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SOME RESULTS ON TOP LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

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Abstract. Let I and J be ideals of a Noetherian local ring $(R,\mathfrak{m})$ and let M be a nonzero finitely generated R-module. We study the relation between the vanishing of $H^{\dim M}_{I,J}(M)$ and the comparison of certain ideal topologies. Also, we characterize when the integral closure of an ideal relative to the Noetherian R-module M/JM is equal to its integral closure relative to the Artinian R-module $H^{\dim M}_{I,J}(M)$.

Keywords: Artinian module; integral closure; local cohomology; quasi-unmixed module

MSC 2010: 13B22, 13D45, 13E05

1. Introduction

As a generalization of the usual local cohomology modules, Takahashi, Yoshino and Yoshizawa introduced the local cohomology modules with respect to a pair of ideals in [16]. To be more precise, let I and J be ideals of a commutative Noetherian ring R and let M be an R-module. Suppose that

$$\Gamma_{I,J}(M) := \{x \in M \colon\ I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

For an integer i, the ith right derived functor of $\Gamma_{I,J}$ is denoted by $H^i_{I,J}$ and the R-module $H^i_{I,J}(M)$ is called the ith local cohomology module of M with respect to ideals (I,J). Assume that M is finitely generated and $\dim M = d$. In this paper, we are interested in the structure of $H^d_{I,J}(M)$.

It is well known that $H^d_{I,J}(M)$ is Artinian (see [5], Theorem 2.1). In [4], Theorem 2.1, Chu computed its attached prime ideals. As the first result of this paper, we compute the attached prime ideals of $H^d_{I,J}(M)$ using another proof.

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In [14], Theorem 3.4, Sharp, Tiraş and Yassi proved that if $(R,\mathfrak{m})$ is a homomorphic image of a Gorenstein local ring, then the integral closure $\overline{I}$ of I is equal to the integral closure of I relative to the Artinian R-module $H^{\dim R}_{\mathfrak{m}}(R)$ for all ideals I of R if and only if $\dim R/\mathfrak{p} = \dim R$ for every minimal prime $\mathfrak{p}$ of R. Recall that $\overline{I} = \{x \in R\colon x$ satisfies an equation of the form $x^n + i_1x^{n-1} + \ldots + i_n = 0$, where $i_j \in I^j$ for $j = 1,\ldots,n\}$. In Section 2, we generalize their result to the local cohomology modules with respect to a pair of ideals (see Corollary 2.6).

Let $I :_R \langle J\rangle$ be the ultimate constant value of the ascending chain of ideals $(I :_R J) \subseteq (I :_R J^2) \subseteq \ldots$ Martí-Farré showed in [8], Remark 2.5, that if R is quasi-unmixed, then $H^d_I(R) = 0$ if and only if the topology defined by the filtration $\{I^n :_R \langle\mathfrak{m}\rangle\}_{n\geqslant 0}$ is finer than the topology defined by the integral closures of the powers of I; furthermore, if R is unmixed, then $H^d_I(R) = 0$ if and only if the topology defined by the filtration $\{I^n :_R \langle\mathfrak{m}\rangle\}_{n\geqslant 0}$ is equivalent to the I-adic topology. Recall that a nonzero finitely generated module M over a commutative Noetherian local ring $(R,\mathfrak{m})$ is said to be quasi-unmixed (or unmixed) if for every $\mathfrak{p} \in \mathrm{mAss}_{R^*}(M^*)$ (or $\mathfrak{p} \in \mathrm{Ass}_{R^*}(M^*)$, respectively) the condition $\dim R^*/\mathfrak{p} = \dim M$ is satisfied, here M^* is the completion of M with respect to the $\mathfrak{m}$-adic topology. More generally, if R is not necessarily local, M is locally quasi-unmixed (or unmixed) module if, for every $\mathfrak{p} \in \mathrm{Supp}_R(M)$, $M_{\mathfrak{p}}$ is quasi-unmixed (or unmixed, respectively) module over $R_{\mathfrak{p}}$. In Section 3, we generalize the above-mentioned work of Martí-Farré to local cohomology modules with respect to a pair of ideals (see Theorem 3.5).

Throughout this paper, all rings considered are commutative and have nonzero identity elements. We freely use the conventions of the notation for commutative algebra from the books Bruns and Herzog (see [3]), Brodmann and Sharp (see [2]) and Matsumura (see [9]).

2. Integral closures and local cohomology

The main result in this section characterizes when the integral closure of an arbitrary ideal K relative to the Noetherian R-module M/JM is equal to the integral closure of K relative to the Artinian R-module $H^{\dim M}_{I,J}(M)$. We begin by recalling the definitions and results needed. For each R-module M, we denote by $\mathrm{mAss}_R(M)$ the set of minimal primes of $\mathrm{Ass}_R(M)$.

Definition 2.1. Let I be an ideal of a commutative ring R.

(1) Let M be a Noetherian R-module. An element $x \in R$ is said to be *integrally dependent on I relative to M* if there exists a positive integer n such that

$$x^n M \subseteq \sum_{i=1}^{n} x^{n-i} I^i M.$$

Then the set

$$I^{-(M)} := \{x \in R\colon\ x \text{ is integrally dependent on } I \text{ relative to } M\}$$

is an ideal of R, called the *integral closure of I relative to M*.

(2) (See [13].) Let N be an Artinian R-module. An element $x \in R$ is said to be *integrally dependent on I relative to N* if there exists a positive integer n such that

$$\left(0 :_N \sum_{i=1}^{n} x^{n-i} I^i\right) \subseteq (0 :_N x^n).$$

Then the set

$$I^{*(N)} := \{x \in R\colon\ x \text{ is integrally dependent on } I \text{ relative to } N\}$$

is an ideal of R, called the *integral closure of I relative to N*.

Remark 2.2.

(1) Let M be a Noetherian module over a ring R and I an ideal of R. Then $I^{-(M)}$ is the unique ideal J of R such that $\operatorname{Ann}_R(M) \subseteq J$ and

$$J/\operatorname{Ann}_R(M) = \overline{(I + \operatorname{Ann}_R(M))/\operatorname{Ann}_R(M)},$$

the integral closure of the ideal $(I + \operatorname{Ann}_R(M))/\operatorname{Ann}_R(M)$ in the ring $R/\operatorname{Ann}_R(M)$.

(2) Let N be a module over commutative Noetherian ring R and suppose that N is Artinian and an injective cogenerator (see [2], page 130) for R. Let D denote the functor $\operatorname{Hom}_R(-, N)$. Then $I^{-(M)} = I^{*(D(M))}$, for all ideals I of R and for all Noetherian R-modules M.

(3) Let M be a Noetherian module over a ring R and I an ideal of R. Then

$$I^{-(M)} = \bigcap_{\mathfrak{p} \in \operatorname{mAss}_R(M)} I^{-(R/\mathfrak{p})} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(M)} I^{-(R/\mathfrak{p})}.$$

Proof. (1) and (2) hold by [14], Remark 1.6 and Theorem 2.1 (iii). (3) follows from (1) and [15], Proposition 1.1.5 (2). □

The next two results will be useful in the proof of main result in this section. Recall that for an R-module T, a prime ideal $\mathfrak{p}$ of R is said to be an attached prime ideal of T if $\mathfrak{p} = \operatorname{Ann}_R(T/L)$ for some submodule L of T. We denote the set of attached prime ideals of T by $\operatorname{Att}_R(T)$. It should be noted that Theorem 2.3 is also proved in a different version by Merighe and Jorge Pérez (see [11]).

Theorem 2.3. *Let R be a Noetherian complete local ring. Let M be a finitely generated R-module and N an Artinian R-module such that $\operatorname{Att}_R(N) \subseteq \operatorname{mAss}_R(M)$. Then the following statements are equivalent:*

(i) $I^{-(M)} = I^{*(N)}$ *for every ideal I of R;*
(ii) $(0)^{-(M)} = (0)^{*(N)}$*;*
(iii) $\operatorname{Ann}_R(M)$ *and* $\operatorname{Ann}_R(N)$ *have the same radicals;*
(iv) $\operatorname{mAss}_R(M) = \operatorname{Att}_R(N)$.

Proof. The conclusion (i) ⇒ (ii) is clear. In order to prove that (ii) ⇒ (iii) as $(0)^{-(M)} = \sqrt{\operatorname{Ann}_R(M)}$ and $\operatorname{Ann}_R(N) \subseteq (0)^{*(N)}$, it follows from [2], Proposition 7.2.11 (ii), that $\operatorname{Ann}_R(M)$ and $\operatorname{Ann}_R(N)$ have the same radicals. The conclusion (iii) ⇒ (iv) follows from [2], Proposition 7.2.11 (ii). Finally, we prove the implication (iv) ⇒ (i). To this end, let $E := E(R/\mathfrak{m})$, the injective envelope of the R-module $R/\mathfrak{m}$, and let D denote the Matlis duality functor $\operatorname{Hom}_R(-, E)$. As N is an Artinian R-module, it follows from [2], Theorem 10.2.12 (iii), that $D(N)$ is Noetherian and $D(D(N)) \cong N$. Therefore by Remark 2.2 (2) and (3),

$$I^{*(N)} = I^{*(D(D(N)))} = I^{-(D(N))} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(D(N))} I^{-(R/\mathfrak{p})}.$$

On the other hand, by virtue of [2], Exercise 7.2.10 (iv), $\operatorname{Ass}_R(D(N)) = \operatorname{mAss}_R(M)$. So $I^{*(N)} = I^{-(M)}$ by Remark 2.2 (3). □

For an R-module M, we set $\operatorname{Assh}_R(M) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) \colon \dim R/\mathfrak{p} = \dim M\}$.

Lemma 2.4. *Let I and J be ideals of a Noetherian local ring $(R, \mathfrak{m})$ and let M be a nonzero finitely generated R-module of dimension d. Then*

$$\operatorname{Att}_R(H^d_{I,J}(M)) = \{\mathfrak{p} \in \operatorname{Assh}_R(M/JM) \colon H^d_I(R/\mathfrak{p}) \neq 0\}.$$

Proof. If $\dim M/JM < d$, then the result is clear by [16], Theorem 4.3. So we assume $\dim M/JM = d$. There is a finite filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ of M such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Supp}_R(M)$ and $i = 1, \ldots, n$. Then there are the exact sequences

$$0 \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to 0$$

and hence by [16], Theorem 4.7 (1), we have the exact sequences

$$H^d_{I,J}(M_{i-1}) \to H^d_{I,J}(M_i) \to H^d_{I,J}(R/\mathfrak{p}_i) \to 0$$

for $i = 1, \ldots, n$. Since $d \geqslant \dim R/(J + \mathfrak{p}_i)$ for $i = 1, \ldots, n$, it is readily checked that, if $d > \dim R/(J + \mathfrak{p}_i)$, then $H^d_{I,J}(R/\mathfrak{p}_i) = 0$ by [16], Theorem 4.3; and if $d = \dim R/(J + \mathfrak{p}_i)$, then $J \subseteq \mathfrak{p}_i$, and so it follows from [16], Corollary 2.5, that $H^d_{I,J}(R/\mathfrak{p}_i) \cong H^d_I(R/\mathfrak{p}_i)$.

Before completing the proof, we record a corollary of the above proof. We recall that if I is an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and M is a nonzero finitely generated R-module then $\ell(I, M)$ denotes the analytic spread of I with respect to M, so $\ell(I, M) := \dim \mathscr{M}(I, M)/(\mathfrak{m}, t^{-1})\mathscr{M}(I, M)$, where t is an indeterminate and $\mathscr{M}(I, M)$ is the Rees module $M[t^{-1}, It] := \bigoplus_{n\in\mathbb{Z}} I^n M t^n$ over the Rees ring $\mathscr{R}(I, R)$. In the case $M = R$, $\ell(I, R)$ is the classical analytic spread $\ell(I)$ of I.

Corollary 2.5. *Let I and J be ideals of a Noetherian local ring $(R, \mathfrak{m})$ and let M be a nonzero finitely generated R-module of dimension d. If $H^d_{I,J}(M) \neq 0$, then $\ell(I, M/JM) = \ell(I, M) = d$.*

P r o o f. In view of [16], Theorem 4.3, we have $\dim M/JM = d$. So it follows from the above proof that there is $\mathfrak{p} \in \operatorname{Supp}_R(M)$ with $J \subseteq \mathfrak{p}$ and $H^d_I(R/\mathfrak{p}) \neq 0$. Therefore it follows from [12], Lemma 3.1, that $\ell(I, R/\mathfrak{p}) = d$. Then by [7], Remark 5.5.4, and [12], Lemma 2.2, we have

$$d = \ell(I, R/\mathfrak{p}) \leqslant \ell(I, M/JM) \leqslant \ell(I, M) \leqslant \dim M = d$$

and so $\ell(I, M/JM) = \ell(I, M) = d$. □

P r o o f of Lemma 2.4 (continued). Let

$$\mathcal{F} := \{\mathfrak{p} \in \operatorname{Assh}_R(M/JM)\colon\ H^d_I(R/\mathfrak{p}) \neq 0\}.$$

Then

$$\operatorname{Att}_R(H^d_{I,J}(M)) \subseteq \bigcup_{\mathfrak{p}\in\mathcal{F}} \operatorname{Att}_R(H^d_I(R/\mathfrak{p})).$$

By [6], Theorem 2.5, we have $\operatorname{Att}_R(H^d_I(R/\mathfrak{p})) = \{\mathfrak{p}\}$ for all $\mathfrak{p} \in \mathcal{F}$. Therefore, $\operatorname{Att}_R(H^d_{I,J}(M)) \subseteq \mathcal{F}$. For other inclusion, the exact sequence

$$0 \to JM \to M \to M/JM \to 0$$

induces the exact sequence

$$H^d_{I,J}(M) \to H^d_{I,J}(M/JM) \to 0$$

by [16], Theorem 4.7 (1). Since $H^d_{I,J}(M/JM) \cong H^d_I(M/JM)$ by [16], Corollary 2.5, it yields that $\mathrm{Att}_R\big(H^d_I(M/JM)\big) \subseteq \mathrm{Att}_R(H^d_{I,J}(M))$. Then, in view of [6], Theorem 2.5, $\mathcal{F} \subseteq \mathrm{Att}_R(H^d_{I,J}(M))$. □

Here is the main result of this section, which is a generalization of the main result of Sharp, Tiraş and Yassi (see [14], Theorem 3.4).

Corollary 2.6. *Let I and J be ideals of a Noetherian complete local ring $(R, \mathfrak{m})$ and let M be a nonzero finitely generated R-module of dimension d. Then the following conditions are equivalent:*

(i) $K^{-(M/JM)} = K^{*(H^d_{I,J}(M))}$ *for every ideal K of R;*
(ii) $0^{-(M/JM)} = 0^{*(H^d_{I,J}(M))}$*;*
(iii) $\mathrm{Ann}_R(M/JM)$ *and* $\mathrm{Ann}_R\big(H^d_{I,J}(M)\big)$ *have the same radicals;*
(iv) $\mathrm{mAss}_R(M/JM) = \mathrm{Att}_R\big(H^d_{I,J}(M)\big)$.

P r o o f. Follows from Theorem 2.3 and Lemma 2.4. □

3. Ideal topologies and local cohomology

In this section we generalize the main results of Martí-Farré in [8] to local cohomology modules with respect to a pair of ideals. We begin with a definition.

Definition 3.1.

(1) Let I and $\mathfrak{p}$ be ideals of a Noetherian ring R such that $\mathfrak{p}$ is prime. Then $\mathfrak{p}$ is called a *quintasymptotic* (or *quintessential*) *prime ideal of* I precisely when there exists $z \in \mathrm{mAss}_{R^*_\mathfrak{p}}(R^*_\mathfrak{p})$ (or $z \in \mathrm{Ass}_{R^*_\mathfrak{p}}(R^*_\mathfrak{p})$, respectively) such that $\mathrm{Rad}(IR^*_\mathfrak{p} + z) = \mathfrak{p}R^*_\mathfrak{p}$. The set of quitasymptotic (or quintessential) primes of I is denoted by $\overline{Q^*}(I)$ (or $Q(I)$, respectively).
(2) Let S be a multiplicatively closed subset of R and I an ideal of R. The (S)-*component of* I, denoted by $S(I)$, is defined to be the ideal $\bigcup_{s \in S} (I :_R s)$ of R.

Lemma 3.2. *Let I and J be ideals of a Noetherian local ring $(R, \mathfrak{m})$ and let M be a nonzero finitely generated R-module such that* $\dim M = \dim M/JM = d$. *Then* $H^d_{I,J}(M) = 0$ *if and only if* $H^d_I\big(R/\mathrm{Ann}_R(M/JM)\big) = 0$.

Proof. It follows from Lemma 2.4 and [16], Corollary 2.5. □

Proposition 3.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let I and J be ideals of R. Let M be a nonzero finitely generated R-module such that $\dim M = \dim M/JM = d$ and K denote the ideal $\operatorname{Ann}_R(M/JM)$. Let us consider the following conditions:*

(i) *There exists a multiplicatively closed subset S of R with $\mathfrak{m}\cap S \neq \emptyset$ and such that the topology defined by $\{S(I^n+K)\}_{n\geqslant 0}$ is finer than the topology defined by $\{(I^n)^{-(M/JM)}\}_{n\geqslant 0}$.*

(ii) $H^d_{I,J}(M) = 0$.

Then (i) *always implies* (ii) *and the converse holds if M/JM is a quasi-unmixed R-module.*

Proof. Assume that (i) holds and show that (ii) is true. Denote $\widetilde{S} = \{s+K : s\in S\}$ and $\widetilde{R} = R/K$. Then $\widetilde{S}$ is a multiplicatively closed subset of local ring $(\widetilde{R},\mathfrak{m}\widetilde{R})$ with $\mathfrak{m}\widetilde{R}\cap\widetilde{S}\neq\emptyset$ and such that the topology defined by $\{\widetilde{S}(I^n\widetilde{R})\}_{n\geqslant 0}$ is finer than the topology defined by $\{\overline{I^n\widetilde{R}}\}_{n\geqslant 0}$ by Remark 2.2 (1). So $H^d_{I\widetilde{R}}(\widetilde{R}) = 0$ by [8], Proposition 2.1. Then $H^d_{I,J}(M) = 0$ by the Independence Theorem (see [2], Theorem 4.2.1) and Lemma 3.2.

If M/JM is a quasi-unmixed R-module then $\widetilde{R}$ is a quasi-unmixed ring. Therefore the implication (ii) ⇒ (i) is similar to (i) ⇒ (ii) and follows from [8], Proposition 2.1. □

The following remark is useful in the proof of the main result of this section.

Remark 3.4. Let I and J be ideals in a Noetherian ring R.

(1) It is shown in [1] that the set $\operatorname{Ass}_R(R/I^n)$ is stable for all large n and denoted by $A^*(I)$. Also, $\overline{Q^*}(I)\subseteq Q(I)\subseteq A^*(I)$ by [10], Lemma 2.1.

(2) It is clear from the definition that $\overline{Q^*}(I) = Q(I)$ if R is locally unmixed.

(3) Let S be a multiplicatively closed subset of R. Then $S\subseteq R\setminus \bigcup\limits_{\mathfrak{p}\in Q(I)}\mathfrak{p}$ if and only if for all integers $k\geqslant 0$ there is an integer $m\geqslant 0$ such that $S(I^m)\subseteq I^k$ by [10], Theorem 1.2.

(4) Let S be a multiplicatively closed subset of R. Then the $S\subseteq R\setminus \bigcup\limits_{\mathfrak{p}\in\overline{Q^*}(I)}\mathfrak{p}$ if and only if for all integers $k\geqslant 0$ there is an integer $m\geqslant 0$ such that $S(I^m)\subseteq\overline{I^k}$ by [10], Theorem 1.5.

(5) Let $I = \mathfrak{q}_1\cap\ldots\cap\mathfrak{q}_r\cap\mathfrak{q}_{r+1}\cap\ldots\cap\mathfrak{q}_n$ be a minimal primary decomposition of I with $J\subseteq\sqrt{\mathfrak{q}_i}$ exactly when $i = 1,\ldots,r$, then it is easily seen that $I:\langle J\rangle = \mathfrak{q}_{r+1}\cap\ldots\cap\mathfrak{q}_n$.

(6) Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let S be the multiplicatively closed subset of R defined by

$$S = R \setminus \bigcup_{\substack{\mathfrak{p} \neq \mathfrak{m} \\ \mathfrak{p}/J \in A^*(I(R/J))}} \mathfrak{p}.$$

Then it is easy to see that $(I^n + J) : \langle \mathfrak{m} \rangle = S(I^n + J)$ for all $n \geqslant 0$.

The next theorem is the main result of this section.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I and J be ideals of R. Let M be a nonzero finitely generated R-module such that $\dim M = \dim M/JM = d$ and K denote the $\operatorname{Ann}_R(M/JM)$. If M/JM is a quasi-unmixed R-module then the following conditions are equivalent:*

(i) *The topology defined by $\{(I^n + K) : \langle \mathfrak{m} \rangle\}_{n \geqslant 0}$ is finer than the topology defined by the filtration $\{(I^n)^{-(M/JM)}\}_{n \geqslant 0}$.*

(ii) $H^d_{I,J}(M) = 0$.

Furthermore, if R/K is an unmixed ring, then the above conditions are also equivalent to:

(iii) *The topology defined by $\{(I^n + K) : \langle \mathfrak{m} \rangle\}_{n \geqslant 0}$ is equivalent to the topology defined by $\{I^n + K\}_{n \geqslant 0}$.*

Proof. In order to prove (i) $\Rightarrow$ (ii), let S be the multiplicatively closed subset of R defined by

$$S = R \setminus \bigcup_{\substack{\mathfrak{p} \neq \mathfrak{m} \\ \mathfrak{p}/K \in A^*(I(R/K))}} \mathfrak{p}.$$

Then $H^d_{I,J}(M) = 0$ by Proposition 3.3 and Remark 3.4 (6).

Let $\widetilde{R} = R/K$. In order to prove the implication (ii) $\Rightarrow$ (i), suppose that $H^d_{I,J}(M) = 0$. Then $H^d_{I\widetilde{R}}(\widetilde{R}) = 0$ by the Independence Theorem (see [2], Theorem 4.2.1) and Lemma 3.2. So by the Lichtenbaum-Hartshorne Theorem (see [2], Theorem 8.2.1), we have $\mathfrak{m}\widetilde{R} \notin \overline{Q^*}(I\widetilde{R})$. Now, let $\widetilde{S}$ be the multiplicatively closed subset of $\widetilde{R}$ defined by

$$\widetilde{S} = \widetilde{R} \setminus \bigcup_{\substack{K \subseteq \mathfrak{p} \neq \mathfrak{m} \\ \mathfrak{p}\widetilde{R} \in A^*(I\widetilde{R})}} \mathfrak{p}\widetilde{R}.$$

Then it follows from Remark 3.4 (1) that

$$\widetilde{S} \subseteq \widetilde{R} \setminus \bigcup_{\substack{K \subseteq \mathfrak{p} \\ \mathfrak{p}\widetilde{R} \in \overline{Q^*}(I\widetilde{R})}} \mathfrak{p}\widetilde{R}.$$

So the topology defined by $\{\widetilde{S}(I^n\widetilde{R})\}_{n\geqslant 0}$ is finer than the topology defined by $\{\overline{I^n\widetilde{R}}\}_{n\geqslant 0}$ by Remark 3.4 (4). We can now use Remarks 3.4 (6) and 2.2 (1) to complete the proof of (i).

Finally, assume that $\widetilde{R}$ is unmixed and that (i) holds. We show that (iii) is true. To this end, let $\widetilde{S}$ be the multiplicatively closed subset of $\widetilde{R}$ defined in the above paragraph. So in view of Remarks 3.4 (6) and 2.2 (1), the topology defined by $\{\widetilde{S}(I^n\widetilde{R})\}_{n\geqslant 0}$ is finer than the topology defined by $\{\overline{I^n\widetilde{R}}\}_{n\geqslant 0}$. Therefore,

$$\widetilde{S}\subseteq\widetilde{R}\setminus\bigcup_{\substack{K\subseteq\mathfrak{p}\\ \mathfrak{p}\widetilde{R}\in Q(I\widetilde{R})}}\mathfrak{p}\widetilde{R}$$

by Remark 3.4 (2) and (4). We can now use Remark 3.4 (3) and (6) to complete the proof of (iii). The implication (iii) $\Rightarrow$ (i) is obviously true. $\square$

Martí-Farré in [8], Corollary 2.6, showed that if R is a quasi-unmixed Noetherian local ring and $\ell(I)\neq\dim R$ for an ideal I of R, then $H_I^{\dim R}(R)=0$. This paper will be closed with an extension of this result.

Corollary 3.6. *Let I and J be ideals of a Noetherian local ring $(R,\mathfrak{m})$ and let M be a nonzero finitely generated R-module such that $\dim M=d$. If $\ell(I)\neq d$, then $H^d_{I,J}(M)=0$.*

P r o o f. Since $\ell(I,M)=\ell(I(R/\operatorname{Ann}_R(M)))\leqslant\ell(I)$ by [12], Lemma 2.2, and [7], Remark 5.5.4, then $H^d_{I,J}(M)=0$ by Corollary 2.5. $\square$

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