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DISTANCE MATRICES PERTURBED BY LAPLACIANS

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Abstract. Let T be a tree with n vertices. To each edge of T we assign a weight which is a positive definite matrix of some fixed order, say, s. Let D_{ij} denote the sum of all the weights lying in the path connecting the vertices i and j of T. We now say that D_{ij} is the distance between i and j. Define $D := [D_{ij}]$, where D_{ii} is the $s \times s$ null matrix and for $i \neq j$, D_{ij} is the distance between i and j. Let G be an arbitrary connected weighted graph with n vertices, where each weight is a positive definite matrix of order s. If i and j are adjacent, then define $L_{ij} := -W_{ij}^{-1}$, where W_{ij} is the weight of the edge (i, j). Define $L_{ii} := \sum\limits_{i \neq j, j=1}^{n} W_{ij}^{-1}$. The Laplacian of G is now the $ns \times ns$ block matrix $L := [L_{ij}]$. In this paper, we first note that $D^{-1} - L$ is always nonsingular and then we prove that D and its perturbation $(D^{-1} - L)^{-1}$ have many interesting properties in common.

Keywords: tree; Laplacian matrix; inertia; Haynsworth formula

MSC 2020: 05C50, 15B48

1. Introduction

Consider a finite, simple and undirected graph $G = (V, \mathcal{E})$, where V is the set of vertices and $\mathcal{E}$ is the set of edges. We write $V = \{1, \ldots, n\}$ and $(i, j) \in \mathcal{E}$ if i and j are adjacent. To an edge (i, j) of G we assign a weight W_{ij} which is a positive definite matrix of order s. We now say that G is a weighted graph. Define

$$V_{ij} = \begin{cases} W_{ij}^{-1}, & (i, j) \in \mathcal{E}, \\ 0_s, & \text{else}, \end{cases}$$

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where 0_s is the $s \times s$ null matrix. The Laplacian of G is then the matrix

$$L(G) := \begin{pmatrix} \sum_k V_{1k} & -V_{12} & -V_{13} & \dots & -V_{1n} \\ -V_{21} & \sum_k V_{2k} & -V_{23} & \dots & -V_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ -V_{n1} & -V_{n2} & -V_{n3} & \dots & \sum_k V_{nk} \end{pmatrix}.$$

Recall that a tree is a connected acyclic graph. Let T be a tree with n-vertices. The distance S_{ij} between any two vertices i and j of T is the sum of all the weights lying in the path connecting i and j. Define

$$D_{ij} = \begin{cases} S_{ij}, & i \neq j, \\ 0_s, & i = j. \end{cases}$$

Now the distance matrix of T denoted by $D(T)$ is the $ns \times ns$ block matrix with (i,j)th block equal to D_{ij}. Distance matrices are well studied when $s = 1$, i.e., the weights are positive scalars. These matrices have a wide literature with numerous applications; see for example [3], [4] and references therein. Our objective in this paper is to go beyond the usual scalar case and study much more general class of matrices for which the theory can be extended by some additional matrix theoretical techniques.

Several interesting properties of $D(T)$ and $L(T)$ are known. In addition, there are identities that connect $D(T)$ and $L(T)$. Distance matrices in this weighted setup are introduced in [2] and further investigated extensively in [1]. Some of those important properties are listed below. These will be useful for proving our result. For brevity, we write $D = D(T)$ and $L = L(T)$.

(P1) Let $L^\dagger$ be the Moore-Penrose inverse of L. Then

$$D_{ij} = L^\dagger_{ii} + L^\dagger_{jj} - 2L^\dagger_{ij}.$$

See Theorem 3.4 in [1].

(P2) Let δ_i be the degree of the vertex i and τ be the column vector with ith component equal to $2 - \delta_i$. Suppose I_s is the $s \times s$ identity matrix. Then

$$D^{-1} = -\frac{1}{2}L + \frac{1}{2}\Delta R^{-1}\Delta^T,$$

where R is the sum of all the weights in T and $\Delta := \tau \otimes I_s$; see Theorem 3.7 in [1].

(P3) If J is the block matrix with each block equal to I_s, then D is negative definite on null-space of J. So, D has exactly s positive eigenvalues; see Section 2.3 in [1].

(P4) If G is connected, then $L(G)$ is positive semidefinite, $LJ = 0$ and $\mathrm{rank}(L) = ns - s$. This can be proved easily. Now, it follows that column space of L is contained in $\mathcal{M}$.

1.1. Objective of the paper. Let T be a weighted tree and G be a weighted graph with n vertices. Assume G is connected. As before we shall write D for $D(T)$ and L for $L(G)$. The blocks of D and L will be D_{ij} and L_{ij} which are $s \times s$ matrices. We first show in this paper that $D^{-1} - L$ is always nonsingular. Define $F := (D^{-1} - L)^{-1}$. We say that F is a perturbation of D. By performing certain numerical experiments, we observed that any perturbation F has the following properties.

(a) Each block of F is positive definite.
(b) F has exactly s positive eigenvalues.
(c) F is negative definite on null-space of J.

Items (a), (b), and (c) are satisfied by any distance matrix D. Our objective in this paper is to prove (a), (b), and (c) for any perturbation F of D. When the weights are positive scalars, Bapat, Kirkland and Neumann established a similar result in [3]. It can be noted that the result in this paper is a far reaching generalization of that result.

1.2. Notation. We fix the following notation.

(N1) We say that G is an $ns \times ns$ block matrix if G can be partitioned as

$$\begin{bmatrix} G_{11} & G_{12} & \dots & G_{1n} \\ G_{12} & G_{22} & \dots & G_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ G_{1n} & G_{2n} & \dots & G_{2n} \end{bmatrix},$$

where each G_{ij} is an $s \times s$ matrix. We now write $G = [G_{ij}]$.

(N2) Let I_s denote the identity matrix of order s. Fix a positive integer n. Now, J will be the $ns \times ns$ block matrix $[I_s]$.

(N3) We use $\mathcal{M}$ to denote the null space of J.

(N4) Define $U := e \otimes I_s$, where e is the column vector of all ones in $\mathbb{R}^n$. Thus, J can be written as $[U, \dots, U]$. Let e_i be the n-vector with 1 in the position i and zeros elsewhere and $E_i := e_i \otimes I_s$.

(N5) The transpose of a matrix A is denoted by A'.

(N6) If A is a symmetric matrix, we use $\mathrm{In}(A)$ to denote the inertia of A. We write $\mathrm{In}(A) = (n_-(A), n_z(A), n_+(A))$, where $n_-(A)$ and $n_+(A)$ are the number of negative and positive eigenvalues of A, respectively, and $n_z(A)$ is the nullity of A.

(N7) Suppose m is a positive integer. The notation $[m]$ will denote the finite set $\{1, \ldots, m\}$.

(N8) If $\Delta_1, \Delta_2 \subseteq [n]$, then $G[[\Delta_1, \Delta_2]] = [X_{ij}]$ will denote the $|\Delta_1| \times |\Delta_2|$ block matrix, where $(i, j) \in \Delta_1 \times \Delta_2$. If $\Delta = \Delta_1 = \Delta_2$, then we simply write $G[[\Delta]]$ for $G[[\Delta_1, \Delta_2]]$.

2. Result

We now prove our main result.

Theorem 2.1. *Let T be a weighted tree with n vertices, where each weight is of order s. Let $D = [D_{ij}]$ be the distance matrix of T. Suppose G is a connected and weighted graph with n vertices, where each weight is of order s. Let $L = [L_{ij}]$ be the Laplacian matrix of G. For any $\beta \geqslant 0$, the following are true.*

(i) *$D^{-1} - \beta L$ is nonsingular.*

(ii) *$\mathrm{In}(D^{-1} - \beta L) = (ns - s, 0, s)$.*

(iii) *Let $i \in [n]$ and $\Delta := [n] \setminus \{i\}$. Define $F := D^{-1} - \beta L$. Then $F[[\Delta]]$ is negative definite.*

(iv) *The bordered matrix $G := \begin{bmatrix} (D^{-1} - \beta L)^{-1} & U \\ U' & 0 \end{bmatrix}$ is nonsingular and has exactly s positive eigenvalues.*

(v) *$(D^{-1} - \beta L)^{-1}$ is negative semidefinite on $\mathcal{M}$.*

(vi) *Every block in $(D^{-1} - L)^{-1}$ is positive definite.*

Proof. (i) Let $x \in \mathbb{R}^{ns}$ be such that

$$x'(D^{-1} - \beta L) = 0.$$

Put $y = D^{-1}x$. Then $y' = \beta x'L$ and so, by (P4), $y \in \mathcal{M}$. By (P1), D is negative semidefinite on $\mathcal{M}$, and hence $y'Dy \leqslant 0$. This implies that $x'D^{-1}x \leqslant 0$ and hence, $x'Lx \leqslant 0$. As L is positive semidefinite, $x'Lx = 0$, and therefore, $Lx = 0$. This leads to $x'D^{-1} = 0$ and hence, $x = 0$. Thus we get (i).

(ii) By (P2) and (P3), $\mathrm{In}(D^{-1}) = (ns - s, 0, s)$. In view of (i), $D^{-1} - \delta L$ is nonsingular for any $\delta \geqslant 0$. Using the continuity of eigenvalues, we get

$$\mathrm{In}(D^{-1} - \beta L) = (ns - s, 0, s).$$

(iii) Without loss of generality, we assume $i = n$. Now $\Delta = \{1, \ldots, n-1\}$. We complete the proof by showing that $L[[\Delta]]$ is positive definite and $D^{-1}[[\Delta]]$ is negative semidefinite. Let $L[[\Delta]]x = 0$ for some nonzero x in $\mathbb{R}^{ns-s}$. Define $\widetilde{x} \in \mathbb{R}^{ns}$ by $\widetilde{x} := (x, 0)'$. Then $\widetilde{x}' L \widetilde{x} = 0$. So, $L\widetilde{x} = 0$. Hence, by (P4), $\widetilde{x}$ is an element in the column space of J. This implies that $\widetilde{x}$ is of the form $(p, p, \ldots, p)'$ for some $p \in \mathbb{R}^s$. Since $\widetilde{x} = (x, 0)'$, this means that $x = 0$. Thus, $L[[\Delta]]$ is positive definite.

We now show that $D^{-1}[[\Delta]]$ is negative semidefinite. Since $\mathrm{In}(D^{-1}) = (ns-s, 0, s)$, by interlacing theorem, $D^{-1}[[\Delta]]$ can have at most s nonnegative eigenvalues. As $D_{nn} = 0$, by Theorem 2 of Fiedler and Markham [5], nullity of $D^{-1}[[\Delta]]$ must be s. Therefore, $D^{-1}[[\Delta]]$ has no positive eigenvalues and thus, $D^{-1}[[\Delta]]$ is negative semidefinite. Hence, $F[[\Delta]] = D^{-1}[[\Delta]] - L[[\Delta]]$ is negative definite.

(iv) Define $F := (D^{-1} - \beta L)^{-1}$. Then $G = \begin{bmatrix} F & U \\ U' & 0 \end{bmatrix}$. Let the Schur complement of F in G be G/F. Since $LU = 0$, we have

$$G/F = -U'F^{-1}U = -U'(D^{-1} - \beta L)U = -U'D^{-1}U.$$

By Corollary 2.8 in [1], $U'D^{-1}U$ is positive definite, and hence,

$$\mathrm{In}(G/F) = (s, 0, 0).$$

By (ii), $\mathrm{In}(F) = (ns - s, 0, s)$. In view of Haynsworth inertia additivity formula,

$$\mathrm{In}(G/F) = \mathrm{In}(G) - \mathrm{In}(F),$$

and so

$$\mathrm{In}(G) = \mathrm{In}(G/F) + \mathrm{In}(F) = (s, 0, 0) + (ns - s, 0, s) = (ns, 0, s).$$

This completes the proof of (iv).

(v) We now show that $(D^{-1} - \beta L)^{-1}$ is negative semidefinite on $\mathcal{M}$. Assume the contrary. Let $p \in \mathbb{R}^{ns}$ be such that

$$p \in \mathcal{M} \quad \text{and} \quad p'(D^{-1} - \beta L)^{-1} p > 0.$$

Now consider the following subspace of $\mathbb{R}^{ns+s}$:

$$W := \{(\alpha p, y)' \colon \alpha \in \mathbb{R} \quad \text{and} \quad y \in \mathbb{R}^s\}.$$

Let $v := (\delta p, y)'$ be an arbitrary nonzero vector in W. As before, define

$$G = \begin{bmatrix} (D^{-1} - \beta L)^{-1} & U \\ U' & 0 \end{bmatrix}.$$

Since $U'p = 0$, we see that

$$v'Gv = \delta^2 p'(D^{-1} - \beta L)^{-1} p \geqslant 0.$$

So, G is positive semidefinite on W. Since the dimension of W is $s+1$, G will have at least $s+1$ nonnegative eigenvalues. But from (iv), we see that $\mathrm{In}(G) = (ns, 0, s)$. This is a contradiction. Hence, $(D^{-1} - \beta L)^{-1}$ is negative semidefinite on $\mathcal{M}$.

(vi) Define $A := (D^{-1} - L)^{-1}$ and let the (i,j)th block of A be written A_{ij}. We shall first prove that all the diagonal blocks of A are positive definite. Let $H := A^{-1}$ and let H_{ij} be the (i,j)th block of H.

We claim A_{11} is positive definite. Define $\Delta := \{2, \ldots, n\}$ and $Q := H[[\Delta]]$. We note that

$$H = \begin{bmatrix} H_{11} & K \\ K' & Q \end{bmatrix},$$

and hence by Theorem 2 of Fiedler and Markham [5], nullity of Q and nullity of A_{11} are equal. By (iii), Q is negative definite and so Q is nonsingular. Hence, A_{11} is nonsingular. Now it follows that

$$\mathrm{In}(Q) = \mathrm{In}(A/A_{11}),$$

where A/A_{11} is the Schur complement of A_{11} in A. In view of inertia additivity formula, (ii) and (iii), we see that

$$\mathrm{In}(A_{11}) = \mathrm{In}(A) - \mathrm{In}(Q) = (ns - s, 0, s) - (ns - s, 0, 0) = (0, 0, s).$$

Thus, A_{11} is positive definite. By a similar argument, we conclude that all diagonal blocks of A are positive definite.

For a nonzero vector $x \in \mathbb{R}^s$, define

$$G_x := [x' A_{ij} x].$$

We claim that the off-diagonal entries of G_x are nonzero. Let $y = (y_1, y_2, \ldots, y_n)'$ be an element in $\{e\}^\perp$. For each $i \in [n]$, let $p^i := y_i x$ and $p := (p^1, p^2, \ldots, p^n)'$. As

$$\sum_{i=1}^{n} p^i = \left(\sum_{i=1}^{n} y_i \right) x$$

and $\sum_{i=1}^{n} y_i = 0$, it follows that $Jp = 0$ and hence, $p \in \mathcal{M}$. By (v), A is negative definite on $\mathcal{M}$. Since $p'Ap = y'G_x y$, we now see that G_x is negative definite on $\{e\}^\perp$. This implies G_x has at least $n-1$ negative eigenvalues. As each diagonal block A_{ii} is positive definite, the diagonal entries of G_x are positive. So

$$\mathrm{In}(G_x) = (n-1, 0, 1).$$

Applying the interlacing theorem, we now see that all the off-diagonal entries of G_x are nonzero. This proves our claim.

To this end, we have thus shown that if A_{ij} is an off-diagonal block of A, then either A_{ij} is positive definite or negative definite. We now claim that A_{ij} is positive definite. Define $f\colon (0,\infty) \to \mathbb{R}^{s\times s}$ by

$$f(\alpha) := E_i'(D^{-1} - \alpha L)^{-1} E_j.$$

Note that $f(\alpha)$ is the (i,j)th off-diagonal block of $(D^{-1} - \alpha L)^{-1}$. By a similar argument as above, we see that $f(\alpha)$ is either positive definite or negative definite, for any $\alpha \in (0,\infty)$. We now note that

$$\mathrm{trace}(D_{ij}) = \lim_{\alpha \downarrow 0} \mathrm{trace}(f(\alpha)).$$

As each off-diagonal block D_{ij} is positive definite, $\mathrm{trace}(D_{ij}) > 0$. So $\mathrm{trace}(f(\delta)) > 0$ for some $\delta > 0$. As $f(\alpha)$ is positive definite or negative definite for all $\alpha > 0$, it follows that $\mathrm{trace}(f(\alpha)) \neq 0$ for each $\alpha \in (0,\infty)$. Thus, $\mathrm{trace}(f(\alpha)) > 0$ for each $\alpha > 0$ and hence $f(\alpha)$ is positive definite for all $\alpha > 0$. Therefore each block of $(D^{-1} - L)^{-1}$ is positive definite. $\square$

2.1. Example. To illustrate our result, we give the following example.

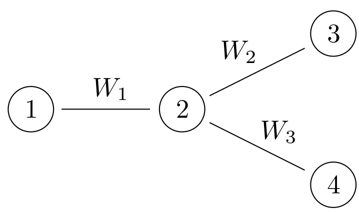

Figure 1. Tree T.

Example 2.1. Consider the following tree T on four vertices, see Figure 1. Define $W_1 = \begin{bmatrix} 8 & 6 \\ 6 & 5 \end{bmatrix}$, $W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$, and $W_3 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$. Now, the distance matrix of T is

$$D = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 8 & 6 & 9 & 7 & 13 & 6 \\ 0 & 0 & 6 & 5 & 7 & 10 & 6 & 10 \\ \hline 8 & 6 & 0 & 0 & 1 & 1 & 5 & 0 \\ 6 & 5 & 0 & 0 & 1 & 5 & 0 & 5 \\ \hline 9 & 7 & 1 & 1 & 0 & 0 & 6 & 1 \\ 7 & 10 & 1 & 5 & 0 & 0 & 1 & 10 \\ \hline 13 & 6 & 5 & 0 & 6 & 1 & 0 & 0 \\ 6 & 10 & 0 & 5 & 1 & 10 & 0 & 0 \end{array}\right].$$

Consider the graph G on four vertices (see Figure 2), where $S_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $S_2 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, $S_3 = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$, and $S_4 = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$. The Laplacian ma-

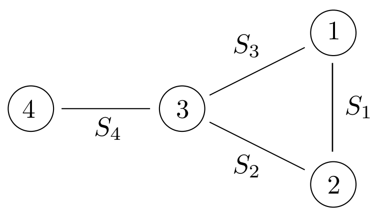

Figure 2. A connected graph G.

trix $L(G)$ of G is:

$$L = \left[\begin{array}{cc|cc|cc|cc} \frac{3}{2} & 2 & -\frac{1}{2} & 0 & -1 & -2 & 0 & 0 \\ 2 & \frac{11}{2} & 0 & -\frac{1}{2} & -2 & -5 & 0 & 0 \\ \hline -\frac{1}{2} & 0 & \frac{5}{8} & 0 & -\frac{1}{8} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{5}{8} & 0 & -\frac{1}{8} & 0 & 0 \\ \hline -1 & -2 & -\frac{1}{8} & 0 & \frac{23}{16} & \frac{35}{16} & -\frac{5}{16} & -\frac{3}{16} \\ -2 & -5 & 0 & -\frac{1}{8} & \frac{35}{16} & \frac{87}{16} & -\frac{3}{16} & -\frac{5}{16} \\ \hline 0 & 0 & 0 & 0 & -\frac{5}{16} & -\frac{3}{16} & \frac{5}{16} & \frac{3}{16} \\ 0 & 0 & 0 & 0 & -\frac{3}{16} & -\frac{5}{16} & \frac{3}{16} & \frac{5}{16} \end{array}\right].$$

Then the matrix $(D^{-1}-L)^{-1}$ is:

$$\left[\begin{array}{cc|cc|cc|cc}
\frac{3419893}{612184} & \frac{2467937}{612184} & \frac{3525525}{612184} & \frac{2430433}{612184} & \frac{3944573}{612184} & \frac{2285161}{612184} & \frac{4731635}{612184} & \frac{1962623}{612184} \\
\frac{2467937}{612184} & \frac{1957213}{306092} & \frac{2255293}{612184} & \frac{935945}{153046} & \frac{2218981}{612184} & \frac{1023631}{153046} & \frac{1853663}{612184} & \frac{2458795}{306092} \\
\hline
\frac{3525525}{612184} & \frac{2255293}{612184} & \frac{3037701}{612184} & \frac{1985813}{612184} & \frac{3651821}{612184} & \frac{2212445}{612184} & \frac{4430931}{612184} & \frac{1803363}{612184} \\
\frac{2430433}{612184} & \frac{935945}{153046} & \frac{1985813}{612184} & \frac{1566953}{306092} & \frac{2093885}{612184} & \frac{1966725}{306092} & \frac{1746783}{612184} & \frac{1159859}{153046} \\
\hline
\frac{3944573}{612184} & \frac{2218981}{612184} & \frac{3651821}{612184} & \frac{2093885}{612184} & \frac{3655573}{612184} & \frac{2328197}{612184} & \frac{4622251}{612184} & \frac{1831995}{612184} \\
\frac{2285161}{612184} & \frac{1023631}{153046} & \frac{2212445}{612184} & \frac{1966725}{306092} & \frac{2328197}{612184} & \frac{2026753}{306092} & \frac{1884375}{612184} & \frac{1242045}{153046} \\
\hline
\frac{4731635}{612184} & \frac{1853663}{612184} & \frac{4430931}{612184} & \frac{1746783}{612184} & \frac{4622251}{612184} & \frac{1884375}{612184} & \frac{3647621}{612184} & \frac{2294033}{612184} \\
\frac{1962623}{612184} & \frac{2458795}{306092} & \frac{1803363}{612184} & \frac{1159859}{153046} & \frac{1831995}{612184} & \frac{1242045}{153046} & \frac{2294033}{612184} & \frac{1968213}{306092}
\end{array}\right]$$

We note that each block in $(D^{-1}-L)^{-1}$ is positive definite, $\mathrm{In}((D^{-1}-L)^{-1}) = (6,0,2)$, and $(D^{-1}-L)^{-1}$ is negative semidefinite on $\mathcal{M}$.

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