Zuming Peng

The study on semicopula based implications

Kybernetika, Vol. 56 (2020), No. 4, 662–694

Persistent URL: http://dml.cz/dmlcz/148378

KYBERNETIKA — VOLUME *56* (2020), NUMBER 4, PAGES 662–694

THE STUDY ON SEMICOPULA BASED IMPLICATIONS

Zuming Peng

Recently, Baczyński et al. (2017) proposed a new family of implication operators called semicopula based implications, which combines a given a priori fuzzy implication and a semicopula. In this paper, firstly, the relationship between the basic properties of the priori fuzzy implication and the semicopula based implication are analyzed. Secondly, the conditions such that the semicopula based implication is a fuzzy implication are studied, the study is carried out mainly in the case that the semicopula is a special family semicopula and the priori fuzzy implication is a (U, N)-implication. Moreover, the case that the semicopula based implication is 2-increasing (directionally decreasing, respectively) is also considered.

Keywords: fuzzy implications, semicopula based implications, (U, N)-implications, semicopula, 2-increasing

Classification: 03E72, 03B52

1. INTRODUCTION

Fuzzy implications are the generalization of the classical (Boolean) implications on the unit interval $[0, 1]$, and are the basis for fuzzy logic systems, fuzzy control, decision theory, expert systems [2, 6, 11, 15]. The main way of generating fuzzy implications is from basic fuzzy logic connectives [1, 2, 3, 4], t-norms, t-conorms and negations. Other way of generating fuzzy implications is from monotone functions [13, 14, 16, 22, 26], or from convex combination of two fuzzy implications [17, 18, 19, 21, 23, 24, 25]. Moreover, in consideration of imperfect knowledge that involve uncertainty, imprecision and randomness, the probability implications and probability S-implications [10] were proposed.

Recently, Baczyński et al. [5] introduced a new method of constructing implications based on a fuzzy implication I and a semicopula B. The resulting implication J_{IB} is defined by

$$J_{IB}(x, y) = I(x, B(x, y)), \quad x, y \in [0, 1]. \tag{1}$$

The implication J_{IB} can be seen as a generalization of the probabilistic implication and the probabilistic S-implication. However, its monotonicity in the first coordinate may fail, and thus it need not be a fuzzy implication. Due to this, in this paper, as a supplement of this research topic from the theoretical point of view, we attempt a systematic study of J_{IB}.

DOI: 10.14736/kyb-2020-4-0662

The paper is organized as follows. In Section 2, some concepts and results are recalled. In Section 3, we discuss the relationship between the properties of J_{IB} and I. In Section 4, conditions for J_{IB} being a fuzzy implication are deeply studied. In Section 5, some trivial results of Φ-conjugate with J_{IB} are given. The last section is the Conclusion.

2. PRELIMINARIES

For the convenience of reading, in this section, we recall some definitions and results that will be used in the rest of the paper.

Definition 2.1. (Durante et al. [8]) A function $B : [0,1]^2 \to [0,1]$ is said to be a semicopula if, and only if, it satisfies the two following conditions:

(i) $B(x,1) = B(1,x) = x$ for all x in [0,1],

(ii) $B(x,y)$ is increasing in each place.

Definition 2.2. (Klement et al. [12], Nelsen [20]) A function $T : [0,1]^2 \to [0,1]$ is said to be a t-norm if it is an associative and commutative semicopula. A function $S : [0,1]^2 \to [0,1]$ is said to be a t-conorm if it is an associative and commutative dual semicopula. A function $C : [0,1]^2 \to [0,1]$ is said to be a copula if it is a semicopula and satisfies $C(x_1,y_1) + C(x_2,y_2) \geq C(x_1,y_2) + C(x_2,y_1)$ for all x_1, x_2, y_1, $y_2 \in [0,1]$ with $x_1 < x_2$ and $y_1 < y_2$.

Five basic t-norms that will be used throughout this paper are given as follows:

- $T_M(x,y) = \min(x,y)$, is the greatest semicopula.
- $T_P(x,y) = xy$.
- $T_{LK}(x,y) = \max(x+y-1,0)$.
- $T_D(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1)^2, \\ \min(x,y), & \text{otherwise,} \end{cases}$ is the smallest semicopula.
- $T_{nM}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1, \\ \min(x,y), & \text{otherwise.} \end{cases}$

These basic t-norms form the following chains:

$$T_D \leq T_{LK} \leq T_P \leq T_M,\ T_D \leq T_{LK} \leq T_{nM} \leq T_M.$$

Definition 2.3. (Baczyński et al. [2]) A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies, for all x, x_1, x_2, y, y_1, $y_2 \in [0,1]$, the following conditions:

(I1) if $x_1 < x_2$, then $I(x_1,y) \geq I(x_2,y)$, i. e., $I(\cdot,y)$ is decreasing,

(I2) if $y_1 < y_2$, then $I(x,y_1) \leq I(x,y_2)$, i. e., $I(x,\cdot)$ is increasing,

(I3) $I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$.

The set of all fuzzy implications will be denoted by FI.

Important fuzzy implications that will be used throughout this paper are given as follows:

- The Lukasiewicz implication: $I_{LK}(x,y) = \min(1, 1-x+y)$.
- The Rescher implication: $I_{RS}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ 0, & \text{if } x > y. \end{cases}$
- The Weber implication: $I_{WB}(x,y) = \begin{cases} 1, & \text{if } x \in [0,1), \\ y, & \text{if } x = 1. \end{cases}$
- The greatest implication: $I_G(x,y) = \begin{cases} 1, & \text{if } x = 1 \text{ and } y = 0, \\ 0, & \text{otherwise.} \end{cases}$

Remark 2.4. (Baczyński et al. [2]) I satisfies the following properties (called left and right boundary condition, respectively):

(LB) $I(0,y) = 1$, for all $y \in [0,1]$,

(RB) $I(x,1) = 1$, for all $x \in [0,1]$.

Definition 2.5. (Baczyński et al. [2])

(i) A function $N : [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1, N(1) = 0$, and N is decreasing.

(ii) A fuzzy negation N is strong if it is an involution, i. e., $N(N(x)) = x$ for all $x \in [0,1]$.

(iii) Let T be a t-norm. A function $N_T : [0,1] \to [0,1]$ defined as

$$N_T(x) = \sup\{y \in [0,1] | T(x,y) = 0\}, \ x \in [0,1],$$

is called the natural negation of T or the negation induced by T.

(iv) Let $I \in FI$. A function $N_I : [0,1] \to [0,1]$ defined as $N_I(x) = I(x,0)$, $x \in [0,1]$, is called the natural negation of I.

Important fuzzy negations that will be used throughout this paper are given as follows:

- The classical fuzzy negation: $N_C(x) = 1 - x$ for all $x \in [0,1]$.
- The least fuzzy negation: $N_{D_1}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \in (0,1]. \end{cases}$
- The greatest fuzzy negation: $N_{D_2}(x) = \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{if } x = 1. \end{cases}$

Definition 2.6. (Baczyński et al. [2], Fodor [9], Grzegorzewski [10], Pradera et al. [24]) An operator $I : [0,1]^2 \to [0,1]$ is said to satisfy

(i) the left neutrality property, if $I(1,y) = y$ for all $y \in [0,1]$. (NP)

(ii) the identity principle, if $I(x,x) = 1$ for all $x \in [0,1]$. (IP)

(iii) the ordering property, if $I(x,y) = 1 \Leftrightarrow x \leq y$ for all $x, y \in [0,1]$. (OP)

(iv) the contrapositive symmetry with respect to a fuzzy negation N, if

$$I(x,y) = I(N(y), N(x)), \text{ for all } x, y \in [0,1]. \quad \text{(CP(N))}$$

(v) the exchange principle, if $I(x, I(y,z)) = I(y, I(x,z))$ for all $x, y, z \in [0,1]$. (EP)

(vi) the T-conditionality, if $T(x, I(x,y)) \leq y, x, y \in [0,1]$, where T is a t-norm. (TC)

(vii) the lowest truth property, $I(x,y) = 1$ if and only if $x = 0$ or $y = 1$. (LT)

(viii) the lowest falsity property, $I(x,y) = 0$ if and only if $x = 1$ and $y = 0$. (LF)

(ix) the boolean-like law, $I(x, I(y,x)) = 1$ for all $x, y \in [0,1]$. (BL)

Definition 2.7. (Baczyński et al. [2], Ouyang [21], Yager [26], Grzegorzewski [10])

(i) A function $I_{S,N} : [0,1]^2 \to [0,1]$ is called an (S,N)-implication if there exist a t-conorm S and a fuzzy negation N such that $I_{S,N}(x,y) = S(N(x), y)$ for all $x, y \in [0,1]$.

(ii) A function $I_T : [0,1]^2 \to [0,1]$ is called an R-implication if there exists a t-norm T such that $I_T(x,y) = \sup\{t \in [0,1] | T(x,t) \leq y\}$ for all $x, y \in [0,1]$.

(iii) Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1)=0$. The function $I_f : [0,1]^2 \to [0,1]$ defined by $I_f(x,y) = f^{-1}(x \cdot f(y))$ for all x, $y \in [0,1]$, with the convention $0 \cdot \infty = 0$, is called an f-generated implication.

(iv) Let $g : [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I_g : [0,1]^2 \to [0,1]$ defined by

$$I_g(x,y) = g^{-1}\left(\min\left(\frac{1}{x} \cdot g(y), g(1)\right)\right) \text{ for all } x,\ y \in [0,1],$$

with the convention $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a g-generated implication.

(v) Let C be a copula. A function $\widetilde{I}_C : [0,1]^2 \to [0,1]$ given by

$$\widetilde{I}_C(x,y) = C(x,y) - x + 1, \text{ for all } x, y \in [0,1],$$

is called a probabilistic $S-$implication.

(vi) Let C be a copula. A function $I_C : [0,1]^2 \to [0,1]$ given by

$$I_C(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{C(x,y)}{x}, & \text{if } x > 0, \end{cases}$$

is called a probabilistic implication.

Definition 2.8. (Baczyński et al. [6])

(i) A function $\varphi : [0,1] \to [0,1]$ is an automorphism if it is continuous and strictly increasing and satisfies the boundary conditions $\varphi(0) = 0$ and $\varphi(1) = 1$. By Φ we denote the family of all automorphism from [0, 1] to [0, 1].

(ii) The functions $f, g : [0,1]^n \to [0,1]$ are called $\Phi-$conjugate, if there exists a $\varphi \in \Phi$ such that $g = f_\varphi$, where $f_\varphi(x_1, x_2, \cdots, x_n) = \varphi^{-1}(f(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n)))$, for all $x_1, x_2, \ldots, x_n \in [0,1]$.

Proposition 2.9. (Baczyński et al. [5]) For any fuzzy implication I and any semicopula B the following conditions hold:

(i) J_{IB} is increasing with respect to the second variable.

(ii) $J_{IB}(1,0) = 0$, $J_{IB}(0,y) = 1$ for all $y \in [0,1]$.

(iii) $J_{IB}(1,y) = I(1,y)$ for all $y \in [0,1]$.

(iv) $J_{IB}(x,0) = I(x,0)$, i.e., $N_{J_{IB}}(x) = N_I(x)$ for all $x \in [0,1]$.

(v) $J_{IB}(x,1) = I(x,x)$ for all $x \in [0,1]$.

3. RELATIONSHIP BETWEEN THE PROPERTIES OF J_{IB} AND I

In this section, we discuss the relationship between the basic properties of J_{IB} and I.

Proposition 3.1. Let I be a fuzzy implication and B a semicopula. Then, the implication J_{IB} defined by (1) satisfies (NP) ((RB), respectively) if and only if I satisfies (NP) ((IP), respectively).

Proof. Straightforward from Proposition 2.9 (iii) and (v). □

Proposition 3.2. Let I be a fuzzy implication and B a semicopula. If the implication J_{IB} defined by (1) satisfies (IP)((OP), respectively), then I satisfies (IP).

Proof. J_{IB} satisfies (IP)((OP), respectively) $\Rightarrow J_{IB}(x,x) = 1 \Rightarrow I(x, B(x,x)) = 1$. Since $I(x,x) \geq I(x, B(x,x))$, then $I(x,x) = 1$ for all $x \in [0,1]$. □

The fact that the converse of Proposition 3.2 does not hold can be easily shown in the following example.

Example 3.3. Consider the Fodor implication I_{FD}:

$$I_{FD}(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \min(1-x, y), & \text{if } x > y. \end{cases}$$

It satisfies (IP). Taking a semicopula $B(x,y) = xy$. From (1) we get

$$J_{I_{FD}B}(0.5, 0.5) = I_{FD}(0.5, B(0.5, 0.5)) = I_{FD}(0.5, 0.25) = 0.25 \neq 1,$$

that is, $J_{I_{FD}B}$ does not satisfy (IP) ((OP), respectively).

Nevertheless, if I satisfies (OP), we have the following proposition.

Proposition 3.4. Let I be a fuzzy implication that satisfies (OP). Then J_{IB} satisfies (IP) if and only if $B = T_M$.

Proof. (Necessity). J_{IB} satisfies (IP) $\Rightarrow J_{IB}(x,x) = 1$ for all $x \in [0,1] \Rightarrow I(x, B(x,x)) = 1$ for all $x \in [0,1]$. Since I satisfies (OP), then $x = B(x,x)$ for all $x \in [0,1]$. Hence $B(x,y) = \min(x,y)$, i.e., $B = T_M$.

(Sufficiency). Obvious. □

Remark 3.5. (i) Let I be a fuzzy implication that satisfies (IP). If $B = T_M$, then J_{IB} satisfies (IP).

(ii) Let I be a fuzzy implication that satisfies (OP). Then J_{IB} satisfies (OP) if and only if $B = T_M$.

(iii) Let I_T be an R-implication of a left-continuous t-norm T. Then $J_{I_T B}$ satisfies (IP)((OP), respectively) if and only if $B = T_M$.

From Remark 3.5, we have the following problem.

Problem 3.6. Let I be a fuzzy implication that satisfies (IP) but not (OP), what conditions does B satisfy such that J_{IB} satisfies (IP)?

Proposition 3.7. Let $I \in FI$ and B be a semicopula, let S, $\widehat{S}$ be sets defined as

$$S = \{(x,y)|I(x,y) = 1,\ x, y \in [0,1]\},$$
$$\widehat{S} = \{(x,y)|y \geq B(x,x),\ x, y \in [0,1]\}.$$

Then J_{IB} satisfies (IP) if and only if $\widehat{S} \subseteq S$.

Proof. (Sufficiency) Let $x \in [0,1]$ and $(x, B(x,x)) \in \widehat{S}$. Since $\widehat{S} \subseteq S$, then $(x, B(x,x)) \in S$. Hence $J_{IB}(x,x) = I(x, B(x,x)) = 1$, i.e., J_{IB} satisfies (IP).

(Necessity) Let $(x,y) \in \widehat{S}$, then $y \geq B(x,x)$. Since J_{IB} satisfies (IP), then $I(x,y) \geq I(x, B(x,x)) = 1$, i.e., $(x,y) \in S$. Hence $\widehat{S} \subseteq S$. □

Remark 3.8. From Proposition 3.7, for a given semicopula B, there always exists a $I \in FI$ such that J_{IB} satisfies (IP). Moreover, I may not be unique.

Example 3.9. Let $B = T_P$. Note that $\widehat{S} = \{(x,y)|y \geq B(x,x),\ x, y \in [0,1]\}$, then

$$\hat{S} = \{(x,y)|y \geq x^2,\ x, y \in [0,1]\}.$$

Consider the following fuzzy implication

$$I_1(x,y) = \begin{cases} 1, & \text{if } y \geq x^2, \\ 1 - x + y, & \text{if } y < x^2. \end{cases}$$

Since $S_1 = \{(x,y)|I_1(x,y) = 1,\ x,y \in [0,1]\}$, then $S_1 = \{(x,y)|y \geq x^2,\ x,y \in [0,1]\}$. Thus $\widehat{S} \subseteq S_1$. Hence J_{I_1B} satisfies (IP) by Proposition 3.7.

Consider the following fuzzy implication

$$I_2(x,y) = \begin{cases} 1, & \text{if } y \geq x^3, \\ 1 - x + y, & \text{if } y < x^3. \end{cases}$$

It is easy to see that $S_2 = \{(x,y)|y \geq x^3,\ x,y \in [0,1]\}$, then $\widehat{S} \subseteq S_2$. Hence J_{I_2B} satisfies (IP) by Proposition 3.7.

Proposition 3.10. Let $I \in FI$ and B be a semicopula that is strictly increasing in second variable, and let S, $\widehat{S}$ be sets defined as

$$\begin{aligned} S &= \{(x,y)|I(x,y) = 1, x,y \in [0,1]\}, \\ \widehat{S} &= \{(x,y)|y \geq B(x,x), x,y \in [0,1]\}. \end{aligned}$$

Then J_{IB} satisfies (OP) if and only if $\widehat{S} = S$.

Proof. (Sufficiency) Let x, $y \in [0,1]$. If $x = 0$, then $J_{IB}(x,y) = 1 \Rightarrow x \leq y$.

If $x > 0$, then
$$\begin{aligned} J_{IB}(x,y) = 1 &\Rightarrow I(x,B(x,y)) = 1 \\ &\Rightarrow (x,B(x,y)) \in S \\ &\Rightarrow (x,B(x,y)) \in \widehat{S} \\ &\Rightarrow B(x,y) \geq B(x,x). \end{aligned}$$
Note that B is strictly increasing in second variable, thus $x \leq y$.

On the other hand, if $x \leq y$, then $J_{IB}(x,y) = I(x,B(x,y)) \geq I(x,B(x,x))$. Since $B(x,x) \geq B(x,x)$, then $(x,B(x,x)) \in \widehat{S}$. By $\widehat{S} = S$, we have $(x,B(x,x)) \in S$, i.e., $I(x,B(x,x)) = 1$. Hence $J_{IB}(x,y) = 1$.

(Necessity) Let J_{IB} satisfy (OP), follows that J_{IB} satisfies (IP), then $\widehat{S} \subseteq S$ by Proposition 3.7. If $(x,y) \in S$, then $I(x,y) = 1$. Assume that $(x,y) \in \overline{\widehat{S}}$, where $\overline{\widehat{S}}$ is the complement of $\widehat{S}$ in $[0,1]^2$, then $y < B(x,x) \leq x$. Hence

$$I(x,y) \leq I(x,B(x,x)) \leq I(x,B(x,y)) = J_{IB}(x,y).$$

Note that $I(x,y) = 1$, then $J_{IB}(x,y) = 1$. Since J_{IB} satisfies (OP), then $x \leq y$, this contradicts the fact that $y < B(x,x) \leq x$. Hence $S \subseteq \widehat{S}$, thus $S = \widehat{S}$. □

Lemma 3.11. (Baczyński et al. [2]) Let $I \in FI$ and N be a fuzzy negation. If I satisfies (NP) and (CP(N)), then $N = N_I$ is a strong fuzzy negation.

Proposition 3.12. Let I be a fuzzy implication and B a semicopula. If the implication J_{IB} satisfies (CP(N)), then

(i) I satisfies (IP).

(ii) J_{IB} is a fuzzy implication.

(iii) If I satisfies (NP), then $N = N_I$ is a strong negation.

(iv) If I satisfies (OP), then $x = B(x, y) \Leftrightarrow N(y) = B(N(y), N(x))$, $x, y \in [0, 1]$.

Proof.

(i) Assume that J_{IB} satisfies (CP(N)), then

$$I(x, B(x, y)) = I(N(y), B(N(y), N(x))) \text{ for all } x, y \in [0, 1].$$

Taking $y = 1$, then $I(x, x) = I(0, B(0, N(x))) = 1$ for all $x \in [0, 1]$, i. e., I satisfies (IP).

(ii) From [2] Lemma 1.5.4 (ii) and Proposition 2.9 (i), we have, J_{IB} satisfies (I1), then J_{IB} is a fuzzy implication.

(iii) Straightforward from Lemma 3.11, Proposition 2.9 (iii) and (iv).

(iv) Suppose that there exists some $(x_0, y_0) \in [0, 1]^2$ such that $x_0 = B(x_0, y_0)$ and $N(y_0) > B(N(y_0), N(x_0))$, then

$$J_{IB}(x_0, y_0) = I(x_0, B(x_0, y_0)) = 1,$$
$$J_{IB}(N(y_0), N(x_0)) = I(N(y_0), B(N(y_0), N(x_0))) < 1.$$

Hence $J_{IB}(x_0, y_0) > J_{IB}(N(y_0), N(x_0))$. This contradicts the fact that J_{IB} satisfies (CP(N)). Thus completes the proof. □

Remark 3.13. (i) Let I be a fuzzy implication that satisfies (NP). If N_I, the natural negation of I, is not strong, by Proposition 3.12 (iii), then J_{IB} does not satisfy (CP) with respect to any fuzzy negation N and any semicopula B. Therefore,

- for the (S, N)-implication $I_{S,N}$: if N is not strong, then $J_{I_{S,N}B}$ does not satisfy (CP) with respect to any fuzzy negation N and any semicopula B.
- for the R-implication I_T: if N_T, the natural negation of T, is not strong (strict, continuous, strictly decreasing, respectively), then $J_{I_T B}$ does not satisfy (CP) with respect to any fuzzy negation N and any semicopula B.
- for the g-implication I_g: since I_g satisfies (NP) and $N_{I_g} = N_{D_1}$, then $J_{I_g B}$ does not satisfy (CP) with respect to any fuzzy negation N and any semicopula B.
- for the probabilistic implication I_C: since I_C satisfies (NP) and $N_{I_C} = N_{D_1}$, then $J_{I_C B}$ does not satisfy (CP) with respect to any fuzzy negation N and any semicopula B.

(ii) For the f-implication I_f: note that I_f does not satisfy (IP), then $J_{I_f B}$ does not satisfy (CP) with respect to any fuzzy negation N and any semicopula B.

(iii) Let $I \in FI$. Even if I satisfies (IP) and (CP), J_{IB} may not satisfy (CP).

Example 3.14. Consider the Lukasiewicz implication I_{LK}. It satisfies (IP) and (CP). Let $B = T_D$.

From (1) we get

$$J_{I_{LK}B}(x,y) = \begin{cases} y, & \text{if } x = 1, \\ 1, & \text{if } y = 1, \\ 1-x, & \text{otherwise.} \end{cases}$$

Note that $J_{I_{LK}B}(0.9, 0.9) = 0.1 < 0.9 = J_{I_{LK}B}(1, 0.9)$, i. e., $J_{I_{LK}B}$ does not satisfy (I1), then $J_{I_{LK}B}$ does not satisfy (CP) with respect to any fuzzy negation N by Lemma 1.5.4 in [2].

Problem 3.15. Let I be a fuzzy implication that satisfies (IP) and (NP), but not (CP), and its natural negation N_I is strong. Does J_{IB} not satisfy (CP)?

Unfortunately, the answer is negative.

Example 3.16. Consider the following fuzzy implication

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \max(\frac{y}{x}, 1-x), & \text{if } x > y. \end{cases}$$

It satisfies (IP) and (NP), but not (CP).

Taking $B = T_P$, from (1) we get

$$J_{IB}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ \max(y, 1-x), & \text{otherwise.} \end{cases}$$

Obviously, $J_{IB}(x,y) = J_{IB}(N_I(y), N_I(x))$, i. e., J_{IB} satisfies (CP) with respect to N_I.

Proposition 3.17. Let I be a fuzzy implication that satisfies (IP) and (CP), then there exists a semicopula B such that J_{IB} satisfies (CP).

Proof. Consider $B = T_M$. Let $x, y \in [0,1]$. If $x \le y$, then $I(x,y) = 1 = I(x,x) = I(x, T_M(x,y)) = J_{IB}(x,y)$ by I satisfies (IP). If $x > y$, then $I(x,y) = I(x, T_M(x,y)) = J_{IB}(x,y)$. Hence $J_{IB} = I$. Note that I satisfies (CP), then J_{IB} satisfies (CP). $\square$

Proposition 3.18. Let $f : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function with $f(1) = 0$, let T be a continuous Archimedean t-norm with additive generator f and I_T its R-implication. Then the following statements are equivalent:

(i) $J_{I_T B}$ satisfies (CP) with respect to a fuzzy negation N.

(ii) $N = N_{I_T}$, and the triple (f, B, N) satisfies the following equation:

$$f(B(x,y)) - f(B(N(y), N(x))) = f(x) - f(N(y)).$$

Proof. From [2] Theorem 2.1.5 and Definition 2.7 (ii), we get

$$I_T(x,y) = f^{-1}(\max(f(y) - f(x), 0)) \text{ for all } x, y \in [0,1],$$

then the rest of the proof comes directly from calculation. □

Proposition 3.19. Let I be a continuous fuzzy implication that satisfies (OP) and (EP). Then the following statements are equivalent:

(i) J_{IB} satisfies (CP) with respect to a fuzzy negation N.

(ii) There exists a $\varphi \in \Phi$, such that $N(x) = \varphi^{-1}(1 - \varphi(x))$ and

$$\varphi(B(x,y)) - \varphi(B(N(y), N(x))) = \varphi(x) - \varphi(N(y)) \text{ for all } x, y \in [0,1].$$

Proof. The proof comes directly from calculation. □

Proposition 3.20. Let B be a semicopula and N a fuzzy negation, let C be a copula and $\widetilde{I}_C$ a probabilistic $S-$implication, let $J_{\widetilde{I}_C B}$ be an implication defined by (1). Then the following statements are equivalent:

(i) $J_{\widetilde{I}_C B}$ satisfies (CP) with respect to N.

(ii) $\widetilde{I}_C = I_{LK}$, $N = N_C$ and $B(x,y) - B(N(y), N(x)) = x - N(y)$ for all x, $y \in [0,1]$.

Proof. The proof comes directly from calculation. □

Proposition 3.21. Let $I \in FI$ and B be a semicopula. If the implication J_{IB} satisfies (EP), then I satisfies (IP).

Proof. J_{IB} satisfies (EP) $\Rightarrow J_{IB}(x, J_{IB}(y,z)) = J_{IB}(y, J_{IB}(x,z))$ for all $x,y,z \in [0,1]$.

Taking $x = 0$, then $J_{IB}(0, J_{IB}(y,z)) = J_{IB}(y, J_{IB}(0,z))$

$$\begin{aligned} &\Rightarrow I(0, B(0, J_{IB}(y,z))) = J_{IB}(y, I(0, B(0,z)) \\ &\Rightarrow 1 = J_{IB}(y,1) \\ &\Rightarrow I(y, B(y,1)) = 1 \\ &\Rightarrow I(y,y) = 1 \text{ for all } y \in [0,1]. \end{aligned}$$

That is, I satisfies (IP). □

Remark 3.22. (i) Let N_I, the natural negation of I, be a continuous negation. If I does not satisfy (NP), then J_{IB} does not satisfy (EP).

In fact, suppose that J_{IB} satisfies (EP), then for all x, y, $z \in [0,1]$, we get

$$J_{IB}(x, J_{IB}(y,z)) = J_{IB}(y, J_{IB}(x,z)).$$

Taking $z = 0$, $x = 1$, then $I(1, N_I(y)) = I(y,0) = N_I(y)$. A contradiction to the fact that I does not satisfy (NP).

(ii) Let I be a fuzzy implication that satisfies (IP). Even if I does not satisfy (EP), J_{IB} may satisfy (EP).

(iii) Let I be a fuzzy implication that satisfies (IP). Even if I satisfies (EP), J_{IB} may not satisfy (EP).

Example 3.23. Consider the Rescher implication I_{RS}. It satisfies (IP) but not (EP). Let $B = T_P$. From (1) we get

$$J_{I_{RS}B}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let x, y, $z \in [0,1]$. By calculations, we have

$$J_{I_{RS}B}(x, J_{I_{RS}B}(y,z)) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \text{ or } z = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$J_{I_{RS}B}(y, J_{I_{RS}B}(x,z)) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \text{ or } z = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $J_{I_{RS}B}(x, J_{I_{RS}B}(y,z)) = J_{I_{RS}B}(y, J_{I_{RS}B}(x,z))$, i. e., $J_{I_{RS}B}$ satisfies (EP).

On the other hand, consider the Lukasiewicz implication I_{LK}. It satisfies (IP) and (EP).

Let $B = T_D$. Taking $x = 0.9$, $y = 0.5$ and $z = 0.3$, we have

$$J_{I_{LK}B}(0.5, 0.3) = I_{LK}(0.5, T_D(0.5, 0.3)) = I_{LK}(0.5, 0) = 0.5,$$
$$J_{I_{LK}B}(0.9, 0.3) = I_{LK}(0.9, T_D(0.9, 0.3)) = I_{LK}(0.9, 0) = 0.1.$$

Thus

$$J_{I_{LK}B}(0.9, J_{I_{LK}B}(0.5, 0.3)) = J_{I_{LK}B}(0.9, 0.5) = I_{LK}(0.9, T_D(0.9, 0.5)) = 0.1,$$
$$J_{I_{LK}B}(0.5, J_{I_{LK}B}(0.9, 0.3)) = J_{I_{LK}B}(0.5, 0.1) = I_{LK}(0.5, T_D(0.5, 0.1)) = 0.5.$$

This fact shows that $J_{I_{LK}B}$ does not satisfy (EP).

Proposition 3.24. Let $I \in FI$ and N_I, the natural negation of I, be a continuous negation. Then J_{IB} satisfies (EP) if and only if J_{IB} is an (S, N_I)-implication.

Proof. The proof comes directly from Theorem 2.4.10, Remark 2.4.13 (ii) in [2] and Proposition 2.9 (i). □

Corollary 3.25. Let $I \in FI$ and N_I, the natural negation of I, be a strong negation. If the implication J_{IB} satisfies (EP), then J_{IB} satisfies (CP) with respect to N_I.

Proposition 3.26. Let I be a fuzzy implication and B a semicopula, and let T, $T^{'}$ be t-norms such that $T^{'} \leq T$. If I satisfies (TC) with T, then J_{IB} satisfies (TC) with $T^{'}$.

P r o o f. $T'(x, J_{IB}(x,y)) \le T(x, J_{IB}(x,y)) = T(x, I(x, B(x,y))) \le T(x, I(x,y)) \le y$ for all $x, y \in [0,1]$. $\square$

Remark 3.27. (i) Note that $N_{J_{IB}} = N_I$, if $N_I > N_T$, the nature negation of T, then, for any semicopula B, J_{IB} does not satisfy (TC) with t-norm T by Proposition 7.4.3 in [2].

(ii) Let $N_I \le N_T$, and let A, $\overline{A}$ be sets defined as

$$A = \{(x,y) | T(x, I(x,y)) \le y \text{ for all } x, y \in [0,1]\},$$
$$\overline{A} = \{(x,y) | T(x, I(x,y)) > y \text{ for all } x, y \in [0,1]\}.$$

If $\overline{A} = \emptyset$, J_{IB} satisfies (TC) with T. If $\overline{A} \ne \emptyset$, then J_{IB} satisfies (TC) with T if only if $(x, B(x,y)) \in A$ for all $(x,y) \in \overline{A}$.

(iii) Even if I does not satisfy (TC) with a t-norm T, J_{IB} may satisfy (TC) with T.

Example 3.28. Let $I = I_{LK}$ and $T = T_{nM}$. Since $T_{nM}(0.6, I(0.6, 0.3)) = 0.6 > 0.3$, then I_{LK} does not satisfy (TC) with T_{nM}. However, taking $B = T_D$, we get

$$J_{IB}(x,y) = J_{I_{LK}T_D}(x,y) = \begin{cases} 1 - x, & \text{if } x,\, y \in [0,1), \\ y, & \text{if } x = 1, \\ 1, & \text{if } y = 1. \end{cases}$$

For $x, y \in [0,1)$, $T_{nM}(x, J_{IB}(x,y)) = 0 \le y$. For $x = 1$ and $y \in [0,1]$, $T_{nM}(x, J_{IB}(x,y)) = y \le y$. For $x \in [0,1]$ and $y = 1$, $T_{nM}(x, J_{IB}(x,y)) = x \le 1 = y$. Hence $T_{nM}(x, J_{IB}(x,y)) \le y$ for all $x, y \in [0,1]$, i. e., J_{IB} satisfies (TC) with T_{nM}.

Proposition 3.29. Let $I \in FI$ and B be a semicopula. Then the implication J_{IB} satisfies (LF) if and only if I satisfies (LF).

P r o o f. Let J_{IB} satisfy (LF), i. e., $J_{IB}(x,y) = 0 \Leftrightarrow x = 1$ and $y = 0$. Assume that $I(x,y) = 0$. Note that $J_{IB} \le I$, then $J_{IB}(x,y) = 0$, thus $x = 1$ and $y = 0$. Reversely, assume that $x = 1$ and $y = 0$, then $I(x,y) = 0$. Thus $I(x,y) = 0 \Leftrightarrow x = 1$ and $y = 0$, i. e., I satisfies (LF).

On the other hand, let I satisfy (LF), i. e., $I(x,y) = 0 \Leftrightarrow x = 1$ and $y = 0$. Assume that $J_{IB}(x,y) = 0$, i. e., $I(x, B(x,y)) = 0$, then $x = 1$ and $B(x,y) = 0$. Hence $x = 1$ and $y = 0$. Reversely, assume that $x = 1$ and $y = 0$, then $J_{IB}(x,y) = 0$. Thus $J_{IB}(x,y) = 0 \Leftrightarrow x = 1$ and $y = 0$, i. e., J_{IB} satisfies (LF). $\square$

Proposition 3.30. Let $I \in FI$ and B be a semicopula. If J_{IB} satisfies (LT), then I satisfies (IP).

P r o o f. Let J_{IB} satisfy (LT), then $J_{IB}(x,1) = 1$ for all $x \in [0,1]$, i. e., $I(x, B(x,1)) = 1$. Thus $I(x,x) = 1$ for all $x \in [0,1]$. Hence I satisfies (IP). $\square$

Let $FI_{(\mathrm{IP})}$ be the set of all fuzzy implications that satisfies (IP).

Remark 3.31. (i) Let I be a fuzzy implication that satisfies (LT). Then J_{IB} does not satisfy (LT) for any semicopula B. Actually, if I satisfies (LT), then $I(x,x) \neq 1$ for $x \in (0,1)$. Hence, I does not satisfy (IP). Thus J_{IB} does not satisfy (LT) by Proposition 3.30.

(ii) There exists some $I \in FI_{(\mathrm{IP})}$ such that J_{IB} satisfy (LT). See Example 3.32 (i).

(iii) There exists some $I \in FI_{(\mathrm{IP})}$ such that J_{IB} does not satisfy (LT) for any semicopula B. See Example 3.32 (ii).

Example 3.32. (i) Consider the Lukasiewicz implication I_{LK}. It satisfies (IP) but not (LT). Taking $B = T_P$, then the implication $J_{I_{LK}B}$ has the following form:

$$J_{I_{LK}B}(x,y) = 1 - x + xy, \text{ for all } x,\ y \in [0,1].$$

Since $J_{I_{LK}B}(x,y) = 1 \Leftrightarrow 1 - x + xy = 1 \Leftrightarrow x = 0$ or $y = 1$, then $J_{I_{LK}B}$ satisfies (LT).

(ii) Consider the Weber implication I_{WB}. It satisfies (IP) but not (LT). For any semicopula B, by calculations, we get $J_{I_{WB}B} = I_{WB}$. This shows that there exists some $I \in FI_{(\mathrm{IP})}$ such that J_{IB} does not satisfy (LT) for any semicopula B.

Problem 3.33. Let I be a fuzzy implication that satisfies (IP), what conditions does semicopula B have to ensure that J_{IB} satisfy (LT)?

In the following, we give a partial answer.

Proposition 3.34. Let I be a fuzzy implication that satisfies (OP), and let B be a semicopula. If $x = B(x,y) \Leftrightarrow x = 0$ or $y = 1$, then J_{IB} satisfies (LT).

Proof. Let I satisfy (OP), then $J_{IB}(x,y) = 1 \Leftrightarrow I(x, B(x,y)) = 1 \Leftrightarrow x = B(x,y) \Leftrightarrow x = 0$ or $y = 1$. $\square$

Proposition 3.35. Let $I \in FI$ and B be a semicopula. If J_{IB} satisfies (BL), then I satisfies (IP) and (BL).

Proof. Let J_{IB} satisfy (BL), then $J_{IB}(x, J_{IB}(y,x)) = 1$ for all $x,y \in [0,1]$, i. e.,

$$I(x, B(x, I(y, B(y,x)))) = 1 \text{ for all } x,y \in [0,1].$$

Taking $y = 0$, then $I(x,x) = 1$ for all $x \in [0,1]$. That is, I satisfies (IP).

Since $I(x, I(y,x)) \geq I(x, I(y, B(y,x))) \geq I(x, B(x, I(y, B(y,x))))$, then I satisfies (BL). $\square$

Remark 3.36. Let $I \in FI$. Even if I satisfies (IP) and (BL), J_{IB} may not satisfy (BL).

Example 3.37. Consider the following fuzzy implication

$$I(x,y)=\begin{cases}1, & \text{if } x^2\le y,\\ y, & \text{if } x^2>y.\end{cases}$$

Obviously, it satisfies (IP). Let $x,\ y\in[0,1]$. If $y^2\le x$, then $I(x,I(y,x))=I(x,1)=1$. If $y^2>x$, then $I(x,I(y,x))=I(x,x)=1$. Hence $I(x,I(y,x))=1$ for all $x,\ y\in[0,1]$. That is, I satisfies (BL).

Let semicopula $B=T_D$. From (1) we get

$$J_{IB}(x,y)=\begin{cases}1, & \text{if } x=0 \text{ or } y=1,\\ y, & \text{if } x=1,\\ 0, & \text{otherwise.}\end{cases}$$

Since $J_{IB}(0.5,J_{IB}(0.5,0.5))=J_{IB}(0.5,0)=0\ne1$, i.e., J_{IB} does not satisfy (BL).

Lemma 3.38. Let $I\in FI$. If I satisfies (IP) and (NP), then I satisfies (BL).

Proof. Let $x,\ y\in[0,1]$. If $x\ge y$, then $I(y,x)=1$ by (IP). Hence $I(x,I(y,x))=1$. If $x<y$, since I satisfies (NP), then $I(y,x)\ge I(1,x)=x$. Hence $I(x,I(y,x))\ge I(x,x)=1$. Thus $I(x,I(y,x))=1$ for all $x,\ y\in[0,1]$, i.e., I satisfies (BL). □

Proposition 3.39. Let B be a semicopula and I a fuzzy implication that satisfies (NP). Let S, $\widehat{S}$ be sets defined by

$$S=\{(x,y)|I(x,y)=1,\ x,y\in[0,1]\},$$
$$\widehat{S}=\{(x,y)|y\ge B(x,x),\ x,y\in[0,1]\}.$$

If $\widehat{S}\subseteq S$, then J_{IB} satisfies (BL).

Proof. Let I satisfy (NP), then J_{IB} satisfies (NP) by Proposition 3.1. Let $\widehat{S}\subseteq S$, then J_{IB} satisfies (IP) by Proposition 3.7. Thus J_{IB} satisfies (BL) by Lemma 3.38. □

Proposition 3.40. Let I be a fuzzy implication that satisfies (OP) and (NP), and let B be a semicopula. Then J_{IB} satisfies (BL) if and only if $B=T_M$.

Proof. (Necessity) Let J_{IB} satisfy (BL), then $J_{IB}(x,J_{IB}(y,x))=1$ for all $x,y\in[0,1]$, i.e.,

$$I(x,B(x,I(y,B(y,x))))=1, \text{ for all } x,y\in[0,1].$$

Since I satisfies (OP), then $x=B(x,I(y,B(y,x)))$ for all $x,y\in[0,1]$. Taking $y=1$, then $x=B(x,I(1,x))$. Note that I satisfies (NP), hence $x=B(x,x)$ for all $x\in[0,1]$, i.e., $B=T_M$.

(Sufficiency) Obvious. □

4. CONDITIONS FOR J_{IB} BEING A FUZZY IMPLICATION

4.1. Sufficient conditions

In this section, the sufficient conditions such that J_{IB} is a fuzzy implication are studied.

Definition 4.1. (Nelsen [19]) A function $I : [0,1]^2 \to [0,1]$ is 2-increasing if it satisfies, for all x_1, x_2, y_1, $y_2 \in [0,1]$ with $x_1 < x_2$ and $y_1 < y_2$, the following inequation:

$$I(x_1, y_1) + I(x_2, y_2) \geq I(x_1, y_2) + I(x_2, y_1).$$

Lemma 4.2. Let $I : [0,1]^2 \to [0,1]$ be a function. If I satisfies the following conditions:

(i) I is 2-increasing,

(ii) $I(0,x) = 1$, $I(x,1) = 1$, $I(1,0) = 0$.

then I is a fuzzy implication.

Proof. It suffices to prove that I satisfies (I1) and (I2). Let x_1, x_2, y_1, $y_2 \in [0,1]$ such that $x_1 < x_2$, $y_1 < y_2$. Since I is 2-increasing, then

$$I(x_1, y_1) + I(x_2, y_2) \geq I(x_1, y_2) + I(x_2, y_1).$$

Taking $y_2 = 1$, then $I(x_1, y_1) \geq I(x_2, y_1)$ for $x_1 < x_2$ and x_1, x_2, $y_1 \in [0,1]$. Therefore, I satisfies (I1). Taking $x_1 = 0$, then $I(x_2, y_2) \geq I(x_2, y_1)$ for $y_1 < y_2$ and x_2, y_1, $y_2 \in [0,1]$, thus I satisfies (I2). □

Remark 4.3. (i) A fuzzy implication I may not be 2-increasing. See Example 4.4.

(ii) A fuzzy implication I is 2-increasing, which may or not satisfy (IP). See Example 4.5.

Example 4.4. Consider the Rescher implication I_{RS}. Since $I_{RS}(0.7, 0.6)+I_{RS}(0.8, 0.75) = 0$, $I_{RS}(0.7, 0.75)+I_{RS}(0.8, 0.6) = 1$, then $I_{RS}(0.7, 0.6)+I_{RS}(0.8, 0.75) < I_{RS}(0.7, 0.75) +I_{RS}(0.8, 0.6)$. That is, I_{RS} is not 2-increasing.

Example 4.5. Let x_1, x_2, y_1, $y_2 \in [0,1]$ and $x_1 < x_2$, $y_1 < y_2$. Consider the Reichenbach implication I_{RC}:

$$I_{RC}(x, y) = 1 - x + xy \text{ for all } x, y \in [0,1].$$

Since $I_{RC}(x_1, y_1) + I_{RC}(x_2, y_2) - (I_{RC}(x_1, y_2) + I_{RC}(x_2, y_1)) = (x_1 - x_2)(y_1 - y_2) \geq 0$, then I_{RC} is 2-increasing.

Consider the Weber implication I_{WB}. If $x_2 < 1$, then $I_{WB}(x_1, y_1) + I_{WB}(x_2, y_2) = 1 = I_{WB}(x_1, y_2) + I_{WB}(x_2, y_1)$. If $x_2 = 1$, then $I_{WB}(x_1, y_1) + I_{WB}(x_2, y_2) = 1 + y_2 > 1 + y_1 = I_{WB}(x_1, y_2) + I_{WB}(x_2, y_1)$. Hence

$$I_{WB}(x_1, y_1) + I_{WB}(x_2, y_2) \geq I_{WB}(x_1, y_2) + I_{WB}(x_2, y_1),$$

that is, I_{WB} is 2-increasing. However, I_{RC} does not satisfy (IP), I_{WB} satisfies (IP).

Theorem 4.6. Let I be a fuzzy implication that satisfies (IP). If J_{IB} is 2-increasing, then J_{IB} is a fuzzy implication.

Proof. Straightforward from Proposition 2.9 (ii), (v), and Lemma 4.2. $\square$

Corollary 4.7. Let I be a fuzzy implication that satisfies both (IP) and (NP), and let $N_I = N_C$. If J_{IB} is 2-increasing, then J_{IB} is a probabilistic S-implication.

Proof. Let $C : [0,1]^2 \to [0,1]$ be a function defined by

$$C(x,y) = J_{IB}(x,y) + x - 1, \text{ for all } x, y \in [0,1].$$

It suffices to show that C is a copula.

Assume that J_{IB} is 2-increasing. Since I satisfies (IP), then J_{IB} is a fuzzy implication by Theorem 4.6. Thus, for all $x \in [0,1]$, we get

$$\begin{aligned} C(x,1) &= J_{IB}(x,1) + x - 1 = 1 + x - 1 = x, \\ C(0,x) &= J_{IB}(0,x) + 0 - 1 = 1 + 0 - 1 = 0. \end{aligned}$$

Note that I satisfies (NP) and $N_I = N_C$, by Proposition 2.9 (iii), (iv), for all $x \in [0,1]$, we get

$$\begin{aligned} C(1,x) &= J_{IB}(1,x) + 1 - 1 = x, \\ C(x,0) &= J_{IB}(x,0) = N_I(x) + x - 1 = 1 - x + x - 1 = 0. \end{aligned}$$

On the other hand, let x_1, x_2, y_1, $y_2 \in [0,1]$ and $x_1 < x_2$, $y_1 < y_2$. Note that J_{IB} is 2-increasing, i. e.,

$$J_{IB}(x_1,y_1) + J_{IB}(x_2,y_2) \geq J_{IB}(x_1,y_2) + J_{IB}(x_2,y_1),$$

then $C(x_1,y_1) + C(x_2,y_2) \geq C(x_1,y_2) + C(x_2,y_1)$, i. e., C is 2-increasing.

Taking $y_1 = 0$, then $C(x_2,y_2) \geq C(x_1,y_2)$, i. e., C is increasing in the first variable. Taking $x_1 = 0$, then $C(x_2,y_2) \geq C(x_2,y_1)$, i. e., C is increasing in the second variable.

Hence C is a copula. $\square$

Remark 4.8. Let I be a fuzzy implication that satisfies (IP). Even if I is 2-increasing, J_{IB} may not be 2-increasing.

Example 4.9. Consider the Lukasiewicz implication I_{LK}. It satisfies (IP), and it is 2-increasing. Let $B = T_D$. From (1) we get

$$J_{I_{LK}B}(x,y) = \begin{cases} 1 - x, & \text{if } x,\ y \in [0,1), \\ y, & \text{if } x = 1, \\ 1, & \text{if } y = 1. \end{cases}$$

Since $J_{I_{LK}B}(0.9, 0.3) + J_{I_{LK}B}(1,1) = 1.1 < 1.3 = J_{I_{LK}B}(0.9,1) + J_{I_{LK}B}(1,0.3)$, then $J_{I_{LK}B}$ is not 2-increasing.

An open problem: Let I be a fuzzy implication that satisfies (IP). If I is 2-increasing, what conditions does the semicopula B have to ensure that J_{IB} be 2-increasing?

Definition 4.10. (Bustince et al. [7]) Let (r_1, r_2) be a real 2-dimensional vector, $(r_1, r_2) \neq (0, 0)$. A function $I : [0,1]^2 \to [0,1]$ is (r_1, r_2)-decreasing if it satisfies, for every point $(x_1, x_2) \in [0,1]^2$ and every real number $c \in R^+$ such that $(x_1 + cr_1, x_2 + cr_2) \in [0,1]^2$, the following inequation:

$$I(x_1, x_2) \geq I(x_1 + cr_1, x_2 + cr_2).$$

Theorem 4.11. Let I be a fuzzy implication that satisfies (IP), and let B be a semicopula. If there exist r_1, $r_2 \in R^+$, such that J_{IB} is (r_1, r_2)-decreasing, then J_{IB} is a fuzzy implication.

Proof. Let J_{IB} be a (r_1, r_2)-decreasing function, and let x, $y \in [0,1]$. Then, for all $c \in R^+$, such that $(x + cr_1, y + cr_2) \in [0,1]^2$, we get $J_{IB}(x, y) \geq J_{IB}(x + cr_1, y + cr_2)$, i. e.,

$$I(x, B(x, y)) \geq I(x + cr_1, B(x + cr_1, y + cr_2)).$$

Since $B(x + cr_1, y + cr_2) \geq B(x + cr_1, y)$, then $I(x, B(x, y)) \geq I(x + cr_1, B(x + cr_1, y))$, i. e., $J_{IB}(x, y) \geq J_{IB}(x + cr_1, y)$. Hence J_{IB} is a fuzzy implication. □

In the following, we study the necessary and sufficient conditions such that J_{IB} is a fuzzy implication when $I = I_{RS}$ (I_{LK}, I_{WB}, I_G, respectively).

Proposition 4.12. Let $I = I_{RS}$ and B be a semicopula. If $|B(x_2, y) - B(x_1, y)| \leq |x_2 - x_1|$ for all x_1, x_2, $y \in [0,1]$, then J_{IB} is a fuzzy implication.

Proof. Let x_1, x_2, $y \in [0,1]$ and $x_1 < x_2$. It suffices to prove that $J_{IB}(x_1, y) \geq J_{IB}(x_2, y)$.

Since $I = I_{RS}$, then

$$J_{IB}(x, y) = \begin{cases} 1, & \text{if } x = B(x, y), \\ 0, & \text{if } x > B(x, y). \end{cases}$$

If $x_1 = 0$ or $y = 1$, then $J_{IB}(x_1, y) = 1 \geq J_{IB}(x_2, y)$.

If $x_1 > 0$, $y < 1$ and $x_1 = B(x_1, y)$, then $J_{IB}(x_1, y) = 1 \geq J_{IB}(x_2, y)$.

If $x_1 > 0$, $y < 1$ and $x_1 > B(x_1, y)$, then $J_{IB}(x_1, y) = 0$. Since $|B(x_2, y) - B(x_1, y)| \leq |x_2 - x_1|$, then $B(x_2, y) - B(x_1, y) \leq x_2 - x_1$. Thus

$$x_2 \geq (B(x_2, y) - B(x_1, y)) + x_1 = B(x_2, y) + (x_1 - B(x_1, y)) > B(x_2, y).$$

Hence $J_{IB}(x_2, y) = 0$. Therefore $J_{IB}(x_1, y) \geq J_{IB}(x_2, y)$. □

Proposition 4.13. Let $I = I_{LK}$ and B be a semicopula. Then J_{IB} is a fuzzy implication if and only if $|B(x_2, y) - B(x_1, y)| \leq |x_2 - x_1|$ for all x_1, x_2, $y \in [0,1]$.

Proof. Straightforward from calculation. □

Proposition 4.14. If $I = I_{WB}$ (I_G, respectively), then J_{IB} is a fuzzy implication for any semicopula B.

Proof. Assume that $I = I_{WB}$. From (1) we get

$$\begin{aligned} J_{IB}(x,y) &= \begin{cases} 1, & \text{if } x \in [0,1), \\ B(x,y), & \text{if } x = 1, \end{cases} \\ &= \begin{cases} 1, & \text{if } x \in [0,1), \\ y, & \text{if } x = 1, \end{cases} \\ &= I_{WB}(x,y). \end{aligned}$$

Similarly, assume that $I = I_G$, then $J_{IB} = I_G$. Hence J_{IB} is a fuzzy implication for any semicopula B. □

Remark 4.15. (i) From Proposition 4.14, there exists some $I \in FI$ such that J_{IB} is a fuzzy implication for any semicopula B. Hence the conclusion in [5] (Line 18, page 146) is incorrect.

(ii) J_{IB} may not be 2-increasing even if I is 2-increasing and satisfies (IP) by Propositions 4.13, 4.14 and Theorem 4.6.

Inspired by Proposition 4.14, for any semicopula B, it is interesting to find which fuzzy implication I satisfies the equation $I = J_{IB}$.

Proposition 4.16. Let I be a fuzzy implication that satisfies (IP), then $I = J_{IB}$ for any semicopula B if, and only if there exists an increasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, such that

$$I(x,y) = \begin{cases} 1, & \text{if } x < 1, \\ f(y), & \text{if } x = 1. \end{cases}$$

Proof. (Necessity) Firstly, consider the following semicopula B:

$$B(x,y) = \begin{cases} 0, & \text{if } x, y \in [0,1), \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Let $I = J_{IB}$, then

$$I(x,y) = \begin{cases} I(x,0), & \text{if } x, y \in [0,1), \\ I(1,y), & \text{if } x = 1, \\ 1, & \text{if } y = 1. \end{cases} \tag{2}$$

Secondly, let us consider the semicopula $B(x,y) = \min(x,y)$. By $I = J_{IB}$, we obtain

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ I(x,y), & \text{if } x > y. \end{cases} \tag{3}$$

From (2) and (3), we obtain $I(x,0)=1$ for all $x\in[0,1)$. Hence

$$I(x,y)=\begin{cases}1, & \text{if } x<1,\\ I(1,y), & \text{if } x=1.\end{cases}$$

Let $f(y)=I(1,y)$. Obviously, $f:[0,1]\to[0,1]$ is an increasing function with $f(0)=0$ and $f(1)=1$. Thus

$$I(x,y)=\begin{cases}1, & \text{if } x<1,\\ f(y), & \text{if } x=1.\end{cases}$$

(Sufficiency) The proof comes directly from calculation. □

Remark 4.17. (i) Let $f:[0,1]\to[0,1]$ be an increasing function with $f(0)=0$ and $f(1)=1$, and let I be a fuzzy implication defined by

$$I(x,y)=\begin{cases}1, & \text{if } x<1,\\ f(y), & \text{if } x=1.\end{cases}\tag{4}$$

Then J_{IB} is a fuzzy implication for any semicopula B, and J_{IB} is 2-increasing.

(ii) Let I be a fuzzy implication that satisfies (NP) and (IP). If $I=J_{IB}$ for any semicopula B, then $I=I_{WB}$.

(iii) An open problem: Does there exist some $I\in FI$, which is not an implication defined as (4), such that J_{IB} is a fuzzy implication for any semicopula B?

Proposition 4.18. Let I be a continuous fuzzy implication that satisfies (OP) and (EP). Then J_{IB} is a fuzzy implication if and only if there exists a $\varphi\in\Phi$, such that the semicopula $B_{\varphi^{-1}}$ is 1-Lipschitz.

Proof. Let I be a continuous fuzzy implication that satisfies (OP) and (EP), then I is Φ-conjugate with I_{LK}, i.e., there exists a $\varphi\in\Phi$, such that

$$I(x,y)=\varphi^{-1}(I_{LK}(\varphi(x),\varphi(y)))=\varphi^{-1}(\min(1-\varphi(x)+\varphi(y),1)),\ \text{ for all } x,y\in[0,1].$$

Let x_1, x_2, $y\in[0,1]$ and $x_1\le x_2$. J_{IB} is a fuzzy implication implies the following equivalences:

$$\begin{aligned}
&J_{IB}(x_1,y)\ge J_{IB}(x_2,y)\\
&\Leftrightarrow 1-\varphi(x_1)+\varphi(B(x_1,y))\ge 1-\varphi(x_2)+\varphi(B(x_2,y))\\
&\Leftrightarrow \varphi(B(x_2,y))-\varphi(B(x_1,y))\le\varphi(x_2)-\varphi(x_1)\\
&\Leftrightarrow \varphi(B(\varphi^{-1}(\varphi(x_2)),\varphi^{-1}(\varphi(y))))-\varphi(B(\varphi^{-1}(\varphi(x_1)),\varphi^{-1}(\varphi(y))))\le\varphi(x_2)-\varphi(x_1)\\
&\Leftrightarrow B_{\varphi^{-1}} \text{ is 1-Lipschitz,}
\end{aligned}$$

i.e., J_{IB} is a fuzzy implication if and only if there exists a $\varphi\in\Phi$, such that the semicopula $B_{\varphi^{-1}}$ is 1-Lipschitz.

□

4.2. Conditions for J_{IB} being a fuzzy implication with a special kind of semicopula

In this section, we study the sufficient conditions such that J_{IB} is a fuzzy implication with B defined by

$$B(x,y) = \begin{cases} \min(x,y), & \text{if } y > N(x), \\ 0, & \text{if } y \le N(x), \end{cases} \tag{5}$$

where N is a continuous fuzzy negation. A semicopula of form (5) is denoted by $B^N_{\min}$.

Definition 4.19. (Baczyński et al. [2]) Let N be a fuzzy negation, the range of N is given by $\mathrm{Ran}(N) = \{N(x)|x \in [0,1]\}$, the pseudo-inverse of N is given by $N^{(-1)}(y) = \sup\{x \in [0,1]|N(x) \ge y\}$, $y \in [0,1]$.

Proposition 4.20. Let I be a fuzzy implication that satisfies (IP), and $B = B^N_{\min}$. Let $s \in (0,1)$ be a fixed point of N, and $n = \min\{x \in [0,1]|N(x) = s\}$. Then J_{IB} is a fuzzy implication if and only if

(i) $N_I(x) = 1$ for all $x \in [0,n)$, and

(ii) for each $y \in (0,s]$, $I(x,y) \le N_I(N^{(-1)}(y))$ for all $x \in (N^{(-1)}(y),1]$.

Proof. (Necessity) (i) Suppose that there exists an $x_0 \in (0,n)$ such that $N_I(x_0) < 1$. Let $N(x_0) = y_0$, then $J_{IB}(x_0,y_0) = I(x_0,B(x_0,y_0)) = I(x_0,0) = N_I(x_0) < 1$.

Let $\varepsilon > 0$ such that $x_0 + \varepsilon < s$, then $N(x_0+\varepsilon) < N(x_0) = y_0$. Note that $y_0 = N(x_0) > N(n) = s$, thus $J_{IB}(x_0+\varepsilon,y_0) = I(x_0+\varepsilon,B(x_0+\varepsilon,y_0)) = I(x_0+\varepsilon,x_0+\varepsilon) = 1 > J_{IB}(x_0,y_0)$, this contradicts the fact that J_{IB} is a fuzzy implication.

(ii) Suppose that there exist an $y_0 \in (0,s]$ and an $x_0 \in (N^{(-1)}(y_0),1]$ such that

$$I(x_0,y_0) > N_I(N^{(-1)}(y_0)). \tag{6}$$

Let $x' = N^{(-1)}(y_0)$, then $x' \ge s$ and $N(x') = y_0$. Hence

$$J_{IB}(x',y_0) = I(x',B(x',y_0)) = I(x',0) = N_I(N^{(-1)}(y_0)). \tag{7}$$

Let $\varepsilon > 0$ such that $x'+\varepsilon = x_0$, then $N(x_0) < N(x') = y_0$. Note that $y_0 \le s < x_0$, then

$$J_{IB}(x'+\varepsilon,y_0) = I(x_0,B(x_0,y_0)) = I(x_0,y_0). \tag{8}$$

From (6), (7) and (8), we obtain $J_{IB}(x',y_0) < J_{IB}(x'+\varepsilon,y_0)$, this contradicts the fact that J_{IB} is a fuzzy implication.

(Sufficiency). Let x_1, x_2, $y \in [0,1]$ with $x_1 < x_2$. Note that $J_{IB}(x,y) = 1$ for all $x \le y$ and $y \ge s$, hence it suffices to prove that

$$J_{IB}(x_1,y) \ge J_{IB}(x_2,y) \text{ for all } y \in (0,s).$$

Actually, let $y \in (0,s)$ and $x'' = N^{(-1)}(y)$. If $x_1 < x_2 \le x''$, then

$$J_{IB}(x_1,y) = I(x_1,0) = N_I(x_1), \;\; J_{IB}(x_2,y) = I(x_2,0) = N_I(x_2),$$

thus $J_{IB}(x_1, y) \geq J_{IB}(x_2, y)$.

If $x_1 \leq x'' < x_2$, then $J_{IB}(x_1, y) = N_I(x_1) \geq N_I(x'') = N_I(N^{(-1)}(y))$, $J_{IB}(x_2, y) = I(x_2, y)$. Thus $J_{IB}(x_1, y) \geq J_{IB}(x_2, y)$.

If $x'' < x_1 < x_2$, then $J_{IB}(x_1, y) = I(x_1, y)$, $J_{IB}(x_2, y) = I(x_2, y)$. Thus $J_{IB}(x_1, y) \geq J_{IB}(x_2, y)$.

From above discussion, we get $J_{IB}(x_1, y) \geq J_{IB}(x_2, y)$ for all x_1, x_2, $y \in [0, 1]$ with $x_1 < x_2$. Hence J_{IB} is a fuzzy implication, see also Figure 1. □

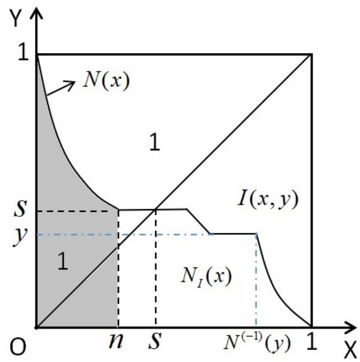

Fig. 1. The fuzzy implication J_{IB} in Proposition 4.20.

Corollary 4.21. Let I be a fuzzy implication that satisfies (IP), let $B = B^N_{\min}$ and N be a strict fuzzy negation with a fixed point $s \in (0, 1)$. Then J_{IB} is a fuzzy implication if and only if

(i) $N_I(x) = 1$ for all $x \in [0, s)$, and

(ii) for each $y \in (0, s]$, $I(x, y) \leq N_I(N^{-1}(y))$ for all $x \in (N^{-1}(y), 1]$.

Proof. Let $s \in (0, 1)$ be the fixed point of N, and let $n = \min\{x \in [0, 1] | N(x) = s\}$. Since N is strict, then $n = s$. Therefore, the proof comes directly from Proposition 4.20. □

Corollary 4.22. Let I be a fuzzy implication that satisfies (IP), let $B = B^N_{\min}$ and N be a continuous negation with a fixed point $s \in (0, 1)$. Then J_{IB} satisfies (IP) if and only if $N_I(s) = 1$.

Proof. (Sufficiency). Since s is a fixed point of N, then $s = N(s)$. If $x \leq N(x)$, then $x \leq s$. Hence $N_I(x) \geq N_I(s) = 1$, i.e., $I(x, 0) = 1$ for all $x \leq N(x)$. Note that

$$J_{IB}(x, x) = I(x, B(x, x)) = \begin{cases} I(x, 0), & \text{if } x \leq N(x), \\ 1, & \text{if } x > N(x), \end{cases}$$

then $J_{IB}(x, x) = 1$ for all $x \in [0, 1]$. Hence J_{IB} satisfies (IP).

(Necessity). Obvious. □

4.3. Conditions for J_{IB} being a fuzzy implication with $I = I_{UN}$

In this section, we discuss the conditions such that J_{IB} is a fuzzy implication when I is a (U, N)-implication.

Definition 4.23. (Baczyński et al. [2]) An associative, commutative and increasing operator $U : [0,1]^2 \to [0,1]$ is called a uninorm if it has a neutral element $e \in [0,1]$, i. e., $U(e,x) = x$, for all $x \in [0,1]$.

Obviously, if $e = 0$, then U is a t-conorm and if $e = 1$, then U is a t-norm.

Definition 4.24. (Baczyński et al. [2]) A uninorm U is called conjunctive if $U(0,1) = 0$. A uninorm U is called disjunctive if $U(0,1) = 1$.

Theorem 4.25. (Baczyński et al. [2]) Let U be a uninorm with neutral element $e \in (0,1)$, such that the functions $U(x,1)$ and $U(x,0)$ are continuous except at the point $x = e$. If U is disjunctive, then there exist a t-norm T and a t-conorm S such that

$$U(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}), & \text{if } x, y \in [0,e], \\ e + (1-e) \cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & \text{if } x, y \in [e,1], \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{9}$$

A uninorm of form (9) will be denoted by $U^{\max}_{T,S,e}$.

Definition 4.26. (Baczyński et al. [2]) A function $I : [0,1]^2 \to [0,1]$ is called a (U, N)-implication, if there exist a disjunctive uninorm U and a fuzzy negation N such that

$$I(x,y) = U(N(x), y), \;\; x, y \in [0,1].$$

If I is a (U, N)-implication generated from a disjunctive uninorm U and a negation N, then we will denote it by I_{UN}.

Lemma 4.27. Let U be a uninorm with a neutral element $e \in (0,1)$. If I_{UN} satisfies (IP), then $N(x) = 1$ for all $x \in [0,e]$.

Proof. Assume that I_{UN} satisfy (IP), then $I_{UN}(x,x) = 1$ for all $x \in [0,1]$, i.e, $U(N(x),x) = 1$ for all $x \in [0,1]$. Let $x = e$, then $U(N(e),e) = 1$, thus $N(e) = 1$. Since $N(x)$ is decreasing, then $N(x) \geq N(e) = 1$ for $x \in [0,e]$. Hence $N(x) = 1$ for all $x \in [0,e]$. □

Lemma 4.28. Let $U = U^{\max}_{T,S,e}$. If I_{UN} satisfies (IP), then

(i) $N^{(-1)}(e) = 1$.

(ii) $S\left(\frac{N(x)-e}{1-e}, \frac{x-e}{1-e}\right) = 1$ for all $x \in (e, N^{(-1)}(e))$.

(iii) If S is positive, i. e., $S(x,y) = 1 \Rightarrow$ either $x = 1$ or $y = 1$, then $N = N_{D_2}$.

Proof. Let $x \in [0, e]$, then $N(x) = 1$ by Lemma 4.27. Thus

$$U(N(x), x) = \max(N(x), x) = N(x) = 1.$$

Let $x \in (e, 1]$, then $0 \leq N(x) \leq N(e) = 1$.

If $e \leq N(x) \leq 1$, namely, $e < x < N^{(-1)}(e)$, then

$$U(N(x), x) = e + (1 - e) \cdot S\left(\frac{N(x) - e}{1 - e}, \frac{x - e}{1 - e}\right).$$

If $0 \leq N(x) \leq e$, namely, $N^{(-1)}(e) \leq x \leq 1$, then

$$U(N(x), x) = \max(N(x), x) = x.$$

From above discussion, we get

$$I_{UN}(x, x) = \begin{cases} 1, & \text{if } x \in [0, e], \\ e + (1 - e) \cdot S\left(\frac{N(x)-e}{1-e}, \frac{x-e}{1-e}\right), & \text{if } x \in \left(e, N^{(-1)}(e)\right), \\ x, & \text{if } x \in [N^{(-1)}(e), 1]. \end{cases}$$

Assume that I_{UN} satisfies (IP), then

$$N^{(-1)}(e) = 1, \text{ and } S\left(\frac{N(x) - e}{1 - e}, \frac{x - e}{1 - e}\right) = 1 \text{ for all } x \in \Big(e, N^{(-1)}(e)\Big).$$

Assume that S is positive, then we get

$$\frac{N(x) - e}{1 - e} = 1 \text{ or } \frac{x - e}{1 - e} = 1.$$

Note that $e < x < 1$, then $\frac{N(x)-e}{1-e} = 1$, thus $N(x) = 1$ for all $x \in (e, 1)$. Hence $N = N_{D_2}$. □

Proposition 4.29. (Baczyński et al. [5]) Let $I \in FI$ and B be a semicopula. If J_{IB} is a fuzzy implication, then I satisfies (IP).

Proposition 4.30. Let $I = I_{UN}$. If J_{IB} is a fuzzy implication, then

(i) $N(x) = 1$ for all $x \in [0, e]$.

(ii) If $U = U^{\max}_{T,S,e}$, then $N^{(-1)}(e) = 1$, and $S\left(\frac{N(x)-e}{1-e}, \frac{x-e}{1-e}\right) = 1$ for all $x \in (e, 1)$.

(iii) If s is a fixed point of N, then $s > e$.

Proof. Straightforward from Lemmas 4.27, 4.28 and Proposition 4.29. □

Remark 4.31. Let $I = I_{UN}$. If N is strict, then there is no semicopula B such that J_{IB} is a fuzzy implication. Actually, suppose that J_{IB} is a fuzzy implication, then $N(x) = 1$ for all $x \in [0, e]$ by Proposition 4.30 (i), a contradiction to the hypothesis that N is strict.

Proposition 4.32. Let $U = U^{\max}_{T,S,e}$ and $I = I_{UN}$. If the t-conorm S is positive, then J_{IB} is a fuzzy implication if and only if $N = N_{D_2}$.

Proof. (Sufficiency) Since $J_{IB}(1,1) = U(N(1), B(1,1)) = U(0,1) = 1$, then it suffices to prove that J_{IB} satisfies (I1).

Let x_1, x_2, $y \in [0,1]$ with $x_1 < x_2$. Since $N_{D_2}(x) = 1$ for all $x \in [0,1)$ and $U(1,x) = 1$ for all $x \in [0,1]$, then

$$\begin{aligned} J_{IB}(x_1,y) - J_{IB}(x_2,y) &= U(N_{D_2}(x_1), B(x_1,y)) - U(N_{D_2}(x_2), B(x_2,y)) \\ &= 1 - U(N_{D_2}(x_2), B(x_2,y)) \\ &\geq 0, \end{aligned}$$

that is, J_{IB} satisfies (I1).

(Necessity) Straightforward from Lemma 4.28 (iii). □

Theorem 4.33. Let $I = I_{UN}$, $U = U^{\max}_{T,S,e}$, and N be a fuzzy negation that satisfies $N(x) = 1$ for all $x \in [0,e]$, and $N^{(-1)}(e) = 1$. Let B be a semicopula that satisfies $B(x,y) \geq e$ for all x, $y \in [e,1]$. Then J_{IB} is a fuzzy implication if and only if the triple (S, N, B) satisfies the following conditions:

(a) $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) \geq \frac{y-e}{1-e}$ for all $x \in (e,1)$ and $y \in (e,1]$,

(b) $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right)$ is decreasing with respect to $x \in (e,1)$ for every $y \in (e,1]$.

Proof. Since $J_{IB}(1,1) = U(N(1), B(1,1)) = U(0,1) = 1$, then it suffices to prove that J_{IB} satisfies (I1) if and only if the conditions (a) and (b) hold.

Let x, $y \in [0,1]$. Note that $U_{\max}(1,a) = 1$ for all $a \in [0,1]$, then, for $x \in [0,e]$ and $y \in [0,1]$, $J_{IB}(x,y) = U(N(x), B(x,y)) = U(1, B(x,y)) = 1$.

Since $N^{(-1)}(e) = 1$, then $N(x) \geq e$ for all $x \in [e,1)$. Actually, suppose that there exists an $x_0 \in (e,1)$ such that $N(x_0) < e$, then $N(x) < e$ for all $x \in [x_0,1]$. Thus

$$\begin{aligned} N^{(-1)}(e) &= \sup\{x \in [0,1] | N(x) \geq e\} \\ &= \sup\{\{x \in [0,x_0) | N(x) \geq e\} \cup \{x \in [x_0,1] | N(x) \geq e\}\} \\ &= \sup\{x \in [0,x_0) | N(x) \geq e\}. \end{aligned}$$

Note that $\{x \in [e,x_0) | N(x) \geq e\} \subseteq [e,x_0)$, then $\sup\{x \in [e,x_0) | N(x) \geq e\} \leq \sup[e,x_0)$. Since $\sup[e,x_0) \leq x_0$, then

$$\sup\{x \in [e,x_0) | N(x) \geq e\} \leq x_0 < 1, \text{ i.e., } N^{(-1)}(e) \leq x_0 < 1,$$

this contradicts the fact that $N^{(-1)}(e) = 1$. That is, $N(x) \geq e$ for all $x \in [e,1)$.

For $x \in (e, 1)$ and $y \in [0, e]$. Note that $B(x, y) \leq \min(x, y)$, then $B(x, y) \leq e$. Hence

$$J_{IB}(x, y) = U(N(x), B(x, y)) = \max(N(x), B(x, y)) = N(x).$$

For $x \in (e, 1)$ and $y \in (e, 1]$. Since $B(x, y) \geq e$ for all $x, y \in [e, 1]$, then

$$J_{IB}(x, y) = U(N(x), B(x, y)) = e + (1 - e) \cdot S\left(\frac{N(x) - e}{1 - e}, \frac{B(x,\ y) - e}{1 - e}\right).$$

For $x = 1$ and $y \in [0, e]$. Note that $B(1, y) = y \leq e$, then

$$J_{IB}(x, y) = U(N(x), B(x, y)) = U(0, y) = e \cdot T\left(\frac{0}{e}, \frac{y}{e}\right) = 0.$$

For $x = 1$ and $y \in (e, 1]$. Note that $B(1, y) = y > e$, then

$$J_{IB}(x, y) = U(N(x), B(x, y)) = U(0, y) = \max(0, y) = y.$$

From above discussion, we get

$$J_{IB}(x, y) = \begin{cases} 1, & \text{if } x \in [0, e] \text{ and } y \in [0, 1], \\ N(x), & \text{if } x \in (e, 1) \text{ and } y \in [0, e], \\ e + (1 - e) \cdot S\left(\frac{N(x)-e}{1-e}, \frac{B(x,\ y)-e}{1-e}\right), & \text{if } x \in (e, 1) \text{ and } y \in (e, 1], \\ 0, & \text{if } x = 1 \text{ and } y \in [0, e], \\ y, & \text{if } x = 1 \text{ and } y \in (e, 1]. \end{cases}$$

Let x_1, x_2, $y \in [0, 1]$ with $x_1 < x_2$. Consider the following cases:

Case 1. If $x_1 \leq e$, then $J_{IB}(x_1, y) = 1 \geq J_{IB}(x_2, y)$ for all $y \in [0, 1]$.

Case 2. If $e < x_1 < x_2 < 1$ and $y \in [0, e]$, then
$$J_{IB}(x_1, y) = N(x_1) \geq N(x_2) = J_{IB}(x_2, y).$$

Case 3. If $e < x_1 < x_2 < 1$ and $y \in (e, 1]$, then
$$J_{IB}(x_1, y) \geq J_{IB}(x_2, y) \Leftrightarrow S\left(\frac{N(x_1)-e}{1-e}, \frac{B(x_1,y)-e}{1-e}\right) \geq S\left(\frac{N(x_2)-e}{1-e}, \frac{B(x_2,y)-e}{1-e}\right),$$
i. e., $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right)$ is decreasing with respect to $x \in [e, 1]$ for every $y \in (e, 1]$.

Case 4. If $e < x_1 < x_2 = 1$ and $y \in [0, e]$, then $J_{IB}(x_1, y) = N(x_1) \geq e$, $J_{IB}(x_2, y) = 0$. Hence $J_{IB}(x_1, y) \geq J_{IB}(x_2, y)$.

Case 5. If $e < x_1 < x_2 = 1$ and $y \in (e, 1]$, then $J_{IB}(x_1, y) = S\left(\frac{N(x_1)-e}{1-e}, \frac{B(x_1,y)-e}{1-e}\right)$, $J_{IB}(x_2, y) = y$. Hence $J_{IB}(x_1, y) \geq J_{IB}(x_2, y) \Leftrightarrow S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) \geq \frac{y-e}{1-e}$ for all $x \in (e, 1)$, $y \in (e, 1]$.

From above cases, J_{IB} satisfies (I1) if only if the triple (S, N, B) satisfies the following conditions:

(a) $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) \geq \frac{y-e}{1-e}$ for all $x \in (e, 1)$ and $y \in (e, 1]$,

(b) $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right)$ is decreasing with respect to $x \in (e, 1)$ for every $y \in (e, 1]$.

Thus we complete the proof. $\square$

To illustrate there exists a triple (S, N, B) such that the conditions (a) and (b) of Theorem 4.33 hold, an example is given.

Example 4.34. Let N be a fuzzy negation defined as

$$N(x) = \begin{cases} 1, & \text{if } x \in [0, e], \\ -x + e + 1, & \text{if } x \in (e, \frac{1+e}{2}], \\ \frac{1+e}{2}, & \text{if } x \in (\frac{1+e}{2}, 1), \\ 0, & \text{if } x = 1, \end{cases}$$

where $e \in (0.5, 1)$. Obviously, N satisfies $N(x) = 1$ for all $x \in [0, e]$, and $N^{(-1)}(e) = 1$.

Let $B = T_{nM}$. It is easy to see that $T_{nM}(x, y) \geq e$ for all x, $y \in [e, 1]$.

Let $S = S_{nM}^{N_1}$, that is

$$S(x, y) = S_{nM}^{N_1}(x, y) = \begin{cases} 1, & \text{if } x \geq N_1(y), \\ \max(x, y), & \text{if } x < N_1(y), \end{cases}$$

where $N_1(y) = 1 - y$. Note that $\frac{N(x)-e}{1-e} \geq N_1(\frac{x-e}{1-e})$ for all $x \in (e, 1)$, then, for $x \in (e, 1)$, $y \in (e, 1]$ with $x \leq y$, we get

$$\begin{aligned} S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) &= S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{T_{nM}(x,y)-e}{1-e}\right) \\ &= S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{x-e}{1-e}\right) \\ &= 1. \end{aligned}$$

Similarly, for $x \in (e, 1)$, $y \in (e, 1]$ with $x > y$, we get

$$S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) = S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{y-e}{1-e}\right).$$

Hence

$$S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) = \begin{cases} 1, & \text{if } x \leq y, x \in (e, 1), y \in (e, 1], \\ S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{y-e}{1-e}\right), & \text{if } x > y, x \in (e, 1), y \in (e, 1]. \end{cases}$$

It easy to see that $S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right)$ is decreasing with respect to $x \in (e, 1)$ for every $y \in (e, 1]$, and $S_{nM}^{N_1}\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) \geq \frac{y-e}{1-e}$, i. e., conditions (a) and (b) of Theorem 4.33 hold.

If the conditions on B are weakened, we obtain Theorem 4.38.

Definition 4.35. (Klement et al. [12]) Let $e \in (0,1)$ be a constant. A semicopula B is $G(e)$-continuous if for every $y \in [e,1]$, there exists $x \in [e,1]$ such that $B(x,y) = e$.

Remark 4.36. (i) If a semicopula B is continuous, then B is $G(e)$-continuous.

(ii) A semicopula B may not be continuous even is $G(e)$-continuous.

Example 4.37. (i) Let $e \in [0,1]$ and $B = T_P$. Obviously, B is continuous. For each $y \in [e,1]$, there exists $x = \frac{e}{y} \in [e,1]$ such that $B(x,y) = e$, hence B is $G(e)$-continuous.

(ii) Consider $B = T_{nM}$, which is not continuous. However, let $e = 0.7$, for each $y \in [0.7,1]$, there exists $x = 0.7$ such that $B(x,y) = 0.7$, i. e., B is $G(0.7)$-continuous.

Theorem 4.38. Let $I = I_{UN}$, $U = U^{\max}_{T,S,e}$ and N be a fuzzy negation that satisfies $N(x) = 1$ for all $x \in [0,e]$, and $N^{(-1)}(e) = 1$. Let B be a semicopula that is $G(e)$-continuous. Then J_{IB} is a fuzzy implication if and only if the triple (S,N,B) satisfies following conditions:

(a) $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) \geq \frac{y-e}{1-e}$ for $y \in (e,1]$ and $x \in \{t \in [e,1) | B(t,y) \geq e\}$,

(b) $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right)$ is decreasing with respect to $x \in \{t \in [e,1) | B(t,y) \geq e\}$ for every $y \in (e,1]$.

Proof. Similar to the proof of Theorem 4.33, it suffices to prove that J_{IB} satisfies (I1) if and only if the conditions (a) and (b) hold.

Let $x, y \in [0,1]$. Note that $U_{\max}(1,a) = 1$ for all $a \in [0,1]$, then, for $x \in [0,e]$, we get

$$J_{IB}(x,y) = U(N(x), B(x,y)) = U(1, B(x,y)) = 1 \text{ for all } y \in [0,1].$$

Note that $N^{(-1)}(e) = 1$, then $N(x) \geq e$ for all $x \in [e,1)$.

(1) For $x \in (e,1)$ and $y \in [0,e]$, since $B(x,y) \leq \min(x,y) = y \leq e$, then

$$J_{IB}(x,y) = U(N(x), B(x,y)) = \max(N(x), B(x,y)) = N(x).$$

(2) For $x \in (e,1)$ and $y \in (e,1]$ such that $B(x,y) \leq e$, then

$$J_{IB}(x,y) = U(N(x), B(x,y)) = \max(N(x), B(x,y)) = N(x).$$

(3) For $x \in (e,1)$ and $y \in (e,1]$ such that $B(x,y) > e$, then

$$J_{IB}(x,y) = U(N(x), B(x,y)) = e + (1-e) \cdot S\left(\frac{N(x)-e}{1-e}, \frac{B(x,\, y)-e}{1-e}\right).$$

(4) For $x = 1$ and $y \in [0, e]$, since $B(x, y) = y \leq e$, then

$$J_{IB}(x, y) = U(N(x), B(x, y)) = U(0, y) = 0.$$

(5) For $x = 1$ and $y \in (e, 1]$, since $B(x, y) = y > e$, then

$$J_{IB}(x, y) = U(N(x), B(x, y)) = U(0, y) = \max(0, y) = y.$$

From above discussion, we get

$$J_{IB}(x, y) = \begin{cases} 1, & \text{if } x \in [0, e] \text{ and } y \in [0, 1], \\ N(x), & \text{if } x \in (e, 1), y \in [0, 1] \text{ and } B(x, y) \leq e, \\ e + (1 - e) \cdot S(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}), & \text{if } x \in (e, 1), y \in (e, 1] \text{ and } B(x, y) > e, \\ 0, & \text{if } x = 1 \text{ and } y \in [0, e], \\ y, & \text{if } x = 1 \text{ and } y \in (e, 1]. \end{cases}$$

Let x_1, x_2, $y \in [0, 1]$ and $x_1 < x_2$. Consider the following cases:

Case 1. If $x_1 \leq e$, then $J_{IB}(x_1, y) = 1 \geq J_{IB}(x_2, y)$ for all $y \in [0, 1]$.

Case 2. If $e < x_1 < x_2 < 1$ and $y \in [0, e]$, note that $B(x, y) \leq \min(x, y) \leq y \leq e$, then $J_{IB}(x_1, y) = N(x_1) \geq N(x_2) = J_{IB}(x_2, y)$.

Case 3. If $e < x_1 < x_2 < 1$ and $y \in (e, 1]$. Given $y_0 \in (e, 1]$, since B is $G(e)$-continuous, then there exists an $x_0 \in [e, 1]$ such that $B(x_0, y_0) = e$.

Let $A(y) = \{t \in [e, 1) | B(t, y) \geq e\}$, $y \in (e, 1]$. Obviously, $A \neq \emptyset$.

Case 3.1. $x_2 \leq x_0$. Note that $B(x_1, y_0) \leq B(x_2, y_0) \leq B(x_0, y_0) = e$, then

$$J_{IB}(x_1, y_0) = N(x_1) \geq N(x_2) = J_{IB}(x_2, y_0).$$

Case 3.2. $x_1 \leq x_0 < x_2 < 1$. Note that $B(x_1, y_0) \leq e$ and $B(x_2, y_0) \geq e$, then $x_2 \in A(y_0)$, and

$$J_{IB}(x_1, y_0) = N(x_1),$$

$$J_{IB}(x_0, y_0) = N(x_0) = e + (1 - e) \cdot S\left(\frac{N(x_0)-e}{1-e}, \frac{B(x_0,y_0)-e}{1-e}\right),$$

$$J_{IB}(x_2, y_0) = e + (1 - e) \cdot S\left(\frac{N(x_2)-e}{1-e}, \frac{B(x_2,y_0)-e}{1-e}\right).$$

Hence, for all $x_2 \in A(y_0)$, we have

$$J_{IB}(x_1, y_0) \geq J_{IB}(x_2, y_0) \Leftrightarrow S\left(\frac{N(x_0)-e}{1-e}, \frac{B(x_0,y_0)-e}{1-e}\right) \geq S\left(\frac{N(x_2)-e}{1-e}, \frac{B(x_2,y_0)-e}{1-e}\right)$$

Case 3.3. $x_0 \leq x_1 < x_2 < 1$. Note that $e = B(x_0, y_0) \leq B(x_1, y_0) \leq B(x_2, y_0)$, then $x_1 \in A(y_0)$, $x_2 \in A(y_0)$, and

$$J_{IB}(x_1, y_0) = e + (1 - e) \cdot S\left(\frac{N(x_1)-e}{1-e}, \frac{B(x_1,y_0)-e}{1-e}\right),$$

$$J_{IB}(x_2, y_0) = e + (1 - e) \cdot S\left(\frac{N(x_2)-e}{1-e}, \frac{B(x_2,y_0)-e}{1-e}\right).$$

Thus $J_{IB}(x_1, y_0) \geq J_{IB}(x_2, y_0) \Leftrightarrow S\left(\frac{N(x_1)-e}{1-e}, \frac{B(x_1,y_0)-e}{1-e}\right) \geq S\left(\frac{N(x_2)-e}{1-e}, \frac{B(x_2,y_0)-e}{1-e}\right)$, i. e., $S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y_0)-e}{1-e}\right)$ is decreasing with respect to $x \in A(y_0)$.

From Cases 3.2 and 3.3, and the arbitrary of y_0, we get $J_{IB}(x_1, y) \geq J_{IB}(x_2, y) \Leftrightarrow S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right)$ is decreasing with respect to $x \in \{t \in [e, 1) | B(t, y) \geq e\}$ for every $y \in (e, 1]$.

Case 4. If $e \leq x_1 < x_2 = 1$ and $y \in [0, e]$, then $J_{IB}(x_1, y) \geq 0 = J_{IB}(x_2, y)$.

Case 5. If $e \leq x_1 < x_2 = 1$ and $y \in (e, 1]$. Similarly, let $y_0 \in (e, 1]$, since B is $G(e)$-continuous, then there exists an $x_0 \in (e, 1)$ such that $B(x_0, y_0) = e$.

Let $A(y) = \{t \in [e, 1) | B(t, y) \geq e\}$, $y \in (e, 1]$.

Case 5.1. $x_0 \leq x_1$. Note that $B(x_1, y_0) \geq B(x_0, y_0) = e$, i. e., $x_1 \in A(y_0)$, then

$$J_{IB}(x_1, y_0) = e + (1-e) \cdot S\left(\frac{N(x_1)-e}{1-e}, \frac{B(x_1,y_0)-e}{1-e}\right).$$

Hence $J_{IB}(x_1, y_0) \geq J_{IB}(x_2, y_0) \Leftrightarrow S\left(\frac{N(x_1)-e}{1-e}, \frac{B(x_1,y_0)-e}{1-e}\right) \geq \frac{y_0-e}{1-e}$.

Case 5.2. $e \leq x_1 < x_0$. Note that $B(x_1, y) \leq B(x_0, y) = e$, then

$$J_{IB}(x_1, y_0) = N(x_1) \geq N(x_0) = e + (1-e) \cdot S\left(\frac{N(x_0)-e}{1-e}, \frac{B(x_0,y_0)-e}{1-e}\right),$$

$$J_{IB}(x_2, y_0) = y_0.$$

Hence $J_{IB}(x_1, y_0) \geq J_{IB}(x_2, y_0) \Leftrightarrow S\left(\frac{N(x_0)-e}{1-e}, \frac{B(x_0,y_0)-e}{1-e}\right) \geq \frac{y_0-e}{1-e}$.

From Cases 5.1 and 5.2, and the arbitrary of y_0, we get

$J_{IB}(x_1, y) \geq J_{IB}(x_2, y) \Leftrightarrow S\left(\frac{N(x_1)-e}{1-e}, \frac{B(x_1,y)-e}{1-e}\right) \geq \frac{y-e}{1-e}$ for all $x_1 \in A(y)$,

i. e.,

$$S\left(\frac{N(x)-e}{1-e}, \frac{B(x,y)-e}{1-e}\right) \geq \frac{y-e}{1-e} \text{ for } y \in (e, 1] \text{ and } x \in \{t \in [e, 1) | B(t, y) \geq e\}.$$

From above discussion, J_{IB} satisfies (I1) if and only if the conditions (a) and (b) hold.□

To show the application of Theorem 4.38, an example is given.

Example 4.39. Let $e \in (0, 1)$ and N be a fuzzy negation defined as

$$N(x) = \begin{cases} 1, & \text{if } x \in [0, e], \\ -x + e + 1, & \text{if } x \in [e, 1), \\ 0, & \text{if } x = 1. \end{cases}$$

Obviously, N satisfies $N(x) = 1$ for all $x \in [0, e]$, and $N^{(-1)}(e) = 1$.

Let $S = S_{LK}$, the Lukasiewicz t-conorm, i. e., $S(x, y) = \min(x + y, 1)$. Consider $B = T_P$. Obviously, B is $G(e)$-continuous.

For $e \le x < 1, e < y \le 1$, we have

$$S\left(\frac{N(x)-e}{1-e}, \frac{xy-e}{1-e}\right) = 1 + x \cdot \frac{y-1}{1-e}.$$

Then, given $y \in (e,1]$, the function $S\left(\frac{N(x)-e}{1-e}, \frac{xy-e}{1-e}\right)$ is decreasing with respect to x. Thus

$$S\left(\frac{N(x)-e}{1-e}, \frac{xy-e}{1-e}\right) > 1 + \frac{y-1}{1-e} = \frac{y-e}{1-e},$$

for all $x \in [\frac{e}{y}, 1)$. This fact shows that there exists a triple (S, N, B) such that the conditions (a) and (b) hold.

From (1) we get

$$J_{IB}(x,y) = \begin{cases} 1, & \text{if } x \in [0,e] \text{ and } y \in [0,1], \\ 1+e-x, & \text{if } x \in (e,1),\ y \in [0,1] \text{ and } xy \le e, \\ 1-x+xy, & \text{if } x \in (e,1),\ y \in [e,1] \text{ and } xy \ge e, \\ 0, & \text{if } x = 1 \text{ and } y \in [0,e], \\ y, & \text{if } x = 1 \text{ and } y \in (e,1]. \end{cases}$$

Hence J_{IB} is a fuzzy implication by Theorem 4.38, see also Figure 2.

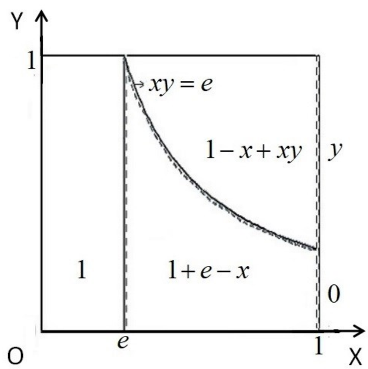

Fig. 2. The fuzzy implication J_{IB} in Example 4.39.

5. PROPERTIES OF Φ-CONJUGATE WITH J_{IB}

In this section, some trivial properties of Φ-conjugate with J_{IB} are presented.

Proposition 5.1. Let $\varphi \in \Phi$, $I \in FI$ and B be a semicopula, then $(J_{IB})_\varphi = J_{I_\varphi B_\varphi}$.

Proof. Let x, $y \in [0,1]$, then $(J_{IB})_\varphi(x,y) = \varphi^{-1}(J_{IB}(\varphi(x), \varphi(y)))$

$$
\begin{aligned}
&= \varphi^{-1}(I(\varphi(x), B(\varphi(x), \varphi(y)))) \\
&= \varphi^{-1}(I(\varphi(x), \varphi \circ \varphi^{-1} B(\varphi(x), \varphi(y)))) \\
&= I_\varphi(x, \varphi^{-1}(B(\varphi(x), \varphi(y)))) \\
&= I_\varphi(x, B_\varphi(x, y)) \\
&= J_{I_\varphi B_\varphi}(x, y).
\end{aligned}
$$

□

Proposition 5.2. Let $\varphi \in \Phi$. If I satisfies (NP), then $(J_{IB})_\varphi$ satisfies (NP).

Proof. Straightforward from Proposition 2.9 (iii) and Proposition 1.3.6 in [2]. □

Proposition 5.3. Let $\varphi \in \Phi$ and $N_{(J_{IB})_\varphi}$ be the natural negation of $(J_{IB})_\varphi$. If $I \in FI$, then $N_{(J_{IB})_\varphi}$ is a fuzzy negation, and $N_{(J_{IB})_\varphi} = (N_I)_\varphi$.

Proof. Note that $(J_{IB})_\varphi(x, y) = \varphi^{-1}(I(\varphi(x), B(\varphi(x), \varphi(y))))$, then

$$
\begin{aligned}
N_{(J_{IB})_\varphi} &= (J_{IB})_\varphi(x, 0) \\
&= \varphi^{-1}(I(\varphi(x), B(\varphi(x), \varphi(0)))) \\
&= \varphi^{-1}(I(\varphi(x), 0)) \\
&= \varphi^{-1}(N_I(\varphi(x))) \\
&= (N_I)_\varphi.
\end{aligned}
$$

Since I is a fuzzy implication, then N_I is a fuzzy negation by Lemma 1.4.14 in [2]. Note that $\varphi \in \Phi$, thus $(N_I)_\varphi$ is a fuzzy negation by Proposition 1.4.8 in [2]. Hence $N_{(J_{IB})_\varphi}$ is a fuzzy negation. □

Proposition 5.4. Let $\varphi \in \Phi$ and I be a fuzzy implication that satisfies (IP). If J_{IB} is 2-increasing, then $J_{I_\varphi B_\varphi}$ is a fuzzy implication.

Proof. Let I satisfy (IP) and J_{IB} is 2-increasing, then J_{IB} is a fuzzy implication by Theorem 4.6. Note that $\varphi \in \Phi$, then $(J_{IB})_\varphi$ is a fuzzy implication by Proposition 1.1.8 in [2]. Thus $J_{I_\varphi B_\varphi}$ is a fuzzy implication by Proposition 5.1. □

6. CONCLUSION

In this paper, the research on the implication J_{IB} is mainly carried out in two ways, one is studying the relationship between the properties of J_{IB} and I, another is studying the conditions such that J_{IB} is a fuzzy implication. The main results and research contents are as follows:

- Implication J_{IB} satisfies (NP) if and only if I satisfies (NP), so is (LF), but not (IP), (OP), (CP), (EP), (LT) and (BL).
- Sufficient conditions under which J_{IB} satisfies (IP) ((OP), (CP), (EP), (LT), (BL), respectively) are studied and introduced.

- If the implication J_{IB} is 2-increasing ((r_1, r_2)-decreasing, respectively)), then J_{IB} is a fuzzy implication.
- Conditions under which J_{IB} is a fuzzy implication when B is a special semicopula (I is an (U, N)-implication, respectively)) are studied/introduced.

In our future work, we want to study the following problems:

⋆ The open problems proposed in Section 4.1.

⋆ The conditions under which J_{IB} is a fuzzy implication when I is a QL-implication.

⋆ The distributivity of J_{IB}.

ACKNOWLEDGEMENT

The author is deeply grateful to the Associate Editor and to the referee for their careful reading, useful remarks and constructive suggestions that substantially improved the quality of the paper. This work was partially supported by NSF of Chongqing (cstc2018jcyjAX0540).

(Received October 29, 2018)

REFERENCES

[1] M. Baczyński and B. Jayaram: On the characterization of (S, N)-implications. Fuzzy Sets Systems. *158* (2007), 1713–1727. DOI:10.1016/j.fss.2007.02.010

[2] M. Baczyński and B. Jayaram: Fuzzy Implications. Studies in Fuzziness and Soft Computing *231*, Springer, Berlin Heidelberg 2008.

[3] M. Baczyński and B. Jayaram: (S, N)-and R-implications: a state-of-the-artsurvey. Fuzzy Sets Systems *159* (2008), 1836–1859. DOI:10.1016/j.fss.2007.11.015

[4] M. Baczyński and B. Jayaram: QL-implications: some properties and intersections. Fuzzy Sets Systems *161* (2010), 158–188. DOI:10.1016/j.fss.2008.09.021

[5] M. Baczyński, P. Grzegorzewski, R. Mesiar, P. Helbin, and W. Niemyska: Fuzzy implications based on semicopulas. Fuzzy Sets Systems *323* (2017), 138–151. DOI:10.1016/j.fss.2016.09.009

[6] M. Baczyński, G. Beliakov, H. B. Sola, and A. Pradera: Advances in Fuzzy Implication Functions. Springer-Verlag, Berlin 2013. DOI:10.1007/978-3-642-35677-3

[7] H. Bustince, J. Fernandez, A. Kolesárová, and R. Mesiar: Directional monotonicity of fusion functions. Europ. J. Oper. Res. *244* (2015), 300–308. DOI:10.1016/j.ejor.2015.01.018

[8] F. Durante and C. Sempi: Semicopulæ. Kybernetika *41* (2005), 315–328.

[9] J. C. Fodor: Contrapositive symmetry of fuzzy implications. Fuzzy Sets Systems *69* (1995), 141–156. DOI:10.1016/0165-0114(94)00210-x

[10] P. Grzegorzewski: Probabilistic implications. Fuzzy Sets Systems *226* (2013), 53–66. DOI:10.1016/j.fss.2013.01.003

[11] S. Jenei: A new approach for interpolation and extrapolation of compact fuzzy quantities. In: Proc. 21th Linz Seminar on Fuzzy Set Theory 2000, pp. 13–18.

[12] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.

[13] H. Liu: On a new class of implications: (g, min)-implications and several classical tautologies. Int. J. Uncert. Fuzziness Knowl. Based Systems *20* (2012), 1–20. DOI:10.1142/s0218488512500018

[14] H. Liu: Fuzzy implications derived from generalized additive generators of representable uninorms. IEEE Trans. Fuzzy Systems *21* (2013), 555–566. DOI:10.1109/tfuzz.2012.2222892

[15] M. Mas, M. Monserrat, J. Torrens, and E. Trillas: A survey on fuzzy implication functions. IEEE Trans. Fuzzy Systems *15* (2007), 1107–1121. DOI:10.1109/tfuzz.2007.896304

[16] S. Massanet and J. Torrens: On a new class of fuzzy implication: h-implication and generalization. Inform. Sci. *181* (2011), 2111–2127. DOI:10.1016/j.ins.2011.01.030

[17] S. Massanet and J. Torrens: Threshold generation method of construction of a new implication from two given ones. Fuzzy Sets Systems *205* (2012), 50–75. DOI:10.1016/j.fss.2012.01.013

[18] S. Massanet and J. Torrens: On some properties of threshold generated implications. Fuzzy Sets Systems *205* (2012), 30–49. DOI:10.1016/j.fss.2012.01.018

[19] S. Massanet and J. Torrens: On the vertical threshold generation method of fuzzy implication and its properties. Fuzzy Sets Systems *226* (2013), 32–52. DOI:10.1016/j.fss.2013.03.003

[20] R. B. Nelsen: An Introduction to Copulas (Second edition). Springer, New York 2006. DOI:10.1007/0-387-28678-0

[21] Y. Ouyang: On fuzzy implications determined by aggregation operators. Inform. Sci. *193* (2012), 153–162. DOI:10.1016/j.ins.2012.01.001

[22] Z. Peng: A new family of (A, N)-implications: construction and properties. Iran. J. Fuzzy Systems *17* (2020), 129–145. DOI:10.1016/0165-0114(94)90082-5

[23] A. Pradera: A review of the relationships between aggregation, implication and negation functions. Adv. Intell. Systems Comput. *228* (2013), 31–36. DOI:10.1007/978-3-642-39165-1_6

[24] A. Pradera, G. Beliakov, H. Bustince, and B. De Baets: A review of the relationships between implication, negation and aggregation functions from the point of view of material implication. Inform. Sci. *329* (2016), 357–380. DOI:10.1016/j.ins.2015.09.033

[25] Y. Su, A. Xie, and H. Liu: On ordinal sum implications. Inform. Sci. *293* (2015), 251–262. DOI:10.1016/j.ins.2014.09.021

[26] R. R. Yager: On some new classes of implication operators and their role in approximate reasoning. Inform. Sci. *167* (2004), 193–216. DOI:10.1016/j.ins.2003.04.001

Zuming Peng, College of Mathematics and Statistics, Yangtze Normal University, Fuling 408100, Chongqing. P. R. China.
e-mail: 364886982@qq.com