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THE BICROSSED PRODUCTS OF H_4 AND H_8

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Abstract. Let H_4 and H_8 be the Sweedler's and Kac-Paljutkin Hopf algebras, respectively. We prove that any Hopf algebra which factorizes through H_8 and H_4 (equivalently, any bicrossed product between the Hopf algebras H_8 and H_4) must be isomorphic to one of the following four Hopf algebras: $H_8 \otimes H_4, H_{32,1}, H_{32,2}, H_{32,3}$. The set of all matched pairs $(H_8, H_4, \triangleright, \triangleleft)$ is explicitly described, and then the associated bicrossed product is given by generators and relations.

Keywords: Kac-Paljutkin Hopf algebra; Sweedler's Hopf algebra; bicrossed product; factorization problem

MSC 2020: 16T10, 16T05, 16S40

1. Introduction

The factorization problem stemmed from the group theory and was first considered in [11] by Maillet. This problem aims at the description and classification of all groups G which factor through two given groups N and H, i.e. $G = NH$, and $N \cap H = \{1\}$. In [7], the group X is called the bicrossed product of N and G. However, although the statement of the problem seems very simple and natural, no major progress has been made so far since we still have not commanded exhaustive methods to handle it. For example, even the description and classification of groups which factor through two finite cyclic groups is still an open problem.

An important step in dealing with the factorization problem for groups was the bicrossed product construction introduced in the paper by Zappa, see [21], later on, Takeuchi discovered the same construction in the paper [16], where the terminology bicrossed product was brought up for the first time. The main ingredients in this

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construction are the so-called matched pairs of groups. Subsequently, Majid in [12] generalized this notion to the context of Hopf algebras, and considered a more computational approach of the problem. The present paper is a contribution to the factorization problem for Hopf algebras.

In papers [3], [4] the authors proposed a strategy for classifying the bicrossed products of finite groups and Hopf algebras following Majids construction. The method proposed in [3] was followed in [5] to classify bicrossed products of two Sweedler's Hopf algebras, in [9], [10] to compute the automorphism of Drinfeld doubles of a purely nonabelian finite group and quasitriangular structure of the doubles of a finite group, respectively; then in [1] to classify bicrossed products of two Taft algebras, and finally in [2] to classify bicrossed products of Taft algebras and group algebras, where the group is a finite cyclic group.

In the 1960's, Kac and Paljutkin in [8] discovered a non-commutative, non-cocommutative semisimple Hopf algebra H_8 which is 8-dimensional. Later, in [13] Masouka showed that there is only one (up to isomorphisms) semisimple Hopf algebra of dimension 8, and presented it under the perspective of biproducts and bicrossed products. Ore extension is an important tool to study Hopf algebras, see [14], [17], [18], [19], [20], recently Pansera in [15] constructed H_8 from the point of view of Ore extension in the classification of the inner faithful Hopf actions of H_8 on the quantum plane.

As we all know, the 4-dimensional Sweedler's Hopf algebra H_4 is the simplest non-commutative, non-cocommutative semisimple Hopf algebra. In this paper, we will describe all the bicrossed products between H_8 and H_4, and prove that any Hopf algebra which factorizes through H_8 and H_4 must be isomorphic to one of the four Hopf algebras $H_8 \otimes H_4, H_{32,1}, H_{32,2}, H_{32,3}$ given explicitly in Theorem 5.1.

This paper is organized as follows. In Section 1, we recall the basic definitions and facts needed in our computations. In Section 2, we determine when H_4 becomes a left H_8-module coalgebra, and in Section 3, we determine when H_8 becomes a right H_4-module coalgebra. In Section 4, we find the suitable mutual actions between H_4 and H_8 such that they could make up matched pairs. And the bicrossed products are also given.

Throughout this paper, k will be an arbitrary algebraically closed field of characteristic zero. Unless otherwise specified, all algebras, coalgebras, bialgebras, Hopf algebras, tensor products and homomorphisms are over k.

2. Preliminaries

In this section, we will recall some basic definitions and facts.

Recall that Sweedler's 4-dimensional Hopf algebra, H_4, is generated by two elements G and X subject to the relations

$$G^2 = 1, \quad X^2 = 0, \quad XG = -GX.$$

The coalgebra structure and the antipode are given by

$$\Delta(G) = G \otimes G, \quad \varepsilon(G) = 1, \quad S(G) = G,$$
$$\Delta(X) = X \otimes G + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = GX.$$

It is well known that the set of group-like elements $G(H_4)$ and the set of primitive elements $P_{g,h}(H_4)$ are given as

$$G(H_4) = \{1, G\}, \quad P_{1,1}(H_4) = P_{G,G}(H_4) = \{0\},$$
$$P_{G,1}(H_4) = k(G-1) \oplus kX, \quad P_{1,G}(H_4) = k(G-1) \oplus kGX.$$

The Hopf algebra H_8 is generated by three elements g, h and z subject to the relations

$$g^2 = 1, \quad h^2 = 1, \quad gh = hg,$$
$$z^2 = \tfrac{1}{2}(1 + g + h - gh), \quad gz = zh, \quad hz = zg.$$

The coalgebra structure and antipode are given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \varepsilon(g) &= 1, & S(g) &= g, \\ \Delta(h) &= h \otimes h, & \varepsilon(h) &= 1, & S(h) &= h, \\ \Delta(z) &= J(z \otimes z), & \varepsilon(z) &= 1, & S(z) &= z, \end{aligned}$$

where $J = \frac{1}{2}(1 \otimes 1 + g \otimes 1 + 1 \otimes h - g \otimes h)$. Obviously H_8 is 8-dimensional with the basis $\{g^i h^j z^k\colon\ 0 \leqslant i, j, k \leqslant 1,\ i, j, k \in \mathbb{N}\}$. It is easy to verify that the element z is invertible with $z^4 = 1$.

A matched pair of Hopf algebras is a quadruple $(A, H, \triangleright, \triangleleft)$, where A and H are Hopf algebras, $\triangleright\colon\ H \otimes A \to A$ and $\triangleleft\colon\ H \otimes A \to H$ are coalgebra maps such that A is a left H-module coalgebra, H is a right A-module coalgebra and the following compatible conditions hold:

$$h \triangleright 1_A = \varepsilon(h)1_A, \quad 1_H \triangleleft a = \varepsilon(a)1_H, \tag{2.1}$$
$$h \triangleright (ab) = (h_1 \triangleright a_1)((h_2 \triangleleft a_2) \triangleright b), \tag{2.2}$$
$$(gh) \triangleleft a = (g \triangleleft (h_1 \triangleright a_1))(h_2 \triangleleft a_2), \tag{2.3}$$
$$h_1 \triangleleft a_1 \otimes h_2 \triangleright a_2 = h_2 \triangleleft a_2 \otimes h_1 \triangleright a_1, \tag{2.4}$$

for all $a, b \in A$, $g, h \in H$.

In this section we will find all the actions $\triangleright : H_8 \otimes H_4 \to H_4$ such that as a vector space, H_4 is made into a left H_8-module coalgebra satisfying $x \triangleright 1 = \varepsilon(x)1$ for all $x \in H_8$.

First of all, $g \triangleright 1 = h \triangleright 1 = z \triangleright 1 = 1$. Then $g \triangleright G \in G(H_4)$. We have $g \triangleright G \neq 1$ for otherwise $G = g^2 \triangleright G = 1$. Therefore $g \triangleright G = G$. Similarly $h \triangleright G = G$. Since

$$\begin{aligned}\Delta(z \triangleright G) &= \tfrac{1}{2}[z \triangleright G \otimes z \triangleright G + gz \triangleright G \otimes z \triangleright G + z \triangleright G \otimes hz \triangleright G - gz \triangleright G \otimes hz \triangleright G] \\ &= \tfrac{1}{2}[z \triangleright G \otimes z \triangleright G + zh \triangleright G \otimes z \triangleright G + z \triangleright G \otimes zg \triangleright G - zh \triangleright G \otimes zg \triangleright G] \\ &= z \triangleright G \otimes z \triangleright G,\end{aligned}$$

hence $z \triangleright G \in G(H_4)$. We have $z \triangleright G \neq 1$ for otherwise

$$1 = z \triangleright (z \triangleright G) = z^2 \triangleright G = \tfrac{1}{2}(1 + g + h - gh) \triangleright G = G.$$

Therefore $z \triangleright G = G$.

By computation we have $g \triangleright X \in P_{G,1}(H_4)$, thus $g \triangleright X = \alpha_1(G-1) + \beta_1 X$ for any $\alpha_1, \beta_1 \in k$. Since the action of g is compatible with $g^2 = 1$, we have

$$X = g^2 \triangleright X = g \triangleright (g \triangleright X) = (1 + \beta_1)\alpha_1(G-1) + \beta_1^2 X.$$

Thus $(1 + \beta_1)\alpha_1 = 0$ and $\beta_1^2 = 1$. Therefore

$$g \triangleright X = \alpha_1(G-1) - X \quad \text{or} \quad g \triangleright X = X. \tag{3.1}$$

Similarly for any $\alpha_2 \in k$

$$h \triangleright X = \alpha_2(G-1) - X \quad \text{or} \quad h \triangleright X = X. \tag{3.2}$$

Because the actions of g and h on X should be compatible with $gh = hg$, we need to check the actions given in (3.1) and (3.2). For example if $g \triangleright X = \alpha_1(G-1) - X$, $h \triangleright X = \alpha_2(G-1) - X$,

$$\begin{aligned} gh \triangleright X &= g \triangleright (\alpha_2(G-1) - X) = \alpha_2(G-1) - \alpha_1(G-1) + X, \\ hg \triangleright X &= h \triangleright (\alpha_1(G-1) - X) = \alpha_1(G-1) - \alpha_2(G-1) + X. \end{aligned}$$

Then $gh = hg$ forces $\alpha_1 = \alpha_2$. The other three cases can also be checked easily. Now we give the actions of g, h on G, X in the following tables:

$\triangleright^1$	1	G	X
1	1	G	X
g	1	G	X
h	1	G	X
gh	1	G	X

$\triangleright^2$	1	G	X
1	1	G	X
g	1	G	X
h	1	G	$\alpha(G-1)-X$
gh	1	G	$\alpha(G-1)-X$

$\triangleright^3$	1	G	X
1	1	G	X
g	1	G	$\alpha(G-1)-X$
h	1	G	X
gh	1	G	$\alpha(G-1)-X$

$\triangleright^4$	1	G	X
1	1	G	X
g	1	G	$\alpha(G-1)-X$
h	1	G	$\alpha(G-1)-X$
gh	1	G	X

For the action $z \triangleright X$, we have

$$\begin{aligned}\Delta(z \triangleright X) &= \tfrac{1}{2}[z \triangleright X \otimes G + 1 \otimes z \triangleright X + gz \triangleright X \otimes G + 1 \otimes z \triangleright X + z \triangleright X \otimes G \\ &\qquad + 1 \otimes hz \triangleright X - gz \triangleright X \otimes G - 1 \otimes hz \triangleright X] \\ &= z \triangleright X \otimes G + 1 \otimes z \triangleright X,\end{aligned}$$

hence $z \triangleright X \in P_{G,1}(H_4)$, and $z \triangleright X = \beta(G-1) + \gamma X$ for $\beta, \gamma \in k$. On one hand,

$$z^2 \triangleright X = z \triangleright (\beta(G-1) + \gamma X) = \beta(1+\gamma)(G-1) + \gamma^2 X.$$

On the other hand,

$$z^2 \triangleright X = \tfrac{1}{2}(1 + g + h - gh) \triangleright X.$$

Therefore the element $z \triangleright X$ is determined by the actions of g, h on X, and we will consider every case.

For the action $\triangleright^1$, we obtain $\beta(1+\gamma)(G-1) + \gamma^2 X = X$. Then

$$z \triangleright X = X, \quad \text{or} \quad z \triangleright X = \beta(G-1) - X.$$

It is straightforward to see that both the actions are compatible with the relation

$$gz = zh, \quad hz = zg. \tag{3.3}$$

For the action $\triangleright^2$, we have

$$z \triangleright X = X, \quad \text{or} \quad z \triangleright X = \beta(G-1) - X.$$

However, both the actions are not compatible with the relation (3.3). Similarly the action $\triangleright^3$ does not hold either.

For the action $\triangleright^4$, we have

$$\alpha(G-1) - X = \beta(1+\gamma)(G-1) + \gamma^2 X.$$

Thus $\gamma = \mathrm{i}$, $\alpha = (1+\mathrm{i})\beta$ or $\gamma = -\mathrm{i}$, $\alpha = (1-\mathrm{i})\beta$. Since

$$gz \triangleright X = \mathrm{i}[\beta(G-1) \pm X] = zh \triangleright X,$$

the action $\triangleright^4$ is well defined. So till now we have four actions, which are redenoted by $\triangleright^1$ and $\triangleright^2$, as follows:

$\triangleright^1$	1	G	X
1	1	G	X
g	1	G	X
h	1	G	X
z	1	G	X

$\triangleright^2$	1	G	X
1	1	G	X
g	1	G	X
h	1	G	X
z	1	G	$\alpha(G-1) - X$

$\triangleright^3$	1	G	X
1	1	G	X
g	1	G	$\alpha(G-1) - X$
h	1	G	$\alpha(G-1) - X$
z	1	G	$\frac{\alpha}{1+\mathrm{i}}(G-1) + \mathrm{i}X$

$\triangleright^4$	1	G	X
1	1	G	X
g	1	G	$\alpha(G-1) - X$
h	1	G	$\alpha(G-1) - X$
z	1	G	$\frac{\alpha}{1-\mathrm{i}}(G-1) - \mathrm{i}X$

Similarly we have the action of g, h and z on GX as follows:

$\triangleright^a$	GX
1	GX
g	GX
h	GX
z	GX

$\triangleright^b$	GX
1	GX
g	GX
h	GX
z	$\beta(G-1) - GX$

$\triangleright^c$	GX
1	GX
g	$\beta(G-1) - GX$
h	$\beta(G-1) - GX$
z	$\frac{\alpha}{1+\mathrm{i}}(G-1) + \mathrm{i}X$

$\triangleright^d$	GX
1	GX
g	$\beta(G-1) - GX$
h	$\beta(G-1) - GX$
z	$\frac{\beta}{1-\mathrm{i}}(G-1) - \mathrm{i}GX$

Proposition 3.1. *There are* 16 *kinds of actions of* H_8 *on* H_4 *defined as above.*

4. The right H_4-module coalgebra structures on H_8

In this section we will find all actions $\triangleleft : H_8 \otimes H_4 \to H_8$ such that as a vector space, H_8 is made into a right H_4-module coalgebra satisfying $1 \triangleleft x = \varepsilon(x)1$ for all $x \in H_4$.

Lemma 4.1.

(1) $G(H_8) = \{1, g, h, gh\}$.

(2) $P_{g^ih^j, g^mh^n}(H_8) = \alpha_{ijmn}(g^ih^j - g^mh^n)$ *for* $0 \leqslant i, j, m, n \leqslant 1$ *and* $\alpha_{ijmn} \in k$.

P r o o f. (1) Suppose that $x = \sum\limits_{i,j=0}^{1} f_{ij} g^i h^j + \sum\limits_{i,j=0}^{1} e_{ij} g^i h^j z$ is a group-like element of H_8. Then by $\Delta(x) = x \otimes x$, we have

$$\sum_{i,j,m,n=0}^{1} f_{ij} f_{mn} g^i h^j \otimes g^m h^n + \sum_{i,j,m,n=0}^{1} e_{ij} f_{mn} g^i h^j z \otimes g^m h^n + \sum_{i,j,m,n=0}^{1} e_{mn} f_{ij} g^i h^j \otimes g^m h^n z + \sum_{i,j,m,n=0}^{1} e_{ij} e_{mn} g^i h^j z \otimes g^m h^n z = \sum_{i,j=0}^{1} f_{ij} g^i h^j \otimes g^i h^j + \sum_{i,j=0}^{1} e_{ij} (g^i h^j \otimes g^i h^j) J(z \otimes z).$$

By comparison, we obtain that

$$\sum_{i,j,m,n=0}^{1} e_{ij} f_{m,n} g^i h^j z \otimes g^m h^n = 0, \qquad \sum_{i,j,m,n=0}^{1} e_{m,n} f_{ij} g^i h^j \otimes g^m h^n z = 0.$$

Hence for all $0 \leqslant i, j, m, n \leqslant 1$, $e_{ij} f_{m,n} = 0$, and

$$f_{00}^2 = f_{00}, \quad f_{10}^2 = f_{10}, \quad f_{01}^2 = f_{01}, \quad f_{11}^2 = f_{11},$$
$$f_{00} f_{01} = 0, \quad f_{00} f_{10} = 0, \quad f_{00} f_{11} = 0, \quad f_{10} f_{01} = 0, \quad f_{01} f_{11} = 0, \quad f_{10} f_{11} = 0.$$

From the above relations, we can see that $f_{ij} = 0$ or $f_{ij} = 1$, and if $f_{ij} = 1$ for some pair i, j, the rest must be 0.

If $f_{ij} = 0$ for $0 \leqslant i, j \leqslant 1$, since z is invertible, we have

$$\sum_{i,j,m,n=0}^{1} e_{ij} e_{mn} g^i h^j \otimes g^m h^n = \sum_{i,j=0}^{1} e_{ij} (g^i h^j \otimes g^i h^j) J.$$

By comparison of the coefficients, we get the relations

$$\begin{aligned}
&e_{00} = 2e_{00}^2, \quad e_{00} = 2e_{00}e_{10}, \quad e_{00} = 2e_{00}e_{01}, \quad e_{00} = -2e_{10}e_{01},\\
&e_{10} = 2e_{10}^2, \quad e_{10} = 2e_{00}e_{10}, \quad e_{10} = 2e_{10}e_{11}, \quad e_{10} = -2e_{00}e_{11},\\
&e_{01} = 2e_{01}^2, \quad e_{01} = 2e_{01}e_{11}, \quad e_{01} = 2e_{01}e_{00}, \quad e_{01} = -2e_{11}e_{00},\\
&e_{11} = 2e_{11}^2, \quad e_{11} = 2e_{11}e_{01}, \quad e_{11} = 2e_{11}e_{10}, \quad e_{11} = -2e_{10}e_{01}.
\end{aligned}$$

We claim that $e_{00} = 0$ for otherwise $e_{10} = e_{00} = e_{01} = \frac{1}{2}$, which contradicts $e_{00} = -2e_{10}e_{01}$. In the same manner, we obtain that $e_{01} = e_{10} = e_{11} = 0$.

(2) Suppose that $x = \sum_{i,j=0}^{1} f_{ij}g^ih^j + \sum_{i,j=0}^{1} e_{ij}g^ih^jz$ is a (g^sh^t, g^mh^n)-primitive element of H_8. We have

$$\begin{aligned}
&\sum_{i,j=0}^{1} f_{ij}g^ih^j \otimes g^sh^t + \sum_{i,j=0}^{1} f_{ij}g^mh^n \otimes g^ih^j + \sum_{i,j=0}^{1} e_{ij}g^ih^jz \otimes g^sh^t\\
&\quad + \sum_{i,j=0}^{1} e_{ij}g^mh^n \otimes g^ih^jz = \sum_{i,j=0}^{1} f_{ij}g^ih^j \otimes g^ih^j + \sum_{i,j=0}^{1} e_{ij}(g^ih^j \otimes g^ih^j)J(z \otimes z).
\end{aligned}$$

By comparison of the coefficients, $e_{ij} = 0$ $(0 \leqslant i, j \leqslant 1)$, and $f_{ij} = 0$ except f_{st}, f_{mn} with $f_{st} = -f_{mn}$. Therefore $x = f_{st}(g^sh^t - g^mh^n)$. The proof is completed. □

Now we will analyze the actions of G on the basis $(1, g, h, gh, z, gz, hz, ghz)$. It is obvious that elements $g \triangleleft G$, $h \triangleleft G$ and $gh \triangleleft G$ are all group-like elements of H_8. First, $g \triangleleft G \neq 1$ for otherwise $1 = g \triangleleft G^2 = g$. Also $h \triangleleft G \neq 1$, $gh \triangleleft G \neq 1$. Now we will list all possible cases:

(a) If $g \triangleleft G = g$, then $h \triangleleft G \neq g$ for otherwise $h = h \triangleleft G^2 = g \triangleleft G = g$. On the one hand assume that $h \triangleleft G = h$, since $(gh \triangleleft G) \triangleleft G = gh$, we obtain $gh \triangleleft G = gh$. On the other hand, if $h \triangleleft G = gh$, then $gh \triangleleft G = h \triangleleft G^2 = h$.

(b) If $g \triangleleft G = h$, then $h \triangleleft G = g$ and $gh \triangleleft G = gh$.

(c) If $g \triangleleft G = gh$, then $gh \triangleleft G = g$ and $h \triangleleft G = h$.

Assume that

$$z \triangleleft G = \sum_{i,j=0}^{1} f_{ij}g^ih^j + \sum_{i,j=0}^{1} e_{ij}g^ih^jz$$

for $f_{ij}, e_{ij} \in k$, $0 \leqslant i, j \leqslant 1$. Then we have

$$z = \sum_{i,j=0}^{1} (f_{ij}g^ih^j \triangleleft G) + \sum_{i,j=0}^{1} (e_{ij}g^ih^jz \triangleleft G).$$

Clearly $f_{ij} = 0$ $(0 \leqslant i, j \leqslant 1)$, and z could be linearly represented by $z \triangleleft G$, $gz \triangleleft G$, $hz \triangleleft G$, $ghz \triangleleft G$ which are linearly independent. So are gz, hz, ghz. Therefore there exists an invertible matrix $A = (a_{ij}) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of order 4 such that

$$(z \triangleleft G, gz \triangleleft G, hz \triangleleft G, ghz \triangleleft G) = (z, gz, hz, ghz)A.$$

Since $G^2 = 1$, we get $A^2 = E$. For $z \triangleleft G = a_{11}z + a_{21}gz + a_{31}hz + a_{41}ghz$, on the one hand

$$\begin{aligned}
\Delta(z \triangleleft G) &= a_{11}J(z \otimes z) + a_{21}(g \otimes g)J(z \otimes z) + a_{31}(h \otimes h)J(z \otimes z) \\
&\quad + a_{41}(gh \otimes gh)J(z \otimes z) \\
&= \tfrac{1}{2}a_{11}(z \otimes z + gz \otimes z + z \otimes hz - gz \otimes hz) \\
&\quad + \tfrac{1}{2}a_{21}(gz \otimes gz + z \otimes gz + gz \otimes ghz - z \otimes ghz) \\
&\quad + \tfrac{1}{2}a_{31}(hz \otimes hz + ghz \otimes hz + hz \otimes z - ghz \otimes z) \\
&\quad + \tfrac{1}{2}a_{41}(ghz \otimes ghz + hz \otimes ghz + ghz \otimes gz - hz \otimes gz) \\
&= [(z, gz, hz, ghz) \otimes (z, gz, hz, ghz)] \\
&\quad \times \tfrac{1}{2}(a_{11}, a_{21}, a_{11}, -a_{21}, a_{11}, a_{21}, -a_{11}, a_{21}, a_{31}, -a_{41}, a_{31}, a_{41}, -a_{31}, a_{41}, a_{31}, a_{41})^{\top}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\Delta(z \triangleleft G) &= \tfrac{1}{2}(z \triangleleft G \otimes z \triangleleft G + gz \triangleleft G \otimes z \triangleleft G + z \triangleleft G \otimes hz \triangleleft G - gz \triangleleft G \otimes hz \triangleleft G) \\
&= [(z, gz, hz, ghz) \otimes (z, gz, hz, ghz)] \\
&\quad \times \tfrac{1}{2}[\alpha_1 \otimes \alpha_1 + \alpha_2 \otimes \alpha_1 + \alpha_1 \otimes \alpha_3 - \alpha_2 \otimes \alpha_3].
\end{aligned}$$

Hence we have the relation

$$\begin{aligned}
(4.1)\quad &(a_{11}, a_{21}, a_{11}, -a_{21}, a_{11}, a_{21}, -a_{11}, a_{21}, a_{31}, -a_{41}, a_{31}, a_{41}, -a_{31}, a_{41}, a_{31}, a_{41})^{\top} \\
&= \alpha_1 \otimes \alpha_1 + \alpha_2 \otimes \alpha_1 + \alpha_1 \otimes \alpha_3 - \alpha_2 \otimes \alpha_3.
\end{aligned}$$

Similarly for $gz \triangleleft G$, $hz \triangleleft G$, $ghz \triangleleft G$, we have the relations

$$\begin{aligned}
(4.2)\quad &(a_{12}, a_{22}, a_{12}, -a_{22}, a_{12}, a_{22}, -a_{12}, a_{22}, a_{32}, -a_{42}, a_{32}, a_{42}, -a_{32}, a_{42}, a_{32}, a_{42})^{\top} \\
&= \alpha_2 \otimes \alpha_2 + \alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \alpha_4 - \alpha_1 \otimes \alpha_4, \\
(4.3)\quad &(a_{13}, a_{23}, a_{13}, -a_{23}, a_{13}, a_{23}, -a_{13}, a_{23}, a_{33}, -a_{43}, a_{33}, a_{43}, -a_{33}, a_{43}, a_{33}, a_{43})^{\top} \\
&= \alpha_3 \otimes \alpha_3 + \alpha_4 \otimes \alpha_3 + \alpha_3 \otimes \alpha_1 - \alpha_4 \otimes \alpha_1, \\
(4.4)\quad &(a_{14}, a_{24}, a_{14}, -a_{24}, a_{14}, a_{24}, -a_{14}, a_{24}, a_{34}, -a_{44}, a_{34}, a_{44}, -a_{34}, a_{44}, a_{34}, a_{44})^{\top} \\
&= \alpha_4 \otimes \alpha_4 + \alpha_3 \otimes \alpha_4 + \alpha_4 \otimes \alpha_2 - \alpha_3 \otimes \alpha_2.
\end{aligned}$$

Lemma 4.2. *In order to make H_8 be a right H_4 module coalgebra, the matrix A associated to the actions of G on the basis (z, gz, hz, ghz) must satisfy the conditions $A^2 = E$ and the identities* (4.1)–(4.4).

Example 4.3.

(1) Let $A = E$, then clearly A must satisfy the relations (4.1)–(4.4).

(2) Let A be the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

That is, $z \triangleleft G = ghz$, $gz \triangleleft G = hz$, $hz \triangleleft G = gz$, $ghz \triangleleft G = z$. By a long and tedious verification, $A^2 = E$ and satisfies the relations (4.1)–(4.4).

Next we will consider the actions of X on the basis of H_8. Since

$$\begin{aligned} \Delta(g \triangleleft X) &= g \triangleleft X \otimes g \triangleleft G + g \otimes g \triangleleft X, \\ \Delta(h \triangleleft X) &= h \triangleleft X \otimes h \triangleleft G + h \otimes h \triangleleft X, \\ \Delta(gh \triangleleft X) &= gh \triangleleft X \otimes gh \triangleleft G + gh \otimes gh \triangleleft X, \end{aligned}$$

we need to consider all the cases for the actions of G on g, h, gh.

(a) When $g \triangleleft G = g$, $h \triangleleft G = h$, $gh \triangleleft G = gh$, then $g \triangleleft X = h \triangleleft X = gh \triangleleft X = 0$.

(b) When $g \triangleleft G = g$, $h \triangleleft G = gh$, $gh \triangleleft G = h$, then $g \triangleleft X = 0$, and

$$h \triangleleft X = \alpha(h - gh), \quad gh \triangleleft X = \beta(h - gh).$$

Moreover, $GX = -XG$, $X^2 = 0$ implies $\alpha = \beta$.

(c) When $g \triangleleft G = h$, $h \triangleleft G = g$, $gh \triangleleft G = gh$, then $gh \triangleleft X = 0$, and

$$g \triangleleft X = \alpha(g - h), \quad h \triangleleft X = \alpha(g - h).$$

(d) When $g \triangleleft G = gh$, $h \triangleleft G = h$, $gh \triangleleft G = g$, then $h \triangleleft X = 0$, and

$$g \triangleleft X = \alpha(g - gh), \quad gh \triangleleft X = \alpha(g - gh).$$

Suppose that $z \triangleleft X = \sum_{i,j=0}^{1} f_{ij} g^i h^j + \sum_{i,j=0}^{1} e_{ij} g^i h^j z$.

$$\begin{aligned} &\sum_{i,j,m,n=0}^{1} f_{ij} f_{mn} g^i h^j \otimes g^m h^n + \sum_{i,j,m,n=0}^{1} e_{ij} f_{mn} g^i h^j z \otimes g^m h^n \\ &\quad + \sum_{i,j,m,n=0}^{1} e_{mn} f_{ij} g^i h^j \otimes g^m h^n z + \sum_{i,j,m,n=0}^{1} e_{ij} e_{mn} g^i h^j z \otimes g^m h^n z \\ &= \tfrac{1}{2}[z \triangleleft X \otimes z \triangleleft G + gz \triangleleft X \otimes z \triangleleft G + z \triangleleft X \otimes hz \triangleleft G - gz \triangleleft X \otimes hz \triangleleft G \\ &\quad + z \otimes z \triangleleft X + gz \otimes z \triangleleft X + z \otimes hz \triangleleft X - gz \otimes hz \triangleleft X]. \end{aligned}$$

Because the items like $g^i h^j \otimes g^m h^n$ will not appear on the right hand side, $f_{ij} = 0$ for all $0 \leqslant i, j \leqslant 1$. Thus $z \triangleleft X$ is a linear combination of z, gz, hz, ghz; so are $gz \triangleleft X$, $hz \triangleleft X$, $ghz \triangleleft X$. Therefore there exists a matrix $B = (\beta_1, \beta_2, \beta_3, \beta_4)$ of order 4 such that

$$(z \triangleleft X, gz \triangleleft X, hz \triangleleft X, ghz \triangleleft X) = (z, gz, hz, ghz)B.$$

The relations $GX = -XG$, $X^2 = 0$ imply $AB = -BA$, $B^2 = 0$, respectively.

Now we will consider $\Delta(z \triangleleft X), \Delta(gz \triangleleft X), \Delta(hz \triangleleft X)$ and $\Delta(ghz \triangleleft X)$. On the one hand,

$$\begin{aligned}
\Delta(z \triangleleft X) = {} & b_{11}J(z \otimes z) + b_{21}(g \otimes g)J(z \otimes z) + b_{31}(h \otimes h)J(z \otimes z) \\
& + b_{41}(gh \otimes gh)J(z \otimes z) \\
= {} & \tfrac{1}{2}b_{11}(z \otimes z + gz \otimes z + z \otimes hz - gz \otimes hz) \\
& + \tfrac{1}{2}b_{21}(gz \otimes gz + z \otimes gz + gz \otimes ghz - z \otimes ghz) \\
& + \tfrac{1}{2}b_{31}(hz \otimes hz + ghz \otimes hz + hz \otimes z - ghz \otimes z) \\
& + \tfrac{1}{2}b_{41}(ghz \otimes ghz + hz \otimes ghz + ghz \otimes gz - hz \otimes gz) \\
= {} & [(z, gz, hz, ghz) \otimes (z, gz, hz, ghz)] \\
& \times \tfrac{1}{2}(b_{11}, b_{21}, b_{11}, -b_{21}, b_{11}, b_{21}, -b_{11}, b_{21}, b_{31}, -b_{41}, b_{31}, b_{41}, -b_{31}, b_{41}, b_{31}, b_{41})^\top.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Delta(z \triangleleft X) = {} & \tfrac{1}{2}[z \triangleleft X \otimes z \triangleleft G + gz \triangleleft X \otimes z \triangleleft G + z \triangleleft X \otimes hz \triangleleft G - gz \triangleleft X \otimes hz \triangleleft G \\
& + z \otimes z \triangleleft X + gz \otimes z \triangleleft X + z \otimes hz \triangleleft X - gz \otimes hz \triangleleft X] \\
= {} & [(z, gz, hz, ghz) \otimes (z, gz, hz, ghz)] \\
& \times \tfrac{1}{2}[\beta_1 \otimes \alpha_1 + \beta_2 \otimes \alpha_1 + \beta_1 \otimes \alpha_3 - \beta_2 \otimes \alpha_3 + e_1 \otimes \beta_1 + e_2 \otimes \beta_1 \\
& + e_1 \otimes \beta_3 - e_2 \otimes \beta_3].
\end{aligned}$$

Therefore

$$\begin{aligned}
(4.5) \qquad & (b_{11}, b_{21}, b_{11}, -b_{21}, b_{11}, b_{21}, -b_{11}, b_{21}, b_{31}, -b_{41}, b_{31}, b_{41}, -b_{31}, b_{41}, b_{31}, b_{41})^\top \\
& = \beta_1 \otimes \alpha_1 + \beta_2 \otimes \alpha_1 + \beta_1 \otimes \alpha_3 - \beta_2 \otimes \alpha_3 + e_1 \otimes \beta_1 + e_2 \otimes \beta_1 \\
& \quad + e_1 \otimes \beta_3 - e_2 \otimes \beta_3, \\
(4.6) \qquad & (b_{12}, b_{22}, b_{12}, -b_{22}, b_{12}, b_{22}, -b_{12}, b_{22}, b_{32}, -b_{42}, b_{32}, b_{42}, -b_{32}, b_{42}, b_{32}, b_{42})^\top \\
& = \beta_2 \otimes \alpha_2 + \beta_1 \otimes \alpha_2 + \beta_2 \otimes \alpha_4 - \beta_1 \otimes \alpha_4 + e_2 \otimes \beta_2 + e_1 \otimes \beta_2 \\
& \quad + e_2 \otimes \beta_4 - e_1 \otimes \beta_4,
\end{aligned}$$

$$(b_{13}, b_{23}, b_{13}, -b_{23}, b_{13}, b_{23}, -b_{13}, b_{23}, b_{33}, -b_{43}, b_{33}, b_{43}, -b_{33}, b_{43}, b_{33}, b_{43})^\top \\ = \beta_3 \otimes \alpha_3 + \beta_4 \otimes \alpha_3 + \beta_3 \otimes \alpha_1 - \beta_4 \otimes \alpha_1 + e_3 \otimes \beta_3 + e_4 \otimes \beta_3 \\ + e_3 \otimes \beta_1 - e_4 \otimes \beta_1, \tag{4.7}$$

$$(b_{14}, b_{24}, b_{14}, -b_{24}, b_{14}, b_{24}, -b_{14}, b_{24}, b_{34}, -b_{44}, b_{34}, b_{44}, -b_{34}, b_{44}, b_{34}, b_{44})^\top \\ = \beta_4 \otimes \alpha_4 + \beta_3 \otimes \alpha_4 + \beta_4 \otimes \alpha_2 - \beta_3 \otimes \alpha_2 + e_4 \otimes \beta_4 + e_3 \otimes \beta_4 \\ + e_4 \otimes \beta_2 - e_3 \otimes \beta_2. \tag{4.8}$$

Lemma 4.4. *In order to make H_8 be a right H_4 module coalgebra, the matrix B associated to the actions of X on the basis (z, gz, hz, ghz) must satisfy the conditions $B^2 = 0$ and the identities (4.5)–(4.8).*

5. The matched pairs between H_8 and H_4

In this section, we will find the suitable actions which make H_8 and H_4 matched pairs. We first check the relation (2.4) for the pairs $(g, X), (h, X), (gh, X)$, which should satisfy the identities

$$g \triangleleft X \otimes G + g \otimes g \triangleright X = g \triangleleft X \otimes 1 + g \triangleleft G \otimes g \triangleright X,$$
$$h \triangleleft X \otimes G + h \otimes h \triangleright X = h \triangleleft X \otimes 1 + h \triangleleft G \otimes h \triangleright X,$$
$$gh \triangleleft X \otimes G + gh \otimes gh \triangleright X = gh \triangleleft X \otimes 1 + gh \triangleleft G \otimes gh \triangleright X.$$

When only $g \triangleleft G = g$, $h \triangleleft G = h$, $gh \triangleleft G = gh$, the above identities hold. At this moment, $g \triangleleft X = h \triangleleft X = gh \triangleleft X = 0$.

$$1 = z \triangleright G^2 = \tfrac{1}{2}[(z \triangleright G)((z \triangleleft G) \triangleright G) + (gz \triangleright G)((z \triangleleft G) \triangleright G) + (z \triangleright G)((hz \triangleleft G) \triangleright G) \\ - (gz \triangleright G)((hz \triangleleft G) \triangleright G)] \\ = G((z \triangleleft G) \triangleright G) = a_{11} + a_{21} + a_{31} + a_{41}.$$

For the pairs $(g^i h^j, G)$, $(g^i h^j z, G)$ the relation (2.4) is trivial.

On the one hand

$$z^2 \triangleleft G = \tfrac{1}{2}(1 + g + h - gh) \triangleleft G = z^2.$$

On the other hand by relation (2.3)

$$z^2 \triangleleft G = \tfrac{1}{2}[(z \triangleleft (z \triangleright G))(z \triangleleft G) + (z \triangleleft (gz \triangleright G))(z \triangleleft G) + (z \triangleleft (z \triangleright G))(hz \triangleleft G) \\ - (z \triangleleft (gz \triangleright G))(hz \triangleleft G)] \\ = (z \triangleleft G)(z \triangleleft G) = (a_{11}z + a_{21}gz + a_{31}hz + a_{41}ghz)^2.$$

By the comparison of coefficients, we have

$$\begin{cases} a_{11}^2 + 2a_{21}a_{31} + a_{41}^2 = 1, \\ (a_{11} + a_{41})(a_{21} + a_{31}) = 0, \\ a_{21}^2 + 2a_{11}a_{41} + a_{31}^2 = 0. \end{cases} \tag{5.1}$$

$$gz \triangleleft G = (g \triangleleft G)(z \triangleleft G) = a_{11}gz + a_{21}z + a_{31}ghz + a_{41}hz = (z \triangleleft G)h = zh \triangleleft G,$$

which is compatible with $gz = zh$. Thus

$$a_{11}gz + a_{21}z + a_{31}ghz + a_{41}hz = a_{12}z + a_{22}gz + a_{32}hz + a_{42}ghz.$$

Similarly, we have

(a) $hz \triangleleft G = zg \triangleleft G$, $a_{11}hz + a_{21}ghz + a_{31}z + a_{41}gz = a_{13}z + a_{23}gz + a_{33}hz + a_{43}ghz$.

(b) $ghz \triangleleft G = zgh \triangleleft G$, $a_{11}ghz + a_{21}hz + a_{31}gz + a_{41}z = a_{14}z + a_{24}gz + a_{34}hz + a_{44}ghz$.

So we get

$$a_{11} = a_{22} = a_{33} = a_{44}, \quad a_{21} = a_{12} = a_{43} = a_{34},$$

$$a_{31} = a_{42} = a_{13} = a_{24}, \quad a_{41} = a_{32} = a_{23} = a_{14}.$$

Therefore the matrix A has the form

$$\begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix},$$

where $a, b, c, d \in k$. Since $A^2 = E$, plus the relation (5.1), we have the set of equations

$$\begin{cases} a + b + c + d = 1, \\ a^2 + b^2 + c^2 + d^2 = 1, \\ ac + bd = 0, \\ ab + cd = 0, \\ ad + bc = 0, \\ a^2 + 2bc + d^2 = 1, \\ (a + d)(b + c) = 0, \\ b^2 + 2ad + c^2 = 0. \end{cases}$$

We obtain four solutions for the above set of equations:

(1) $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = \frac{1}{2}$, $d = -\frac{1}{2}$,

(2) $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $c = \frac{1}{2}$, $d = \frac{1}{2}$,

(3) $a = 1$, $b = c = d = 0$,

(4) $a = b = c = 0$, $d = 1$.

Now we will verify the actions $g^ih^j \triangleright GX$ and $g^ih^jz \triangleright GX$. For the action $\triangleright^1$

$$g \triangleright GX = (g \triangleright G)((g \triangleleft G) \triangleright X) = GX,$$
$$g \triangleright XG = (g \triangleright X)((g \triangleleft G) \triangleright G) + (g \triangleright 1)((g \triangleleft X) \triangleright G) = XG,$$

which is compatible with the relation $XG = -GX$. Similarly $h \triangleright GX = gh \triangleright GX = GX$.

$$\begin{aligned} z \triangleright GX &= \tfrac{1}{2}[(z \triangleright G)((z \triangleleft G) \triangleright X) + (gz \triangleright G)((z \triangleleft G) \triangleright X) + (z \triangleright G)((hz \triangleleft G) \triangleright X) \\ &\quad - (gz \triangleright G)((hz \triangleleft G) \triangleright X)] \\ &= \tfrac{1}{2}G[(z \triangleleft G) \triangleright X + (z \triangleleft G) \triangleright X + (hz \triangleleft G) \triangleright X - (hz \triangleleft G) \triangleright X] \\ &= G((z \triangleleft G) \triangleright X) = (a_{11} + a_{21} + a_{31} + a_{41})GX = GX, \end{aligned}$$

and

$$\begin{aligned} z \triangleright XG &= \tfrac{1}{2}[(z \triangleright X)((z \triangleleft G) \triangleright G) + ((z \triangleleft X) \triangleright G) + (gz \triangleright X)((z \triangleleft G) \triangleright G) \\ &\quad + ((z \triangleleft X) \triangleright G) + (z \triangleright X)((hz \triangleleft G) \triangleright G) + ((hz \triangleleft X) \triangleright G) \\ &\quad - (gz \triangleright X)((hz \triangleleft G) \triangleright G) - ((hz \triangleleft X) \triangleright G)] \\ &= X((z \triangleleft G) \triangleright G) + (z \triangleleft X) \triangleright G \\ &= (a_{11} + a_{21} + a_{31} + a_{41})XG + (b_{11} + b_{21} + b_{31} + b_{41})G \\ &= XG + (b_{11} + b_{21} + b_{31} + b_{41})G. \end{aligned}$$

Hence we obtain $b_{11} + b_{21} + b_{31} + b_{41} = 0$. That is, $z \triangleright GX = GX$.

$$\begin{aligned} g^ih^j \triangleright X^2 &= (g^ih^j \triangleright X)((g^ih^j \triangleleft G) \triangleright X) + (g^ih^j \triangleright 1)((g^ih^j \triangleleft X) \triangleright X) = X^2 = 0, \\ z \triangleright X^2 &= \tfrac{1}{2}[(z \triangleright X)((z \triangleleft G) \triangleright X) + ((z \triangleleft X) \triangleright X) + (gz \triangleright X)((z \triangleleft G) \triangleright X) \\ &\quad + ((z \triangleleft X) \triangleright X) + (z \triangleright X)((hz \triangleleft G) \triangleright X) + ((hz \triangleleft X) \triangleright X) \\ &\quad - (gz \triangleright X)((hz \triangleleft G) \triangleright X) - ((hz \triangleleft X) \triangleright X)] \\ &= \tfrac{1}{2}[X((z \triangleleft G) \triangleright X) + (z \triangleleft X) \triangleright X + X((z \triangleleft G) \triangleright X) + (z \triangleleft X) \triangleright X \\ &\quad + X((hz \triangleleft G) \triangleright X) + ((hz \triangleleft X) \triangleright X) - X((hz \triangleleft G) \triangleright X) \\ &\quad - ((hz \triangleleft X) \triangleright X)] \\ &= X((z \triangleleft G) \triangleright X) + (z \triangleleft X) \triangleright X = (b_{11} + b_{21} + b_{31} + b_{41})X = 0, \end{aligned}$$

which naturally holds.

Since $z^2 \triangleleft X = \frac{1}{2}(1 + g + h - gh) \triangleleft X = 0$,

$$\begin{aligned} z^2 \triangleleft X &= (z \triangleleft X)(z \triangleleft G) + z(z \triangleleft X) \\ &= (b_{11}z + b_{21}gz + b_{31}hz + b_{41}ghz)(az + bgz + chz + dghz) \\ &\quad + z(b_{11}z + b_{21}gz + b_{31}hz + b_{41}ghz) = 0. \end{aligned}$$

When the solution of matrix A is (1), by the comparison of coefficients we have

$$\begin{cases} 2b_{11} + b_{21} + b_{31} = 0, \\ b_{11} = b_{41}. \end{cases} \tag{5.2}$$

$$gz \triangleleft X = (g \triangleleft X)(z \triangleleft G) + g(z \triangleleft X) = g(z \triangleleft X) = (z \triangleleft X)h = zh \triangleleft X,$$
$$b_{12}z + b_{22}gz + b_{32}hz + b_{42}ghz = b_{11}gz + b_{21}z + b_{31}ghz + b_{41}hz.$$

Similarly it is straightforward to verify that $hz \triangleleft X = zg \triangleleft X$, $ghz \triangleleft X = zgh \triangleleft X$, and we obtain the relations

$$b_{13}z + b_{23}gz + b_{33}hz + b_{43}ghz = b_{11}hz + b_{21}ghz + b_{31}z + b_{41}gz,$$
$$b_{14}z + b_{24}gz + b_{34}hz + b_{44}ghz = b_{11}ghz + b_{21}hz + b_{31}gz + b_{41}z.$$

Hence we have

$$b_{11} = b_{22} = b_{33} = b_{44}, \quad b_{21} = b_{12} = b_{43} = b_{34},$$
$$b_{31} = b_{42} = b_{13} = b_{24}, \quad b_{41} = b_{14} = b_{23} = b_{32}.$$

So the matrix B has the form

$$\begin{pmatrix} p & q & r & s \\ q & p & s & r \\ r & s & p & q \\ s & r & q & p \end{pmatrix},$$

where $p, q, r, s \in k$. Since $B^2 = 0$, plus the relation (5.3), we have the set of equations

$$\begin{cases} 2p + q + r = 0, \\ p = s, \\ p^2 + q^2 + r^2 + s^2 = 0, \\ pq + rs = 0, \\ ps + qr = 0. \end{cases} \tag{5.3}$$

Then $p = q = r = s = 0$, that is, $B = 0$.

Now when the pair (z, X) satisfies the relation (2.4), we have

$$z \otimes X = z \triangleleft G \otimes X.$$

The above identity implies $z \triangleleft G = z$, which is a contradiction to our assumption.

By the same analysis, the second solution of A does not hold either. When $A = E$, it is easy to see that $B = 0$, and the relation (2.4) holds for the pair (z, X). The Hopf algebras (H_8, H_4) form a matched pair under the matrix $A = E, B = 0$.

For the fourth solution of A, it is easy to see that $B = 0$. However, since $z \triangleleft G = ghz$, the relation (2.4) is not valid for the pair (z, X).

For action $\triangleright^2$, $g \triangleright GX = h \triangleright GX = gh \triangleright GX = GX$.

$$
\begin{aligned}
z \triangleright GX &= G((z \triangleleft G) \triangleright X) = (a_{11} + a_{21} + a_{31} + a_{41})G(z \triangleright X) = \alpha(1 - G) - GX, \\
z \triangleright XG &= (z \triangleright X)((z \triangleleft G) \triangleright G) + (z \triangleleft X) \triangleright G \\
&= \alpha(1 - G) - XG + (b_{11} + b_{21} + b_{31} + b_{41})G,
\end{aligned}
$$

therefore $b_{11} + b_{21} + b_{31} + b_{41} = 0$.

$$
\begin{aligned}
z \triangleright X^2 &= (z \triangleright X)((z \triangleleft G) \triangleright X) + ((z \triangleleft X) \triangleright X) = (z \triangleright X)(z \triangleright X) + (z \triangleleft X) \triangleright X \\
&= (z \triangleright X)^2 + (b_{11} + b_{21} + b_{31} + b_{41})z \triangleright X = (z \triangleright X)^2 = 2\alpha^2(1 - G) + 2\alpha X.
\end{aligned}
$$

Since $X^2 = 0$, we have $\alpha = 0$. That is, $z \triangleright X = -X$.

Whatever solutions of A, we can get $B = 0$, and by the relation (2.3), when $A = E$, H_8 and H_4 make a matched pair.

For the action $\triangleright^3$,

$$
\begin{aligned}
g \triangleright GX &= (g \triangleright G)((g \triangleleft G) \triangleright X) = G(\alpha(G - 1) - X) = \alpha(1 - G) - GX, \\
g \triangleright XG &= (g \triangleright X)((g \triangleleft G) \triangleright G) + (g \triangleright 1)((g \triangleleft X) \triangleright G) = \alpha(1 - G) - XG.
\end{aligned}
$$

Then $\alpha(1 - G) - GX = -\alpha(1 - G) - GX$, which implies that $\alpha = 0$ and

$$
g \triangleright X = -X, \quad h \triangleright X = -X, \quad z \triangleright X = \mathrm{i}X, \quad g \triangleright GX = -GX.
$$

Similarly we can get $h \triangleright GX = -GX$.

$$
\begin{aligned}
z \triangleright GX &= G((z \triangleleft G) \triangleright X) = G(az \triangleright X + bgz \triangleright X + chz \triangleright X + dghz \triangleright X) \\
&= G(a\mathrm{i}X - b\mathrm{i}X - c\mathrm{i}X + d\mathrm{i}X) = (a - b - c + d)\mathrm{i}GX, \\
z \triangleright XG &= \tfrac{1}{2}[(z \triangleright X)((z \triangleleft G) \triangleright G) + ((z \triangleleft X) \triangleright G) - (z \triangleright X)((z \triangleleft G) \triangleright G)) \\
&\qquad + ((z \triangleleft X) \triangleright G + (z \triangleright X)((hz \triangleleft G) \triangleright G) + ((hz \triangleleft X) \triangleright G) \\
&\qquad + (z \triangleright X)((hz \triangleleft G) \triangleright G) - ((hz \triangleleft X) \triangleright G)] \\
&= (z \triangleleft X) \triangleright G + (z \triangleright X)((hz \triangleleft G) \triangleright G) \\
&= (b_{11} + b_{21} + b_{31} + b_{41})G + (a + b + c + d)\mathrm{i}XG.
\end{aligned}
$$

Therefore $b_{11}+b_{21}+b_{31}+b_{41}=0$, and $a+b+c+d=a-b-c+d$, which implies $b+c=0$. Hence we have $a=1$, $b=c=d=0$, or $a=b=c=0$, $d=1$. This is easy to check that $z \triangleright X^2 = (z \triangleright X)((hz \triangleleft G) \triangleright X) = 0$.

$$z^2 \triangleleft X = z(z \triangleleft X) + \mathrm{i}(z \triangleleft X)(hz \triangleleft G) = 0.$$

When $a=1$, $b=c=d=0$,

$$z(b_{11}z + b_{21}gz + b_{31}hz + b_{41}ghz) + \mathrm{i}(b_{11}z + b_{21}gz + b_{31}hz + b_{41}ghz)hz = 0,$$

which implies $b_{11}+b_{41}=0, b_{21}+b_{31}=0$. By the relation (2.3)

$$\begin{aligned} g(z \triangleleft X) &= gz \triangleleft X = zh \triangleleft X = -(z \triangleleft X)h, \\ h(z \triangleleft X) &= hz \triangleleft X = zg \triangleleft X = -(z \triangleleft X)g, \\ gh(z \triangleleft X) &= ghz \triangleleft X = zgh \triangleleft X = (z \triangleleft X)gh, \end{aligned}$$

we obtain that $B=0$. However, the pair (z, X) does not satisfy the relation (2.4).

When $a=b=c=0$, $d=1$, we also get $B=0$, and it is routine to verify that all the pairs satisfy the relation (2.4). Therefore H_8 and H_4 make a matched pair under the matrix A.

For the action $\triangleright^4$, by a similar computation, we have

$$\begin{gathered} g \triangleright X = h \triangleright X = -X, \quad z \triangleright X = -\mathrm{i}X, \\ g \triangleright GX = h \triangleright GX = -GX, \quad z \triangleright GX = -\mathrm{i}X, \\ a=b=c=0, \quad d=1. \end{gathered}$$

In summary, by direct computations we have the main result.

Theorem 5.1. *A Hopf algebra E factories through H_8 and H_4 if and only if*

(1) $E \cong H_8 \otimes H_4$.

(2) $E \cong H_{32,1}$ *subject to the relations:*

$$\begin{gathered} g^2 = h^2 = G^2 = 1, \quad gh = hg, \quad gz = zh, \quad hz = zg, \\ z^2 = \tfrac{1}{2}(1+g+h-gh), \quad X^2 = 0, \quad GX = -XG, \\ gG = Gg, \quad hG = Gh, \quad zG = Gz, \quad gX = Xg, \quad hX = Xh, \quad zX = -Xz. \end{gathered}$$

(3) $E \cong H_{32,2}$ *subject to the relations:*

$$\begin{gathered} g^2 = h^2 = G^2 = 1, \quad gh = hg, \quad gz = zh, \quad hz = zg, \\ z^2 = \tfrac{1}{2}(1+g+h-gh), \quad X^2 = 0, \quad GX = -XG, \\ gG = Gg, \quad hG = Gh, \quad gzG = Ghz, \quad gX = -Xg, \quad hX = -Xh, \quad zX = \mathrm{i}Xgz. \end{gathered}$$

(4) *$E \cong H_{32,3}$ subjecting to the relations:*

$$g^2 = h^2 = G^2 = 1, \quad gh = hg, \quad gz = zh, \quad hz = zg,$$
$$z^2 = \tfrac{1}{2}(1 + g + h - gh), \quad X^2 = 0, \quad GX = -XG,$$
$$gG = Gg, \quad hG = Gh, \quad gzG = Ghz, \quad gX = -Xg, \quad hX = -Xh, \quad zX = -\mathrm{i}Xgz.$$

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