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GUARANTEED TWO-SIDED BOUNDS ON ALL EIGENVALUES OF PRECONDITIONED DIFFUSION AND ELASTICITY PROBLEMS SOLVED BY THE FINITE ELEMENT METHOD

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Abstract. A method of characterizing all eigenvalues of a preconditioned discretized scalar diffusion operator with Dirichlet boundary conditions has been recently introduced in Gergelits, Mardal, Nielsen, and Strakoš (2019). Motivated by this paper, we offer a slightly different approach that extends the previous results in some directions. Namely, we provide bounds on all increasingly ordered eigenvalues of a general diffusion or elasticity operator with tensor data, discretized with the conforming finite element method, and preconditioned by the inverse of a matrix of the same operator with different data. Our results hold for mixed Dirichlet and Robin or periodic boundary conditions applied to the original and preconditioning problems. The bounds are two-sided, guaranteed, easily accessible, and depend solely on the material data.

Keywords: bound on eigenvalues; preconditioning; elliptic differential equation

MSC 2020: 65F08, 65N30

1. Introduction

In 2009, Nielsen, Tveito, and Hackbusch studied in [12] the spectra of elliptic differential operators of the type $\nabla \cdot k \nabla$ defined on infinite-dimensional spaces which are preconditioned using the inverse of the Laplacian. They proved that the range of the scalar coefficient k is contained in the spectrum of the preconditioned operator, provided that k is continuous. Ten years later, Gergelits, Mardal, Nielsen, and Strakoš

showed in [4] without any assumptions on the continuity of the scalar function k that there exists a one-to-one pairing between the eigenvalues of the discretized operator of the type $\nabla \cdot k\nabla$ preconditioned by the inverse of the discretized Laplacian and the intervals determined by the images under k of the supports of the conforming finite element (FE) nodal basis functions used for the discretization.

The present paper contributes to the results of [4] and generalizes some of them. While in [4], a one-to-one pairing between the eigenvalues and images of the scalar data k defined on supports of the FE basis function is proved, we introduce guaranteed two-sided bounds on all individual eigenvalues. Our approach is based on the Courant-Fischer min-max principle. Similarly to [4], the bounds can be obtained solely from the data of the original and preconditioning problems defined on supports of the FE basis functions. While in [12] and [4] only the diffusion operator with scalar data is considered and the Laplacian operator is used for preconditioning, we treat also the diffusion operator with tensor data and with Dirichlet or Robin boundary conditions for both the original and preconditioning operators. Our theory also applies to operators with non-zero null spaces and to operators with vector valued unknown functions; as an example we study the elasticity operator with general tensor data. Any kind of conforming FE basis functions can be employed for discretization; the sets of the FE basis functions must be the same for the original and preconditioning operators. For the sake of brevity, the name 'preconditioning matrix' (or 'operator') will be used for the matrix $\widetilde{\mathsf{M}}$ (or operator) which is (spectrally) close to the original matrix M (or operator, respectively) rather than for the inverse of $\widetilde{\mathsf{M}}$. In contrast, in literature, including [4], $\widetilde{\mathsf{M}}^{-1}$ is often called the preconditioning matrix.

For numerical solution of sparse discretized elliptic partial differential equations, the conjugate gradient method (or Krylov subspace methods for symmetric problems, in general) is a method of choice; see, e.g., [8], [17], [13]. It is well known that its convergence depends on distribution (clustering) of eigenvalues of the related matrices and on the sizes of components of the initial residual in directions of the associated invariant subspaces. For example, well separated clusters of eigenvalues are favorable for the convergence rate, see, e.g., [8], [16] or the example in [4], Section 2. Using finite precision arithmetic, however, similar types of spectra can slow down the convergence; see, e.g. [9], [15] and the recent comprehensive paper [6]. Therefore, being aware of the bounds on the individual eigenvalues we can better estimate the quality of the preconditioner. Our approach can also provide guaranteed easily accessible lower bounds on the smallest eigenvalue of the preconditioned problem, which is required, for example, for accurate algebraic error estimates; see, e.g., [10].

The structure of the paper is as follows. In the subsequent section, we introduce the diffusion and linear elasticity equations as examples of scalar and vector valued

elliptic differential equations which our approach can be applied to. In Section 3, the discretization and the preconditioning setting are described. In Section 4, the main part of the paper, we suggest a method of estimating the eigenvalues of the preconditioned matrices. The theoretical developments are accompanied by illustrative examples. Finally, we compare our method with the recent results from [4]. A short conclusion summarizes the paper.

2. Diffusion and elasticity problems

Our theory of estimating the eigenvalues will be applied to two frequent types of scalar and vector valued elliptic partial differential equations: the diffusion and linear elasticity equations, respectively. To this end, let us briefly introduce the associated definitions and notation; see, e.g., [1], [2], [3], [11] for further details. We assume general mixed boundary conditions for the diffusion equation, and for simplicity of exposition, homogeneous Dirichlet boundary conditions for the elasticity equation.

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain, where $d = 2$ or 3. We consider the *diffusion equation* with Dirichlet and Robin boundary conditions

$$-\nabla \cdot \boldsymbol{A}\nabla u = f \quad \text{in } \Omega, \quad u = g_1 \quad \text{on } \partial\Omega_1, \quad \boldsymbol{n} \cdot \boldsymbol{A}\nabla u = g_2 - g_3 u \quad \text{on } \partial\Omega_2,$$

where $\partial\Omega_1$ and $\partial\Omega_2$ are two disjoint and relatively open parts of the boundary $\partial\Omega$, $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$, and $\boldsymbol{n}$ denotes the outer normal to $\partial\Omega_2$. After lifting the solution u by a differentiable function u_0 that fulfills the non-homogeneous Dirichlet boundary condition and substituting $u := u + u_0$, the weak form of the new problem reads: find $u \in V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \partial\Omega_1\}$ such that

$$(2.1) \qquad\qquad (u, v)_A = l_{A,f}(v), \quad v \in V,$$

where

$$(u, v)_A = \int_\Omega \nabla v \cdot \boldsymbol{A}\nabla u \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} g_3 uv \, \mathrm{d}S,$$

$$l_{A,f}(v) = \int_\Omega fv \, \mathrm{d}\boldsymbol{x} - \int_\Omega \nabla v \cdot \boldsymbol{A}\nabla u_0 \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} g_2 v \, \mathrm{d}S + \int_{\partial\Omega_2} \boldsymbol{n} \cdot \boldsymbol{A}\nabla u_0 v \, \mathrm{d}S,$$

for $u, v \in V$; see, e.g., [3] for details. We assume $f \in L^2(\Omega)$, $g_2 \in L^2(\partial\Omega_2)$, and $g_3 \in L^\infty(\partial\Omega_2)$, $g_3(\boldsymbol{x}) \geqslant 0$ on $\partial\Omega_2$. The material data $\boldsymbol{A} \colon \Omega \to \mathbb{R}^{d \times d}$ are assumed to be essentially bounded, i.e. $\boldsymbol{A} \in L^\infty(\Omega; \mathbb{R}^{d \times d})$, symmetric, and uniformly elliptic (positive definite) in Ω. Thus there exist constants $0 < c_A \leqslant C_A < \infty$ such that

$$(2.2) \qquad c_A \|\boldsymbol{v}\|_{\mathbb{R}^d}^2 \leqslant (\boldsymbol{A}(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{v})_{\mathbb{R}^d} \leqslant C_A \|\boldsymbol{v}\|_{\mathbb{R}^d}^2, \quad \boldsymbol{x} \in \Omega, \ \boldsymbol{v} \in \mathbb{R}^d.$$

The weak form of the *linear elasticity problem* with homogeneous boundary conditions reads: find $\boldsymbol{u} \in V_0^d$, $V_0 = \{v \in H^1(\Omega); \; v = 0 \text{ on } \partial\Omega\}$, such that

$$(2.3) \qquad (\boldsymbol{u}, \boldsymbol{v})_C = l_{C,F}(\boldsymbol{v}), \quad \boldsymbol{v} \in V_0^d,$$

where

$$(\boldsymbol{u}, \boldsymbol{v})_C = \int_\Omega \sum_{i,j,k,l=1}^d c_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x},$$

$$l_{C,F}(\boldsymbol{v}) = \int_\Omega \sum_{i=1}^d F_i v_i \, \mathrm{d}\boldsymbol{x},$$

for $\boldsymbol{u}, \boldsymbol{v} \in V_0^d$, where $\boldsymbol{F} \in (L^2(\Omega))^d$ are body forces. Due to the homogeneous Dirichlet boundary conditions on $\partial\Omega_1 = \partial\Omega$, we use the special notation V_0 of the solution space. Let

$$(2.4) \qquad \tau_{ij} = \sum_{k,l=1}^d c_{ijkl} e_{kl}(\boldsymbol{u}), \quad i, j = 1, \ldots, d,$$

be the components of the Cauchy stress tensor $\boldsymbol{\tau}$ with the strain components e_{ij} obtained from the displacement vector $\boldsymbol{u}$ as

$$e_{kl}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \quad k, l = 1, \ldots, d.$$

Assuming $d = 3$ and denoting $e_i = e_{ii}$, $i = 1, \ldots, d$, we can write

$$\boldsymbol{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \boldsymbol{\partial u}.$$

We assume that the coefficients c_{ijkl} of the tensor $\boldsymbol{c}$ in (2.4) are bounded measurable functions defined in Ω, $c_{ijkl} \in L^\infty(\Omega)$, fulfilling the symmetry conditions

$$(2.5) \qquad c_{ijkl} = c_{jikl} = c_{klij}, \quad i, j, k, l = 1, \ldots, d.$$

Further, we assume there exists a constant $\mu > 0$ such that

$$\mu \sum_{i,j=1}^d \xi_{ij}^2 \leqslant \sum_{i,j,k,l=1}^d c_{ijkl}(\boldsymbol{x})\xi_{ij}\xi_{kl} \quad \text{for all symmetric tensors } \boldsymbol{\xi} \in \mathbb{R}^{d \times d}, \; \boldsymbol{x} \in \Omega.$$

Assuming $d = 3$ and denoting $\tau_i := \tau_{ii}$, $i = 1, \ldots, d$, due to the symmetries (2.5) of c, there exist coefficients $c_{ij} \in L^\infty(\Omega)$, $i, j = 1, \ldots, 6$, such that the stress vector τ can be obtained from the strain vector as

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{pmatrix} = \boldsymbol{C}\boldsymbol{e}.$$

Starting from this place, we will use only the new set of material coefficients c_{ij}, $i, j = 1, \ldots, 6$, (instead of c_{ijkl}, $i, j, k, l = 1, \ldots, d$) and call the associated matrix $\boldsymbol{C}$. Certain material symmetries imply special structures of $\boldsymbol{C}$. For example, homogeneous cubic 3D materials correspond to $c_{11} = c_{22} = c_{33}$, $c_{44} = c_{55} = c_{66}$, $c_{12} = c_{13} = c_{23}$, and annihilate the other components, where $c_{11} > c_{12}$, $c_{11} + 2c_{12} > 0$ and $c_{44} > 0$. In particular, for isotropic material, we have

$$c_{11} = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}, \quad c_{12} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad c_{44} = \frac{E}{2(1 + \nu)},$$

where $E > 0$ is the Young modulus and $\nu \in (-1, \frac{1}{2})$ is the Poisson ratio [11].

The vector $\boldsymbol{F}$ of external forces fulfills

$$-\boldsymbol{\partial}^\top \boldsymbol{\tau} = -\begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \boldsymbol{F},$$

yielding

$$-\boldsymbol{\partial}^\top \boldsymbol{C} \boldsymbol{\partial} \boldsymbol{u} = \boldsymbol{F}.$$

Thus $(\boldsymbol{u}, \boldsymbol{v})_C$ and $l_{C,F}(\boldsymbol{v})$ can be equivalently written as

$$(\boldsymbol{u}, \boldsymbol{v})_C = \int_\Omega (\boldsymbol{\partial}\boldsymbol{v})^\top \boldsymbol{C} \boldsymbol{\partial}\boldsymbol{u} \, \mathrm{d}\boldsymbol{x},$$

$$l_{C,F}(\boldsymbol{v}) = \int_\Omega^\top \boldsymbol{F} \, \mathrm{d}\boldsymbol{x}.$$

If $d = 2$, the dimensions of the arrays naturally reduce. For example, for cubic materials we get

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_{12} \end{pmatrix}, \quad \boldsymbol{\partial} = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{11} & 0 \\ 0 & 0 & c_{44} \end{pmatrix}.$$

3. Discretization and preconditioning

We assume that a conforming FE method is employed to discretize the diffusion and elasticity problems defined by (2.1) and (2.3), respectively. The domain Ω is thus decomposed into a finite number of elements $\mathcal{E}_j$, $j = 1, \ldots, N_e$. Some continuous FE basis functions (with compact supports) denoted by φ_k, $k = 1, \ldots, N$, are used as approximation and test functions. By $\mathcal{P}_k$ we denote the smallest patch of elements covering the support of φ_k. Correspondingly to Section 2, we denote the material data of the diffusion and elasticity operators by $\boldsymbol{A}$ and $\boldsymbol{C}$, respectively, and the data of the associated preconditioning operators by $\tilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{C}}$, respectively. The function g_3 entering the Robin boundary conditions is allowed to be different in the original and preconditioning operators; therefore, it is denoted by $\tilde{g}_3$ in the latter.

The stiffness matrices A and C of the systems of linear equations of the discretized problems (2.1) and (2.3), respectively, have elements

$$\mathsf{A}_{kl} = \int_\Omega \nabla \varphi_l(\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x}) \nabla \varphi_k(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} g_3(\boldsymbol{x}) \varphi_l(\boldsymbol{x}) \varphi_k(\boldsymbol{x}) \, \mathrm{d}S$$

and

$$(3.1) \qquad \mathsf{C}_{\boldsymbol{kl}} = \int_\Omega (\boldsymbol{\partial}(\varphi_{l_1}(\boldsymbol{x}), \ldots, \varphi_{l_d}(\boldsymbol{x}))^\top)^\top \boldsymbol{C}(\boldsymbol{x}) \boldsymbol{\partial}(\varphi_{k_1}(\boldsymbol{x}), \ldots, \varphi_{k_d}(\boldsymbol{x}))^\top \, \mathrm{d}\boldsymbol{x},$$

respectively, where $k, l = 1, \ldots, N$, and $\boldsymbol{k}, \boldsymbol{l} \in \{1, \ldots, N\}^d$. The preconditioning matrices $\tilde{\mathsf{A}}$ and $\widetilde{\mathsf{C}}$ obtained for the material data $\tilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{C}}$, respectively, have elements

$$\tilde{\mathsf{A}}_{kl} = \int_\Omega \nabla \varphi_l(\boldsymbol{x}) \cdot \tilde{\boldsymbol{A}}(\boldsymbol{x}) \nabla \varphi_k(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} \tilde{g}_3(\boldsymbol{x}) \varphi_l(\boldsymbol{x}) \varphi_k(\boldsymbol{x}) \, \mathrm{d}S$$

and

$$\widetilde{\mathsf{C}}_{\boldsymbol{kl}} = \int_\Omega (\boldsymbol{\partial}(\varphi_{l_1}(\boldsymbol{x}), \ldots, \varphi_{l_d}(\boldsymbol{x}))^\top)^\top \widetilde{\boldsymbol{C}}(\boldsymbol{x}) \boldsymbol{\partial}(\varphi_{k_1}(\boldsymbol{x}), \ldots, \varphi_{k_d}(\boldsymbol{x}))^\top \, \mathrm{d}\boldsymbol{x},$$

respectively. All integrals are supposed to be carried out exactly.

The idea of preconditioning, see, e.g. [7], Section 10.3 or [13], Chapters 9 and 10, is based on the assumptions that a system of linear equations with a matrix $\widetilde{\mathsf{M}}$ is relatively easily solvable and that the spectrum of $\widetilde{\mathsf{M}}^{-1}\mathsf{M}$ is more favorable than that of M regarding some iterative solution method, which does not necessarily mean a smaller condition number [4]. Substituting the equation $\mathsf{Mu} = \mathsf{B}$ with

$$\widetilde{\mathsf{M}}^{-1}\mathsf{Mu} = \widetilde{\mathsf{M}}^{-1}\mathsf{B} \quad \text{or} \quad \widetilde{\mathsf{M}}^{-1/2}\mathsf{M}\widetilde{\mathsf{M}}^{-1/2}\mathsf{v} = \widetilde{\mathsf{M}}^{-1/2}\mathsf{B}, \ \mathsf{u} = \widetilde{\mathsf{M}}^{-1/2}\mathsf{v},$$

thus leads to equivalent problems that can be solved more efficiently than the original one.

The main results of the paper are introduced in this section. Instead of presenting our results for a general elliptic second order partial differential equation with tensor data and a vector valued unknown function $\boldsymbol{u}$, we first develop our theory for the (scalar) diffusion equation with tensor data in full detail. Then we apply the same approach to the elasticity equation. The section is concluded by some general remarks, mainly on the relationship between our results and the recent results from [4].

4.1. Diffusion equation. The lower and upper bounds on the eigenvalues $0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_N$ of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ for any uniformly positive definite measurable data $\boldsymbol{A}, \tilde{\boldsymbol{A}} \colon \Omega \to \mathbb{R}^{d \times d}$ are introduced in this part. The boundary conditions of the original and preconditioning problems may differ at most in the function g_3, i.e. instead of g_3, the function $\tilde{g}_3$ can be used in the Robin boundary condition of the preconditioning problem. We assume, however, that there exist constants $0 < c_g \leqslant C_g < \infty$ such that

$$0 \leqslant c_g \tilde{g}_3(\boldsymbol{x}) \leqslant g_3(\boldsymbol{x}) \leqslant C_g \tilde{g}_3(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_2.$$

Since N is the number of the FE basis functions we have $\mathsf{A}, \tilde{\mathsf{A}} \in \mathbb{R}^{N \times N}$. We now build two sequences of positive real numbers λ_k^{L} and λ_k^{U}, $k = 1, \ldots, N$. Let us first set

$$\alpha_j^{\min} = \operatorname*{ess\,inf}_{\boldsymbol{x} \in \mathcal{E}_j} \lambda_{\min}(\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})),$$

$$\alpha_j^{\max} = \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathcal{E}_j} \lambda_{\max}(\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})),$$

if no edge of $\mathcal{E}_j$ lies in $\partial\Omega_2$, and

$$\alpha_j^{\min} = \min \left\{ \operatorname*{ess\,inf}_{\substack{\boldsymbol{x} \in \partial\Omega_2 \cap \overline{\mathcal{E}}_j, \\ g_3(\boldsymbol{x}) \neq 0}} \tilde{g}_3^{-1}(\boldsymbol{x})g_3(\boldsymbol{x}), \ \operatorname*{ess\,inf}_{\boldsymbol{x} \in \mathcal{E}_j} \lambda_{\min}(\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})) \right\},$$

$$\alpha_j^{\max} = \max \left\{ \operatorname*{ess\,sup}_{\substack{\boldsymbol{x} \in \partial\Omega_2 \cap \overline{\mathcal{E}}_j, \\ g_3(\boldsymbol{x}) \neq 0}} \tilde{g}_3^{-1}(\boldsymbol{x})g_3(\boldsymbol{x}), \ \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathcal{E}_j} \lambda_{\max}(\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})) \right\}$$

if at least one edge of $\mathcal{E}_j$ lies in $\partial\Omega_2$, $j = 1, \ldots, N_e$. If $\boldsymbol{A}(\boldsymbol{x})$ and $\tilde{\boldsymbol{A}}(\boldsymbol{x})$ are element-wise constant and if g_3 and $\tilde{g}_3$ are constant on every edge (of any element) lying in $\partial\Omega_2$, the computation of $\alpha_j^{\min}$ and $\alpha_j^{\max}$ reduces to calculating the extreme eigenvalues of $d \times d$ matrices on all individual elements $\mathcal{E}_j$, $j = 1, \ldots, N_e$, and eventually comparing them with $\tilde{g}_3^{-1}(\boldsymbol{x})g_3(\boldsymbol{x})$ on some of the attached edges. For every function φ_k, supported on the patch $\mathcal{P}_k$, let us set

$$(4.1) \qquad \lambda_k^{\mathrm{L}} = \min_{\mathcal{E}_j \subset \mathcal{P}_k} \alpha_j^{\min}, \quad \lambda_k^{\mathrm{U}} = \max_{\mathcal{E}_j \subset \mathcal{P}_k} \alpha_j^{\max}, \quad j = 1, \ldots, N.$$

Thus λ_k^{L} and λ_k^{U} are in the above sense the smallest and the largest, respectively, eigenvalues of $\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})$ on the patch $\mathcal{P}_k$, or the extremes of $\tilde{g}_3^{-1}g_3$ along the parts of the boundary of $\mathcal{P}_k$ lying in $\partial\Omega_2$. After inspecting all patches, we sort the two sequences in (4.1) in non-decreasing order. Thus we obtain two bijections

$$r, s \colon \{1, \ldots, N\} \to \{1, \ldots, N\}$$

such that

$$(4.2) \qquad \lambda_{r(1)}^{\mathrm{L}} \leqslant \lambda_{r(2)}^{\mathrm{L}} \leqslant \ldots \leqslant \lambda_{r(N)}^{\mathrm{L}}, \quad \lambda_{s(1)}^{\mathrm{U}} \leqslant \lambda_{s(2)}^{\mathrm{U}} \leqslant \ldots \leqslant \lambda_{s(N)}^{\mathrm{U}}.$$

Note that we could define and compute λ_k^{L} and λ_k^{U} directly without defining $\alpha_j^{\min}$ and $\alpha_j^{\max}$. However, dealing with the constants $\alpha_j^{\min}$ and $\alpha_j^{\max}$ is more algorithmically acceptable, because it allows avoiding multiple evaluation of eigenvalues of $\tilde{\boldsymbol{A}}^{-1}\boldsymbol{A}$ on every element.

Next we prove an auxiliary lemma. Let $\sigma(\mathsf{M})$ denote the spectrum of the matrix M.

Lemma 4.1. *Let $\boldsymbol{A}(\boldsymbol{x})$, $\tilde{\boldsymbol{A}}(\boldsymbol{x}) \in \mathbb{R}^{d \times d}$ be symmetric and positive definite for all $\boldsymbol{x} \in \mathcal{D} \subset \Omega$. Let there exist constants $0 < c_1 \leqslant c_2 < \infty$ and $0 < c_3 \leqslant c_4 < \infty$ such that*

$$(4.3) \qquad \sigma(\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})) \subset [c_1, c_2], \quad \boldsymbol{x} \in \mathcal{D},$$

and

$$0 \leqslant c_3 \tilde{g}_3(\boldsymbol{x}) \leqslant g_3(\boldsymbol{x}) \leqslant c_4 \tilde{g}_3(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_2 \cap \overline{\mathcal{D}}.$$

Then for $u \in H_0^1(\Omega)$ we get

$$(4.4) \qquad c_1 \int_{\mathcal{D}} \nabla u \cdot \tilde{\boldsymbol{A}} \nabla u \,\mathrm{d}\boldsymbol{x} \leqslant \int_{\mathcal{D}} \nabla u \cdot \boldsymbol{A} \nabla u \,\mathrm{d}\boldsymbol{x} \leqslant c_2 \int_{\mathcal{D}} \nabla u \cdot \tilde{\boldsymbol{A}} \nabla u \,\mathrm{d}\boldsymbol{x}$$

and

$$(4.5) \qquad \min\{c_1, c_3\} \left(\int_{\mathcal{D}} \nabla u \cdot \tilde{\boldsymbol{A}} \nabla u \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2 \cap \overline{\mathcal{D}}} \tilde{g}_3 u^2 \,\mathrm{d}S \right)$$
$$\leqslant \int_{\mathcal{D}} \nabla u \cdot \boldsymbol{A} \nabla u \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2 \cap \overline{\mathcal{D}}} g_3 u^2 \,\mathrm{d}S$$
$$\leqslant \max\{c_2, c_4\} \left(\int_{\mathcal{D}} \nabla u \cdot \tilde{\boldsymbol{A}} \nabla u \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2 \cap \overline{\mathcal{D}}} \tilde{g}_3 u^2 \,\mathrm{d}S \right).$$

P r o o f.   Since for all $\boldsymbol{v} \in \mathbb{R}^d$ and $\boldsymbol{x} \in \mathcal{D}$ it follows from (4.3) that

$$c_1 \boldsymbol{v}^\top \tilde{\boldsymbol{A}}(\boldsymbol{x}) \boldsymbol{v} \leqslant \boldsymbol{v}^\top \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{v} \leqslant c_2 \boldsymbol{v}^\top \tilde{\boldsymbol{A}}(\boldsymbol{x}) \boldsymbol{v},$$

we get (4.4) by setting $\boldsymbol{v} = \nabla u$ and integrating all three terms over $\mathcal{D}$. Inequalities (4.5) follow obviously using $g_3 \geqslant 0$. $\square$

Now we introduce the first part of the main results of this paper.

Theorem 4.2. *Let us assume that the $(d-1)$-dimensional measure of $\partial\Omega_1$ is positive. The lower and upper bounds on the eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$ of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ are given by (4.2), i.e.,*

$$(4.6) \qquad \lambda^{\mathrm{L}}_{r(k)} \leqslant \lambda_k \leqslant \lambda^{\mathrm{U}}_{s(k)}, \quad k = 1, \ldots, N.$$

P r o o f.   Due to the positive measure of $\partial\Omega_1$, the matrices $\tilde{\mathsf{A}}$ and A are positive definite. We only prove the lower bounds of (4.6); the upper bounds can be proved analogously. Due to the Courant-Fischer min-max theorem, e.g. [7], Theorem 8.1.2,

$$\lambda_k = \max_{\substack{S, \\ \dim S = N-k+1}} \min_{\substack{\mathsf{v} \in S, \\ \mathsf{v} \neq 0}} \frac{\mathsf{v}^\top \mathsf{A}\mathsf{v}}{\mathsf{v}^\top \tilde{\mathsf{A}}\mathsf{v}},$$

where S denotes a subspace of $\mathbb{R}^N$. Then we have

$$\lambda_1 = \max_{\substack{S, \\ \dim S = N}} \min_{\substack{\mathsf{v} \in S, \\ \mathsf{v} \neq 0}} \frac{\mathsf{v}^\top \mathsf{A}\mathsf{v}}{\mathsf{v}^\top \tilde{\mathsf{A}}\mathsf{v}} = \min_{\substack{\mathsf{v} \in \mathbb{R}^N, \\ \mathsf{v} \neq 0}} \frac{\mathsf{v}^\top \mathsf{A}\mathsf{v}}{\mathsf{v}^\top \tilde{\mathsf{A}}\mathsf{v}} \geqslant \lambda^{\mathrm{L}}_{r(1)},$$

where the inequality follows from Lemma 4.1. Indeed, using $u = \sum_{i=1}^{N} \mathsf{v}_i \varphi_i$, definition (4.1) and Lemma 4.1 with $\mathcal{D} = \Omega$, we get

$$\frac{\mathsf{v}^\top \mathsf{A}\mathsf{v}}{\mathsf{v}^\top \tilde{\mathsf{A}}\mathsf{v}} = \frac{\int_\Omega \nabla u \cdot \boldsymbol{A} \nabla u \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} g_3 u^2 \, \mathrm{d}S}{\int_\Omega \nabla u \cdot \tilde{\boldsymbol{A}} \nabla u \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} \tilde{g}_3 u^2 \, \mathrm{d}S} \geqslant \min_{\mathcal{E}_j \subset \Omega} \alpha_j^{\min} = \min_{\mathcal{P}_k \subset \Omega} \lambda^{\mathrm{L}}_k = \lambda^{\mathrm{L}}_{r(1)}.$$

Then we proceed to

$$\lambda_2 = \max_{\substack{S, \\ \dim S = N-1}} \min_{\substack{\mathsf{v} \in S, \\ \mathsf{v} \neq 0}} \frac{\mathsf{v}^\top \mathsf{A}\mathsf{v}}{\mathsf{v}^\top \tilde{\mathsf{A}}\mathsf{v}} \geqslant \min_{\substack{\mathsf{v} \in \mathbb{R}^N, \\ \mathsf{v} \neq 0, \ \mathsf{v}_{r(1)} = 0}} \frac{\mathsf{v}^\top \mathsf{A}\mathsf{v}}{\mathsf{v}^\top \tilde{\mathsf{A}}\mathsf{v}} \geqslant \lambda^{\mathrm{L}}_{r(2)},$$

where the last inequality follows from Lemma 4.1, where (due to $v_{r(1)} = 0$) $\mathcal{D}$ contains only the patches associated to the FE basis functions φ_j, $j \neq r(1)$,

$$\mathcal{D} = \bigcup_{j \in \{1,\dots,N\} \setminus \{r(1)\}} \mathcal{P}_j,$$

and from

$$\min_{\substack{v \in \mathbb{R}^N, \\ v \neq 0,\ v_{r(1)} = 0}} \frac{v^\top A v}{v^\top \tilde{A} v} = \min_{\substack{u = \sum_{i=1}^{N} v_i \varphi_i, \\ v_{r(1)} = 0}} \frac{\int_{\mathcal{D}} \nabla u \cdot \boldsymbol{A} \nabla u \, d\boldsymbol{x} + \int_{\partial \Omega_2 \cap \overline{\mathcal{D}}} g_3 u^2 \, dS}{\int_{\mathcal{D}} \nabla u \cdot \tilde{\boldsymbol{A}} \nabla u \, d\boldsymbol{x} + \int_{\partial \Omega_2 \cap \overline{\mathcal{D}}} \tilde{g}_3 u^2 \, dS}$$

$$\geqslant \min_{\mathcal{E}_j \subset \mathcal{D}} \alpha_j^{\min} = \min_{\mathcal{P}_k \subset \mathcal{D}} \lambda_k^{\mathrm{L}} = \lambda_{r(2)}^{\mathrm{L}}.$$

We can proceed further in the same manner to get all inequalities $\lambda_{r(k)}^{\mathrm{L}} \leqslant \lambda_k$ of (4.6). $\qquad \square$

In Theorem 4.2, we consider positive definite problems with homogeneous Dirichlet and/or general Robin boundary conditions (with $g_3 \geqslant 0$). Neumann boundary condition is a special type of Robin boundary condition with $g_3 = 0$. In practical implementation of non-homogeneous Dirichlet boundary conditions, the lifting function u_0 does not necessarily have to be employed. If the same non-homogeneous Dirichlet boundary conditions are considered for the original and preconditioning problems, the method of getting the lower and upper bounds (4.2) can be used unchanged. Our theory, however, does not cover the settings where the original and preconditioning problems are considered under different non-homogeneous Dirichlet boundary conditions or different functions g_2 in Robin boundary conditions, or if $\partial \Omega_1$ in the preconditioning problem does not coincide with $\partial \Omega_1$ used for the original problem.

If periodic or Neumann boundary conditions are applied along $\partial \Omega$ and if they are the same for the original and preconditioning problems, then A and $\tilde{A}$ are singular; they share the smallest eigenvalue $\lambda_1 = 0$ and the associated eigenvector. Then we can use the same method again to get the bounds on all of the eigenvalues of the preconditioned matrix; however, we must omit the null space of A (which is the same as the null space of $\tilde{A}$) from the respective formulas. To justify the method, we can proceed analogously as in the proof of Theorem 4.2, where the vectors v are now additionally considered fulfilling $\tilde{A} v \neq 0$. Then

$$\lambda_2 \geqslant \min_{\substack{v \in \mathbb{R}^N, \\ \tilde{A} v \neq 0}} \frac{v^\top A v}{v^\top \tilde{A} v} \geqslant \lambda_{r(1)}^{\mathrm{L}}.$$

We can proceed further, analogously to the proof of Theorem 4.2,

$$\lambda_3 \geqslant \min_{\substack{\mathsf{v} \in \mathbb{R}^N, \\ \tilde{\mathsf{A}}\mathsf{v} \neq 0, \mathsf{v}_{r(1)} = 0}} \frac{\mathsf{v}^\top \mathsf{A}\mathsf{v}}{\mathsf{v}^\top \tilde{\mathsf{A}}\mathsf{v}} \geqslant \lambda^{\mathrm{L}}_{r(2)}.$$

In this way we get $N - 1$ lower bounding numbers on the non-zero eigenvalues of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$, where both A and $\tilde{\mathsf{A}}$ are now considered restricted to the subspace of $\mathbb{R}^N$ that is orthogonal to the null space of A. Analogously, we get the upper bounds; thus finally,

$$\lambda^{\mathrm{L}}_{r(k-1)} \leqslant \lambda_k \leqslant \lambda^{\mathrm{U}}_{s(k)}, \quad k = 2, \dots, N.$$

Let us now apply our method to some examples.

E x a m p l e 4.3. Assume $d = 2$, $\Omega = (-\pi, \pi)^2$, $\partial\Omega_2 = \{\boldsymbol{x}; x_1 = \pi\}$,

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} 1 + 0.3\,\mathrm{sign}(\sin x_2) & 0.3 + 0.1\cos x_1 \\ 0.3 + 0.1\cos x_1 & 1 + 0.3\,\mathrm{sign}(\sin x_2) \end{pmatrix},$$

and a simple and a more sophisticated preconditioning operators with

$$\tilde{\boldsymbol{A}}_1(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \tilde{\boldsymbol{A}}_2(\boldsymbol{x}) = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix},$$

respectively. Let us consider one of the following settings:

(a) uniform grid with piece-wise bilinear FE functions, $N = 10^2$ or 30^2, $g_3 = 0$; see Figure 1;

(b) uniform grid with piece-wise bilinear FE functions, periodic boundary conditions, $N = 21^2$; see Figure 2;

(c) non-uniform grid and triangular elements with piece-wise linear FE functions, $g_3 = \tilde{g}_3 = 1 + x_2^2$, $N = 400$; see Figure 3.

The numerical experiments illustrate the results of Theorem 4.2, i.e. that the bounds on the eigenvalues are guaranteed for different types of boundary conditions. We can also notice that since $\boldsymbol{A}$ is point-wise closer to $\tilde{\boldsymbol{A}}_2$ than to $\tilde{\boldsymbol{A}}_1$, the spectrum of the second preconditioned problem (together with its bounds) is closer to unity than the spectrum of the problem preconditioned by using $\tilde{\boldsymbol{A}}_1$. Note also that refining the mesh does not lead to more accurate bounds in general. This is caused by the difference between the extreme eigenvalues of $\tilde{\boldsymbol{A}}_i^{-1}\boldsymbol{A}$, $i = 1, 2$, on individual elements; see also Section 4.3.

The number of CG steps needed to reduce the energy norm of the errors by the factor 10^{-9} (starting with zero initial vectors) for setting (a) with $f = 1$ in Ω is 17 and 13 for $\tilde{\boldsymbol{A}}_1$ and $\tilde{\boldsymbol{A}}_2$, respectively, for $N = 10^2$, and 20 and 15 for $\tilde{\boldsymbol{A}}_1$ and $\tilde{\boldsymbol{A}}_2$, respectively, for $N = 30^2$.
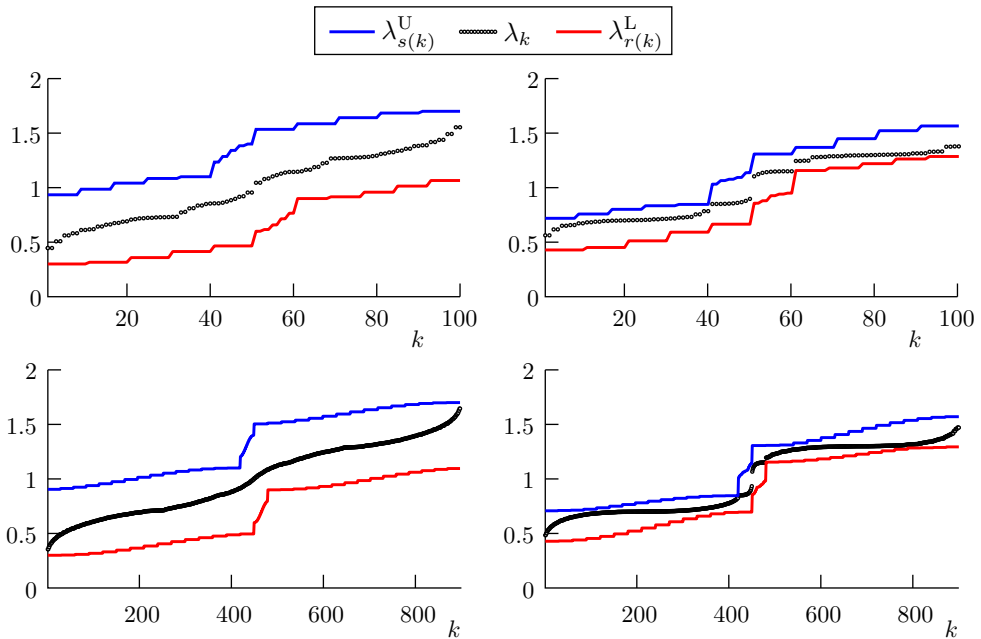
Figure 1. Lower $(\lambda_{r(k)}^{\mathrm{L}})$ and upper $(\lambda_{s(k)}^{\mathrm{U}})$ bounds on eigenvalues λ_k of Example 4.3 (a) with $N = 10^2$ (top graphs) and $N = 30^2$ (bottom graphs) preconditioned by operators with $\tilde{A}_1$ (left) and $\tilde{A}_2$ (right).
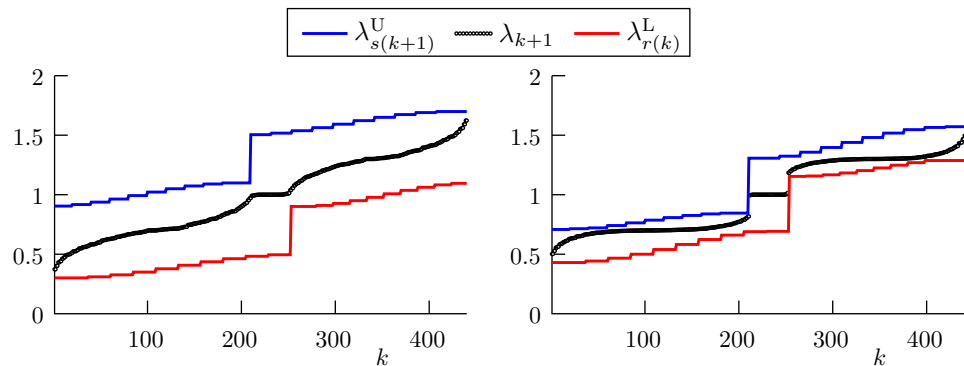

Figure 2. Lower $(\lambda_{r(k)}^{\mathrm{L}})$ and upper $(\lambda_{s(k)}^{\mathrm{U}})$ bounds on eigenvalues λ_k of Example 4.3 (b) with $N = 21^2$ preconditioned by operators with $\tilde{A}_1$ (left) and $\tilde{A}_2$ (right).

Let us emphasize that the error analysis of CG requires not only the eigenvalue distribution, but also (an estimate of) the components of the initial residual in directions of the associated eigenvectors; see, e.g., [4], Formula (2.7) and Remark 4.1. In some cases, however, the eigenvalue distribution can lead to a quite accurate estimate of the number of CG steps:
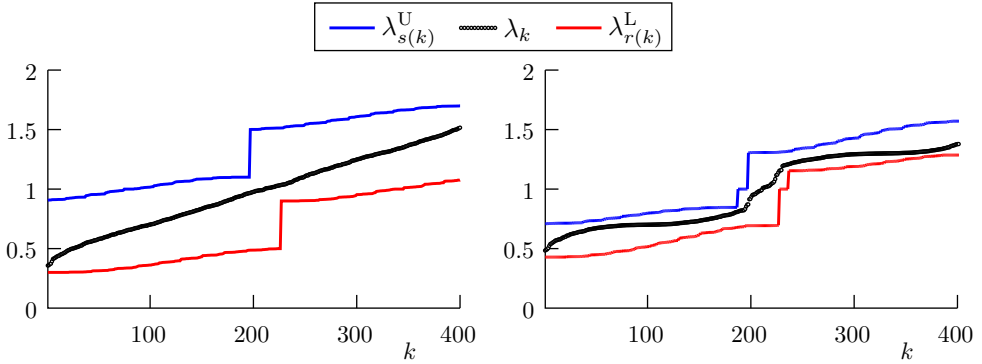
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Figure 3. Lower $(\lambda^{\mathrm{L}}_{r(k)})$ and upper $(\lambda^{\mathrm{U}}_{s(k)})$ bounds on eigenvalues λ_k of Example 4.3 (c) with $N = 400$ preconditioned by operators with $\tilde{A}_1$ (left) and $\tilde{A}_2$ (right) with $g_3 = \tilde{g}_3 = 1 + x_2^2$.

E x a m p l e 4.4.   Assume $d = 2$, $\Omega = (-\pi, \pi)^2$, homogeneous Dirichlet boundary conditions, a uniform grid, $N = 18^2$, and bilinear FE functions. Let Ω_1 and Ω_2 be two small subdomains in Ω (each covering four elements). Let $\boldsymbol{A}(\boldsymbol{x}) = b(\boldsymbol{x})\boldsymbol{I}$, where

(4.7) $b(\boldsymbol{x}) = 1 + z, \ \boldsymbol{x} \in \Omega_1, \quad b(\boldsymbol{x}) = 1 - z, \ \boldsymbol{x} \in \Omega_2, \quad b(\boldsymbol{x}) = 1, \ \boldsymbol{x} \in \Omega \setminus (\Omega_1 \cup \Omega_2),$

where z is some constant in $(-1, 1)$. For preconditioning we use the Laplacian, i.e. $\tilde{\boldsymbol{A}} = \boldsymbol{I}$. In Figure 4, it is shown that the spectrum of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ contains only a few outlying eigenvalues; their number does not depend on z. In accordance with this, the number of CG steps to reduce the energy norm of the error by the factor 10^{-9} is constant (equal to 11) independently of $z \in [0.9, 0.999]$. Note that such a z yields the condition numbers of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ varying from 19 to 1999.
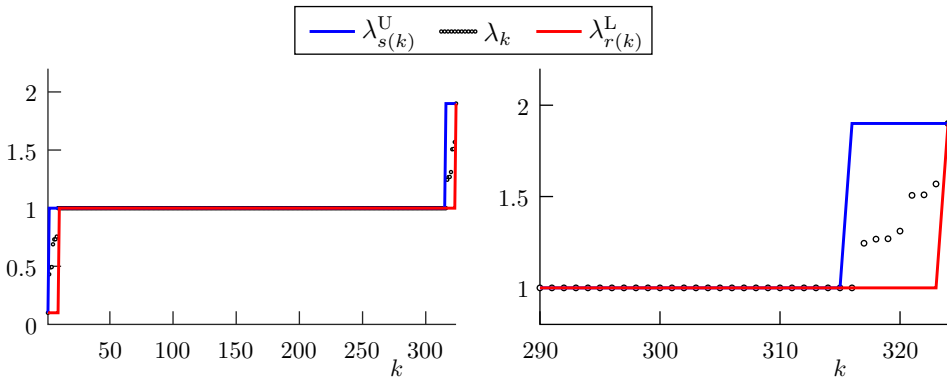

Figure 4. Lower $(\lambda^{\mathrm{L}}_{r(k)})$ and upper $(\lambda^{\mathrm{U}}_{s(k)})$ bounds on eigenvalues λ_k of Example 4.4 for $z = 0.9$ (left) and the detail view (right).

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4.2. Elasticity equation. In the linear elasticity problem, or in vector valued problems in general, the searched function has multiple components,

$$\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \dots, u_d(\boldsymbol{x}))^\top,$$

where individual components are coupled within the equation. For approximation of the scalar functions u_j, $j = 1, \dots, d$, we use the same sets of the FE basis functions φ_k, $k = 1, \dots, N$, supported again inside the patches $\mathcal{P}_k$. Recall that for the sake of simplicity, we consider homogeneous Dirichlet boundary conditions only.

Lemma 4.5. *Let* $\boldsymbol{C}(\boldsymbol{x}), \widetilde{\boldsymbol{C}}(\boldsymbol{x}) \in \mathbb{R}^{m \times m}$, *where* $m = 3$ *if* $d = 2$, *and* $m = 6$ *if* $d = 3$. *Let* $\boldsymbol{C}$ *and* $\widetilde{\boldsymbol{C}}$ *be symmetric and positive definite for all* $\boldsymbol{x} \in \mathcal{D} \subset \Omega$. *Let there exist constants* $0 < c_1 \leqslant c_2 < \infty$ *such that*

$$(4.8) \qquad\qquad \sigma(\widetilde{\boldsymbol{C}}^{-1}(\boldsymbol{x})\boldsymbol{C}(\boldsymbol{x})) \subset [c_1, c_2], \quad \boldsymbol{x} \in \mathcal{D}.$$

Then for $\boldsymbol{u} \in V_0^d$ *we get*

$$(4.9) \qquad c_1 \int_{\mathcal{D}} (\boldsymbol{\partial u})^\top \widetilde{\boldsymbol{C}} \boldsymbol{\partial u} \, \mathrm{d}\boldsymbol{x} \leqslant \int_{\mathcal{D}} (\boldsymbol{\partial u})^\top \boldsymbol{C} \boldsymbol{\partial u} \, \mathrm{d}\boldsymbol{x} \leqslant c_2 \int_{\mathcal{D}} (\boldsymbol{\partial u})^\top \widetilde{\boldsymbol{C}} \boldsymbol{\partial u} \, \mathrm{d}\boldsymbol{x}.$$

P r o o f. From (4.8) for all $\boldsymbol{v} \in \mathbb{R}^d$, $\boldsymbol{x} \in \mathcal{D}$, we get

$$c_1 \boldsymbol{v}^\top \widetilde{\boldsymbol{C}}(\boldsymbol{x})\boldsymbol{v} \leqslant \boldsymbol{v}^\top \boldsymbol{C}(\boldsymbol{x})\boldsymbol{v} \leqslant c_2 \boldsymbol{v}^\top \widetilde{\boldsymbol{C}}(\boldsymbol{x})\boldsymbol{v}.$$

Then by setting $\boldsymbol{v} = \boldsymbol{\partial u}$ and integrating over $\mathcal{D}$, we obtain (4.9). $\qquad\square$

We now show how to obtain the guaranteed bounds on all individual eigenvalues $0 < \lambda_1 \leqslant \dots \leqslant \lambda_{dN}$ of the preconditioned elasticity problem $\widetilde{\mathsf{C}}^{-1}\mathsf{C}$ for any positive definite material data $\boldsymbol{C}$ and $\widetilde{\boldsymbol{C}}$. Since N is the number of the FE basis functions defined on Ω used to approximate each component of $\boldsymbol{u}$, the number of unknowns is dN. We now build two sequences λ_k^{L} and λ_k^{U}, $k = 1, \dots, dN$, to bound the eigenvalues of $\widetilde{\mathsf{C}}^{-1}\mathsf{C}$. In contrast to Section 4.1, for the sake of brevity, we do not define $\alpha_j^{\min}$ and $\alpha_j^{\max}$, but we directly set

$$\hat{\lambda}_k^{\mathrm{L}} = \operatorname*{ess\,inf}_{\boldsymbol{x} \in \mathcal{P}_k} \lambda_{\min}(\widetilde{\boldsymbol{C}}^{-1}(\boldsymbol{x})\boldsymbol{C}(\boldsymbol{x})), \quad \hat{\lambda}_k^{\mathrm{U}} = \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathcal{P}_k} \lambda_{\max}(\widetilde{\boldsymbol{C}}^{-1}(\boldsymbol{x})\boldsymbol{C}(\boldsymbol{x})),$$

$k = 1, \dots, N$. Similarly to the case of the diffusion equation in Section 4.1, we sort these two series in non-decreasing order, and thus get bijections

$$R, S \colon \{1, \dots, N\} \to \{1, \dots, N\},$$

such that

$$\hat{\lambda}_{R(1)}^{\mathrm{L}} \leqslant \dots \leqslant \hat{\lambda}_{R(N)}^{\mathrm{L}}, \quad \hat{\lambda}_{S(1)}^{\mathrm{U}} \leqslant \dots \leqslant \hat{\lambda}_{S(N)}^{\mathrm{U}}.$$

Moreover, we double (if $d = 2$) or triple (if $d = 3$) all items in the two series of $\hat{\lambda}_k^{\mathrm{L}}$ and $\hat{\lambda}_k^{\mathrm{U}}$ and get two new d-times longer sequences

$$\lambda_{(k-1)d+1}^{\mathrm{L}} = \ldots = \lambda_{kd}^{\mathrm{L}} = \hat{\lambda}_k^{\mathrm{L}}, \quad \lambda_{(k-1)d+1}^{\mathrm{U}} = \ldots = \lambda_{kd}^{\mathrm{U}} = \hat{\lambda}_k^{\mathrm{U}}, \quad k = 1, \ldots, N,$$

that can be sorted in non-decreasing order. Thus we obtain two bijections

$$r, s \colon \{1, \ldots, dN\} \to \{1, \ldots, dN\},$$

such that

$$(4.10) \quad \lambda_{r(1)}^{\mathrm{L}} = \ldots = \lambda_{r(d)}^{\mathrm{L}}$$
$$\leqslant \lambda_{r(d+1)}^{\mathrm{L}} = \ldots = \lambda_{r(2d)}^{\mathrm{L}} \leqslant \ldots \leqslant \lambda_{r(dN-d+1)}^{\mathrm{L}} = \ldots = \lambda_{r(dN)}^{\mathrm{L}},$$
$$(4.11) \quad \lambda_{s(1)}^{\mathrm{U}} = \ldots = \lambda_{s(d)}^{\mathrm{U}}$$
$$\leqslant \lambda_{s(d+1)}^{\mathrm{U}} = \ldots = \lambda_{s(2d)}^{\mathrm{U}} \leqslant \ldots \leqslant \lambda_{s(dN-d+1)}^{\mathrm{U}} = \ldots = \lambda_{s(dN)}^{\mathrm{U}}.$$

Note that for $k = 1, \ldots, N$,

$$\hat{\lambda}_{R(k)}^{\mathrm{L}} = \lambda_{r((k-1)d+1)}^{\mathrm{L}} = \ldots = \lambda_{r(kd)}^{\mathrm{L}},$$
$$\hat{\lambda}_{S(k)}^{\mathrm{U}} = \lambda_{s((k-1)d+1)}^{\mathrm{U}} = \ldots = \lambda_{s(kd)}^{\mathrm{U}}.$$

Now we can introduce the second part of the main results of this paper.

Theorem 4.6. *The lower and upper bounds on all eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{dN}$ of $\widetilde{\mathsf{C}}^{-1}\mathsf{C}$ can be obtained from (4.10) and (4.11), namely*

$$(4.12) \qquad \lambda_{r(k)}^{\mathrm{L}} \leqslant \lambda_k \leqslant \lambda_{s(k)}^{\mathrm{U}}, \quad k = 1, \ldots, dN.$$

P r o o f. The proof is similar to the proof of Theorem 4.2. By the Courant-Fischer min-max theorem,

$$\lambda_k = \max_{\substack{S, \\ \dim S = dN-k+1}} \min_{\substack{\mathsf{v} \in S, \\ \mathsf{v} \neq 0}} \frac{\mathsf{v}^\top \mathsf{C} \mathsf{v}}{\mathsf{v}^\top \widetilde{\mathsf{C}} \mathsf{v}}.$$

Then

$$\lambda_d \geqslant \ldots \geqslant \lambda_1 = \min_{\substack{\mathsf{v} \in \mathbb{R}^{dN}, \\ \mathsf{v} \neq 0}} \frac{\mathsf{v}^\top \mathsf{C} \mathsf{v}}{\mathsf{v}^\top \widetilde{\mathsf{C}} \mathsf{v}} \geqslant \lambda_{r(1)}^{\mathrm{L}} = \ldots = \lambda_{r(d)}^{\mathrm{L}},$$

where the last inequality follows from Lemma 4.5. Indeed, representing the coefficients of the components of $\boldsymbol{u} = (u_1, \ldots, u_d)$ with respect to the FE basis functions

in a single vector $\mathsf{v} = (\mathsf{v}_{(1)}^\top, \ldots, \mathsf{v}_{(d)}^\top)^\top = (\mathsf{v}_1, \ldots, \mathsf{v}_{Nd})^\top$, $\mathsf{v}_{(j)} \in \mathbb{R}^N$, $j = 1, \ldots, d$, we get

$$\frac{\mathsf{v}^\top C \mathsf{v}}{\mathsf{v}^\top \widetilde{C} \mathsf{v}} = \frac{\int_\Omega (\boldsymbol{\partial u})^\top C \boldsymbol{\partial u}\, \mathrm{d}\boldsymbol{x}}{\int_\Omega (\boldsymbol{\partial u})^\top \widetilde{C} \boldsymbol{\partial u}\, \mathrm{d}\boldsymbol{x}} \geqslant \min_{\mathcal{P}_k \subset \Omega} \hat{\lambda}_k^{\mathrm{L}} = \hat{\lambda}_{R(1)}^{\mathrm{L}} = \lambda_{r(1)}^{\mathrm{L}} = \ldots = \lambda_{r(d)}^{\mathrm{L}}.$$

Next, we remove $\varphi_{R(1)}$ from all d bases approximating the components of $\boldsymbol{u} = (u_1, \ldots, u_d)$. Then

$$\lambda_{2d} \geqslant \ldots \geqslant \lambda_{d+1} \geqslant \min_{\substack{\mathsf{v} \in \mathbb{R}^N,\ \mathsf{v} \neq 0, \\ \mathsf{v}_{R(1)} = 0, \ldots, \mathsf{v}_{(d-1)N+R(1)} = 0,}} \frac{\mathsf{v}^\top C \mathsf{v}}{\mathsf{v}^\top \widetilde{C} \mathsf{v}} \geqslant \lambda_{r(d+1)}^{\mathrm{L}} = \ldots = \lambda_{r(2d)}^{\mathrm{L}},$$

where the last inequality follows from

$$\frac{\mathsf{v}^\top C \mathsf{v}}{\mathsf{v}^\top \widetilde{C} \mathsf{v}} = \frac{\int_\mathcal{D} (\boldsymbol{\partial u})^\top C \boldsymbol{\partial u}\, \mathrm{d}\boldsymbol{x}}{\int_\mathcal{D} (\boldsymbol{\partial u})^\top \widetilde{C} \boldsymbol{\partial u}\, \mathrm{d}\boldsymbol{x}} \geqslant \min_{\mathcal{P}_k \subset \mathcal{D}} \hat{\lambda}_k^{\mathrm{L}} = \hat{\lambda}_{R(2)}^{\mathrm{L}} = \lambda_{r(d+1)}^{\mathrm{L}} = \ldots = \lambda_{r(2d)}^{\mathrm{L}},$$

where $\mathsf{v}_{R(1)} = 0, \ldots, \mathsf{v}_{(d-1)N+R(1)} = 0$, and correspondingly,

$$\mathcal{D} = \bigcup_{j \in \{1, \ldots, N\} \setminus \{R(1)\}} \mathcal{P}_j.$$

Continuing further in this way, we can prove the lower bounds in (4.6). Analogously, we can get the upper bounds. $\qquad\square$

Example 4.7. Assume the elasticity equation with homogeneous Dirichlet boundary conditions, $d = 2$, $\Omega = (-\pi, \pi)^2$, $N = 21^2$, and the data

$$(4.13) \qquad C(\boldsymbol{x}) = \frac{E(\boldsymbol{x})}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5\,(1-2\nu) \end{pmatrix},$$

where

$$E(\boldsymbol{x}) = 1 + 0.3\,\mathrm{sign}(x_1 x_2), \quad \nu = 0.2.$$

Preconditioning is performed with the constant (homogeneous) data of the type (4.13) with $E = 1$ and either $\nu = 0$ or $\nu = 0.2$, denoted by $\widetilde{C}_1$ and $\widetilde{C}_2$, respectively. A uniform grid with piece-wise bilinear FE functions is employed. We can see in Figure 5 that the preconditioning matrix using the data $\widetilde{C}_2$, which are closer to C, yields the spectrum of the preconditioned matrix closer to unity. Moreover, we can notice two clusters of eigenvalues approximately equal to 0.7 and 1.3, respectively. The number of CG steps to reduce the energy norms of the errors by the factor of 10^{-9} is 14 and 11 for $\widetilde{C}_1$ and $\widetilde{C}_2$, respectively, when we consider $\boldsymbol{F} = (1,0)^\top$. In this example, $\widetilde{C}_1$ is diagonal, while $\widetilde{C}_2$ is more dense. Therefore, the overall efficiency strongly depends on the implementation of the preconditioner. These considerations are, however, beyond the scope of this paper.
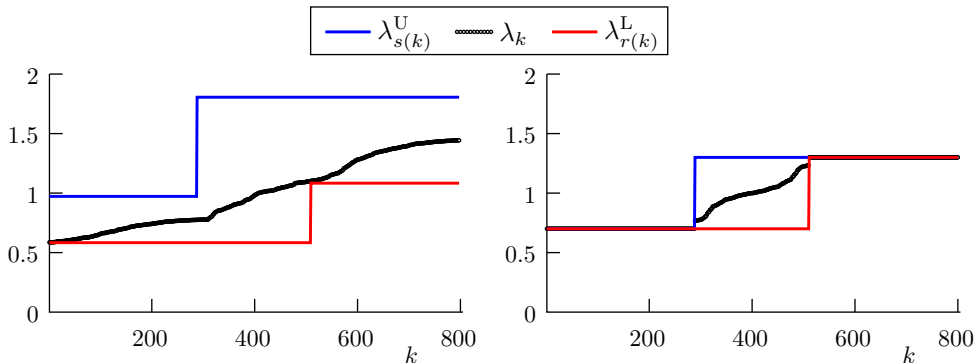
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Figure 5. Lower ($\lambda^{\mathrm{L}}_{r(k)}$) and upper ($\lambda^{\mathrm{U}}_{s(k)}$) bounds on eigenvalues λ_k of the elasticity problem of Example 4.7 with $N = 21^2$ preconditioned by operators with $\widetilde{\boldsymbol{C}}_1$ (left) and $\widetilde{\boldsymbol{C}}_2$ (right).

R e m a r k 4.8. The bilinear form $(\boldsymbol{u}, \boldsymbol{v})_C$ associated with the linear elasticity operator is equivalent to the following bilinear forms defined in V_0^d, see [1],

$$(\boldsymbol{u}, \boldsymbol{v})_{C,\triangle} = \int_\Omega \sum_{i,j=1}^d \frac{\partial v_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x},$$

$$(\boldsymbol{u}, \boldsymbol{v})_{C,\varepsilon} = \int_\Omega (\boldsymbol{\partial v})^\top \boldsymbol{\partial u} \, \mathrm{d}\boldsymbol{x},$$

$$(\boldsymbol{u}, \boldsymbol{v})_{C,d} = \int_\Omega \sum_{i=1}^d (\boldsymbol{\partial}(0, \ldots, 0, v_i, 0, \ldots, 0)^\top)^\top \boldsymbol{C} \boldsymbol{\partial}(0, \ldots, 0, u_i, 0, \ldots, 0)^\top \, \mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{v} = (v_1, \ldots, v_d)^\top$. The equivalence constants and the proofs can be found in [1] and in the references therein. We may notice that our preconditioning matrix $\widetilde{\mathsf{C}}$ with the data in the form $\widetilde{\boldsymbol{C}}(\boldsymbol{x}) = \boldsymbol{I}$ is the same as the matrix of the discretized form $(\boldsymbol{u}, \boldsymbol{v})_{C,\varepsilon}$. Therefore, using our method for obtaining the bounds on the eigenvalues of preconditioned problems can be used to estimate the equivalence constants of the above forms defined in finite-dimensional subspaces of V_0^d spanned by the FE basis functions; for example, we can immediately get

$$\lambda^{\mathrm{L}}_{r(1)}(\boldsymbol{u}, \boldsymbol{u})_{C,\varepsilon} \leqslant (\boldsymbol{u}, \boldsymbol{u})_C \leqslant \lambda^{\mathrm{U}}_{s(dN)}(\boldsymbol{u}, \boldsymbol{u})_{C,\varepsilon}.$$

4.3. General remarks. Let us now compare our results obtained for the diffusion equation with the recent results from [4]. Analogies for the elasticity equation can be considered straightforwardly. In [4], the existence of a pairing between the eigenvalues of the preconditioned matrix and the intervals obtained from the scalar data defined on the patches is proved. Especially, in any of the intervals, some eigenvalue must be found. This allows us to estimate the accuracy of the bounds provided

that the scalar data are continuous or mildly changing in (parts of) Ω. In our paper, instead, we get that $\lambda_k \in [\lambda^{\mathrm{L}}_{r(k)}, \lambda^{\mathrm{U}}_{s(k)}]$, or $\lambda_k \in [\lambda^{\mathrm{L}}_{r(k-1)}, \lambda^{\mathrm{U}}_{s(k)}]$ if the operator is semi-definite with the null space of dimension 1. Let us note that

$$\lambda^{\mathrm{L}}_k \leqslant \lambda^{\mathrm{U}}_k, \quad \lambda^{\mathrm{L}}_{r(k)} \leqslant \lambda^{\mathrm{U}}_{s(k)}, \quad r(k) \leqslant s(k), \quad k = 1, \ldots, N,$$

but $r(k) \neq s(k)$ in general, thus the intervals containing the individual eigenvalues are different from the intervals obtained in [4]. Sometimes, however, the intervals obtained by our method and by the method of [4] (ordered appropriately) coincide; see the following example.

E x a m p l e 4.9. Let us consider the test problem from [4], Section 4: the diffusion equation, $\Omega = (0,1)^2$, $\boldsymbol{A}(\boldsymbol{x}) = \sin(x_1 + x_2)\boldsymbol{I}$, and homogeneous Dirichlet boundary conditions on $\partial\Omega$. Let us use a uniform grid with piece-wise bilinear FE functions, $N = 9^2$ or $N = 19^2$. For preconditioning we use $\tilde{\boldsymbol{A}}(\boldsymbol{x}) = \boldsymbol{I}$. The appropriately ordered bounds provided by [4] and the bounds obtained by our method coincide; they are displayed in Figure 6.
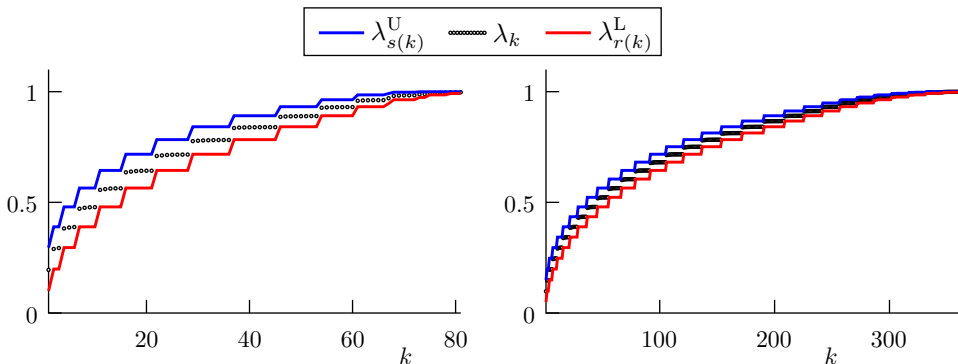

Figure 6. Lower ($\lambda^{\mathrm{L}}_{r(k)}$) and upper ($\lambda^{\mathrm{U}}_{s(k)}$) bounds on eigenvalues λ_k of Example 4.9 with $N = 9^2$ (left) and $N = 19^2$ (right).

The approach developed in [4] can be modified to the case of tensor data and the existence of a permutation $p\colon \{1, \ldots, N\} \to \{1, \ldots, N\}$ can be proved, such that

(4.14) $$\lambda_k \in [\lambda^{\mathrm{L}}_{p(k)}, \lambda^{\mathrm{U}}_{p(k)}], \quad k = 1, \ldots, N.$$

Weyl's inequality (see, e.g., [14], Section 3.5) is used in the proof in the same way as in [4]; the only change is in substituting the extremes of the scalar material data on every patch $\mathcal{P}_j$ by the extremes of the eigenvalues of $\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})$ on $\mathcal{P}_j$. Therefore, we do not provide the proof here. The bounds obtained from (4.6) and from (4.14) are compared in Example 4.11.

Using (4.14), under some special conditions, analogously to the results of [4], some eigenvalues can be identified exactly including their multiplicity. Since we do not present the proof of (4.14), let us formulate and prove this statement separately. For the sake of brevity, we formulate it for the case of non-singular diffusion equation with tensor data only. Generalization to problems with vector valued unknowns is straightforward; see also Example 4.7.

Lemma 4.10. *Let there exist $c > 0$ such that $\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) = c\boldsymbol{I}$ on a union of m patches $\mathcal{D} = \bigcup\limits_{k=1}^{m} \mathcal{P}_{j_k}$. Let none of the patches $\mathcal{P}_{j_k}$, $k = 1, \ldots, m$, attach to $\partial\Omega_2$ where $g_3 \neq 0$, and let the patches be associated with m linearly independent FE functions $\varphi_{j_1}, \ldots, \varphi_{j_m}$. Let A be non-singular. Then c is an eigenvalue of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ of multiplicity at least m.*

P r o o f.   Let $\mathsf{e}^{(j)} \in \mathbb{R}^N$, $(\mathsf{e}^{(j)})_i = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol. Then for every $j = j_1, \ldots, j_m$,

$$\frac{\mathsf{v}^\top \mathsf{A} \mathsf{e}^{(j)}}{\mathsf{v}^\top \tilde{\mathsf{A}} \mathsf{e}^{(j)}} = \frac{\int_\Omega \nabla v \cdot \boldsymbol{A} \nabla \varphi_j \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} g_3 \varphi_j v \, \mathrm{d}S}{\int_\Omega \nabla v \cdot \tilde{\boldsymbol{A}} \nabla \varphi_j \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_2} \tilde{g}_3 \varphi_j v \, \mathrm{d}S} = \frac{c \int_\Omega \nabla v \cdot \tilde{\boldsymbol{A}} \nabla \varphi_j \, \mathrm{d}\boldsymbol{x}}{\int_\Omega \nabla v \cdot \tilde{\boldsymbol{A}} \nabla \varphi_j \, \mathrm{d}\boldsymbol{x}} = c$$

for all $\mathsf{v} \in \mathbb{R}^N$, $\mathsf{v} \neq 0$. This means that c is an eigenvalue of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ associated with the eigenvectors $\mathsf{e}^{(j)}$, $j = j_1, \ldots, j_m$. Since the eigenvectors are linearly independent, the multiplicity of c is at least m. $\qquad\square$

E x a m p l e 4.11.   In this example, we compare our method of estimating the eigenvalues of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ with the method of [4] adapted for tensor data. Especially, we compare the bounds (4.6) with the intervals (4.14). Since we do not know the permutation p, we order the intervals according to the permutation r given by (4.2). Let us consider $d = 2$, $\Omega = (-1,1)^2$, $N = 18^2$, and bilinear FE basis functions. Let

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} 1.2 + 0.5\,(1 + \operatorname{sign} x_1)\,x_1 & 0 \\ 0 & 1.1 - 0.5\,(1 + \operatorname{sign} x_2)\,x_2 \end{pmatrix},$$

and let us use $\tilde{\boldsymbol{A}} = \boldsymbol{I}$ for preconditioning. The eigenvalues of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ and their bounds are displayed in Figure 7. The guaranteed bounds (4.6) are found on the left, while the guaranteed (unordered) intervals from (4.14) are displayed on the right. In this example, the bounds do not provide sharp localization of the eigenvalues (left). The intervals, however, provide very sharp localization of a half of the spectrum (right).
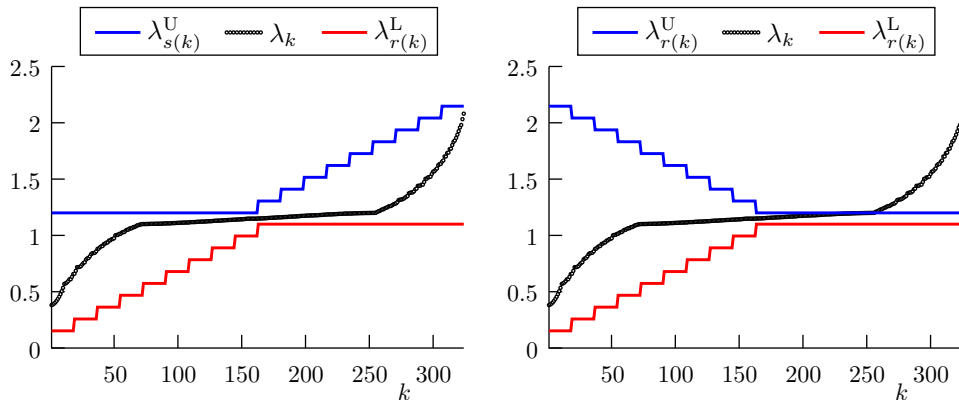
Figure 7. Lower ($\lambda^{\mathrm{L}}_{r(k)}$) and upper ($\lambda^{\mathrm{U}}_{s(k)}$) bounds on eigenvalues λ_k of Example 4.11 (left) and intervals $[\lambda^{\mathrm{L}}_{r(k)}, \lambda^{\mathrm{U}}_{r(k)}]$ (right).

Let us finally focus on limitations of our theory. We could see that in some examples the bounds did not get closer to the true eigenvalues when the mesh size decreases. As a representative 2D example we can take the diffusion equation with constant data, preconditioned by the Laplacian, say,

$$(4.15) \qquad \boldsymbol{A} = \mathrm{diag}(2,1), \quad \tilde{\boldsymbol{A}} = \mathrm{diag}(1,1).$$

While the constant lower and upper bounds are obtained

$$\lambda^{\mathrm{L}}_k = 1, \quad \lambda^{\mathrm{U}}_k = 2, \quad k = 1, \ldots, N,$$

the true eigenvalues of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ are distributed between these two bounds almost achieving both extremes 1 and 2. We could conclude that if the data are of the tensor type and if the preconditioner is poor, i.e. $\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})$ is not close enough to a multiple of the identity $\boldsymbol{I}$ in Ω, the bounds $\lambda^{\mathrm{L}}_{r(k)}$ and $\lambda^{\mathrm{U}}_{s(k)}$ may not say much about the true eigenvalues; the types of the FE basis functions and of the mesh affect the distribution of the true eigenvalues as well. Interestingly, from very recent results of Gergelits et al. [5] we can conclude that the spectrum of the operator $\triangle^{-1}[\nabla \cdot (\boldsymbol{A}\nabla)]$, i.e. the continuous form of example (4.15), is equal to $[1,2]$. We hope that further study elucidates a relationship between the eigenvalues of $\tilde{\mathsf{A}}^{-1}\mathsf{A}$ and the continuous case.

5. Conclusion

To the best of our knowledge, [4] is the first paper on estimating all eigenvalues of a preconditioned discretized diffusion operator. Motivated by [4], we contribute to this theory by introducing another approach based on the Courant-Fisher min-max principle. This allows generalizing some of the results of [4] to vector valued equations with tensor data and with more general boundary conditions preconditioned by arbitrary operators of the same type. We provide guaranteed bounds (defined by (4.2) and by (4.10)–(4.11) for scalar and vector problems, respectively) to every particular eigenvalue. On the other hand, the approach of [4] can provide more accurate estimates of (parts of) the spectra in general. Analogously to [4], the bounds are easily accessible and obtained solely from the data defined on supports of the FE basis functions. If the data are element-wise constant, only $O(N)$ arithmetic operations and sorting of two series of N numbers must be performed. Although we applied our method to only two types of differential equations, we are convinced that the same approach can be used in a wide variety of problems.

References

[1] *R. Blaheta*: Displacement decomposition - incomplete factorization preconditioning techniques for linear elasticity problems. Numer. Linear Algebra Appl. *1* (1994), 107–128. zbl MR doi

[2] *P. G. Ciarlet*: Mathematical Elasticity. Volume I: Three-Dimensional Elasticity. Studies in Mathematics and Its Applications 20. North-Holland, Amsterdam, 1988. zbl MR doi

[3] *A. Ern, J.-L. Guermond*: Theory and Practice of Finite Elements. Applied Mathematical Sciences 159. Springer, New York, 2004. zbl MR doi

[4] *T. Gergelits, K.-A. Mardal, B. F. Nielsen, Z. Strakoš*: Laplacian preconditioning of elliptic PDEs: Localization of the eigenvalues of the discretized operator. SIAM J. Numer. Anal. *57* (2019), 1369–1394. zbl MR doi

[5] *T. Gergelits, B. F. Nielsen, Z. Strakoš*: Generalized spectrum of second order differential operators. SIAM J. Numer. Anal. *58* (2020), 2193–2211. zbl MR doi

[6] *T. Gergelits, Z. Strakoš*: Composite convergence bounds based on Chebyshev polynomials and finite precision conjugate gradient computations. Numer. Algorithms *65* (2014), 759–782. zbl MR doi

[7] *G. H. Golub, C. F. Van Loan*: Matrix Computations. Johns Hopkins Studies in the Mathematical Sciences. The John Hopkins University Press, Baltimore, 1996. zbl MR

[8] *J. Liesen, Z. Strakoš*: Krylov Subspace Methods: Principles and Analysis. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2013. zbl MR doi

[9] *G. Meurant, Z. Strakoš*: The Lanczos and conjugate gradient algorithms in finite precision arithmetic. Acta Numerica *15* (2006), 471–542. zbl MR doi

[10] *G. Meurant, P. Tichý*: On computing quadrature-based bounds for the A-norm of the error in conjugate gradients. Numer. Algorithms *62* (2013), 163–191. zbl MR doi

[11] *J. Nečas, I. Hlaváček*: Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction. Studies in Applied Mechanics 3. Elsevier, Amsterdam, 1981.   zbl MR doi

[12] *B. F. Nielsen, A. Tveito, W. Hackbusch*: Preconditioning by inverting the Laplacian: An analysis of the eigenvalues. IMA J. Numer. Anal. *29* (2009), 24–42.   zbl MR doi

[13] *Y. Saad*: Iterative Methods for Sparse Linear Systems. Society for Industrial and Applied Mathematics, Philadelphia, 2003.   zbl MR doi

[14] *D. Serre*: Matrices: Theory and Applications. Graduate Texts in Mathematics 216. Springer, New York, 2010.   zbl MR doi

[15] *Z. Strakoš*: On the real convergence rate of the conjugate gradient method. Linear Algebra Appl. *154-156* (1991), 535–549.   zbl MR doi

[16] *A. van der Sluis, H. A. van der Vorst*: The rate of convergence of conjugate gradients. Numer. Math. *48* (1986), 543–560.   zbl MR doi

[17] *H. A. van der Vorst*: Iterative Krylov Methods for Large Linear Systems. Cambridge Monographs on Applied and Computational Mathematics 13. Cambridge University Press, Cambridge, 2003.   zbl MR doi

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