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\mathcal{F} -HYPERCYCLIC AND DISJOINT \mathcal{F} -HYPERCYCLIC PROPERTIES
OF BINARY RELATIONS OVER TOPOLOGICAL SPACES

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Abstract. We examine various types of \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) and disjoint \mathcal{F} -hypercyclic (disjoint \mathcal{F} -topologically transitive) properties of binary relations over topological spaces. We pay special attention to finite structures like simple graphs, digraphs and tournaments, providing a great number of illustrative examples.

Keywords: \mathcal{F} -hypercyclic binary relation; \mathcal{F} -topologically transitive binary relation; disjoint \mathcal{F} -hypercyclic binary relation; disjoint \mathcal{F} -topologically transitive binary relation; digraph

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1. INTRODUCTION AND PRELIMINARIES

The notion of a continuous linear frequently hypercyclic operator acting on a separable Fréchet space was introduced by Bayart and Grivaux in 2006 (see [1]). From then on, a great number of authors working in the field of linear topological dynamics have analyzed the notion of frequent hypercyclicity, various generalizations of this concept and certain applications to abstract differential equations. Recently, upper frequent hypercyclic linear operators and \mathcal{F} -transitive linear operators have been investigated by Bonilla, Grosse-Erdmann in [6] and Bès, Menet, Peris, Puig in [3]. For more details on the subject, we refer the reader to [2], [3], [5], [??], [13], [17] and references cited therein.

The main aim of this paper is to continue the research studies [8], [9] and [14], [15]. We analyze \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) and disjoint \mathcal{F} -hypercyclic

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(disjoint \mathcal{F} -topologically transitive) properties of binary relations over topological spaces, focusing special attention on finite topological spaces which do not have a linear vector structure. Concerning similar problematic, one may refer e.g. to the papers by Martínez-Avendaño (see [16]), where the author investigated hypercyclic shifts on weighted L^p spaces of directed trees, Namayanja (see [20]), where chaotic phenomena in a transport equation on a network have been studied with the use of adjacency matrices of infinite line graphs. We present plenty of results and illustrative examples for simple graphs, digraphs and tournaments. With the exception of paper [15], where we have recently analyzed \mathcal{F} -hypercyclic extensions and disjoint \mathcal{F} -hypercyclic extensions of binary relations over topological spaces, the notions of \mathcal{F} -hypercyclicity and \mathcal{F} -topological transitivity have not been considered elsewhere in such a general framework. And, more to the point, with the exception of paper [15], disjoint \mathcal{F} -hypercyclicity and disjoint \mathcal{F} -topological transitivity seem to be not considered elsewhere even for linear continuous operators acting on Banach spaces.

The organization and main ideas of the paper are briefly described as follows. In Section 2, we introduce the notions of \mathcal{F} -hypercyclicity and disjoint \mathcal{F} -hypercyclicity for binary relations, giving also a few noteworthy observations and elementary consequences of definitions. We divide the third section of the paper in three separate subsections. In Subsection 3.1, we analyze \mathcal{F} -hypercyclicity and disjoint \mathcal{F} -hypercyclicity for general binary relations, on finite or infinite topological spaces, having or not a certain number of loops. We slightly extend the implications (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) of a recent result by Bonilla, Grosse-Erdmann (see [6], Theorem 15) in this context, and reformulate the notions introduced in the second section in terms of appropriate conditions on adjacency matrices (it is worth noting that [6], Theorem 15 is exceptional in the existing theory of linear topological dynamics because it is a rare result in which the pivot spaces need not be equipped with linear vector structures). In Subsection 3.2, we focus our attention on the simple graphs. In Proposition 3.5, we first prove that the notions of $d\mathcal{F}$ -hypercyclicity ($d\mathcal{F}$ -topological transitivity) and strong $d\mathcal{F}$ -hypercyclicity (strong $d\mathcal{F}$ -topological transitivity) coincide in the case that $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$, which is unquestionably the best explored in the existing literature. For a simple graph G , we introduce the index S_G and give some upper bounds for S_G in Theorem 3.7, concerning connected bipartite graphs, and Theorem 3.9, concerning connected non-bipartite graphs. In Theorem 3.11, we prove that connected non-bipartite graphs G_1, G_2, \dots, G_N are always (strongly) $d\mathcal{F}$ -hypercyclic (strongly $d\mathcal{F}$ -topologically transitive). Excluding Proposition 3.18, almost all structural results from Subsection 3.3 are devoted to the study of case $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$. We pay a special attention to the question whether the $d\mathcal{F}$ -hypercyclicity ($d\mathcal{F}$ -topological transitivity) of a given digraph G (digraphs G_1, G_2, \dots, G_N) automatically implies the strong $d\mathcal{F}$ -hypercyclicity (strong $d\mathcal{F}$ -

topological transitivity) of G (G_1, G_2, \dots, G_N). In Proposition 3.14, we prove that, if the number of nodes of a digraph G is less than or equal to 4, and G is equipped with an arbitrary topology, then the \mathcal{F} -hypercyclicity of G always implies the strong \mathcal{F} -hypercyclicity of G . For disjointness, we prove that the $d\mathcal{F}$ -hypercyclicity of G_1, G_2, \dots, G_N always implies the strong $d\mathcal{F}$ -hypercyclicity of G_1, G_2, \dots, G_N provided that the number of nodes of each digraph G_i is less than or equal to 3, $1 \leq i \leq N$. The main result of the paper is Theorem 3.17, where we completely solve the above question for tournaments. In addition to the above, we propose several open problems.

We use the standard notation henceforth. For any $s \in \mathbb{R}$, we set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$. Suppose that X, Y, Z and T are given nonempty sets. Let us recall that a binary relation between X into Y is any subset $\varrho \subseteq X \times Y$. If $\varrho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\varrho^{-1} \subseteq Y \times X$ and $\sigma \circ \varrho \subseteq X \times T$ by $\varrho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \varrho\}$ and

$$\sigma \circ \varrho := \{(x, t) \in X \times T : \exists y \in Y \cap Z \text{ such that } (x, y) \in \varrho \text{ and } (y, t) \in \sigma\},$$

respectively. The domain and range of ϱ are defined by $D(\varrho) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in \varrho\}$ and $R(\varrho) := \{y \in Y : \exists x \in X \text{ such that } (x, y) \in \varrho\}$, respectively; $\varrho(x) := \{y \in Y : (x, y) \in \varrho\}$ ($x \in X$), $x \varrho y \Leftrightarrow (x, y) \in \varrho$. Assuming ϱ is a binary relation on X and $n \in \mathbb{N}$, we define ϱ^n inductively; $\varrho^{-n} := (\varrho^n)^{-1}$ and $\varrho^0 := \{(x, x) : x \in X\}$. Set $D_\infty(\varrho) := \bigcap_{n \in \mathbb{N}} D(\varrho^n)$, $\varrho(X') := \{y : y \in \varrho(x) \text{ for some } x \in X'\}$ ($X' \subseteq X$) and $\mathbb{N}_n := \{1, \dots, n\}$ ($n \in \mathbb{N}$). By $P(A)$ and $\chi_A(\cdot)$, we denote the power set of A and the characteristic function of A , respectively.

If X and Y are topological spaces and $\varrho \subseteq X \times Y$, then we say that ϱ is *continuous* if for every open subset V of Y there exists an open subset U of X such that $\varrho^{-1}(V) = U \cap D(\varrho)$. This clearly holds provided that X is equipped with discrete topology.

2. \mathcal{F} -HYPERCYCLICITY AND DISJOINT \mathcal{F} -HYPERCYCLICITY OF BINARY RELATIONS: MAIN DEFINITIONS

Throughout the paper, we assume that X and Y are topological spaces, as well as that $N \in \mathbb{N}$ and $N \geq 2$. Suppose that \mathcal{F} is a nonempty collection of certain subsets of \mathbb{N} , i.e. $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. Observe that we do not require here that $|A| = \infty$ for all $A \in \mathcal{F}$ as well as that \mathcal{F} satisfies the following property:

- (I) $B \in \mathcal{F}$ whenever there exists $A \in \mathcal{F}$ such that $A \subseteq B$.

Let us recall that, if $\mathcal{F} \in P(P(\mathbb{N})) \setminus \{\emptyset\}$ satisfies (I), then it is said that \mathcal{F} is a *Fürstenberg family* (see [12]); if so, then we say that \mathcal{F} is a *proper Fürstenberg family* if $\emptyset \notin \mathcal{F}$.

For the sequel, we also need the notion of an *upper Fürstenberg family*; that is, any proper Fürstenberg family \mathcal{F} satisfying the following two conditions:

- (II) There exist a set D and a countable set M such that $\mathcal{F} = \bigcup_{\delta \in D} \bigcap_{\nu \in M} \mathcal{F}_{\delta, \nu}$, where for each $\delta \in D$ and $\nu \in M$ the following holds: If $A \in \mathcal{F}_{\delta, \nu}$, then there exists a finite subset $F \subseteq \mathbb{N}$ such that the implication $A \cap F \subseteq B \Rightarrow B \in \mathcal{F}_{\delta, \nu}$ holds true.
- (III) If $A \in \mathcal{F}$, then there exists $\delta \in D$ such that, for every $n \in \mathbb{N}$, we have $A - n \equiv \{k - n : k \in A, k > n\} \in \mathcal{F}_\delta$, where $\mathcal{F}_\delta \equiv \bigcap_{\nu \in M} \mathcal{F}_{\delta, \nu}$.

We would like to propose the following definition (observe that the introduced notion can be further generalized by assuming that two nonempty families $\tau_X \in P(P(X))$ and $\tau_Y \in P(P(Y))$ are given satisfying that (X, τ_X) and (Y, τ_Y) are not necessarily topological spaces).

Definition 2.1. Let $(\varrho_n)_{n \in \mathbb{N}}$ be a sequence of binary relations between the spaces X and Y , let ϱ be a binary relation on X , and let $x \in X$. Suppose that $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. Then we say that:

- (i) x is a *strong \mathcal{F} -hypercyclic element* of the sequence $(\varrho_n)_{n \in \mathbb{N}}$ if $x \in \bigcap_{n \in \mathbb{N}} D(\varrho_n)$ and for each $n \in \mathbb{N}$ there exists an element $y_n \in \varrho_n(x)$ such that for each open nonempty subset V of Y we have $\{n \in \mathbb{N} : y_n \in V\} \in \mathcal{F}$; $(\varrho_n)_{n \in \mathbb{N}}$ is said to be *strongly \mathcal{F} -hypercyclic* if there exists a strong \mathcal{F} -hypercyclic element of $(\varrho_n)_{n \in \mathbb{N}}$;
- (ii) ϱ is *strong \mathcal{F} -hypercyclic* if the sequence $(\varrho^n)_{n \in \mathbb{N}}$ is strong \mathcal{F} -hypercyclic; x is said to be a *strong \mathcal{F} -hypercyclic element* of ϱ if x is a strong \mathcal{F} -hypercyclic element of the sequence $(\varrho^n)_{n \in \mathbb{N}}$;
- (iii) x is an *\mathcal{F} -hypercyclic element* of the sequence $(\varrho_n)_{n \in \mathbb{N}}$ if $x \in \bigcap_{n \in \mathbb{N}} D(\varrho_n)$ and for each open nonempty subset V of Y we have

$$S(x, V) := \{n \in \mathbb{N} : \varrho_n x \cap V \neq \emptyset\} \in \mathcal{F};$$

- $(\varrho_n)_{n \in \mathbb{N}}$ is said to be *\mathcal{F} -hypercyclic* if there exists an \mathcal{F} -hypercyclic element of $(\varrho_n)_{n \in \mathbb{N}}$;
- (iv) ϱ is *\mathcal{F} -hypercyclic* if the sequence $(\varrho^n)_{n \in \mathbb{N}}$ is \mathcal{F} -hypercyclic; x is said to be an *\mathcal{F} -hypercyclic element* of ϱ if x is an \mathcal{F} -hypercyclic element of the sequence $(\varrho^n)_{n \in \mathbb{N}}$;
- (v) $(\varrho_n)_{n \in \mathbb{N}}$ is said to be *strongly \mathcal{F} -topologically transitive* if for every open nonempty subset $U \subseteq X$ and for every integer $n \in \mathbb{N}$ there exists an element $y_n \in \varrho_n(U)$ such that for each open nonempty subset V of Y we have $\{n \in \mathbb{N} : y_n \in V\} \in \mathcal{F}$;

- (vi) ϱ is *strongly \mathcal{F} -topologically transitive* if the sequence $(\varrho^n)_{n \in \mathbb{N}}$ is strongly \mathcal{F} -topologically transitive;
- (vii) $(\varrho_n)_{n \in \mathbb{N}}$ is said to be *\mathcal{F} -topologically transitive* if for every two open nonempty subsets $U \subseteq X$ and $V \subseteq Y$ we have

$$S(U, V) := \{n \in \mathbb{N} : \varrho_n(U) \cap V \neq \emptyset\} \in \mathcal{F};$$

- (viii) ϱ is *\mathcal{F} -topologically transitive* if the sequence $(\varrho^n)_{n \in \mathbb{N}}$ is \mathcal{F} -topologically transitive.

In any case set out above, the validity of (I) for \mathcal{F} yields that the strong \mathcal{F} -hypercyclicity (topological transitivity) implies, in turn, the \mathcal{F} -hypercyclicity (topological transitivity) of the considered sequence of binary relations (binary relation, element). This condition also ensures that, for every dynamical property introduced above, say \mathcal{F} -hypercyclicity, any extension of an \mathcal{F} -hypercyclic binary relation ϱ is likewise \mathcal{F} -hypercyclic (a similar statement holds for sequences of binary relations).

The notion introduced in [9] is recovered by setting that \mathcal{F} is a collection of all nonempty subsets of \mathbb{N} (in this case, generally, the notion of \mathcal{F} -hypercyclicity cannot be connected to that of \mathcal{F} -topological transitivity in any reasonable way). It is worth noting that, in [9], the notion of strong \mathcal{F} -hypercyclicity and strong \mathcal{F} -topological transitivity (as well as their disjoint analogues) are called hypercyclicity and topological transitivity, respectively. So, the notions introduced in parts (iii)–(iv) of Definition 2.1 as well as the notions introduced in parts (v)–(vi) of Definition 2.1 are new.

Definition 2.1 is rather general and covers some pathological cases completely unambiguous to be further explored. Furthermore, the following holds:

- (1) The validity of (i), or (iii) ((ii), or (iv)), implies that $\bigcap_{n \in \mathbb{N}} D(\varrho_n) \neq \emptyset$ ($D_\infty(\varrho) \neq \emptyset$).
- (2) The validity of (v) ((vi)) implies that $D(\varrho_n) \neq \emptyset$ for all $n \in \mathbb{N}$ ($D(\varrho^n) \neq \emptyset$ for all $n \in \mathbb{N}$) but not $\bigcap_{n \in \mathbb{N}} D(\varrho_n) \neq \emptyset$ ($D_\infty(\varrho) \neq \emptyset$). To illustrate this, consider first the case that $X = Y = \{x, y\}$ is equipped with discrete topology, $\varrho_{2n-1} := \{(x, y)\}$, $\varrho_{2n} := \{(y, x)\}$ ($n \in \mathbb{N}$) and $\{\mathbb{N}, 2\mathbb{N}, 2\mathbb{N} + 1\} \subseteq \mathcal{F}$. Then it can be easily seen that the sequence $(\varrho_n)_{n \in \mathbb{N}}$ is strongly \mathcal{F} -topologically transitive, $D(\varrho_n) \neq \emptyset$ for all $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} D(\varrho_n) = \emptyset$. For (vi), it is sufficient to consider any binary relation ϱ on $\mathbb{N} = X = Y$ satisfying that $D(\varrho^n) \neq \emptyset$ for all $n \in \mathbb{N}$ and $D_\infty(\varrho) = \emptyset$; then we can endow X and Y with the anti-discrete topology $\tau = \{\emptyset, \mathbb{N}\}$ and ϱ will be strongly \mathcal{F} -topologically transitive provided that $\mathbb{N} \in \mathcal{F}$.
- (3) In the case of considering parts (vii) and (viii), we do not need to have that $D(\varrho_n) = \emptyset$ for all $n \in \mathbb{N}$ ($D(\varrho^n) \neq \emptyset$ for all $n \in \mathbb{N}$); to see this, assume

that $X = Y = \{x, y\}$ is equipped with discrete topology, $\varrho := \{(x, y)\}$ and $\{\emptyset, \{1\}\} \subseteq \mathcal{F}$. Then ϱ is (not strongly) \mathcal{F} -topologically transitive, $D(\varrho^n) = \emptyset$ for all $n \geq 2$ and therefore $D_\infty(\varrho) = \emptyset$.

- (4) Assume $\mathbb{N} \notin \mathcal{F}$ and $D(\varrho_n) \neq \emptyset$ for all $n \in \mathbb{N}$ ($D_\infty(\varrho) \neq \emptyset$). Then we can easily see by plugging ($U = X$ and) $V = Y$ that the sequence $(\varrho_n)_{n \in \mathbb{N}}$ cannot satisfy any of the above introduced properties.

If $X = Y$ and $(\varrho_n)_{n \in \mathbb{N}}$ is a sequence of symmetric binary relations on X , then for each $x \in \bigcap_{n \in \mathbb{N}} D(\varrho_n)$ and for each open nonempty subset V of Y we have $S(x, V) + 2\mathbb{N} \subseteq S(x, V)$, so that $(\varrho_n)_{n \in \mathbb{N}}$ cannot be \mathcal{F} -hypercyclic if for each subset $A \in \mathcal{F}$ the assumption $A \neq \emptyset$ implies that $A + 2\mathbb{N}$ is not a subset of A ; furthermore, in this case, for every two open nonempty subsets $U \subseteq X$ and $V \subseteq X$, we have $S(U, V) = S(V, U)$.

Remark 2.2. Assume that x is a (strong) \mathcal{F} -hypercyclic element of a binary relation ϱ on X , $l \in \mathbb{N}$ and $x \in \varrho^l z$ for some element $z \in X$. If (for each open nonempty subset V of Y and for each sequence (ω_n) in X) for each open nonempty subset V of Y and for each $\omega \in \bigcap_{n \in \mathbb{N}} D(\varrho_n)$ the supposition $(\{n \in \mathbb{N} : \omega_{n+l} \in V\} \in \mathcal{F})$ implies $(\{n \in \mathbb{N} : \omega_n \in V\} \in \mathcal{F})$ $(\{n \in \mathbb{N} : \varrho^{n+l}\omega \in V\} \in \mathcal{F})$ implies $(\{n \in \mathbb{N} : \varrho^n \omega \in V\} \in \mathcal{F})$, then z is likewise a (strong) \mathcal{F} -hypercyclic element for ϱ . This, in particular, holds if \mathcal{F} is a collection of all subsets of \mathbb{N} which do have at least m elements, where $m \in \mathbb{N}_0$.

In the next two definitions, we consider disjoint analogues of the notions introduced in Definition 2.1:

Definition 2.3. Suppose that $\mathcal{F} \in P(P(\mathbb{N}))$, $\mathcal{F} \neq \emptyset$, $N \geq 2$, $(\varrho_{j,n})_{n \in \mathbb{N}}$ is a sequence of binary relations between the spaces X and Y , $1 \leq j \leq N$, ϱ_j is a binary relation on X , $1 \leq j \leq N$ and $x \in X$. Then we say:

- (i) x is a *strong $d\mathcal{F}$ -hypercyclic element of the sequences* $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ if for each $n \in \mathbb{N}$ there exist elements $y_{j,n} \in \varrho_{j,n}(x)$, $1 \leq j \leq N$ such that for every open nonempty subsets V_1, \dots, V_N of Y , we have $\{n \in \mathbb{N} : y_{1,n} \in V_1, y_{2,n} \in V_2, \dots, y_{N,n} \in V_N\} \in \mathcal{F}$; the *sequences* $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ are called *strongly $d\mathcal{F}$ -hypercyclic* if there exists a strong $d\mathcal{F}$ -hypercyclic element of $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$;
- (ii) x is a *strong $d\mathcal{F}$ -hypercyclic element of the binary relations* $\varrho_1, \dots, \varrho_N$ if x is a strong $d\mathcal{F}$ -hypercyclic element of the sequences $(\varrho_1^n)_{n \in \mathbb{N}}, \dots, (\varrho_N^n)_{n \in \mathbb{N}}$; the *binary relations* $\varrho_1, \dots, \varrho_N$ are called *strongly $d\mathcal{F}$ -hypercyclic* if there exists a strong $d\mathcal{F}$ -hypercyclic element of $\varrho_1, \dots, \varrho_N$;
- (iii) x is a *$d\mathcal{F}$ -hypercyclic element of the sequences* $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ if $x \in \bigcap_{1 \leq j \leq N, n \in \mathbb{N}} D_\infty(\varrho_{j,n})$ and for every open nonempty subsets V_1, \dots, V_N of Y , we

have $(V = (V_1, V_2, \dots, V_N))$

$S(x, V) := \{n \in \mathbb{N} : \varrho_{1,n}x \cap V_1 \neq \emptyset, \varrho_{2,n}x \cap V_2 \neq \emptyset, \dots, \varrho_{N,n}x \cap V_N \neq \emptyset\} \in \mathcal{F}$;

the sequences $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ are called *dF-hypercyclic* if there exists a *dF-hypercyclic element* of $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$;

- (iv) x is a *dF-hypercyclic element of the binary relations* $\varrho_1, \dots, \varrho_N$ if x is a *dF-hypercyclic element of the sequences* $(\varrho_1^n)_{n \in \mathbb{N}}, \dots, (\varrho_N^n)_{n \in \mathbb{N}}$; the *binary relations* $\varrho_1, \dots, \varrho_N$ are called *dF-hypercyclic* if there exists a *dF-hypercyclic element of* $\varrho_1, \dots, \varrho_N$.

Definition 2.4. Suppose that $\mathcal{F} \in P(P(\mathbb{N}))$, $\mathcal{F} \neq \emptyset$, $N \geq 2$, $(\varrho_{j,n})_{n \in \mathbb{N}}$ is a sequence of binary relations between the spaces X and Y , $1 \leq j \leq N$, and ϱ_j is a binary relation on X , $1 \leq j \leq N$. Then we say:

- (i) the sequences $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ are *strongly dF-topologically transitive* if for every open nonempty subset $U \subseteq X$ and for every open nonempty subsets V_1, \dots, V_N of Y , there exists an element $x \in U$ such that, for every integers $n \in \mathbb{N}$ and $j \in \mathbb{N}_N$, there exists an element $y_{j,n} \in \varrho_{j,n}x$ such that $\{n \in \mathbb{N} : y_{j,n} \in V_j \text{ for all } j \in \mathbb{N}_N\} \in \mathcal{F}$;
- (ii) the *binary relations* $\varrho_1, \dots, \varrho_N$ are called *strongly dF-topologically transitive* if the sequences $(\varrho_1^n)_{n \in \mathbb{N}}, \dots, (\varrho_N^n)_{n \in \mathbb{N}}$ are *strongly dF-topologically transitive*;
- (iii) the sequences $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ are *dF-topologically transitive* if for every open nonempty subset $U \subseteq X$ and for every open nonempty subsets V_1, \dots, V_N of Y , we have $\{n \in \mathbb{N} : (\exists x \in U) \varrho_{j,n}x \cap V_j \neq \emptyset \text{ for all } j \in \mathbb{N}_N\} \in \mathcal{F}$;
- (iv) the *binary relations* $\varrho_1, \dots, \varrho_N$ are *dF-topologically transitive* if the sequences $(\varrho_1^n)_{n \in \mathbb{N}}, \dots, (\varrho_N^n)_{n \in \mathbb{N}}$ are *dF-topologically transitive*.

If the binary relations $\varrho_1, \dots, \varrho_N$ are (strongly) *dF-hypercyclic* ((strongly) *dF-topologically transitive*), then we also say that the N -tuple $(\varrho_1, \dots, \varrho_N)$ is *strong dF-hypercyclic* ((strongly) *dF-topologically transitive*) and vice versa.

We have the following simple observations:

- (1) Definition 2.3: The validity of (i), or (iii) ((ii), or (iv)), implies that

$$\bigcap_{n \in \mathbb{N}, j \in \mathbb{N}_N} D(\varrho_{j,n}) \neq \emptyset \quad \left(\bigcap_{j \in \mathbb{N}_N} D_\infty(\varrho_j) \neq \emptyset \right).$$

- (2) Definition 2.4: The validity of (i) ((ii)) implies that $\bigcap_{j \in \mathbb{N}_N} D(\varrho_{j,n}) \neq \emptyset$ for all $n \in \mathbb{N}$ ($\bigcap_{j \in \mathbb{N}_N} D(\varrho_j^n) \neq \emptyset$ for all $n \in \mathbb{N}$) but not $\bigcap_{n \in \mathbb{N}} D(\varrho_{j,n}) \neq \emptyset$ for some $j \in \mathbb{N}_N$ ($D_\infty(\varrho_j) \neq \emptyset$ for some $j \in \mathbb{N}_N$).

- (3) Definition 2.4: The validity of (iii) ((iv)) does not imply that there exists $j \in \mathbb{N}_N$ such that $D(\varrho_{j,n}) \neq \emptyset$ for all $n \in \mathbb{N}$ ($D(\varrho_j^n) \neq \emptyset$ for all $n \in \mathbb{N}$); to verify this, let $X = \{x, y\}$ be equipped with the topology $\tau_1 = \{\emptyset, \{x\}, \{x, y\}\}$, let $Y = \{x, y\}$ be equipped with the topology $\tau_2 = \{\emptyset, \{y\}, \{x, y\}\}$, and let $\varrho_1 = \varrho_2 = \{(x, y)\}$. Suppose that $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$. Then $D(\varrho_1^n) = \emptyset$ for $n \geq 2$, ϱ_1 and ϱ_2 are $d\mathcal{F}$ -topologically transitive but not strongly $d\mathcal{F}$ -topologically transitive (see also Remark 3.16 below).
- (4) Assume $\mathbb{N} \notin \mathcal{F}$. If $\bigcap_{j \in \mathbb{N}_N} D(\varrho_{j,n}) \neq \emptyset$ for all $n \in \mathbb{N}$ ($\bigcap_{j \in \mathbb{N}_N} D(\varrho_j^n) \neq \emptyset$ for all $n \in \mathbb{N}$), then the sequences $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ (binary relations $\varrho_1, \dots, \varrho_N$) cannot satisfy any of the above introduced disjoint properties.

Remark 2.5.

- (i) In parts (i) and (iii) of Definition 2.3, the topology on X does not play any role. We can assume that Y is equipped with arbitrary topologies $\tau_1^Y, \dots, \tau_N^Y$ and that, for every $i \in \mathbb{N}_N$, V_i is open for the topology τ_i^Y .
- (ii) In parts (ii) and (iv) of Definition 2.3, we can assume that $X = Y$ is equipped with arbitrary topologies $\tau_1^X, \dots, \tau_N^X$ and that, for every $i \in \mathbb{N}_N$, V_i is open for the topology τ_i^X .

Similar observations can be given for Definition 2.4; albeit a great number of our results continues to hold with this extended notion, we will analyze henceforth only the usually considered case that X is equipped with exactly one topology and Y is equipped with exactly one topology.

We round off this section by stating the following simple proposition, stated here without a corresponding proof which can be left to the interested reader:

Proposition 2.6.

- (i) Suppose that the sequences $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ of binary relations between the spaces X and Y , or the binary relations $\varrho_1, \varrho_2, \dots, \varrho_N$ on X , are $d\mathcal{F}$ -hypercyclic (or $d\mathcal{F}$ -topologically transitive). Then for each $j \in \mathbb{N}_N$ the sequence $(\varrho_{j,n})_{n \in \mathbb{N}}$, or the binary relation ϱ_j , is \mathcal{F} -hypercyclic (or \mathcal{F} -topologically transitive) provided that $(\varrho_{1,n})_{n \in \mathbb{N}} = (\varrho_{2,n})_{n \in \mathbb{N}} = \dots = (\varrho_{N,n})_{n \in \mathbb{N}}$, or $\varrho_1 = \varrho_2 = \dots = \varrho_N$, or that the condition (I) holds for \mathcal{F} .
- (ii) Suppose that the sequences $(\varrho_{1,n})_{n \in \mathbb{N}}, \dots, (\varrho_{N,n})_{n \in \mathbb{N}}$ of binary relations between the spaces X and Y , or the binary relations $\varrho_1, \varrho_2, \dots, \varrho_N$ on X , are strongly $d\mathcal{F}$ -hypercyclic (or strongly $d\mathcal{F}$ -topologically transitive). Then for each $j \in \mathbb{N}_N$ the sequence $(\varrho_{j,n})_{n \in \mathbb{N}}$, or the binary relation ϱ_j , is strongly \mathcal{F} -hypercyclic (or strongly \mathcal{F} -topologically transitive) provided that the condition (I) holds for \mathcal{F} .

3. RESULTS FOR GENERAL BINARY RELATIONS,
SIMPLE GRAPHS AND DIGRAPHS

As already mentioned, we will divide this section into three separate subsections. In the first, we will present a few results about \mathcal{F} -hypercyclicity of general binary relations.

3.1. Results for general binary relations. For the sequel, set $D := \bigcap_{n \in \mathbb{N}} D(\varrho_n)$ and $\check{\varrho}_n := \{(x, y) \in \varrho_n : x \in D\}$, $n \in \mathbb{N}$; ($\check{\varrho} := \{(x, y) \in \varrho : x \in D_\infty(\varrho)\}$). Then $\check{\varrho}_n$ ($\check{\varrho}$) is a binary relation between D and Y , with $D(\check{\varrho}_n) = D$ for all $n \in \mathbb{N}$ ($D_\infty(\varrho)$) and Y , with $D(\check{\varrho}) = D_\infty(\varrho)$.

The following proposition holds true:

Proposition 3.1. *Let $(\varrho_n)_{n \in \mathbb{N}}$ be a sequence of binary relations between the spaces X and Y , let ϱ be a binary relation on X , and let $x \in X$. Then x is a (strong) \mathcal{F} -hypercyclic element of the sequence $(\varrho_n)_{n \in \mathbb{N}}$ if and only if x is a (strong) \mathcal{F} -hypercyclic element of the sequence $(\check{\varrho}_n)_{n \in \mathbb{N}}$; in particular, $(\varrho_n)_{n \in \mathbb{N}}$ is (strongly) \mathcal{F} -hypercyclic if and only if $(\check{\varrho}_n)_{n \in \mathbb{N}}$ is (strongly) \mathcal{F} -hypercyclic.*

In a certain sense, the above proposition shows that it is sufficient to introduce the notions of (strong) \mathcal{F} -hypercyclicity only for binary relations whose domain is the whole space X . But, this is actually not the case because we need to know some further properties of D in X ; for example, in the next generalization of implications (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) of [6], Theorem 15, we impose the condition that the subspace D of X is a Baire space, which particularly holds in the following two special cases: X is a Baire space and D is open in X or X is a complete metric space and D is a closed subspace of X :

Theorem 3.2. *Let $(\varrho_n)_{n \in \mathbb{N}}$ be a sequence of binary relations between the topological spaces X and Y , let the subspace D of X be a Baire space, and let Y be second-countable. Assume that $\check{\varrho}_n \subseteq D \times Y$ is continuous for all $n \in \mathbb{N}$. If \mathcal{F} is a Fürstenberg family and (II) holds, then we have (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv), where:*

- (i) *For any nonempty open subset V of Y there is some $\delta \in D$ such that for any nonempty open subset U of X such that $U \cap D \neq \emptyset$ there is some $x \in U \cap D$ such that $\{n \in \mathbb{N} : \varrho_n x \cap V \neq \emptyset\} \in \mathcal{F}_\delta$.*
- (ii) *For any nonempty open subset V of Y there is some $\delta \in D$ such that, for any nonempty open subset U of X such that $U \cap D \neq \emptyset$ and for every $\nu \in M$ there is some $x \in U \cap D$ such that $\{n \in \mathbb{N} : \varrho_n x \cap V \neq \emptyset\} \in \mathcal{F}_{\delta, \nu}$.*
- (iii) *The set consisting of all \mathcal{F} -hypercyclic vectors of $(\varrho_n)_{n \in \mathbb{N}}$ is residual in D .*
- (iv) *The sequence $(\varrho_n)_{n \in \mathbb{N}}$ is \mathcal{F} -hypercyclic.*

Proof. The implications (i) \Rightarrow (ii) and (iii) \Rightarrow (iv) are trivial and all that we need to show is that (ii) implies (iii). For this, we can repeat almost literally the arguments given in the proof of the corresponding implication (b) \Rightarrow (c) of [6], Theorem 15, with the sequence (T_n) and term $T_n x \in V_k$ replaced therein by the sequence of continuous relations $(\check{\varrho}_n)_{n \in \mathbb{N}}$ and the term $\varrho_n x \cap V_k \neq \emptyset$, showing that the set consisting of all \mathcal{F} -hypercyclic vectors of $(\check{\varrho}_n)_{n \in \mathbb{N}}$ is residual in D . After that, we can apply Proposition 3.1. \square

In contrast to [6], we do not use the conditions that X and Y are metric spaces, as well as the condition (III). Keeping in mind Proposition 3.1 and Theorem 3.2, it seems plausible that the assertions of \mathcal{A} -Hypercyclicity Criterion (see [6], Theorem 22) and \mathcal{A} -Hypercyclicity Criterion, second version (see [6], Theorem 26), can be extended to continuous multivalued linear operators (see [8] for the notion of a multivalued linear operator; the continuity is understood in the sense of continuity of a general binary relation). It also seems plausible that a great number of other Hypercyclicity Criteria known in the existing literature can be formulated for continuous multivalued linear operators. We will not discuss these questions in more detail here.

Now we will turn our attention in another direction, by giving a few useful observations in the case that X is equipped with discrete topology or anti-discrete topology. Suppose first that X carries the anti-discrete topology $\tau = \{\emptyset, X\}$ and binary relations $\varrho, \varrho_1, \dots, \varrho_N$ on X are given. Then any vector $x \in D_\infty(\varrho)$ is a strong \mathcal{F} -hypercyclic vector for ϱ (strong $d\mathcal{F}$ -hypercyclic vector for $\varrho_1, \dots, \varrho_N$), so that the notions of \mathcal{F} -hypercyclicity and strong \mathcal{F} -hypercyclicity ($d\mathcal{F}$ -hypercyclicity and strong $d\mathcal{F}$ -hypercyclicity) coincide; the same holds for the notions of \mathcal{F} -topological transitivity and strong \mathcal{F} -topological transitivity ($d\mathcal{F}$ -topological transitivity and strong $d\mathcal{F}$ -topological transitivity). This is no longer true in the case that τ is not the anti-discrete topology and we will illustrate this only for the \mathcal{F} -hypercyclicity: let $X = \{x_1, x_2\}$, $\varrho = \{(x_1, x_2), (x_2, x_2)\}$ and $\tau = \{\emptyset, \{x_2\}, \{x_1, x_2\}\}$. Then both x_1 and x_2 are \mathcal{F} -hypercyclic vectors for ϱ , while x_2 is the only strong \mathcal{F} -hypercyclic vector for ϱ .

If $X = \{x_1, x_2, \dots, x_n\}$ carries the discrete topology, then an element $x \in X$ is an \mathcal{F} -hypercyclic vector for a binary relation ϱ on X if and only if for every nonempty subset V of \mathbb{N}_n we have $\{k \in \mathbb{N} : (\exists i \in V) x_i \in \varrho^k x\} \in \mathcal{F}$. Since

$$\{k \in \mathbb{N} : (\exists i \in V) x_i \in \varrho^k x\} = \bigcup_{i \in V} \{k \in \mathbb{N} : x_i \in \varrho^k x\},$$

we have the following: Assume that \mathcal{F} is closed under finite unions. Then $x \in X$ is an \mathcal{F} -hypercyclic vector for ϱ if and only if $x \in D_\infty(\varrho)$ and for each $i \in \mathbb{N}_n$ we have $\{k \in \mathbb{N} : x_i \in \varrho^k x\} \in \mathcal{F}$. Arguing similarly we can prove that, under the same

assumption on \mathcal{F} , $x \in X$ is a $d\mathcal{F}$ -hypercyclic vector for binary relations $\varrho_1, \varrho_2, \dots, \varrho_N$ on X if and only if $x \in D_\infty(\varrho_j)$ for $1 \leq j \leq N$ and for any choice of elements $x_{i_1}, x_{i_2}, \dots, x_{i_n}$ in X , $1 \leq i_s \leq n$ for $s \in \mathbb{N}_N$ we have $\bigcap_{j \in \mathbb{N}_N} \{k \in \mathbb{N} : x_{i_j} \in \varrho_j^k x\} \in \mathcal{F}$.

Consider the case that $X = Y = \{x_1, x_2, \dots, x_n\}$ is equipped with arbitrary topology. For any binary relation ϱ on X , by $[\varrho]$ we denote the adjacency matrix of ϱ , defined by $a_{ij} := 1$ if $x_i \varrho x_j$, and $a_{ij} := 0$ otherwise. By a ϱ -walk, we mean any finite sequence $x_{i_1} x_{i_2} \dots x_{i_s}$, where $s \in \mathbb{N} \setminus \{1\}$, $1 \leq i_j \leq n$ for $1 \leq j \leq s$ and $x_{i_j} \varrho x_{i_{j+1}}$ for $1 \leq j \leq s-1$; the length of $x_{i_1} x_{i_2} \dots x_{i_s}$ is said to be s , while x_{i_1} and x_{i_s} are said to be the starting and the ending point of $x_{i_1} x_{i_2} \dots x_{i_s}$. We also say that $x_{i_1} x_{i_2} \dots x_{i_s}$ is an $(x_{i_1} - x_{i_s})$ ϱ -walk. Set $[\varrho]^k := [a_{i,j}^k]_{1 \leq i,j \leq n}$, $k \in \mathbb{N}$. Arguing as in the case of simple graphs (see [21]), we can simply conclude that the number of different $(x_i - x_j)$ ϱ -walks of length k equals $a_{i,j}^k$, $1 \leq i, j \leq n$, $k \in \mathbb{N}$. This fact enables one to simply reformulate the notion introduced in Definition 2.1 in terms of appropriate conditions involving the adjacency matrix $[\varrho]$:

Proposition 3.3.

- (i) x_i is an \mathcal{F} -hypercyclic vector for ϱ , $i \in \mathbb{N}_n$ if and only if for every $k \in \mathbb{N}$ there exists a ϱ -walk of length k starting at x_i and for each open nonempty subset V of X we have $\{k \in \mathbb{N} : (\exists j \in V) a_{ij}^k \geq 1\} \in \mathcal{F}$.
- (ii) ϱ is \mathcal{F} -topologically transitive if and only if for each pair of open nonempty subsets U, V of X we have $\{k \in \mathbb{N} : (\exists i \in U) (\exists j \in V) a_{ij}^k \geq 1\} \in \mathcal{F}$.

The situation is quite similar for disjointness. If $[\varrho_s] = [a_{ij}^s]_{1 \leq i,j \leq n}$ is the adjacency matrix of a binary relation ϱ_s on X , then we denote $[\varrho_s]^k = [a_{ij}^{k;s}]_{1 \leq i,j \leq n}$, $s \in \mathbb{N}_N$, $k \in \mathbb{N}$. We have the following:

Proposition 3.4. Let ϱ_s be a binary relation on X , $1 \leq s \leq N$.

- (i) x_i is a $d\mathcal{F}$ -hypercyclic vector for $\varrho_1, \varrho_2, \dots, \varrho_N$, $i \in \mathbb{N}_n$ if and only if for every $k \in \mathbb{N}$ and $s \in \mathbb{N}_N$ there exists a ϱ_s -walk of length k starting at x_i and for each open nonempty subsets V_s of X_s , $1 \leq s \leq N$ we have $\{k \in \mathbb{N} : (\forall s \in \mathbb{N}_N) (\exists j_s \in V_s) a_{ij_s}^{k;s} \geq 1\} \in \mathcal{F}$.
- (ii) $\varrho_1, \varrho_2, \dots, \varrho_N$ are $d\mathcal{F}$ -topologically transitive if and only if for each open nonempty subsets U, V_1, V_2, \dots, V_N of X we have $\{k \in \mathbb{N} : (\exists i \in U) (\forall s \in \mathbb{N}_N) (\exists j_s \in V_s) a_{ij_s}^{k;s} \geq 1\} \in \mathcal{F}$.

3.2. Results for simple graphs. Let $X = G = \{x_1, x_2, \dots, x_n\}$ be finite, let $|G| > 1$, and let ϱ be a symmetric relation on G such that, for every $g \in G$, we have $(g, g) \notin \varrho$. As is well-known, (G, ϱ) is said to be a simple graph (see [4], [7]

and [21] for the basic theory of graphs). By $E(G)$ we denote the set consisting of all unoriented arcs of G . The notion of distance $d(u, v)$ of two nodes $u, v \in G$, as well as the notions of diameter $d(G)$ of a graph G , walks, paths and closed contours in G are defined as usual (let us only recall that $d(u, u) = 0$, $u \in G$). By $[A(G)]$ we denote the adjacency matrix of G . For more details about applications of the matrix theory to graphs, we refer the reader to the monographs [10] and [11].

Suppose, for the time being, that $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$. Then it can be easily seen that the graph G , equipped with discrete topology, is connected if and only if G is (strongly) \mathcal{F} -hypercyclic if and only if G is (strongly) \mathcal{F} -topologically transitive (see [9]); if this is the case, then any element of G is a (strong) hypercyclic element of ϱ . Furthermore, if G is equipped with arbitrary topology, then G is \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) if and only if G is strongly \mathcal{F} -hypercyclic (strongly \mathcal{F} -topologically transitive). For disjointness, a similar statement holds true:

Proposition 3.5. *Let $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$. Suppose that G_1, G_2, \dots, G_N are given graphs with the same set of nodes $X = \{x_1, x_2, \dots, x_n\}$. Then G_1, G_2, \dots, G_N are $d\mathcal{F}$ -hypercyclic ($d\mathcal{F}$ -topologically transitive) if and only if G_1, G_2, \dots, G_N are strongly $d\mathcal{F}$ -hypercyclic (strongly $d\mathcal{F}$ -topologically transitive).*

Proof. We will prove the statement only for $d\mathcal{F}$ -hypercyclicity and strong $d\mathcal{F}$ -hypercyclicity. Since the condition (I) holds, we only need to show that $d\mathcal{F}$ -hypercyclicity of G_1, G_2, \dots, G_N implies their strong $d\mathcal{F}$ -hypercyclicity. Let $x \in X$ be a $d\mathcal{F}$ -hypercyclic vector of G_1, G_2, \dots, G_N ; we will prove that $x \in X$ is a strong $d\mathcal{F}$ -hypercyclic vector of G_1, G_2, \dots, G_N . Let a_n denote the number of open non-empty subsets of G_i , which will be denoted by V_s , $1 \leq s \leq a_n$. Set $r_n := a_n^N$. Let the tuples $(V_1, V_1, \dots, V_1), \dots, (V_{a_n}, V_{a_n}, \dots, V_{a_n})$ be listed in some alphabetic order. For the tuple (V_1, V_1, \dots, V_1) , we know that there exist a positive integer $k_1 \in \mathbb{N}$ and elements $y_{1,k_1} \in \varrho_1^{k_1} x \cap V_1, \dots, y_{N,k_1} \in \varrho_N^{k_1} x \cap V_1$. By the symmetry of relations $\varrho_1, \varrho_2, \dots, \varrho_N$, for the tuple $(V_1, V_1, \dots, V_1, V_2)$, there exist a positive integer $k_2 > k_1$ and elements $y_{1,k_2} \in \varrho_1^{k_2} x \cap V_1, \dots, y_{N-1,k_2} \in \varrho_{N-1}^{k_2} x \cap V_1, y_{N,k_2} \in \varrho_N^{k_2} x \cap V_2$. Repeating this procedure, for the tuple $(V_{a_n}, V_{a_n}, \dots, V_{a_n})$, there exist a positive integer $k_{r_n} > k_{r_n-1}$ and elements $y_{1,k_{r_n}} \in \varrho_1^{k_{r_n}} x \cap V_{a_n}, \dots, y_{N,k_{r_n}} \in \varrho_N^{k_{r_n}} x \cap V_{a_n}$. If $k \notin \{k_1, \dots, k_{r_n}\}$, then we take elements $y_{1,k} \in \varrho_1^k x, \dots, y_{N,k} \in \varrho_N^k x$ arbitrarily (we know that such elements exist because $x \in \bigcap_{1 \leq j \leq N} D_\infty(\varrho_j)$). With the sequence $y_{j,k} \in \varrho_{j,k}(x)$, $1 \leq j \leq N$, the requirements of Definition 2.3(i) are satisfied. \square

We are returning to the case of general $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. If $\emptyset \notin \mathcal{F}$, then the \mathcal{F} -hypercyclicity (\mathcal{F} -topological transitivity) of G implies that G is connected. On the other hand, if $\emptyset \in \mathcal{F}$ and G is not connected, then G cannot be \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive). In the sequel, we will consider only the case that $E(G) \neq \emptyset$,

when we clearly have $\mathbb{N} \in \mathcal{F}$; this will be our standing assumption in the rest of this subsection. Since the associated binary relation ϱ is symmetric, we will also assume that for each $A \in \mathcal{F}$ the assumption $A \neq \emptyset$ implies $A + 2\mathbb{N} \subseteq A$. Observe also that we have $S(U, V) = S(V, U)$ for any open nonempty subsets U and V of G .

As the next illustrative example shows, the notion of \mathcal{F} -hypercyclicity for simple graphs is far from being clear and easy understandable (see also [15]):

Example 3.6.

- (i) Let $G = \{x_1, x_2, x_3, x_4\}$ be equipped with discrete topology, let G be the un-oriented square $x_1x_2x_3x_4$, and let \mathcal{F} be the collection of all nonempty subsets of \mathbb{N} containing only odd elements. Then, for every $i \in \mathbb{N}_4$ and $n \in 2\mathbb{N} + 1$, we have that $x_i \notin \varrho^n x_i$, which simply implies that the corresponding symmetric relation ϱ cannot possess any of the introduced \mathcal{F} -dynamical properties from Definition 2.1; furthermore, G is \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) if and only if $\{2\mathbb{N}, 2\mathbb{N} + 1\} \subseteq \mathcal{F}$.
- (ii) Let the complete graph K_n be equipped with discrete topology. Then the following holds:
 - (a) $n = 2$: K_n is \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) if and only if $\{2\mathbb{N}, 2\mathbb{N} + 1\} \subseteq \mathcal{F}$.
 - (b) $n \geq 3$: K_n is \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) if and only if $\mathbb{N} \setminus \{1\} \in \mathcal{F}$.
- (iii) Let the complete graph K_n be equipped with discrete topology. Then the following holds:
 - (a) $n = 2$: The graphs K_n, \dots, K_n , totally counted N times, are \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) if and only if $\{\emptyset, 2\mathbb{N}, 2\mathbb{N} + 1\} \subseteq \mathcal{F}$.
 - (b) $n \geq 3$: The graphs K_n, \dots, K_n , totally counted N times, are \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) if and only if $\mathbb{N} \setminus \{1\} \in \mathcal{F}$.

Assume that G is equipped with a topology τ on X . Set

$$S_{G,\tau} := \text{card}(\{S(U, V) : \emptyset \neq U, \emptyset \neq V, U, V \subseteq G\}).$$

If τ is the discrete topology on X , then we simply write S_G in place of $S_{G,\tau}$.

For connected bipartite graphs, the following result holds true:

Theorem 3.7. *Let G be a connected bipartite graph. Then, for every two open nonempty subsets U and V of X , we have*

$$(3.1) \quad S(U, V) = L(U, V) := \{d(u, v) + 2k : u \in U, v \in V, k \in \mathbb{N}_0\} \cap \mathbb{N}$$

and

$$(3.2) \quad S_{G,\tau} \leq S_G \leq d(G) + \frac{1}{4}(d(G))^2 - \frac{1}{4}\chi_{2\mathbb{N}+1}(d(G)).$$

Proof. Let U and V be given. If $n \in S(U, V)$, then there exist nodes $u \in U$, $v \in V$ and a walk in G of length $n \in \mathbb{N}$ connecting u and v . If $d(u, v) = n$, then clearly $n \in L(U, V)$; otherwise, $d(u, v) < n$ and $n - d(u, v) \in 2\mathbb{N}$, due to the fact that G is bipartite, and we again have $n \in L(U, V)$. Conversely, if $n \in L(U, V)$, then there exist two nodes $u \in U$, $v \in V$ and a number $k \in \mathbb{N}_0$ such that $n = d(u, v) + 2k$. If $u = v$, then $k \in \mathbb{N}$ and there exists a walk in G of length $2k$ connecting u and $v = u$ because G is connected. Otherwise, $u \neq v$ and there exists a walk in G of length $d(u, v)$ connecting u and v . By the connectivity of G , there exists a walk in G of length $d(u, v) + 2k = n$, so that $n \in S(U, V)$ and (3.1) holds. This implies that $S(U, V)$ is equal to some of the sets $1 + 2\mathbb{N}_0, \dots, d(G) + 2\mathbb{N}_0$ or some of their finite nonempty unions. Since the inequality $S_{G,\tau} \leq S_G$ is trivial, for the proof of (3.2) it suffices to prove that there exist at most $\frac{1}{4}(d(G))^2 - \frac{1}{4}\chi_{2\mathbb{N}+1}(d(G))$ different finite nonempty unions of the sets $1 + 2\mathbb{N}_0, \dots, d(G) + 2\mathbb{N}_0$. But, this simply follows from the fact that any such union is of the form $A_{i,j} := \{\min(i, j), \min(i, j) + 2, \dots, \max(i, j) - 1\} \cup \{s \in \mathbb{N} : s \geq \max(i, j)\}$, where $1 \leq i, j \leq d(G)$, i is even, j is odd, the fact that any such two sets $A_{i,j}$ and $A_{i',j'}$ differ if $(i, j) \neq (i', j')$, $1 \leq i, i', j, j' \leq d(G)$, i, i' are even, j, j' are odd, and the product principle. The proof of the theorem is thereby complete. \square

For the proof of the inclusion $L(U, V) \subseteq S(U, V)$, we have not used the assumption that G is bipartite. Therefore, we have:

Proposition 3.8. *Let G be a connected graph. Then, for every two nonempty subsets U and V of X , we have $L(U, V) \subseteq S(U, V)$.*

As Example 3.6(i) shows, the estimate (3.1) cannot be improved for connected bipartite graphs having four nodes (let us recall that $\mathbb{N} \in \mathcal{F}$ is our standing assumption). The situation is quite similar in the general case because for the path P_n , where $n \geq 2$, we have

$$d(P_n) = n - 1 \quad \text{and} \quad S_G = n - 1 + \frac{1}{4}(n - 1)^2 - \frac{1}{4}\chi_{2\mathbb{N}+1}(n - 1).$$

By $\vartheta(G)$ we denote the smallest number, if such exists, satisfying that any two nodes u and v of G can be connected by an even walk of length $\leq \vartheta(G)$ and an odd walk of length $\leq \vartheta(G)$; otherwise, we set $\vartheta(G) := \infty$.

Theorem 3.9. *Let G be a connected graph. Then we have*

$$(3.3) \quad S_G \leq \left\lfloor \frac{1}{4}(\vartheta^2(G) + 2\vartheta(G) + 1) \right\rfloor.$$

Proof. Clearly, it suffices to examine the case in which $\vartheta(G) < \infty$. In this case, we have

$$(3.4) \quad B := \{s \in \mathbb{N} : s \geq \vartheta(G)\} \subseteq S(U, V)$$

for any two nonempty subsets U and V of G . Set $m_{U,V} := \min(S(U, V))$. If $m_{U,V} = \vartheta(G)$, then $S(U, V) = \{s \in \mathbb{N} : s \geq \vartheta(G)\}$. If $1 \leq i = m_{U,V} < \vartheta(G)$, then $S(U, V) = (i + 2\mathbb{N}_0) \cup B$ or there exists a natural number $i' \in (i, \vartheta(G))$ such that $i' - i$ is an odd number and $S(U, V) = \{s \in \mathbb{N} : s \in i + 2\mathbb{N}_0, s < i'\} \cup \{s \in \mathbb{N} : s \geq i'\}$. Hence, if $\vartheta(G) - i$ is an even number, we have at most $\frac{1}{2}(\vartheta(G) - i + 2)$ different possibilities for $S(U, V)$, while if $\vartheta(G) - i$ is an odd number, we have at most $\frac{1}{2}(\vartheta(G) - i + 1)$ different possibilities for $S(U, V)$. Summa summarum,

$$(3.5) \quad S_G \leq 1 + \sum_{i=1}^{\vartheta(G)-1} \frac{\vartheta(G) - i + 2}{2} - \sum_{i \in I} \frac{1}{2},$$

where $I = \{i \in \mathbb{N}_{\vartheta(G)-1} : \vartheta(G) - 1 - i \text{ is odd}\}$. Then the estimate (3.3) follows from a simple computation involving (3.5) and the equality $\lfloor \frac{1}{4}(\vartheta^2(G) + 2\vartheta(G) + 1) \rfloor = \lfloor \frac{1}{4}(\vartheta^2(G) + 2\vartheta(G)) \rfloor$, holding for even numbers $\vartheta(G)$. \square

If a connected graph G is not bipartite, then it contains a closed contour of odd length as a subgraph. If $l \in \mathbb{N}$ is any number such that the closed contour C_{2l+1} is a subgraph of G , then it is very elementary to prove that $\vartheta(G) \leq \max_{u,v \in G} [2d(u, C_{2l+1}) + d(u, v) + 2l + 1]$, where $d(u, C_{2l+1}) := \inf\{d(u, w) : w \in C_{2l+1}\}$.

Remark 3.10. For a non-bipartite connected graph G , the estimate (3.4) follows immediately from the facts that the index of G is strictly greater than 1, any component of the principal eigenvector of G is strictly positive, and by an application of [11], Theorem 2.2.5; see [11] for the notion.

Let $m, n \in \mathbb{N}$, let $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_m\}$, and let $G_i(X, Y)$ be a bipartite graph with colored classes X and Y , $1 \leq i \leq N$. Suppose that $X \cup Y$ is equipped with discrete topology and $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$. As indicated in [9], the graphs $G_1(X, Y), G_2(X, Y), \dots, G_N(X, Y)$ cannot be strongly $d\mathcal{F}$ -hypercyclic (strongly $d\mathcal{F}$ -topologically transitive); by Proposition 3.5, it readily follows that $G_1(X, Y), G_2(X, Y), \dots, G_N(X, Y)$ cannot be $d\mathcal{F}$ -hypercyclic ($d\mathcal{F}$ -topologically transitive). On the other hand, by our comment from Remark 3.10, for arbitrary non-bipartite connected graphs G_1, G_2, \dots, G_N and for arbitrary nonempty subsets V_1, V_2, \dots, V_N of \mathbb{N}_n , we always have the existence of a positive integer $k_0 \in \mathbb{N}$ such that

$$[k_0, \infty) \cap \mathbb{N} \subseteq \{k \in \mathbb{N} : (\forall i \in U) (\forall s \in \mathbb{N}_N) (\forall j_s \in V_s) a_{ij_s}^{k;s} \geq 1\}.$$

Taking into account Proposition 3.4, the above immediately implies:

Theorem 3.11. *Let G_1, G_2, \dots, G_N be non-bipartite connected graphs, and let $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$. Then G_1, G_2, \dots, G_N are always (strongly) $d\mathcal{F}$ -hypercyclic (strongly $d\mathcal{F}$ -topologically transitive) and any element of X is a (strong) $d\mathcal{F}$ -hypercyclic vector of G_1, G_2, \dots, G_N .*

With the notion introduced in [9], we also have that G_1, G_2, \dots, G_N are d -topologically mixing.

We would like to propose the following problem:

Problem 1. Given a connected non-bipartite graph G with $n \geq 2$ nodes, find as good as possible upper bound for S_G in terms of $d(G)$ and n .

Theorem 3.7 and Proposition 3.9 can be reconsidered for disjointness; Problem 1 can be reformulated in this context, as well. For the sake of brevity, we will skip all related details about these questions.

3.3. Results for digraphs and tournaments. A digraph is any pair (G, ϱ) , where G is a finite nonempty set and $\varrho \subseteq (G \times G) \setminus \Delta_G$; hence, in our definition, we do not allow G to contain any loop. We will consider only finite nontrivial digraphs henceforth ($|G| > 1$). The elements in G and ϱ are called points (vertices, nodes) and arcs respectively; if arc $(x, y) \in \varrho$, then we say that x is adjacent to y and write xy for $\text{arc}(x, y)$. Two vertices x and y of a digraph G are said to be *nonadjacent* if $(x, y) \notin \varrho$ and $(y, x) \notin \varrho$. If we replace each arc (x, y) in G by symmetric pairs (x, y) and (y, x) of arcs, we obtain the underlying simple graph G associated to G . The notions of outdegree $d^+(x)$, indegree $d^-(x)$ and degree $d(x) := d^+(x) + d^-(x)$ of a vertex $x \in G$ as well as the notions of Hamiltonicity of G , a semi-walk in G , a walk in G and their lengths are defined as usual (see [7]). A digraph G is called *asymmetric* if ϱ is an anti-symmetric relation. If G is an asymmetric digraph and G is \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive), then for each set $A \in \mathcal{F}$ we have $2 \notin A$. Let us recall that a tournament T is a digraph in which any two different nodes are connected by exactly one arc. The set of nodes of any digraph G (tournament T) considered below will be $X = V(G) = \{x_1, x_2, \dots, x_n\}$.

Let us recall that a digraph (G, ϱ) is said to be *strongly connected* if for any two different points x and y from G there is an oriented $x - y$ walk, while (G, ϱ) is said to be *weakly connected* if for any two different points x and y from G there is an $x - y$ semi-walk, which is equivalent to saying that the underlying simple graph G associated to G is connected (see [7]). For various generalizations, see [9]. By $[A(G)]_{1 \leq i, j \leq n}$ we denote the adjacency matrix of G .

For any digraph G (digraphs G_1, \dots, G_N), denote by G (G_1, \dots, G_N) the associated simple graphs defined as above. The notions introduced in Definition 2.1

(Definition 2.3 and Definition 2.4) can be used to define \mathcal{F}_w -hypercyclic and \mathcal{F}_w -topologically transitive properties ($d\mathcal{F}_w$ -hypercyclic and $d\mathcal{F}_w$ -topologically transitive properties) of G (G_1, \dots, G_N). For example, we say that G is *strongly \mathcal{F}_w -hypercyclic* if the associated simple graph G is strongly \mathcal{F} -hypercyclic, while G_1, \dots, G_N are said to be *$d\mathcal{F}_w$ -topologically transitive* if G_1, \dots, G_N are $d\mathcal{F}$ -topologically transitive, and so on and so forth. In such a way, we extend the notion of d -weakly connected digraphs (see [9]). It is clear that any \mathcal{F} -hypercyclic property ($d\mathcal{F}$ -hypercyclic property) implies the corresponding \mathcal{F}_w -hypercyclic property ($d\mathcal{F}_w$ -hypercyclic property). The same holds for topological transitivity.

The numbers $S_{G,\tau}$ and S_G are meaningful for digraphs, as well, but calculating upper bounds for the number S_G is not so simple task for digraphs. Let us only note that in a primitive digraph G (this means that there is a positive integer $k \in \mathbb{N}$ such that there is a walk of length k from each vertex u to each vertex v (possibly u again) of G ; the smallest integer k with this property is said to be the exponent of G), for any two open nonempty subsets U and V of X , we have $\{k \in \mathbb{N} : k \geq \exp(G)\} \subseteq S(U, V)$, where $\exp(G)$ denotes the exponent of G . Unless stated otherwise, we assume henceforth that $\mathcal{F} = P(P(\mathbb{N})) \setminus \{\emptyset\}$.

The following result is closely connected with Proposition 3.5:

Proposition 3.12.

- (i) *If G is equipped with discrete topology, then G is \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive) if and only if G is strongly \mathcal{F} -hypercyclic (strongly \mathcal{F} -topologically transitive).*
- (ii) *Suppose that G_1, G_2, \dots, G_N are given digraphs and G_i is equipped with discrete topology on X , $1 \leq i \leq N$. Then G_1, G_2, \dots, G_N are $d\mathcal{F}$ -hypercyclic ($d\mathcal{F}$ -topologically transitive) if and only if G_1, G_2, \dots, G_N are strongly $d\mathcal{F}$ -hypercyclic (strongly $d\mathcal{F}$ -topologically transitive).*

Proof. We present only the main points of the proof of (ii) for $d\mathcal{F}$ -hypercyclicity and strong $d\mathcal{F}$ -hypercyclicity, which is very similar to that of Proposition 3.5. It suffices to show that any $d\mathcal{F}$ -hypercyclic vector of G_1, G_2, \dots, G_N is likewise a strong $d\mathcal{F}$ -hypercyclic vector of G_1, G_2, \dots, G_N . To see this, we can copy the arguments given for simple graphs because we can always find a strictly increasing sequence $k_1 < k_2 < \dots < k_{r_n}$ satisfying the properties stated in the proof of Proposition 3.5, due to our assumption that any digraph G_i is equipped with discrete topology on X , $1 \leq i \leq N$ and the fact that for each $d\mathcal{F}$ -hypercyclic vector x of G_1, G_2, \dots, G_N there exists a positive integer $k \in \mathbb{N}$ such that, for every $j \in \mathbb{N}_N$, there exists an $(x - x)$ walk in G_j of length k . □

Using Proposition 3.12, we can rephrase a great number of our results established in [9] for strong \mathcal{F} -hypercyclicity (strong \mathcal{F} -topological transitivity) and strong $d\mathcal{F}$ -hypercyclicity (strong $d\mathcal{F}$ -topological transitivity). For example, we have the following result:

- (i) Let G be a tournament equipped with discrete topology. Then G is strongly \mathcal{F} -hypercyclic if and only if the indegree of any vertex is strictly positive.
- (ii) Let G be a digraph equipped with discrete topology, satisfying that for any two vertices x, y in G such that xy is not an arc in G one has $d^+(x) + d^-(y) \geq n - 1$. Then ϱ is strongly \mathcal{F} -hypercyclic if and only if the indegree of any vertex is strictly positive.
- (iii) Let $n \geq 4$, and let T_1, T_2, \dots, T_N be tournaments equipped with discrete topologies. Then T_1, T_2, \dots, T_N are strongly $d\mathcal{F}$ -topologically transitive if and only if T_j is strongly connected for all $j \in \mathbb{N}_N$ if and only if T_j is a Hamiltonian tournament for all $j \in \mathbb{N}_N$.

It is worth noting that Proposition 3.12 does not hold if G (some of G'_i 's for $1 \leq i \leq N$) is equipped with a topology that is not discrete:

Example 3.13.

- (i) Let $n = 5$ and let G be a digraph with the associated binary relation $\varrho = \{(x_3, x_2), (x_3, x_5), (x_2, x_1), (x_1, x_4), (x_4, x_1)\}$. Suppose that G is equipped with the topology $\tau = \{\emptyset, \{x_2\}, \{x_5\}, \{x_2, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}\}$. Then x_3 is the only \mathcal{F} -hypercyclic vector of G and therefore G is \mathcal{F} -hypercyclic. On the other hand, G is not \mathcal{F} -topologically transitive because there is no $(x_2 - x_5)$ walk in G . In this concrete example, G is not strongly \mathcal{F} -hypercyclic because x_3 is not a strong \mathcal{F} -hypercyclic vector of G ; this follows from the fact that for the sequence (y_n) satisfying the requirements from Definition 2.1(i) we need to have $y_1 = x_2$ and $y_1 = x_5$, which is a contradiction. If we consider N copies of the digraph G , then it can be simply verified that the obtained tuple is $d\mathcal{F}$ -hypercyclic, not strongly $d\mathcal{F}$ -hypercyclic and not $d\mathcal{F}$ -topologically transitive.
- (ii) Let $n = 4$ and let G be a digraph with the associated binary relation $\varrho = \{(x_1, x_2), (x_1, x_3), (x_3, x_4), (x_4, x_3)\}$. Suppose that G is equipped with the topology $\tau = \{\emptyset, \{x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3, x_4\}\}$. Consider N copies of the digraph G ; then x_1 is the only $d\mathcal{F}$ -hypercyclic vector of the obtained tuple T , which is $d\mathcal{F}$ -hypercyclic and not strongly $d\mathcal{F}$ -hypercyclic. Otherwise, x_1 needs to be a strong $d\mathcal{F}$ -hypercyclic vector of T , which is a contradiction because the choices $V_1 = V_2 = \{x_2\}$ and $V_1 = \{x_2\}, V_2 = \{x_3\}$ impose that, for the sequence $(y_{j,k})_{1 \leq j \leq N, k \in \mathbb{N}}$ in Definition 2.3(i), we must have $y_{2,1} = x_2$ and $y_{2,1} = x_3$. Observe also that G and T are, respectively, not \mathcal{F} -topologically transitive and $d\mathcal{F}$ -topologically transitive.

The values $n = 5$ and $n = 4$ are optimal, as the next two propositions indicate:

Proposition 3.14. *Let $n \leq 4$ and let G be a given digraph. Then G is \mathcal{F} -hypercyclic if and only if G is strongly \mathcal{F} -hypercyclic.*

Proof. We will consider only the most complicated case $n = 4$; the proof for \mathcal{F} -hypercyclicity goes as follows. Let $x = x_1$ be an \mathcal{F} -hypercyclic vector for G and let V_1, \dots, V_m be all nonempty open subsets of X . Then there exist a natural number k_i and an element $y_{k_i} \in \varrho^{k_i}x \cap V_i$, $1 \leq i \leq m$. If there exists an $(x_1 - x_1)$ walk in X , it is clear that k_1, \dots, k_m can be chosen arbitrarily large and it is trivial to show that, in this case, x_1 needs to be a strong \mathcal{F} -hypercyclic vector for G . Suppose that there is no $(x_1 - x_1)$ walk in X and $x_1 \neq y_{k_i}$ for $1 \leq i \leq m$; since $x \in D_\infty(\varrho)$, it readily follows that the unoriented segments x_2x_3 , x_2x_4 or x_3x_4 belong to ϱ , with the meaning clear, or a closed contour connecting x_2 , x_3 and x_4 belongs to ϱ , again with the meaning clear. The last case is trivial because we can reach the points x_2 , x_3 and x_4 by walks of arbitrarily large length, starting from x_1 , so that x_1 needs to be a strong \mathcal{F} -hypercyclic vector for G by an elementary line of reasoning. Otherwise, we may assume that the unoriented segment x_3x_4 belongs to ϱ . If x_3x_2 or x_4x_2 is an arc in G , then it is clear that we can reach the points x_2 , x_3 and x_4 by walks of arbitrarily large lengths, starting from x_1 , so that the proof is complete. If this is not the case, then x_1x_2 may or may not be an arc in G . In the former case, if some of the elements y_1, \dots, y_m is equal to x_2 , then we may assume without loss of generality that at most one of these elements is equal to x_2 (because the set $\{x_2\}$ can be listed in the sequence V_1, \dots, V_m at most once and the points x_3 and x_4 can be reached by walks from x_1). In the latter case, any of elements y_1, \dots, y_m cannot be equal to x_2 and any of numbers k_1, \dots, k_m can be chosen arbitrarily large because the points x_3 and x_4 can be reached by walks of arbitrarily large lengths, starting from x_1 . The proof of proposition is completed. \square

Proposition 3.15. *Let $n \leq 3$ and let G_1, G_2, \dots, G_N be given digraphs. Then G_1, G_2, \dots, G_N are $d\mathcal{F}$ -hypercyclic if and only if G_1, G_2, \dots, G_N are strongly $d\mathcal{F}$ -hypercyclic.*

Proof. The proof is very similar to that of Proposition 3.5. Let x_1 denote a $d\mathcal{F}$ -hypercyclic vector of G_1, G_2, \dots, G_N . Then some of the unoriented arcs x_1x_2 , x_1x_3 , x_2x_3 or a closed oriented contour of length 3 is contained in any G_i , $1 \leq i \leq N$; this implies that, in the proof of Proposition 3.5, we can choose $k_2 > k_1$, because the number k_2 can be replaced therein, optionally, by any number $k_2 + 6s$, where $s \in \mathbb{N}$. Keeping this in mind, we can repeat literally the arguments given in the proof of Proposition 3.5. \square

Remark 3.16. In our previous analyses, we have constructed a tournament (two tournaments) having two nodes and equipped with a certain topology (topologies) which is \mathcal{F} -topologically transitive but not strongly \mathcal{F} -topologically transitive ($d\mathcal{F}$ -topologically transitive but not strongly $d\mathcal{F}$ -topologically transitive). Hence, the statements of Proposition 3.14 and Proposition 3.15 do not hold for \mathcal{F} -topological transitivity and strong \mathcal{F} -topological transitivity.

No tournament (N tournaments) having two nodes can be \mathcal{F} -hypercyclic because the domain of the square of the associated binary relation (relations) is the empty set. In the case that $n \geq 3$, we have the following result closely connected with Proposition 3.14 and Proposition 3.15:

Theorem 3.17.

- (i) *Suppose that $n \geq 3$ and T is a given tournament. Then T is \mathcal{F} -hypercyclic if and only if T is strongly \mathcal{F} -hypercyclic.*
- (ii) *Suppose that $n \leq 4$ and T_1, T_2, \dots, T_N are given tournaments. Then T_1, T_2, \dots, T_N are $d\mathcal{F}$ -hypercyclic if and only if T_1, T_2, \dots, T_N are strongly $d\mathcal{F}$ -hypercyclic.*
- (iii) *Suppose that $n \geq 5$. Then we can always find tournaments T_1, T_2, \dots, T_N that are $d\mathcal{F}$ -hypercyclic and not strongly $d\mathcal{F}$ -hypercyclic.*

Proof. We will prove (i) by induction. If $n = 3$, then T is isomorphic to a Hamiltonian contour, which is clearly strongly \mathcal{F} -hypercyclic, or to the tournament with the set of nodes x_1x_2, x_1x_3 and x_2x_3 , which cannot be \mathcal{F} -hypercyclic because the cube of the associated binary relation has empty domain. Suppose that the statement of proposition holds for each tournament having strictly less than $n > 3$ nodes and let us prove the statement for an arbitrary tournament T having n nodes. Let x_{i_1} be an \mathcal{F} -hypercyclic vector of T . Due to the famous theorem of Rédei (see e.g. [21]), there exists a Hamiltonian path, say $x_1 \mapsto x_2 \mapsto x_3 \mapsto \dots \mapsto x_n$, in T . If there exists adjacent nodes x_j and x_l for some $j \leq i_1$ and $l \geq i_1, j, l \in \mathbb{N}_n$, then there is a closed $(x_{i_1} - x_{i_1})$ -walk in T and the statement trivially holds. Otherwise, the set of nodes which can be reached from x_{i_1} by a closed walk is $X' \equiv \{x_{i_1+1}, \dots, x_n\}$. If $i_1 > 1$, then the subtournament T' induced by the set of nodes X' is \mathcal{F} -hypercyclic with respect to the subspace topology on X' . By induction hypothesis, T' is strongly \mathcal{F} -hypercyclic, which clearly implies that T is strongly \mathcal{F} -hypercyclic, as well. The remaining case to be considered is $i_1 = 1$. The result trivially follows if the initial topology is anti-discrete or the set $\{x_1\}$ is an open set of the initial topology, because then there exists a closed $(x_1 - x_1)$ -walk in T . If this is not the case, set $y_i := x_{i+1}$ for $1 \leq i \leq n - 1$ and take $y_i \in \varrho^i x_1$ arbitrarily for $i \geq n$. Then for any open set V of the initial topology there exists an integer $i \in \mathbb{N}_n \setminus \{1\}$ such that $x_i \in V$. It is clear that the set $\{k \in \mathbb{N} : y_k \in V\}$ is nonempty because it contains the element x_{i-1} . This

completes the proof of (i). The proof of (ii) for $n = 2$ and $n = 3$ is simple. If $n = 4$, assume that x_1 is a $d\mathcal{F}$ -hypercyclic vector of T_1, T_2, \dots, T_N . Then Proposition 2.6 and the fact that $x_1 \in \bigcap_{1 \leq j \leq N, n \in \mathbb{N}} D_\infty(\varrho_j)$ implies that a closed oriented contour of length 3 or a closed oriented contour of length 4 is contained in any G_i , $1 \leq i \leq N$. Then we can argue as in the proofs of Proposition 3.5 and Proposition 3.15 because we can construct a sequence $(k_s)_{1 \leq s \leq r_n}$ such that $k_1 < k_2 < \dots < k_{r_n-1} < k_{r_n}$; this follows from the fact that any point reachable from x_1 by a walk of length k is also reachable from x_1 by a walk of length $k + 12s$, where $s \in \mathbb{N}$. For a counterexample in (iii), let T_1 be any tournament containing the following sets of arcs: $\{x_1x_i : 1 \leq i \leq n\}$, $\{x_2x_i : 3 \leq i \leq n\}$, $\{x_3x_i : i = 4 \text{ or } 6 \leq i \leq n\}$, $\{x_4x_i : 5 \leq i \leq n\}$ and $\{x_5x_i : i = 3 \text{ or } 6 \leq i \leq n\}$. Let T_2 be any tournament containing the following sets of arcs: $\{x_1x_i : 1 \leq i \leq n\}$, $\{x_3x_i : i = 2 \text{ or } 4 \leq i \leq n\}$, $\{x_2x_i : 4 \leq i \leq n\}$, $\{x_4x_i : 5 \leq i \leq n\}$ and $\{x_5x_i : i = 2 \text{ or } 6 \leq i \leq n\}$. Let $\tau = \{\emptyset, \{x_2\}, \{x_3\}, \{x_2, x_3\}, X\}$. Then it can be easily seen that x_1 is a unique $d\mathcal{F}$ -hypercyclic vector of T_1, T_2, \dots, T_N and that T_1, T_2, \dots, T_N are $d\mathcal{F}$ -hypercyclic but not strongly $d\mathcal{F}$ -hypercyclic because x_1 is not a strong $d\mathcal{F}$ -hypercyclic vector of T_1, T_2, \dots, T_N ; this can be seen by plugging $V_1 = V_2 = \{x_2\}$ and $V_1 = V_2 = \{x_3\}$ in Definition 2.3(i), which immediately forces that $y_{1,1} = x_2$ and $y_{1,1} = x_3$, a contradiction. \square

The strong connectivity of a tournament T is equivalent to its irreducibility (see [19]). Applying [19], Theorem 1, we get that the adjacency matrix $[A(T)]$ of T is primitive, i.e. there exists a natural number $q \in \mathbb{N}$ such that any element of the matrix $[A(T)]^q$ is strictly positive. Due to the strong connectivity of T , we get that any element of the matrix $[A(T)]^{q'}$ is strictly positive for all $q' \geq q$ (see also [9]). If we denote by d and e the diameter and exponent of T , then $d \leq e \leq d + 3$ and we can take $e = q \leq d + 3$ due to [18], Theorem 19; see [18], Section 12, Section 13 for the notion and more details on the subject. This immediately implies the following result:

Proposition 3.18.

- (i) *Let T be a strongly connected tournament with exponent e . If T is \mathcal{F} -hypercyclic (\mathcal{F} -topologically transitive), then for each set $A \in \mathcal{F}$ we have $\{n \in \mathbb{N} : n \geq e\} \subseteq A$.*
- (ii) *Let T_1, \dots, T_N be strongly connected tournaments, and let e be the largest value of their exponents. If T_1, \dots, T_N are $d\mathcal{F}$ -hypercyclic ($d\mathcal{F}$ -topologically transitive), then for each set $A \in \mathcal{F}$ we have $\{n \in \mathbb{N} : n \geq e\} \subseteq A$.*

It is a well known fact that the exponent e of a primitive tournament T satisfies $3 \leq e \leq n + 2$ if $n \geq 5$. Furthermore, if $n \geq 6$, then [18], Theorem 20 implies the existence of a primitive tournament T with exponent $e \in [3, n + 2]$ given in advance.

For such a tournament T , we have the existence of two elements x_i and x_j , for some $i, j \in \mathbb{N}_n$, such that there is no $(x_i - x_j)$ walk in T of length $e - 1$. This implies that x_i cannot be an \mathcal{F} -hypercyclic vector of T if each set $A \in \mathcal{F}$ contains the number $e - 1$.

We close the paper by proposing some open problems and observations for tournaments having four vertices. There exist four non-isomorphic tournaments with four vertices. The only one of them is transitive (acyclic) and this tournament is \mathcal{F} -topologically transitive for the discrete topology if and only if $\{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\} \subseteq \mathcal{F}$. The second (third) is obtained as the union of the closed contour $x_2x_3x_4$ and arcs x_1x_2, x_1x_3, x_1x_4 (x_2x_1, x_3x_1, x_4x_1), and these tournaments are \mathcal{F} -topologically transitive for discrete topology if and only if \mathcal{F} contains the sets $\emptyset, 3\mathbb{N}, 1 + 3\mathbb{N}, 2 + 3\mathbb{N}$ and their finite unions. The fourth is obtained as the union of the closed contour $x_1x_2x_3x_4$ and arcs x_1x_3, x_2x_4 . Direct computation of powers of the corresponding adjacency matrix shows that this tournament is \mathcal{F} -topologically transitive for the discrete topology if and only if \mathcal{F} contains the sets $\mathbb{N} \setminus \{2\}, \mathbb{N} \setminus \{1, 4\}, \mathbb{N} \setminus \{2, 3, 6\}, \mathbb{N} \setminus \{1, 2, 3, 6\}, \mathbb{N} \setminus \{1, 3, 4, 7\}, \mathbb{N} \setminus \{1, 2, 4, 5, 8\}$ and their finite unions. The obtained results seem to be very dissociated and, because of that, we would like to propose the following problem:

Problem 2. Let $n \geq 2$, and let $A_i \subseteq \mathbb{N}, 1 \leq i \leq (2^n - 1)(2^n - 1)$ be given sets. Find necessary and sufficient conditions for the existence of a tournament T_n with n vertices such that $\{S(U, V) : \emptyset \neq U, \emptyset \neq V, U, V \subseteq V(T)\} = \{A_i : 1 \leq i \leq (2^n - 1)(2^n - 1)\}$.




It is also meaningful to ask the following question:

Problem 3. Set $a_n := \{S_{T_n} : T_n \text{ is a tournament with } n \text{ vertices}\}, n \geq 2$. Find some upper bounds for $a_n, n \geq 2$ and the asymptotic behaviour of the sequence $(a_n)_{n \geq 2}$.

Problem 4. Let $n \geq 2$. Construct a tournament T_n with n vertices such that $S_{T_n} = a_n$. How many non-isomorphic tournaments with n vertices satisfy this equality?

Problem 5. Reconsider Problem 2, Problem 3 and Problem 4 for strongly connected tournaments and some classes of asymmetric digraphs.

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