g^{(k)})^m / p$, where p is a nonzero polynomial and $m, n \in \mathbb{N}$ such that $n + m + (m-2)k > 1$. Suppose $H \not\equiv 0$. If F, G share $(1, k_1)$ and f, g share ∞ IM, where $0 \leqslant k_1 \leqslant \infty$, then*

$$\begin{aligned}\overline{N}(r, \infty; f) \leqslant{}& \frac{k+1}{k(n+m+(m-2)k-1)} (T(r, f) + T(r, g)) \\ &+ \frac{1}{n+m+(m-2)k-1} \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).\end{aligned}$$

P r o o f. First we suppose ∞ is a Picard exceptional value of both f and g. Then the lemma follows immediately. Next we suppose ∞ is not a Picard exceptional value of both f and g. We claim that $V \not\equiv 0$. If possible, suppose $V \equiv 0$. Then by integration we obtain

$$1 - \frac{1}{F} \equiv A\Big(1 - \frac{1}{G}\Big), \quad A \in \mathbb{C} \setminus \{0\}.$$

Let z_0 be a pole of f with multiplicity q and a pole of g with multiplicity r such that $p(z_0) \neq 0$. Then from the definition of F and G we have $1/F(z_0) = 0$ and $1/G(z_0) = 0$. So $A = 1$ and hence $F \equiv G$. Since $H \not\equiv 0$, it follows that $F \not\equiv G$. Therefore we arrive at a contradiction. Hence $V \not\equiv 0$. Also $m(r, V) = S(r, f) + S(r, g)$.

Clearly z_0 is a pole of F with multiplicity $(n + m)q + mk$ and a pole of G with multiplicity $(n + m)r + mk$. Clearly

$$\frac{F'(z)}{F(z)\big(F(z) - 1\big)} = O((z - z_0)^{(n+m)q+mk-1})$$

and

$$\frac{G'(z)}{G(z)(G(z) - 1)} = O((z - z_0)^{(n+m)r+mk-1}).$$

Consequently,

$$V(z) = O((z - z_0)^{(n+m)t+mk-1}),$$

where $t = \min\{q, r\}$. Since f and g share ∞ IM, from the definition of V it is clear that z_0 is a zero of V with multiplicity at least $n+m+mk-1$. So from the definition of V and using Lemma 2 we have

$$\begin{aligned}
&(n + m + mk - 1)\overline{N}(r, \infty; f) \\
&\quad \leqslant N(r, 0; V) + O(\log r) \leqslant T(r, V) + S(r, f) + S(r, g) \\
&\quad \leqslant N(r, \infty; V) + S(r, f) + S(r, g) \\
&\quad \leqslant \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\quad \leqslant \overline{N}(r, 0; f) + \overline{N}(r, 0; f^{(k)} \mid f \neq 0) + \overline{N}(r, 0; g) + \overline{N}(r, 0; g^{(k)} \mid g \neq 0) \\
&\qquad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\quad \leqslant \overline{N}(r, 0; f) + k\overline{N}(r, \infty; f) + N_k(r, 0; f) + \overline{N}(r, 0; g) + k\overline{N}(r, \infty; g) \\
&\qquad + N_k(r, 0; g) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\quad \leqslant \frac{k+1}{k} N(r, 0; f) + \frac{k+1}{k} N(r, 0; g) + 2k\overline{N}(r, \infty; f) \\
&\qquad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\quad \leqslant \frac{k+1}{k}(T(r, f) + T(r, g)) + 2k\overline{N}(r, \infty; f) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).
\end{aligned}$$

Hence the lemma follows. □

Lemma 9. *Let f be a non-constant meromorphic function and let $F = f^n(f^{(k)})^m$, where $m, n, k \in \mathbb{N}$ such that $n > m$. Then*

$$(n-m)T(r,f) \leqslant T(r,F) - mN(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f).$$

P r o o f. Note that

$$\begin{aligned}N(r,\infty;F) &= N(r,\infty;f^n) + N(r,\infty;(f^{(k)})^m)\\ &= N(r,\infty;f^n) + mN(r,\infty;f) + mk\overline{N}(r,\infty;f) + S(r,f),\end{aligned}$$

i.e.

$$N\big(r,\infty;f^n\big) = N(r,\infty,F) - mN(r,\infty;f) - mk\overline{N}(r,\infty,f) + S(r,f).$$

Also

$$\begin{aligned}m(r,f^n) &= m\Big(r,\frac{F}{(f^{(k)})^m}\Big) \leqslant m(r,F) + m\Big(r,\frac{1}{(f^{(k)})^m}\Big) + S(r,f)\\ &= m(r,F) + T(r,(f^{(k)})^m) - N(r,0;(f^{(k)})^m) + S(r,f)\\ &= m(r,F) + N(r,\infty;(f^{(k)})^m) + m(r,(f^{(k)})^m) - N(r,0;(f^{(k)})^m) + S(r,f)\\ &\leqslant m(r,F) + mN(r,\infty;f) + mk\overline{N}(r,\infty;f) + m\Big(r,\frac{(f^{(k)})^m}{f^m}\Big) + m(r,f^m)\\ &\quad - N(r,0;(f^{(k)})^m) + S(r,f)\\ &= m(r,F) + mT(r,f) + mk\overline{N}(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f).\end{aligned}$$

Now

$$\begin{aligned}nT(r,f) &= N(r,\infty;f^n) + m(r,f^n)\\ &\leqslant T(r,F) + mT(r,f) - mN(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f),\end{aligned}$$

i.e.

$$(n-m)T(r,f) \leqslant T(r,F) - mN(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f).$$

This completes the lemma. □

Lemma 10. *Let f be a transcendental meromorphic function and let $a(z)$ ($a(z) \not\equiv 0$, $a(z) \not\equiv \infty$) be a small function of f. If $n > m+1$, then $f^n(f^{(k)})^m - a$ has infinitely many zeros, where $k, m, n \in \mathbb{N}$.*

P r o o f. Let $F=f^n(f^{(k)})^m$. Now in view of Lemma 9 and the second fundamental theorem for small functions (see [22]) we get

$$\begin{aligned}(n-m)T(r,f)&\leqslant T(r,F)-mN(r,\infty;f)-N(r,0;(f^{(k)})^m)+S(r,f)\\&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,a;F)-mN(r,\infty;f)\\&\quad-N(r,0;(f^{(k)})^m)+(\varepsilon+o(1))T(r,f)\\&\leqslant \overline{N}(r,0;f)+\overline{N}(r,0;(f^{(k)})^m)+\overline{N}(r,\infty;f)+\overline{N}(r,a;F)\\&\quad-mN(r,\infty;f)-N(r,0;(f^{(k)})^m)+(\varepsilon+o(1))T(r,f)\\&\leqslant \overline{N}(r,0;f)+\overline{N}(r,a;F)+(\varepsilon+o(1))T(r,f)\\&\leqslant T(r,f)+\overline{N}(r,a;F)+(\varepsilon+o(1))T(r,f)\end{aligned}$$

for all $\varepsilon>0$. Take $\varepsilon<1$. Since $n>m+1$, from the above one can easily say that $F-a$ has infinitely many zeros. This completes the lemma. □

R e m a r k 7. By Lemma 10, one can easily say that $f^n(f^{(k)})^m a^{-1}-1$ has infinitely many zeros.

Lemma 11 ([12]). *Let f and g be two non-constant meromorphic functions. Suppose that f and g share 0 and ∞ CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM for $k=1,2,\dots,6$. Then f and g satisfy one of the following cases:*

(i) *$f\equiv tg$, where $t\in\mathbb{C}\setminus\{0\}$,*
(ii) *$f(z)=\mathrm{e}^{az+b}$, $g(z)=\mathrm{e}^{cz+d}$, where a, b, c and $d\in\mathbb{C}$, $(a,c\neq0)$,*
(iii) *$f(z)=a/(1-b\mathrm{e}^{\alpha(z)})$, $g(z)=a/(\mathrm{e}^{-\alpha(z)}-b)$, where $a,b\in\mathbb{C}\setminus\{0\}$ and α is a non-constant entire function,*
(iv) *$f(z)=a(1-b\mathrm{e}^{cz})$, $g(z)=d(\mathrm{e}^{-cz}-b)$, where a, b, c and $d\in\mathbb{C}\setminus\{0\}$.*

Lemma 12. *Let f and g be two transcendental meromorphic functions having zeros of multiplicities at least k, where $k\in\mathbb{N}$ and let $m,n\in\mathbb{N}$. Let $f^{(k)}$, $g^{(k)}$ share 0 CM and f, g share ∞ IM. If $f^n(f^{(k)})^m\equiv g^n(g^{(k)})^m$, then $f\equiv tg$, where $t\in\mathbb{C}\setminus\{0\}$ such that $t^{n+m}=1$.*

P r o o f. Suppose

$$f^n(f^{(k)})^m\equiv g^n(g^{(k)})^m, \tag{3.3}$$

i.e.

$$\frac{f^n}{g^n}\equiv\frac{\left(g^{(k)}\right)^m}{\left(f^{(k)}\right)^m}. \tag{3.4}$$

Since f and g share ∞ IM, it follows from (3.3) that f and g share ∞ CM and so $f^{(k)}$ and $g^{(k)}$ share ∞ CM. Again since $f^{(k)}$ and $g^{(k)}$ share 0 CM, it follows that f and g share 0 CM also. Let $h_1 = f/g$ and $h_2 = f^{(k)}/g^{(k)}$. Then $h_1 \neq 0, \infty$ and $h_2 \neq 0, \infty$. From (3.4) we see that

$$h_1^n h_2^m \equiv 1. \tag{3.5}$$

First we suppose h_1 is a non-constant entire function. Clearly h_2 is also a non-constant entire function. Let $F_1 = h_1^n$ and $G_1 = h_2^m$. Also from (3.5) we get

$$F_1 G_1 \equiv 1. \tag{3.6}$$

Clearly $F_1 \not\equiv d_1 G_1$, where $d_1 \in \mathbb{C} \setminus \{0\}$, otherwise F_1 will be a constant and so h_1 will be a constant.

Since $F_1 \neq 0, \infty$ and $G_1 \neq 0, \infty$, then there exist two non-constant entire functions α and β such that $F_1 = \mathrm{e}^{\alpha}$ and $G_1 = \mathrm{e}^{\beta}$. Now from (3.6) we see that $\alpha + \beta = C$, where $C \in \mathbb{C}$. Therefore $\alpha' = -\beta'$. Note that $F_1' = \alpha' \mathrm{e}^{\alpha}$ and $G_1' = \beta' \mathrm{e}^{\beta}$. This shows that F_1' and G_1' share 0 CM. Note that $F_1 \neq 0$, $F_1 \neq \infty$, $G_1 \neq 0$, $G_1 \neq \infty$ and $F_1 \not\equiv d_1 G_1$, where $d_1 \in \mathbb{C} \setminus \{0\}$. Now in view of Lemma 11 we have

$$F_1(z) = c_1 \mathrm{e}^{az} \quad \text{and} \quad G_1(z) = c_2 \mathrm{e}^{-az},$$

where $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1 c_2 = 1$. Since

$$\Big(\frac{f(z)}{g(z)}\Big)^n = c_1 \mathrm{e}^{az} \quad \text{and} \quad \Big(\frac{f^{(k)}(z)}{g^{(k)}(z)}\Big)^m = c_2 \mathrm{e}^{-az},$$

it follows that

$$\frac{f(z)}{g(z)} = t_1 \mathrm{e}^{az/n} = t_1 \mathrm{e}^{cz} \tag{3.7}$$

and

$$\frac{f^{(k)}(z)}{g^{(k)}(z)} = t_2 \mathrm{e}^{-az/m} = t_2 \mathrm{e}^{dz}, \tag{3.8}$$

where $c, d, t_1, t_2 \in \mathbb{C} \setminus \{0\}$ such that $t_1^n = c_1$, $t_2^m = c_2$, $c = a/n$ and $d = -a/m$. Let

$$\Phi_1 = \frac{f^{(k+1)}}{f^{(k)}} - \frac{g^{(k+1)}}{g^{(k)}}. \tag{3.9}$$

From (3.8) we see that

$$\Phi_1(z) = d. \tag{3.10}$$

Again from (3.7) we see that

$$f^{(j)}(z) = t_1 \sum_{i=0}^{j} {}^{j}C_i (\mathrm{e}^{cz})^{(j-i)} g^{(i)}(z),$$

where we define $g^{(0)}(z) = g(z)$. Consequently, we have

$$\begin{aligned} f^{(k+1)}(z) = t_1 \Big(& c^{k+1}\mathrm{e}^{cz} g(z) + (k+1)c^k \mathrm{e}^{cz} g'(z) + \ldots + \frac{k(k+1)}{2} c^2 \mathrm{e}^{cz} g^{(k-1)} \\ & + (k+1)c\mathrm{e}^{cz} g^{(k)}(z) + \mathrm{e}^{cz} g^{(k+1)}(z) \Big) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} f^{(k)}(z) = t_1 \Big(& c^k \mathrm{e}^{cz} g(z) + kc^{k-1}\mathrm{e}^{cz} g'(z) + \ldots + \frac{(k-1)k}{2} c^2 \mathrm{e}^{cz} g^{(k-2)} \\ & + kc\mathrm{e}^{cz} g^{(k-1)}(z) + \mathrm{e}^{cz} g^{(k)}(z) \Big). \end{aligned} \tag{3.12}$$

Now from (3.9), (3.11) and (3.12) we have

$$\Phi_1 = \frac{F_2 - G_2 + (k+1)cg^{(k)}g^{(k)} - kcg^{(k-1)}g^{(k+1)}}{F_3 + g^{(k)}g^{(k)}}, \tag{3.13}$$

where

$$\begin{aligned} F_2 &= c^{k+1} g g^{(k)} + (k+1)c^k g' g^{(k)} + \ldots + \frac{k(k+1)}{2} c^2 g^{(k-1)} g^{(k)}, \\ G_2 &= c^k g g^{(k+1)} + kc^{k-1} g' g^{(k+1)} + \ldots + \frac{(k-1)k}{2} c^2 g^{(k-2)} g^{(k+1)} \end{aligned}$$

and

$$F_3 = c^k g g^{(k)} + \ldots + kcg^{(k-1)} g^{(k)}.$$

Let z_p be a zero of $g(z)$ with multiplicity p ($p \geqslant k$). Then the Taylor expansion of g about z_p is

$$g(z) = b_p(z - z_p)^p + b_{p+1}(z - z_p)^{p+1} + b_{p+2}(z - z_p)^{p+2} + \ldots, \quad b_p \neq 0. \tag{3.14}$$

We now consider the following two cases.

Case 1. Suppose $p = k$. Then

$$g^{(k)}(z) = k!\, b_k + (k+1)!\, b_{k+1}(z - z_k) + \dots \tag{3.15}$$

and

$$g^{(k+1)}(z) = (k+1)!\, b_{k+1} + (k+2)!\, b_{k+2}(z - z_k) + \dots \tag{3.16}$$

Now from (3.13), (3.15) and (3.16) we have

$$\Phi_1(z_k) = c\frac{(k+1)(k!)^2 b_k^2}{(k!)^2 b_k^2} = c(k+1). \tag{3.17}$$

Therefore we arrive at a contradiction from (3.10) and (3.17).

Case 2. Suppose $p \geqslant k+1$. Then

$$\begin{aligned} g^{(k-2)}(z) &= p(p-1)\dots(p-k+3)b_p(z-z_p)^{(p-k+2)} + \dots \\ g^{(k-1)}(z) &= p(p-1)\dots(p-k+2)b_p(z-z_p)^{(p-k+1)} + \dots \\ g^{(k)}(z) &= p(p-1)\dots(p-k+1)b_p(z-z_p)^{(p-k)} + \dots \end{aligned}$$

and

$$g^{(k+1)}(z) = p(p-1)\dots(p-k)b_p(z-z_p)^{(p-k-1)} + \dots$$

Therefore

$$g^{(k)}(z)g^{(k)}(z) = Kb_p^2(z-z_p)^{2p-2k} + \dots, \tag{3.18}$$

$$g^{(k-1)}(z)g^{(k+1)}(z) = \frac{p-k}{p-k+1}Kb_p^2(z-z_p)^{2p-2k} + \dots, \tag{3.19}$$

where $K = (p(p-1)\dots(p-k+1))^2$. Also

$$F_2(z) = O((z-z_p)^{2p-2k+1}), \quad G_2(z) = O((z-z_p)^{2p-2k+1})$$

and

$$F_3(z) = O((z-z_p)^{2p-2k+1}).$$

Now from (3.13), (3.18) and (3.19) we have

$$\Phi_1(z_p) = \frac{(k+1)cKb_p^2 - kc(p-k)(p-k+1)^{-1}Kb_p^2}{Kb_p^2} = c\frac{p+1}{p-k+1}. \tag{3.20}$$

Therefore we arrive at a contradiction from (3.10) and (3.20).

Thus, in either cases one can easily say that g has no zeros. Since f and g share 0 CM, it follows that f and g have no zeros. But this is impossible because the zeros of f and g are of multiplicities at least k. Hence h_1 is constant. Then from (3.3) we get $h_1^{n+m} = 1$. Therefore we have $f \equiv tg$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+m} = 1$. This completes the lemma. $\square$

Lemma 13 ([6])**.** *Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most 1.*

Lemma 14 (Zalcman [19], [27])**.** *Let F be a family of meromorphic functions in the unit disc Δ and α be a real number satisfying $-1 < \alpha < 1$. Then if F is not normal at a point $z_0 \in \Delta$, there exist for each α with $-1 < \alpha < 1$*

(i) *points $z_n \in \Delta$, $z_n \to z_0$,*
(ii) *positive numbers ϱ_n, $\varrho_n \to 0^+$ and*
(iii) *functions $f_n \in F$,*

such that $\varrho_n^{-\alpha} f_n(z_n + \varrho_n\zeta) \to g(\zeta)$ spherically uniformly on a compact subset of $\mathbb{C}$, where g is a non-constant meromorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \leqslant g^{\#}(0) = 1$, $\zeta \in \mathbb{C}$.

Lemma 15. *Let f and g be two transcendental meromorphic functions having zeros of multiplicities at least k, where $k \in \mathbb{N}$. Also let $f^n(f^{(k)})^m - p$, $g^n(g^{(k)})^m - p$ share 0 CM and $f^{(k)}$, $g^{(k)}$ share 0 CM and f, g share ∞ IM, where p is a nonzero polynomial and $m, n \in \mathbb{N}$. Then $f^n(f^{(k)})^m g^n(g^{(k)})^m \not\equiv p^2$.*

P r o o f. Suppose

$$f^n(f^{(k)})^m g^n(g^{(k)})^m \equiv p^2. \tag{3.21}$$

Since f and g share ∞ IM, from (3.21) one can easily say that f and g are transcendental entire functions. We consider the following cases.

Case 1. Let $\deg(p) = l$ ($\geqslant 1$). Now from (3.21) it follows that $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$. Let

$$F = \frac{f^n(f^{(k)})^m}{p} \quad \text{and} \quad G = \frac{g^n(g^{(k)})^m}{p}. \tag{3.22}$$

From (3.21) we get

$$FG \equiv 1. \tag{3.23}$$

If $F \equiv C_1 G$, where $C_1 \in \mathbb{C} \setminus \{0\}$, then F is a constant, which is impossible by Lemma 5. Hence $F \not\equiv C_1 G$. Let

$$\Phi = \frac{f^n(f^{(k)})^m - p}{g^n(g^{(k)})^m - p}. \tag{3.24}$$

Since f and g are transcendental entire functions, it follows that $f^n(f^{(k)})^m - p \neq \infty$ and $g^n(g^{(k)})^m - p \neq \infty$. Also since $f^n(f^{(k)})^m - p$ and $g^n(g^{(k)})^m - p$ share 0 CM, we deduce from (3.24) that

$$\Phi \equiv \mathrm{e}^{\gamma}, \tag{3.25}$$

where γ is an entire function. Let $f_1 = F$, $f_2 = -\mathrm{e}^{\gamma}G$ and $f_3 = \mathrm{e}^{\gamma}$. Here f_1 is transcendental. Now from (3.25) we have $f_1 + f_2 + f_3 \equiv 1$. Hence by Lemma 7 we get

$$\begin{aligned}\sum_{j=1}^{3} N(r,0;f_j) + 2\sum_{j=1}^{3} \overline{N}(r,\infty;f_j) &\leqslant N(r,0;F) + N(r,0;\mathrm{e}^{\gamma}G) + O(\log r)\\ &\leqslant (\lambda + o(1))T_1(r),\end{aligned}$$

as $r \to \infty$, $r \in I$, $\lambda < 1$ and $T_1(r) = \max\limits_{1\leqslant j\leqslant 3} T(r,f_j)$.

So by Lemma 6, we get either $\mathrm{e}^{\gamma}G \equiv -1$ or $\mathrm{e}^{\gamma} \equiv 1$. But here the only possibility is that $\mathrm{e}^{\gamma}G \equiv -1$, i.e. $g^n(g^{(k)})^m \equiv -\mathrm{e}^{-\gamma}p$ and so from (3.21) we obtain

$$F \equiv \mathrm{e}^{\gamma_1}G, \quad \text{i.e.} \quad f^n(f^{(k)})^m \equiv \mathrm{e}^{\gamma_1}g^n(g^{(k)})^m,$$

where γ_1 is a non-constant entire function. Then from (3.21) we get

$$f^n(f^{(k)})^m \equiv c\mathrm{e}^{\gamma_1/2}p \quad \text{and} \quad g^n(g^{(k)})^m \equiv c\mathrm{e}^{-\gamma_1/2}p, \tag{3.26}$$

where $c = \pm 1$. This shows that $f^n(f^{(k)})^m$ and $g^n(g^{(k)})^m$ share 0 CM. Clearly from (3.26) we see F and G are entire functions having no zeros.

Let z_p be a zero of f of multiplicity p ($p \geqslant k$) and z_q be a zero of g of multiplicity q ($q \geqslant k$). Clearly z_p will be a zero of $f^n(f^{(k)})^m$ of multiplicity $(n+m)p - km$ and z_q will be a zero of $g^n(g^{(k)})^m$ of multiplicity $(n+m)q - km$. Since $f^n(f^{(k)})^m$ and $g^n(g^{(k)})^m$ share 0 CM, it follows that $z_p = z_q$ and $p = q$. Consequently, f and g share 0 CM. Since $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$, we can take

$$f(z) = h(z)\mathrm{e}^{\alpha(z)} \quad \text{and} \quad g(z) = h(z)\mathrm{e}^{\beta(z)}, \tag{3.27}$$

where h is a non-constant polynomial and α, β are two non-constant entire functions.

We deduce from (3.27) that

$$f^n(f^{(k)})^m \equiv P_1(h, h', \ldots, h^{(k)}, \alpha', \alpha'', \ldots, \alpha^{(k)})\mathrm{e}^{(n+m)\alpha}, \tag{3.28}$$

where $P_1(h, h', \ldots, h^{(k)}, \alpha', \alpha'', \ldots, \alpha^{(k)})$ is a differential polynomial in $h, h', \ldots, h^{(k)}$, $\alpha', \alpha'', \ldots, \alpha^{(k)}$ and

$$g^n (g^{(k)})^m \equiv P_2(h, h', \ldots, h^{(k)}, \beta', \beta'', \ldots, \beta^{(k)}) \mathrm{e}^{(n+m)\beta}, \tag{3.29}$$

where $P_2(h, h', \ldots, h^{(k)}, \beta', \beta'', \ldots, \beta^{(k)})$ is a differential polynomial in $h, h', \ldots, h^{(k)}$, $\beta', \beta'', \ldots, \beta^{(k)}$.

Let $\mathcal{F} = \{F_\omega\}$ and $\mathcal{G} = \{G_\omega\}$, where $F_\omega(z) = F(z+\omega)$ and $G_\omega(z) = G(z+\omega)$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of entire functions defined on $\mathbb{C}$. We now consider the following two sub-cases.

Sub-case 1.1. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $F^{\#}(\omega) = F_\omega^{\#}(0) \leqslant M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 13 we have that F is of order at most 1. Now from (3.23) we have

$$\varrho(f^n (f^{(k)})^m) = \varrho(F) = \varrho(G) = \varrho(g^n (g^{(k)})^m) \leqslant 1. \tag{3.30}$$

Since F and G are non-constant entire functions having no zeros and $\varrho(F) = \varrho(G) \leqslant 1$, we can take

$$f^n (f^{(k)})^m = c_1 p \mathrm{e}^{az} \quad \text{and} \quad g^n (g^{(k)})^m = c_2 p \mathrm{e}^{bz}, \text{ where } a, b, c_1, c_2 \in \mathbb{C} \setminus \{0\}. \tag{3.31}$$

From (3.21) we see that $a + b = 0$. We claim that both $(n+m)\alpha(z) - az$ and $(n+m)\beta(z) - bz$ are constants. If possible, suppose both $(n+m)\alpha(z) - az$ and $(n+m)\beta(z) - bz$ are non-constants. Let $\alpha_1(z) = (n+m)\alpha(z) - az$ and $\beta_1(z) = (n+m)\beta(z) - bz$. Note that

$$\begin{aligned} T(r, \alpha') = m(r, \alpha') &\leqslant m(r, (n+m)\alpha') + O(1) = m(r, \alpha_1' + a) + O(1) \\ &\leqslant m(r, \alpha_1') + O(1) = m\Big(\frac{(\mathrm{e}^{\alpha_1})'}{\mathrm{e}^{\alpha_1}}\Big) + O(1) = S(r, \mathrm{e}^{\alpha_1}). \end{aligned}$$

Clearly $T(r, \alpha^{(i)}) = S(r, \mathrm{e}^{\alpha_1})$ for $i = 1, 2, \ldots$ Therefore $T(r, P_1) = S(r, \mathrm{e}^{\alpha_1})$ and so $T(r, p/P_1) = S(r, \mathrm{e}^{\alpha_1})$. Similarly we have $T(r, p/P_2) = S(r, \mathrm{e}^{\beta_1})$.

Now from (3.28), (3.29) and (3.31) we conclude that $T(r, \mathrm{e}^{\alpha_1}) = S(r, \mathrm{e}^{\alpha_1})$ and $T(r, \mathrm{e}^{\beta_1}) = S(r, \mathrm{e}^{\beta_1})$. Therefore we arrive at a contradiction. Hence, both α_1 and β_1 are constants. Consequently both α and β are polynomials of degree 1. Finally, we take

$$f(z) = d_1 h(z) \mathrm{e}^{az} \quad \text{and} \quad g(z) = d_1 h(z) \mathrm{e}^{-az}, \text{ where } d_1, d_2 \in \mathbb{C} \setminus \{0\}. \tag{3.32}$$

Now from (3.32) we have

$$f^n (f^{(k)})^m = d_1^{n+m} h^n \Big(\sum_{i=0}^{k} {}^k C_i a^{k-i} h^{(i)}\Big)^m \mathrm{e}^{(n+m)az},$$

where we define $h^{(0)} = h$. Similarly we have

$$g^n(g^{(k)})^m = d_2^{n+m} h^n \left(\sum_{i=0}^{k} {}^kC_i (-1)^{k-i} a^{k-i} h^{(i)} \right) \mathrm{e}^{-(n+m)az}.$$

Since $f^n(f^{(k)})^m$ and $g^n(g^{(k)})^m$ share 0 CM, it follows that

$$\sum_{i=0}^{k} {}^kC_i a^{k-i} h^{(i)} \equiv d^* \sum_{i=0}^{k} {}^kC_i (-1)^{k-i} a^{k-i} h^{(i)}, \tag{3.33}$$

where $d^* \in \mathbb{C} \setminus \{0\}$. But relation (3.33) does not hold.

Sub-case 1.2. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is not normal on $\mathbb{C}$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{F(z+\omega_j)\} \subset \mathcal{F}$, where $z \in \{z\colon\ |z| < 1\}$ and $\{\omega_j\} \subset \mathbb{C}$ is some sequence of complex numbers such that $F^{\#}(\omega_j) \to \infty$, as $|\omega_j| \to \infty$. Then by Lemma 14 there exist

(i) points z_j, $|z_j| < 1$,
(ii) positive numbers ϱ_j, $\varrho_j \to 0^+$,
(iii) a subsequence $\{F(\omega_j + z_j + \varrho_j\zeta)\}$ of $\{F(\omega_j + z)\}$

such that

$$h_j(\zeta) = \varrho_j^{-1/2} F(\omega_j + z_j + \varrho_j\zeta) \to h(\zeta) \tag{3.34}$$

spherically uniformly on a compact subset of $\mathbb{C}$, where $h(\zeta)$ is some non-constant holomorphic function such that $h^{\#}(\zeta) \leqslant h^{\#}(0) = 1$. Now from Lemma 13 we see that $\varrho(h) \leqslant 1$. Also by Hurwitz's theorem we can see that $h(\zeta) \neq 0$. From the proof of Zalcman's lemma (see [19], [27]) we have

$$\varrho_j = \frac{1}{F^{\#}(b_j)} \tag{3.35}$$

and

$$F^{\#}(b_j) \geqslant F^{\#}(\omega_j), \tag{3.36}$$

where $b_j = \omega_j + z_j$. Let

$$\widehat{h}_j(\zeta) = \varrho_j^{1/2} G(\omega_j + z_j + \varrho_j\zeta). \tag{3.37}$$

From (3.23) we have

$$F(\omega_j + z_j + \varrho_j\zeta) G(\omega_j + z_j + \varrho_j\zeta) \equiv 1$$

and so from (3.34) and (3.37) we get

$$h_j(\zeta)\widehat{h}_j(\zeta) \equiv 1. \tag{3.38}$$

Now from (3.34) and (3.38) we can deduce that

$$\widehat{h}_j(\zeta) \to \widehat{h}(\zeta) \tag{3.39}$$

spherically uniformly on a compact subset of $\mathbb{C}$, where $\widehat{h}(\zeta)$ is some non-constant holomorphic function in the complex plane. By Hurwitz's theorem we can see that $\widehat{h}(\zeta) \neq 0$. From (3.34), (3.38) and (3.39) we get $h(\zeta)\widehat{h}(\zeta) \equiv 1$. Since $\varrho(h) \leqslant 1$, we have $\varrho(h) = \varrho(\widehat{h}) \leqslant 1$. Again since h and $\widehat{h}$ are non-constant entire functions having no zeros and $\varrho(h) = \varrho(\widehat{h}) \leqslant 1$, we can take

$$h(z) = c_1 \mathrm{e}^{cz} \quad \text{and} \quad \widehat{h}(z) = \hat{c}_2 \mathrm{e}^{-cz}, \tag{3.40}$$

where $c, c_1, \widehat{c}_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1\widehat{c}_2 = 1$. Also from (3.40) we have

$$\frac{h_j'(\zeta)}{h_j(\zeta)} = \varrho_j \frac{F'(w_j + z_j + \varrho_j\zeta)}{F(w_j + z_j + \varrho_j\zeta)} \to \frac{h'(\zeta)}{h(\zeta)} = c, \tag{3.41}$$

spherically uniformly on a compact subset of $\mathbb{C}$. Now from (3.35) and (3.41) we get

$$\begin{aligned} \Bigl|\frac{h_j'(0)}{h_j(0)}\Bigr| = \varrho_j \Bigl|\frac{F'(\omega_j + z_j)}{F(\omega_j + z_j)}\Bigr| &= \frac{1 + |F(\omega_j + z_j)|^2}{|F'(\omega_j + z_j)|} \frac{|F'(\omega_j + z_j)|}{|F(\omega_j + z_j)|} \\ &= \frac{1 + |F(\omega_j + z_j)|^2}{|F(\omega_j + z_j)|} \to \Bigl|\frac{h'(0)}{h(0)}\Bigr| = |c|, \end{aligned} \tag{3.42}$$

which implies that $\lim\limits_{j\to\infty} F(\omega_j + z_j) \neq 0, \infty$ and so from (3.34) we see that

$$h_j(0) = \varrho_j^{-1/2} F(\omega_j + z_j) \to \infty. \tag{3.43}$$

Again from (3.34) and (3.40) we have

$$h_j(0) \to h(0) = c_1. \tag{3.44}$$

But from (3.43) and (3.44) we arrive at a contradiction.

Case 2. Let $p(z) = b \in \mathbb{C}\setminus\{0\}$. Then from (3.21) we get $f^n(f^{(k)})^m g^n(g^{(k)})^m \equiv b^2$, where f and g are transcendental entire functions. Clearly f and g have no zeros. But this is impossible because zeros of f and g are of multiplicities at least k. This completes the lemma. $\square$

Lemma 16. *Let f and g be two transcendental meromorphic functions having zeros of multiplicities at least k, where $k \in \mathbb{N}$ and let $F = f^n(f^{(k)})^m p^{-1}$, $G = g^n(g^{(k)})^m p^{-1}$, where p is a nonzero polynomial and $m, n \in \mathbb{N}$ such that $n > (mk + k^2 + k + 2)k^{-1}$. Suppose $f^n(f^{(k)})^m - p$, $g^n(g^{(k)})^m - p$ share $(0, k_1)$, where $k_1 \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and f, g share ∞ IM. If $H \equiv 0$, then either $f^n(f^{(k)})^m g^n(g^{(k)})^m \equiv p^2$, where $f^n(f^{(k)})^m - p$, $g^n(g^{(k)})^m - p$ share 0 CM or $f^n(f^{(k)})^m \equiv g^n(g^{(k)})^m$.*

Proof. Since $H \equiv 0$, on integration, we get

$$\frac{F'}{(F-1)^2} \equiv C_1 \frac{G'}{(G-1)^2}, \quad \text{i.e.} \quad \frac{((F_1 - p)p^{-1})'}{((F_1 - p)p^{-1})^2} \equiv C_1 \frac{((G_1 - p)p^{-1})'}{((G_1 - p)p^{-1})^2},$$

where $C_1 \in \mathbb{C} \setminus \{0\}$, $F_1 = f^n(f^{(k)})^m$ and $G_1 = f^n(f^{(k)})^m$. This shows that $(F_1 - p)p^{-1}$ and $(G_1 - p)p^{-1}$ share 0 CM and so $F_1 - p$ and $G_1 - p$ share 0 CM. Finally, by integration we get

$$\frac{1}{F-1} \equiv \frac{bG + a - b}{G - 1}, \tag{3.45}$$

where $a, b \in \mathbb{C}$ $(a \neq 0)$. We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$. If $b = -1$, then from (3.45) we have

$$F \equiv \frac{-a}{G - a - 1}.$$

Therefore $\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p)$. So in view of Lemma 9 and the second fundamental theorem we get

$$\begin{aligned}
(n-m)T(r,g) &\leqslant T(r, g^n(g^{(k)})^m) - mN(r,\infty;g) - N(r,0;(g^{(k)})^m) + S(r,g) \\
&\leqslant T(r,G) - mN(r,\infty;g) - N(r,0;(g^{(k)})^m) + S(r,g) \\
&\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) \\
&\quad - mN(r,\infty;g) - N(r,0;(g^{(k)})^m) + S(r,g) \\
&\leqslant \overline{N}(r,0;g) + \overline{N}(r,0;g^{(k)} \mid g \neq 0) + \overline{N}(r,\infty;f) \\
&\quad - N(r,0;(g^{(k)})^m) + S(r,g) \\
&\leqslant \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g) \\
&\leqslant \frac{1}{k} N(r,0;g) + N(r,\infty;g) + S(r,g) \leqslant \frac{k+1}{k} T(r,g) + S(r,g),
\end{aligned}$$

which is a contradiction since $n > (mk + k + 1)k^{-1}$.

If $b \neq -1$, from (3.45) we obtain that

$$F - \Big(1 + \frac{1}{b}\Big) \equiv \frac{-a}{b^2(G + (a-b)b^{-1})}.$$

So $\overline{N}(r,(b-a)b^{-1};G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+\overline{N}(r,0;p)$. Using Lemma 9 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b\neq 0$ and $a=b$. If $b=-1$, then from (3.45) we have $FG\equiv 1$, i.e. $f^n(f^{(k)})^m g^n(g^{(k)})^m\equiv p^2$, where $f^n(f^{(k)})^m-p$ and $g^n(g^{(k)})^m-p$ share 0 CM.

If $b\neq -1$, from (3.45) we have

$$\frac{1}{F}\equiv\frac{bG}{(1+b)G-1}.$$

Therefore $\overline{N}(r,(1+b)^{-1};G)=\overline{N}(r,0;F)$. So in view of Lemmas 2, 9 and the second fundamental theorem we get

$$\begin{aligned}(n-m)T(r,g)&\leqslant T(r,G)-mN(r,\infty;g)-N(r,0;(g^{(k)})^m)+S(r,g)\\&\leqslant\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\Big(r,\frac{1}{1+b};G\Big)\\&\quad-mN(r,\infty;g)-N(r,0;(g^{(k)})^m)+S(r,g)\\&\leqslant\overline{N}(r,0;g)+\overline{N}(r,0;g^{(k)}\mid g\neq 0)+\overline{N}(r,0;F)\\&\quad-N(r,0;(g^{(k)})^m)+S(r,g)\\&\leqslant\overline{N}(r,0;g)+\overline{N}(r,0;f)+\overline{N}(r,0;f^{(k)}\mid f\neq 0)+S(r,g)\\&\leqslant\overline{N}(r,0;g)+\overline{N}(r,0;f)+k\overline{N}(r,0;f\mid\geqslant k)+k\overline{N}(r,\infty;f)+S(r,g)\\&\leqslant\frac{1}{k}T(r,g)+\frac{1}{k}T(r,f)+T(r,f)+kT(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Without loss of generality, we suppose that $T(r,f)\leqslant T(r,g)$ for $r\in I$. So for $r\in I$ we have

$$(n-m)T(r,g)\leqslant\frac{k^2+k+2}{k}T(r,g)+S(r,g),$$

which is a contradiction since $n>(mk+k^2+k+2)k^{-1}$.

Case 3. Let $b=0$. From (3.45) we obtain

$$F\equiv\frac{G+a-1}{a}. \tag{3.46}$$

If $a\neq 1$, then from (3.46) we obtain $\overline{N}(r,1-a;G)=\overline{N}(r,0;F)$. We can similarly deduce a contradiction as in Case 2. Therefore $a=1$ and from (3.46) we obtain $F\equiv G$, i.e.

$$f^n(f^{(k)})^m\equiv g^n(g^{(k)})^m.$$

This completes the lemma. □

Lemma 17 ([2]). *Let f and g be non-constant meromorphic functions sharing $(1,k_1)$, where $2 \leqslant k_1 \leqslant \infty$. Then*

$$\begin{aligned}\overline{N}(r,1;f \mid= 2) + 2\overline{N}(r,1;f \mid= 3) + \ldots + (k_1-1)\overline{N}(r,1;f \mid= k_1) + k_1\overline{N}_L(r,1;f)\\ + (k_1+1)\overline{N}_L(r,1;g) + k_1\overline{N}_E^{(k_1+1}(r,1;g) \leqslant N(r,1;g) - \overline{N}(r,1;g).\end{aligned}$$

4. Proof of the theorem

Proof of Theorem 1. Let $F = f^n(f^{(k)})^m/p$ and $G = g^n(g^{(k)})^m/p$. Clearly F and G share $(1,k_1)$, except for the zeros of p, and f, g share ∞ IM.

Case 1. Let $H \not\equiv 0$. From (3.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of F' which are not the zeros of $F(F-1)$, (v) zeros of G' which are not the zeros of $G(G-1)$. Since H has only simple poles, we get

$$\begin{aligned}N(r,\infty;H) \leqslant \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F \mid\geqslant 2)\\ + \overline{N}(r,0;G \mid\geqslant 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'),\end{aligned} \tag{4.1}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined. Now from Nevanlinna's fundamental estimate of the logarithmic derivative we obtain $m(r,H) = S(r,F) + S(r,G)$.

Since $T(r,F) \leqslant (n+(k+1)m)T(r,f) + S(r,f)$, $T(r,G) \leqslant (n+(k+1)m)T(r,g) + S(r,g)$, then $m(r,H) = S(r,f) + S(r,g)$. Let z_0 be a simple zero of $F-1$ but $p(z_0) \neq 0$. Clearly z_0 is a simple zero of $G-1$. Then an elementary calculation gives that $H(z) = O((z-z_0))$, which proves that z_0 is a zero of H. Now by the first fundamental theorem of Nevanlinna we get

$$\begin{aligned}N(r,1;F \mid= 1) &\leqslant N(r,0;H) \leqslant T(r,H) + O(1)\\ &= N(r,\infty;H) + m(r,H) + O(1)\\ &\leqslant N(r,\infty;H) + S(r,f) + S(r,g).\end{aligned} \tag{4.2}$$

Using (4.1) and (4.2) we get

$$\begin{aligned}\overline{N}(r,1;F) &\leqslant N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\geqslant 2)\\ &\leqslant \overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;F \mid\geqslant 2) + \overline{N}(r,0;G \mid\geqslant 2) + \overline{N}_*(r,1;F,G)\\ &\quad + \overline{N}(r,1;F \mid\geqslant 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\end{aligned} \tag{4.3}$$

$$\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;F\mid\geqslant 2)+\overline{N}(r,0;G\mid\geqslant 2)+\overline{N}_*(r,1;F,G)$$
$$+\overline{N}(r,1;F\mid\geqslant 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).$$

Now in view of Lemmas 2 and 17 we get

$$\begin{aligned}
(4.4)\quad & \overline{N}_0(r,0;G')+\overline{N}(r,1;F\mid\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G')+\overline{N}(r,1;F\mid=2)+\overline{N}(r,1;F\mid=3)+\ldots+\overline{N}(r,1;F\mid=k_1)\\
&\quad+\overline{N}_E^{(k_1+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G')-\overline{N}(r,1;F\mid=3)-\ldots-(k_1-2)\overline{N}(r,1;F\mid=k_1)\\
&\quad-(k_1-1)\overline{N}_L(r,1;F)-k_1\overline{N}_L(r,1;G)-(k_1-1)\overline{N}_E^{(k_1+1}(r,1;F)\\
&\quad+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G)-(k_1-2)\overline{N}_L(r,1;F)\\
&\quad-(k_1-1)\overline{N}_L(r,1;G)\\
&\leqslant N(r,0;G'\mid G\neq 0)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\\
&\leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\\
&=\overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G).
\end{aligned}$$

Hence using (4.3), (4.4) and Lemma 1 we get from the second fundamental theorem that

$$\begin{aligned}
(4.5)\quad T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F')\\
&\leqslant 2\overline{N}(r,\infty,f)+N_2(r,0;F)+\overline{N}(r,0;G\mid\geqslant 2)+\overline{N}(r,1;F\mid\geqslant 2)\\
&\quad+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f)+N_2(r,0;F)+N_2(r,0;G)-(k_1-2)\overline{N}_*(r,1;F,G)\\
&\quad+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+N_2(r,0;(f^{(k)})^m)+2\overline{N}(r,0;g)\\
&\quad+mN_2(r,0;g^{(k)})-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+N(r,0;(f^{(k)})^m)+2\overline{N}(r,0;g)\\
&\quad+mN_{k+2}(r,0;g)+mk\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_*(r,1;F,G)\\
&\quad+S(r,f)+S(r,g)\\
&\leqslant (3+mk)\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+mN(r,0;g)\\
&\quad+N(r,0;(f^{(k)})^m)-(k_1-2)\overline{N}_*(r,1;F,G)\\
&\quad+S(r,f)+S(r,g).
\end{aligned}$$

Now using Lemmas 8 and 9 we get from (4.5) that

$$
\begin{aligned}
(4.6)\quad (n-m)T(r,f) &\leqslant T(r,f^n(f^{(k)})^m) - mN(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f)\\
&\leqslant T(r,F) - mN(r,\infty;f) - N(r,0;(f^{(k)})^m) + S(r,f)\\
&\leqslant (3+(k-1)m)\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g)\\
&\quad + mN(r,0;g) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant \frac{(k+1)(3+(k-1)m)}{k(n+m+(m-2)k-1)}(T(r,f)+T(r,g))\\
&\quad + \frac{2}{k}(T(r,f)+T(r,g)) + \frac{3+(k-1)m}{n+m+(m-2)k-1}\overline{N}_*(r,1;F,G)\\
&\quad + mT(r,g) - (k_1-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant \frac{(mk+4)n+m^2k^2+(m^2+3m-2)k+2(m+1)}{k(n+m+(m-2)k-1)}T(r) + S(r).
\end{aligned}
$$

In a similar way we can obtain

$$
(4.7)\quad (n-m)T(r,g) \leqslant \frac{(mk+4)n+m^2k^2+(m^2+3m-2)k+2(m+1)}{k(n+m+(m-2)k-1)}T(r) + S(r).
$$

Combining (4.6) and (4.7) we see that

$$
(n-m)T(r) \leqslant \frac{(mk+4)n+m^2k^2+(m^2+3m-2)k+2(m+1)}{k(n+m+(m-2)k-1)}T(r) + S(r),
$$

i.e.

$$
(4.8)\qquad k(n-K_1)(n-K_2)T(r) \leqslant S(r),
$$

where

$$
\begin{aligned}
K_1 &= \frac{(2-m)k^2+(m+1)k+4+\sqrt{L_1}}{2k},\\
K_2 &= \frac{(2-m)k^2+(m+1)k+4-\sqrt{L_1}}{2k}
\end{aligned}
$$

and $L_1 = ((2-m)k^2+(m+1)k+4)^2 + 8k((m^2-m)k^2+(m^2+m-1)k+(m+1))$.

Note that

$$
\begin{aligned}
L_1 &= m^2k^4 + 9m^2k^2 + 2mk^2 + 6m^2k^3 - 6mk^3 + 4k^4(1-m)\\
&\quad + 16k(m+1) + 9k^2 + 4k^3 + 16\\
&< m^2k^4 + 9m^2k^2 + 6m^2k^3 + 10mk^2 - 2mk^3 + 16(3m-1)k\\
&\quad + k^2 + 64 + 8k^2(1-m) + 4k^3(1-m) + 32k(1-m)\\
&\leqslant (mk^2+(3m-1)k+8)^2.
\end{aligned}
$$

Therefore

$$K_1 = \frac{(2-m)k^2 + (m+1)k + 4 + \sqrt{L_1}}{2k} < \frac{(2-m)k^2 + (m+1)k + 4 + mk^2 + (3m-1)k + 8}{2k} = \frac{k^2 + 2mk + 6}{k}.$$

Since $n \geqslant (k^2 + 2mk + 6)k^{-1}$, (4.8) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 16, 12 and 15. $\square$

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