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UNIQUENESS AND TWO SHARED SET PROBLEMS OF MEROMORPHIC FUNCTIONS IN A DIFFERENT ANGLE

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Abstract. The purpose of the paper is to represent the two shared set problems in an elaborative and convenient manner. In the main result of the paper, we have exhaustively treated the two shared set problem on the open complex plane. As a consequence of the main result, we have investigated the same problem in a different perspective, which has yet not been studied. Further, two examples have been exhibited in the paper to show the sharpness of some of these results.

Keywords: meromorphic function; uniqueness; weighted sharing; shared set

MSC 2010: 30D35

1. Introduction, definitions and results

Throughout the paper, by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}^+$ we mean the set of all complex numbers, natural numbers, integers and positive real numbers, respectively. We further denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$. For any meromorphic function f we always mean it is defined on $\mathbb{C}$. For non-negative integers n and m, we define

$$\chi_n = \begin{cases} 0 & \text{when } n = 2m+3, \\ 1 & \text{when } n \neq 2m+3. \end{cases}$$

It is well-known that Gross is the pioneer of the set sharing problem in the uniqueness literature. Henceforth, we recall the following basic definition.

Definition 1.1. Let for a nonconstant meromorphic function f and $S \subset \overline{\mathbb{C}}$, $E_f(S) = \bigcup_{a\in S} \{(z,p) \in \mathbb{C} \times \mathbb{N} \colon f(z) = a \text{ with multiplicity } p\}$ ($\overline{E}_f(S) = \bigcup_{a\in S} \{(z,1) \in \mathbb{C} \times \mathbb{N} \colon f(z) = a\}$). Then we say f, g share the set S *counting multiplicities* (CM) (*ignoring multiplicities* (IM)) if $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$).

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In 2001 Lahiri (see [13], [14]) introduced the scalings between CM and IM which further added essence to the uniqueness literature.

Definition 1.2 ([13], [14]). Let k be a non-negative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also, we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 1.3 ([13]). For $S \subset \overline{\mathbb{C}}$ we define $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$, where k is a non-negative integer $a \in S$ or infinity. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

In connection to the famous question of Gross (see [10]), it was Lin-Yi (see [15]) who initiated the two shared set problems by raising the following question.

Question A. *Can one find two finite sets S_j, $j = 1, 2$, such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical?*

Subsequently a lot of investigations have been carried out by many researchers to find two sets among which one comprises of n elements and the other set contains ∞ and then reduce the value of n as much as possible.

In this respect the introduction of bi-unique range sets can be thought of as the inception of a new direction in set sharing problem. Below we recall the definition.

Definition 1.4 ([4]). A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is called bi-unique range sets for meromorphic (entire) functions with weights m, k if for any two nonconstant meromorphic (entire) functions f and g, $E_f(S_1, m) = E_g(S_1, m)$, $E_f(S_2, k) = E_g(S_2, k)$ imply $f \equiv g$. We say S_i's, $i = 1, 2$, are BURSMm, k (BURSEm, k) in short. As usual, if both $m = k = \infty$, we say S_i's, $i = 1$, 2, are BURSM (BURSE).

We see that the definition of BURSM is actually the study of uniqueness of meromorphic function corresponding to the two shared set problems in $\mathbb{C}$. In this respect it is worthy of mention that the first BURSM prior to its introduction was exhibited by Yi (see [18]) by the following theorem.

Theorem A ([18]). *Let* $S_1 = \{a+b, a+b\omega, \ldots, a+b\omega^{n-1}\}$, $S_2 = \{c_1, c_2\}$, *where* $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$ *and* $b \neq 0$, $c_1 \neq a$, $c_2 \neq a$, $(c_1-a)^n \neq (c_2-a)^n$, $(c_k-a)^n(c_j-a)^n \neq b^{2n}$ $(k, j = 1, 2)$ *are constants. If* $n \geqslant 9$, *then* S_i*'s*, $i = 1, 2$, *are BURSM.*

After that in 2012 Yi-Li (see [17]) improved the above theorem as follows.

Theorem B ([17]). *Let* $S_1 = \{0, 1\}$, $S_2 = \{z\colon \frac{1}{2}(n-1)(n-2)z^n - n(n-2)z^{n-1} + \frac{1}{2}n(n-1)z^{n-2} + 1 = 0\}$, *where* n $(\geqslant 5)$ *is an integer. Then* S_i*'s*, $i = 1, 2$, *are BURSM.*

Observe that the set S_1 in Theorem B is nothing but the set of zeros of the derivatives of the polynomial whose zeros are used to form the set S_2. With the help of this inherited property Banerjee generalized the underlying polynomial used to form S_2 of Theorem B in the following manner.

Theorem C ([4], [5]). *Let* $S_1 = \{0, 1\}$ *and* $S_2 = \{z\colon \frac{1}{2}(n-1)(n-2)z^n - n(n-2)z^{n-1} + \frac{1}{2}n(n-1)z^{n-2} - d = 0\}$, *where* n $(\geqslant 5)$ *is an integer and* $d \neq 0, 1, \frac{1}{2}$ *is a complex number such that* $d^2 - d + 1 \neq 0$. *Then* S_i*'s*, $i = 1, 2$, *are BURSM*$1, 3$, *BURSM*$3, 2$.

It is to be noticed that the polynomials used in Theorems B–C are of the same type. In this respect, we recall the following definitions to proceed further.

Definition 1.5 ([8]). A polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

is called an *initial term gap polynomial* (ITGP) if $a_i = 0$ but $a_j \neq 0$ for at least one j such that $1 \leqslant j < i < n$ and an initial term non gap polynomial (ITNGP) if there does not exist any such i.

Definition 1.6 ([9]). Let $P(z)$ be a polynomial such that $P'(z)$ has mutually k distinct zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively. Then $P(z)$ is said to be a *critically injective polynomial* if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, k\}$.

Since in Theorems B–C the construction of the first set depends upon the choice of the polynomial whose zero set forms the second set and the polynomial is of ITNGP type, it would be interesting to investigate whether all the variants of polynomials can be brought under a single umbrella. This is one of the two motivations for writing this paper. We will show that the generalized polynomial obtained in this paper will improve all the results discussed so far. The second motivation is to find the possible way to proceed from bi-unique range set to two shared set problems in a different angle, which will be discussed in detail in the last section of the paper.

Now we invoke the following definitions which we need for the proof of the main results of the paper.

Definition 1.7 ([12]). For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leqslant m)$ ($N(r, a; f \mid\geqslant m)$) the counting function of those a-points of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leqslant m)$ ($\overline{N}(r, a; f \mid\geqslant m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.8 ([1]). Let f and g be two nonconstant meromorphic functions such that f and g share $(a, 0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the reduced counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$, by $\overline{N}_E^{(2}(r, a; f)$ the reduced counting function of those a-points of f and g where $p = q \geqslant 2$. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$. In a similar manner we can define $\overline{N}_L(r, a; f)$ and $\overline{N}_L(r, a; g)$ for $a \in \mathbb{C} \cup \{\infty\}$.

When f and g share (a, m), $m \geqslant 1$, then $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

Definition 1.9 ([13], [14]). Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Throughout the paper we denote $P(z) = z^n + az^{n-m} + bz^{n-2m} + c$ and $\beta_i = -(c_i^n + ac_i^{n-m} + bc_i^{n-2m})$, where $n, m \in \mathbb{N}$ and $a, b, c \in \mathbb{C}^*$ are such that $a^2 \neq 4b$, $\gcd(m, n) = 1$, $n > 2m$ and c_i's are the roots of the equation

$$nz^{2m} + (n-m)az^m + b(n-2m) = 0 \tag{1.1}$$

for $i = 1, 2, \ldots, 2m$. Note that when $a^2/4b = n(n-2m)/(n-m)^2$, then (1.1) reduces to the equation

$$n\Big(z^m + \frac{a(n-m)}{2n}\Big)^2 - \frac{a^2(n-m)^2}{4n} + b(n-2m) = 0,$$

i.e.

$$n\Big(z^m + \frac{a(n-m)}{2n}\Big)^2 = 0. \tag{1.2}$$

Hence, in this case (1.1) has m distinct roots c_i, $i = 1, 2, \ldots, m$, each being repeated twice. Proceeding similarly it can be easily shown that whenever $a^2/4b \neq$

$n(n-2m)/(n-m)^2$, then (1.1) has exactly $2m$ simple roots c_i for $i=1,2,\dots,2m$. In view of the above discussion, we have the following theorems which are the main results of the paper.

Theorem 1.1. *Let* $S_1=\{0,c_1,c_2,\dots,c_m\}$, $S_2=\{z\colon z^n+az^{n-m}+bz^{n-2m}+c=0\}$, *where* n $(\geqslant 2m+3)$, $\gcd(m,n)=1$, $a^2/4b=n(n-2m)/(n-m)^2$ *and* $a,b,c\in\mathbb{C}^*$ *be such that* $c\neq\beta_i$, $\beta_i\beta_j/(\beta_i+\beta_j)$. *Then*

(i) S_i*'s,* $i=1,2$, *are BURSM*$1,3$*;*

(ii) S_i*'s,* $i=1,2$, *are BURSM*$2,2$.

Note 1.1. Observe that $c\neq 0,\beta_i$ for $i\in\{1,2,\dots,m\}$ imply that S_2 has n distinct elements. So, this condition is essential for the definition of S_2 (set) in the above theorem.

The following example shows that when $m=1$ and $n=5$, then the condition $c\neq\beta_i\beta_j/(\beta_i+\beta_j)$ cannot be removed.

Example 1.1. Let $m=1$ and $n=5$. Then for Theorem 1.1 we have only one c_i and β_i. So we get $b=\frac{4}{15}a^2$, $c_1=-\frac{2}{5}a$, $\beta_1=\frac{16}{625\times 15}a^5$. Now suppose f and g be any two nonconstant meromorphic functions such that $f+g=c_1$. Note that in this case $\beta_i\beta_j/(\beta_i+\beta_j)=\beta_1^2/2\beta_1=\frac{1}{2}\beta_1$. Then for $c=\frac{1}{2}\beta_1$ with the above values of b, c_1 and β_1 we have

$$\begin{aligned}
f^5+af^4+bf^3&=f^3(f^2+af+b)\\
&=(c_1-g)^3((c_1-g)^2+a(c_1-g)+b)\\
&=-(g-c_1)^3(g^2-(2c_1+a)g+c_1^2+ac_1+b)\\
&=-(g^5+(-5c_1-a)g^4+(10c_1^2+4ac_1+b)g^3+(-10c_1^3-6ac_1^2-3bc_1)g^2\\
&\quad+(5c_1^4+4ac_1^3+3bc_1^2)g-c_1^3(c_1^2+ac_1+b))\\
&=-(g^5+ag^4+bg^3+\beta_1),
\end{aligned}$$

i.e.

$$f^5+af^4+bf^3+\tfrac{1}{2}\beta_1=-(g^5+ag^4+bg^3+\beta_1-\tfrac{1}{2}\beta_1),$$

i.e.

$$f^5+af^4+bf^3+c=-(g^5+ag^4+bg^3+c),$$

which implies f and g share S_2. Obviously, we have chosen f and g in such a way that they share the set S_1. So f and g share S_1 and S_2 CM but $f\not\equiv g$.

Observe that the polynomials used for the construction of S_2 in Theorems B–C are all critically injective polynomials. Also from Lemma 2.11, we would see that the polynomial used to form S_2 of Theorem 1.1 is critically injective. Since from Remark 1.1 of [8] we get that the same polynomial is uncertain to be critically injective whenever $a^2(n-m)^2 \neq 4bn(n-2m)$. Therefore it will be interesting to deal the above theorem with S_2 under this supposition. Hence, we have the following theorems.

Theorem 1.2. *Let* $S_1 = \{0, c_1, c_2, \ldots, c_{2m}\}$, $S_2 = \{z\colon\ z^n + az^{n-m} + bz^{n-2m} + c = 0\}$, *where* n $(\geqslant 4m+3)$, $\gcd(m,n) = 1$, $a^2/4b \neq n(n-2m)/(n-m)^2, 1$ *and* $a, b, c \in \mathbb{C}^*$ *be such that* $c \neq \beta_i$, $\beta_i\beta_j/(\beta_i+\beta_j)$. *Then* S_i*'s,* $i = 1, 2$, *are BURSM*$0, 4$.

Theorem 1.3. *Let* $S_1 = \{0, c_1, c_2, \ldots, c_{2m}\}$ *and* $S_2 = \{z\colon\ z^n + bz^{n-2m} + c = 0\}$, *where* $\gcd(n, 2m) = 1$, $b \in \mathbb{C}^*$ *and* $c \neq 0, \beta_i, \beta_i\beta_j/(\beta_i+\beta_j)$. *Then* S_i*'s,* $i = 1, 2$, *are BURSM*$0, 4$ *for* $n \geqslant 4m+3$.

From Theorem 1.2 and Theorem 1.3 it follows that the least cardinality of the second range set is 7 whereas in Theorem 1.1 the least cardinality of the same is 5. So, natural question arises whether it is possible to further reduce the cardinality of S_2 in Theorem 1.2 and Theorem 1.3 so that the least cardinality of S_2 in the two theorems becomes 5. In the next two theorems we have shown that under the additional supposition that the meromorphic functions sharing the sets do not have any simple poles, the above is achievable.

Theorem 1.4. *Let* S_1 *and* S_2 *be two sets as defined in Theorem* 1.2 *for* $n \geqslant 4m+1$. *Also suppose that* f *and* g *be two nonconstant meromorphic functions without having any simple pole such that* $E_f(S_1, 0) = E_g(S_1, 0)$ *and* $E_f(S_2, 2) = E_g(S_2, 2)$. *Then* $f \equiv g$.

Theorem 1.5. *Let* S_1 *and* S_2 *be two sets as defined in Theorem* 1.3 *for* $n \geqslant 4m+1$. *Also suppose that* f *and* g *be two nonconstant meromorphic functions without having any simple pole such that* $E_f(S_1, 0) = E_g(S_1, 0)$ *and* $E_f(S_2, 2) = E_g(S_2, 2)$. *Then* $f \equiv g$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f and g be two nonconstant meromorphic functions and for an integer $n \geqslant 2m+1$

$$F = \frac{P(f) - c}{-c} = \frac{f^{n-2m}(f^{2m} + af^m + b)}{-c}, \tag{2.1}$$
$$G = \frac{P(g) - c}{-c} = \frac{g^{n-2m}(g^{2m} + ag^m + b)}{-c}.$$

Henceforth, we shall denote by H and Ψ the following two functions:

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big), \tag{2.2}$$

$$\Psi=\frac{F'}{F-1}-\frac{G'}{G-1}. \tag{2.3}$$

Lemma 2.1 ([14])**.** *If F, G are two nonconstant meromorphic functions such that they share $(1,1)$ and $H\not\equiv 0$, then*

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leqslant N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. *Let F, G be given by (2.1). Also let $E_f(S_1,p)=E_g(S_1,p)$ and $E_f(S_2,0)=E_g(S_2,0)$, where S_i's, $i=1,2$, are given as in Theorem 1.2 and Theorem 1.1. Suppose $H\not\equiv 0$. Then*

(i) *for $a^2/4b\neq n(n-2m)/(n-m)^2$ we have*

$$\begin{aligned}N(r,H)\leqslant{}&\overline{N}(r,0;f)+\overline{N}(r,0;nf^{2m}+(n-m)af^m+b(n-2m))+\overline{N}(r,\infty;f)\\&+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+\overline{N}_*(r,1;F,G)\end{aligned}$$

and

(ii) *for $a^2/4b=n(n-2m)/(n-m)^2$ we have*

$$\begin{aligned}N(r,H)\leqslant{}&\overline{N}(r,0;f\mid\geqslant p+1)+\overline{N}\Big(r,0;f^m+\frac{a(n-m)}{2n}\mid\geqslant p+1\Big)\\&+\chi_n\Big(\overline{N}(r,0;f\mid\leqslant p)+\overline{N}\Big(r,0;f^m+\frac{a(n-m)}{2n}\mid\leqslant p\Big)\Big)\\&+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+\overline{N}_*(r,1;F,G),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(nf^{2m}+(n-m)af^m+b(n-2m))(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

P r o o f. From (2.1) we get that

$$F'=\frac{f^{n-2m-1}(nf^{2m}+(n-m)af^m+b(n-2m))}{-c}f', \tag{2.4}$$

$$G'=\frac{g^{n-2m-1}(ng^{2m}+(n-m)ag^m+b(n-2m))}{-c}g'. \tag{2.5}$$

From the condition of the lemma we see that

$$nf^{2m}+(n-m)af^m+b(n-2m)=\begin{cases} n\prod_{i=1}^{2m}(f-c_i), & \text{when } \dfrac{a^2}{4b}\neq\dfrac{n(n-2m)}{(n-m)^2},\\ n\prod_{i=1}^{m}(f-c_i)^2, & \text{when } \dfrac{a^2}{4b}=\dfrac{n(n-2m)}{(n-m)^2}.\end{cases}$$

(i) First suppose $a^2/4b\neq n(n-2m)/(n-m)^2$. Then (2.2) reduces to

$$H=\sum_{i=1}^{2m}\frac{f'}{f-c_i}-\sum_{i=1}^{2m}\frac{g'}{g-c_i}+(n-2m-1)\Big(\frac{f'}{f}-\frac{g'}{g}\Big)+\frac{f''}{f'}-\frac{g''}{g'}-\Big(\frac{2F'}{F-1}-\frac{2G'}{G-1}\Big).$$

Since F, G share $(1,0)$, from the construction of H we have

$$\begin{aligned}N(r,H)\leqslant{}&\overline{N}(r,0;f)+\overline{N}(r,0;nf^{2m}+(n-m)af^m+b(n-2m))+\overline{N}(r,\infty;f)\\&+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;g')+\overline{N}_0(r,0;f')+\overline{N}_*(r,1;F,G).\end{aligned}$$

(ii) Next suppose $a^2/4b=n(n-2m)/(n-m)^2$. Then (2.2) reduces to

$$H=\sum_{i=1}^{m}\frac{2f'}{f-c_i}-\sum_{i=1}^{m}\frac{2g'}{g-c_i}+(n-2m-1)\Big(\frac{f'}{f}-\frac{g'}{g}\Big)+\frac{f''}{f'}-\frac{g''}{g'}-\Big(\frac{2F'}{F-1}-\frac{2G'}{G-1}\Big).$$

Let z_0 be a zero of f and a c_i-point of g. Then from the above we can easily conclude that z_0 is not a pole of H for $n=2m+3$ and a pole of H otherwise. So we have

$$\begin{aligned}N(r,H)\leqslant{}&\overline{N}(r,0;f\mid\geqslant p+1)+\overline{N}\Big(r,0;f^m+\frac{a(n-m)}{2n}\mid\geqslant p+1\Big)\\&+\chi_n\Big(\overline{N}(r,0;f\mid\leqslant p)+\overline{N}\Big(r,0;f^m+\frac{a(n-m)}{2n}\mid\leqslant p\Big)\Big)\\&+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+\overline{N}_*(r,1;F,G).\end{aligned}$$

□

Lemma 2.3 ([16]). *Let f be a nonconstant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\ldots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n$ are constants and $a_n\neq0$. Then $T(r,P(f))=nT(r,f)+O(1)$.*

Lemma 2.4 ([6]). *Let f and g be two meromorphic functions sharing $(1,t)$, where $1\leqslant t<\infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\big(t-\tfrac12\big)\overline{N}_*(r,1;f,g)\\&\qquad\leqslant\tfrac12(N(r,1;f)+N(r,1;g)).\end{aligned}$$

Lemma 2.5. *Let S_i, $i = 1, 2$, be defined as in Theorem 1.1, Theorem 1.2 and F, G be given by (2.1). Suppose for two nonconstant meromorphic functions f and g, $E_f(S_1, p) = E_g(S_1, p)$, $E_f(S_2, t) = E_g(S_2, t)$ and $\Psi \not\equiv 0$. Then*
(i) *for $a^2/4b = n(n-2m)/(n-m)^2$ with $n \geqslant 2m+3$ we have*

$$\begin{aligned}(3p+2)\Big(\overline{N}(r,0;f \mid\geqslant p+1) + \overline{N}\Big(r,0;f^m + \frac{a(n-m)}{2n} \mid\geqslant p+1\Big)\Big)\\ \leqslant \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g)\end{aligned}$$

and
(ii) *for $a^2/4b \neq n(n-2m)/(n-m)^2$ with $n > 2m+1$ we have*

$$\begin{aligned}(2p+1)(\overline{N}(r,0;f \mid\geqslant p+1) + \overline{N}(r,0;nf^{2m} + (n-m)af^m + b(n-2m) \mid\geqslant p+1))\\ \leqslant \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g).\end{aligned}$$

Proof. By the given condition clearly F and G share $(1,t)$.
(i) Since $a^2/4b = n(n-2m)/(n-m)^2$, we have

$$\Psi = \frac{nf^{n-2m-1}(f^m + \frac{1}{2}a(n-m)n^{-1})^2 f'}{-c(F-1)} - \frac{ng^{n-2m-1}(g^m + \frac{1}{2}a(n-m)n^{-1})^2 g'}{-c(G-1)}.$$

Let z_0 be a zero or a c_i-point of f with multiplicity r. Since $E_f(S_1,p) = E_g(S_1,p)$, then that would be a zero of Ψ of multiplicity $\min\{(n-2m-1)r + r - 1, 2r + r - 1\}$, i.e. of multiplicity $\min\{(n-2m)r - 1, 3r - 1\}$ if $r \leqslant p$ and a zero of multiplicity at least $\min\{(n-2m-1)(p+1) + p, 2(p+1) + p\}$; i.e. a zero of multiplicity at least $\min\{(n-2m)p + (n-2m-1), 3p+2\} = 3p+2$ if $r > p$. So by a simple calculation we can write

$$\begin{aligned}(3p+2)\Big(\overline{N}(r,0;f \mid\geqslant p+1) + \overline{N}\Big(r,0;f^m + \frac{a(n-m)}{2n} \mid\geqslant p+1\Big)\Big)\\ \leqslant N(r,0;\Psi) \leqslant T(r,\Psi) + O(1) \leqslant N(r,\infty;\Psi) + S(r,F) + S(r,G)\\ \leqslant \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g).\end{aligned}$$

(ii) Since $a^2/4b \neq n(n-2m)/(n-m)^2$, we have

$$\begin{aligned}\Psi = \frac{f^{n-2m-1}(nf^{2m} + (n-m)af^m + b(n-2m))f'}{-c(F-1)}\\ - \frac{g^{n-2m-1}(ng^{2m} + (n-m)ag^m + b(n-2m))g'}{-c(G-1)}.\end{aligned}$$

Let z_0 be a zero or a c_i-point of f with multiplicity r. Since $E_f(S_1,p) = E_g(S_1,p)$, then that would be a zero of Ψ of multiplicity $\min\{(n-2m-1)r + r - 1, r + r - 1\}$, i.e. of multiplicity $\min\{(n-2m)r-1, 2r-1\}$ if $r \leqslant p$ and a zero of multiplicity at least $\min\{(n-2m-1)(p+1)+p, (p+1)+p\}$, i.e. a zero of multiplicity at least $\min\{(n-2m)p+(n-2m-1), 2p+1\} = 2p+1$ if $r > p$. So similarly as above we can have

$$\begin{aligned}
&(2p+1)(\overline{N}(r,0;f \mid\geqslant p+1) + \overline{N}(r,0;nf^{2m}+(n-m)af^m+b(n-2m) \mid\geqslant p+1))\\
&\qquad \leqslant N(r,\infty;\Psi) + S(r,F) + S(r,G)\\
&\qquad \leqslant \overline{N}_*(r,1;F,G) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g).
\end{aligned}$$

□

Lemma 2.6. *Let S_i, $i=1,2$, be defined as in Theorem 1.1, Theorem 1.2 and F, G be given by (2.1). Suppose for two nonconstant meromorphic functions f and g that $E_f(S_1,p) = E_g(S_1,p)$, $E_f(S_2,t) = E_g(S_2,t)$, where $0 \leqslant p < \infty$, $2 \leqslant t < \infty$ and $H \not\equiv 0$. Then*

(i) *for $a^2/4b = n(n-2m)/(n-m)^2$ we have*

$$\begin{aligned}
&(n+m)(T(r,f)+T(r,g))\\
&\quad \leqslant 2\Big(\overline{N}(r,0;f) + \sum_{i=1}^{m}\overline{N}(r,c_i;f)\Big)\\
&\qquad + \overline{N}(r,0;f \mid\geqslant p+1) + \sum_{i=1}^{m}\overline{N}(r,c_i;f \mid\geqslant p+1)\\
&\qquad + \chi_n\Big(\overline{N}(r,0;f \mid\leqslant p) + \overline{N}\Big(r,0;f^m + \frac{a(n-m)}{2n} \mid\leqslant p\Big)\Big)\\
&\qquad + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + \tfrac{1}{2}(N(r,1;F)+N(r,1;G))\\
&\qquad - \big(t-\tfrac{3}{2}\big)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)
\end{aligned}$$

and

(ii) *for $a^2/4b \neq n(n-2m)/(n-m)^2$ we have*

$$\begin{aligned}
&(n+2m)(T(r,f)+T(r,g))\\
&\quad \leqslant 3\Big(\overline{N}(r,0;f) + \sum_{i=1}^{2m}\overline{N}(r,c_i;f)\Big) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g)\\
&\qquad + \tfrac{1}{2}(N(r,1;F)+N(r,1;G)) - \big(t-\tfrac{3}{2}\big)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).
\end{aligned}$$

P r o o f. (i) By the second fundamental theorem we get

$$\begin{aligned}
(2.6)\quad &(n+m)(T(r,f)+T(r,g))\\
&\quad \leqslant \overline{N}(r,1;F) + \overline{N}(r,0;f) + \sum_{i=1}^{m}\overline{N}(r,c_i;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;G)
\end{aligned}$$

$$
\begin{aligned}
&+\overline{N}(r,0;g)+\sum_{i=1}^{m}\overline{N}(r,c_i;g)+\overline{N}(r,\infty;g)-N_0(r,0;f')\\
&-N_0(r,0;g')+S(r,f)+S(r,g).
\end{aligned}
$$

Now the conclusion immediately follows from Lemmas 2.1, 2.2, 2.3 and 2.4 and the fact that $\overline{N}(r,0;f)+\sum\limits_{i=1}^{m}\overline{N}(r,c_i;f)=\overline{N}(r,0;g)+\sum\limits_{i=1}^{m}\overline{N}(r,c_i;g)$.

(ii) By the second fundamental theorem we get

$$
\begin{aligned}
(2.7)\quad &(n+2m)(T(r,f)+T(r,g))\\
&\leqslant \overline{N}(r,1;F)+\overline{N}(r,0;f)+\sum_{i=1}^{2m}\overline{N}(r,c_i;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)\\
&\quad+\overline{N}(r,0;g)+\sum_{i=1}^{2m}\overline{N}(r,c_i;g)+\overline{N}(r,\infty;g)-N_0(r,0;f')\\
&\quad-N_0(r,0;g')+S(r,f)+S(r,g).
\end{aligned}
$$

Now the conclusion immediately follows from Lemmas 2.1, 2.2, 2.3 and 2.4 and the fact that $\overline{N}(r,0;f)+\sum\limits_{i=1}^{2m}\overline{N}(r,c_i;f)=\overline{N}(r,0;g)+\sum\limits_{i=1}^{2m}\overline{N}(r,c_i;g)$. □

Lemma 2.7. *Let S_i, $i=1,2$, be defined as in Theorem 1.1, Theorem 1.2 and F, G be given by (2.1), where $n\geqslant 2m+1$ and they share $(1,t)$ for $1\leqslant t\leqslant\infty$. Then*

(i) *for $a^2/4b=n(n-2m)/(n-m)^2$*

$$
\overline{N}_*(r,1;F,G)\leqslant\frac{1}{t}\left(\overline{N}(r,0;f)+\sum_{i=1}^{m}\overline{N}(r,c_i;f)\right)+S(r,f)
$$

and

(ii) *for $a^2/4b\neq n(n-2m)/(n-m)^2$*

$$
\overline{N}_*(r,1;F,G)\leqslant\frac{1}{t}\left(\overline{N}(r,0;f)+\sum_{i=1}^{2m}\overline{N}(r,c_i;f)\right)+S(r,f).
$$

Proof. The proof is obvious. □

Lemma 2.8. *Let S_i, $i=1,2$, be defined as in Theorem 1.1, Theorem 1.2 and F, G be given by (2.1), where $n\geqslant 2m+1$ and they share $(1,t)$ for $2\leqslant t\leqslant\infty$. Then*

(i)

$$
\overline{N}_*(r,1;F,G)\leqslant\frac{1}{2t-1}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+S(r,f)+S(r,g)
$$

when $a^2/4b=n(n-2m)/(n-m)^2$,

(ii)
$$\overline{N}_*(r,1;F,G) \leqslant \frac{1}{t-1}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+S(r,f)+S(r,g)$$

when $a^2/4b \neq n(n-2m)/(n-m)^2$.

Proof. (i) By Lemma 2.7 and Lemma 2.5 we have

$$\begin{aligned}
\overline{N}_*(r,1;F,G) &\leqslant \frac{1}{t}\left(\overline{N}(r,0;f)+\sum_{i=1}^{m}\overline{N}(r,c_i;f)\right)+S(r,f)\\
&\leqslant \frac{1}{2t}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G))+S(r,f)+S(r,g)\\
&\leqslant \frac{1}{2t-1}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+S(r,f)+S(r,g).
\end{aligned}$$

(ii) Similarly, we can have

$$\begin{aligned}
\overline{N}_*(r,1;F,G) &\leqslant \frac{1}{t}\left(\overline{N}(r,0;f)+\sum_{i=1}^{2m}\overline{N}(r,c_i;f)\right)+S(r,f)\\
&\leqslant \frac{1}{t}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G))+S(r,f)+S(r,g)\\
&\leqslant \frac{1}{t-1}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+S(r,f)+S(r,g).
\end{aligned}$$

$\square$

Lemma 2.9 ([8]). *Let* $\varphi(z)=a^2(z^{n-m}-A)^2-4b(z^{n-2m}-A)(z^n-A)$, *where* $a, A\,(\neq 0), b\,(\neq 0) \in \mathbb{C}$, $\gcd(m,n)=1$, $n>3m$ *and* $a^2 \neq 4b$. *Then the following results hold.*

(i) *If* e^{t_0} *is any multiple zero of* $\varphi(z)$, *then* t_0 *satisfies*

$$\cosh mt_0 = 1 \quad \text{or} \quad \cosh mt_0 = \frac{a^2(n-m)^2}{2bn(n-2m)}-1.$$

(ii) *Each multiple zero of* $\varphi(z)$ *is of multiplicity* 2 *whenever*

$$\frac{a^2}{4b} \neq \frac{n(n-2m)}{(n-m)^2}.$$

Lemma 2.10 ([8]). *Let* $\varphi(z)=a^2(z^{n-m}-A)^2-4b(z^{n-2m}-A)(z^n-A)$, *where* $A,a,b\in\mathbb{C}^*$, $a^2/4b=n(n-2m)/(n-m)^2$, $\gcd(m,n)=1$, $n>2m$. *If* ω^l *is the* m*th root of unity for* $l=0,1,\dots,m-1$, *then*

(i) $\varphi(z)$ *has no multiple zero when* $A\neq\omega^l$,

(ii) $\varphi(z)$ *has exactly one multiple zero when* $A=\omega^l$ *and that is of multiplicity* 4. *In particular, when* $A=1$, *then the multiple zero is* 1.

Lemma 2.11 ([8]). *Let $P(z) = z^n + az^{n-m} + bz^{n-2m} + c$, where $a, b \in \mathbb{C}^*$. Then the following holds.*

(i) *β_i's are nonzero if $a^2 \neq 4b$.*

(ii) *$P(z)$ is critically injective polynomial if $a^2/4b = n(n-2m)/(n-m)^2$.*

3. Proofs of the theorems

Proof of Theorem 1.1. (i) Let f and g be nonconstant meromorphic functions such that $E_f(S_1, 1) = E_g(S_1, 1)$ and $E_f(S_2, 3) = E_g(S_2, 3)$. Suppose F, G be given by (2.1). Then F and G share $(1, 3)$. We consider the following cases.

Case 1. Suppose that $\Psi \not\equiv 0$.

Subcase 1.1. Let $H \not\equiv 0$. Then for n=$2m+3$ using Lemma 2.6 for $p = 1$, $t = 3$, Lemma 2.5 for $p = 1$, $p = 0$ and Lemma 2.3 we obtain

$$\begin{aligned}
&(n+m)(T(r,f)+T(r,g))\\
&\quad\leqslant 2\Big(\overline{N}(r,0;f)+\sum_{i=1}^{m}\overline{N}(r,c_i;f)\Big)\\
&\qquad+\frac{1}{5}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G))\\
&\qquad+2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\frac{1}{2}(N(r,1;F)+N(r,1;G))\\
&\qquad-\frac{3}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\quad\leqslant \frac{16}{5}(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+\frac{n}{2}(T(r,f)+T(r,g))+S(r,f)+S(r,g)\\
&\quad\leqslant \Big(\frac{n}{2}+\frac{16}{5}\Big)(T(r,f)+T(r,g))+S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction.

Next, for $n > 2m+3$, proceeding in the same way as above and using Lemma 2.6 for $t = 3$, Lemma 2.5 for $p = 0$ and Lemma 2.3 we get

$$(n+m)(T(r,f)+T(r,g)) \leqslant \Big(\frac{n}{2}+\frac{7}{2}\Big)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is again a contradiction.

Subcase 1.2. $H \equiv 0$. Then from (2.2) we get

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B, \tag{3.1}$$

where A $(\neq 0)$ and B are two constants. So in view of Lemma 2.3, from (3.1) we get

$$T(r,f) = T(r,g) + O(1). \tag{3.2}$$

Subcase 1.2.1. Suppose $B \neq 0$. Then from (3.1) we get

$$F - 1 \equiv \frac{G-1}{BG + A - B}. \tag{3.3}$$

Subcase 1.2.1.1. If $A - B \neq 0$, then noting that $(B-A)/B \neq 1$, from (3.3) we get

$$\overline{N}\Big(r, \frac{B-A}{B}; G\Big) = \overline{N}(r, \infty; F).$$

Now let us consider the following subcases.

Subcase 1.2.1.1.1. Suppose that $(B-A)/B \neq \beta_i/c$ for all $i = 1, 2, \ldots, m$. Therefore in view of equation (3.2) using the second fundamental theorem we have

$$\begin{aligned}(n+m)T(r,g) &\leqslant \overline{N}(r,0;g) + \sum_{i=1}^{m} \overline{N}(r,c_i;g) + \overline{N}(r,\infty;g) + \overline{N}\Big(r, \frac{B-A}{B}; G\Big) + S(r,g) \\ &\leqslant (m+2)T(r,g) + \overline{N}(r,\infty;f) + S(r,g),\end{aligned}$$

which is a contradiction for $n \geqslant 4$.

Subcase 1.2.1.1.2. Suppose that $(B-A)/B = \beta_i/c$ for one $i \in \{1, 2, \ldots m\}$. Since $a^2/4b = n(n-2m)/(n-m)^2$, then from Lemma 2.2 we know that

$$G' = n\frac{g^{n-2m-1}(g^m + \frac{1}{2}a(n-m)n^{-1})^2}{-c} g'. \tag{3.4}$$

Again $a^2/4b = n(n-2m)/(n-m)^2 \neq 1$ implies $a^2 \neq 4b$. Therefore by Lemma 2.11 we get $\beta_i \neq 0$ and $P(z)$ is critically injective. Since any critically injective polynomial can have at most one multiple zero, $g^n + ag^{n-m} + bg^{n-2m} + \beta_i = (g - c_i)^3 \prod_{j=1}^{n-3}(g - \eta_j)$, where η_j's are $(n-3)$ distinct zeros of $z^n + az^{n-m} + bz^{n-2m} + \beta_i$ such that $\eta_j \neq c_i, 0$. Then from (3.3) we have

$$B(F-1) \equiv \frac{-c(G-1)}{(g-c_i)^3 \prod_{j=1}^{n-3}(g-\eta_j)}. \tag{3.5}$$

Since $E_f(S_1, 0) = E_g(S_1, 0)$, c_i-points of g are not poles of F and hence c_i is an e.v.P of g. Furthermore, each η_j-point of g of multiplicity p is a pole of f of multiplicity q (say). Therefore $p = nq \geqslant n$. So in view of (3.2) and the second fundamental theorem we get

$$\begin{aligned}(n-2)T(r,g) &\leqslant \overline{N}(r,0;g) + \overline{N}(r,c_i;g) + \overline{N}(r,\infty;g) + \sum_{i=1}^{n-3} \overline{N}(r,\eta_j;g) + S(r,g) \\ &\leqslant \Big(2 + \frac{n-3}{n}\Big)T(r,g) + S(r,g),\end{aligned}$$

which is a contradiction for $n \geqslant 2m+3$.

Subcase 1.2.1.2. If $A-B=0$, then from (3.3) we have

$$\frac{G-1}{F-1} \equiv BG = B\frac{g^{n-2m}(g^{2m}+ag^m+b)}{-c}, \tag{3.6}$$

i.e. 0's of g and $(g^{2m}+ag^m+b)$ are poles of F. Since $a^2/4b = n(n-2m)/(n-m)^2$, i.e. $a^2 \neq 4b$, so all the zeros of $w^{2m}+aw^m+b$ are simple. Now let ξ_i be a zero of $w^{2m}+aw^m+b$ for $i \in \{1,2,\dots,2m\}$ and each ξ_i-point of g is of multiplicity p. Then it is a pole of f of multiplicity q for some $q \geqslant 1$. So from (3.6) we get $p = nq$, i.e. $p \geqslant n$. Similarly as in Subcase 1.2.1.1.2 we can prove here that '0' is an e.v.P. of g. Now using the second fundamental theorem we get

$$(2m-1)T(r,g) \leqslant \sum_{i=1}^{2m} \overline{N}(r,\xi_i;g) + \overline{N}(r,0;g) + S(r,g) \leqslant \frac{2m}{n}T(r,g) + S(r,g),$$

which is a contradiction for $n \geqslant 3$.

Subcase 1.2.2. Suppose $B = 0$. Then from (3.1) we get that

$$G-1 = A(F-1),$$

i.e.

$$G' = AF',$$

which implies $\Psi \equiv 0$, a contradiction.

Case 2. Let $\Psi \equiv 0$. Then by integration we get

$$G-1 = A(F-1),$$

i.e.

$$g^n + ag^{n-m} + bg^{n-2m} \equiv A\Big(f^n + af^{n-m} + bf^{n-2m} + c\frac{A-1}{A}\Big) \tag{3.7}$$

and

$$g^n + ag^{n-m} + bg^{n-2m} + c(1-A) \equiv A(f^n + af^{n-m} + bf^{n-2m}). \tag{3.8}$$

Subcase 2.1. Let $A \neq 1$. Then as $c \neq 0$, $c(A-1)/A \neq 0$ and at the same time by Lemma 2.11 we have $\beta_i \neq 0$. Therefore we have the following subcases.

Subcase 2.1.1. Suppose $c(A-1)/A = \beta_i$ for some $i \in \{1,2,\dots,m\}$. Then we claim that $c(1-A) \neq \beta_j$ for any $j \in \{1,2,\dots,m\}$. For if $c(1-A) = \beta_j$, i.e. $A = (c-\beta_j)/c$, and since $c(A-1)/A = \beta_i$, i.e. $A = c/(c-\beta_i)$, it follows that $(c-\beta_j)/c = c/(c-\beta_i)$, i.e. $c = \beta_i\beta_j/(\beta_i+\beta_j)$, a contradiction. Thus $w^n + aw^{n-m} + bw^{n-2m} + c(1-A) = 0$

has only simple roots, say α_i for $i = 1, 2, \ldots, n$. So from (3.8) we get

$$\prod_{i=1}^{n}(g-\alpha_i) \equiv Af^{n-2m}(f^{2m}+af^m+b). \tag{3.9}$$

Since $E_f(S_1, 0) = E_g(S_1, 0)$, from (3.9) obviously '0' is an e.v.P. of f. Now using (3.2) and the second fundamental theorem, in view of (3.9) we get

$$(n-2)T(r,g) \leqslant \sum_{i=1}^{n} \overline{N}(r,\alpha_i; g) + S(r,g) \leqslant 2mT(r,f) + S(r,g),$$

which is a contradiction for $n \geqslant 2m + 3$.

Subcase 2.1.2. Suppose $c(A-1)/A \neq \beta_i$ for all $i \in \{1, 2, \ldots, 2m\}$. So, $w^n + aw^{n-m} + bw^{n-2m} + c(A-1)/A = 0$ has only simple roots, say α_i' for $i = 1, 2, \ldots, n$. Therefore from (3.7) we have

$$g^{n-2m}(g^{2m}+ag^m+b) \equiv A\prod_{i=1}^{n}(f-\alpha_i'). \tag{3.10}$$

Now by the same argument as used in Subcase 2.1.1 we get a contradiction for $n \geqslant 2m + 3$.

Subcase 2.2. Let $A = 1$. Then we get $P(g) \equiv P(f)$, i.e.

$$g^{n-2m}(g^{2m}+ag^m+b) \equiv f^{n-2m}(f^{2m}+af^m+b), \tag{3.11}$$

which implies f, g share ∞ CM. Since $E_f(S_1, 0) = E_g(S_1, 0)$, then equation (3.11) also implies f, g share 0 CM. Now suppose $h = g/f$. Then clearly h does not have any zero and pole. Substituting $g = fh$ into $P(g) \equiv P(f)$ we have

$$f^{2m}(h^n-1) + af^m(h^{n-m}-1) + b(h^{n-2m}-1) = 0. \tag{3.12}$$

Subcase 2.2.1. If h is constant, then as g is nonconstant, $h^n = h^{n-m} = h^{n-2m} = 1$, which implies $h = 1$ as $\gcd(m, n) = 1$. Therefore $f \equiv g$.

Subcase 2.2.2. If h is nonconstant, then in view of Lemma 2.10 we get

$$\Big(f^m + \frac{a}{2}\frac{h^{n-m}-1}{h^n-1}\Big)^2 = \frac{a^2(h-1)^4(h-\delta_1)(h-\delta_2)\ldots(h-\delta_{2n-2m-4})}{4(h^n-1)^2}, \tag{3.13}$$

where δ_i's are the distinct simple zeros of $\varphi(z)$. From (3.13) we conclude that each δ_i-point of h is of multiplicity at least 2. Therefore by the second fundamental theorem we get

$$\begin{aligned}(2n-2m-4)T(r,h) &\leqslant \sum_{i=1}^{2n-2m-4} \overline{N}(r,\delta_i;h) + \overline{N}(r,0;h) + \overline{N}(r,\infty;h) + S(r,h)\\ &\leqslant (n-m-2)T(r,h) + S(r,h),\end{aligned}$$

which is a contradiction for $n \geqslant m+3$.

(ii) Let f and g be two nonconstant meromorphic functions such that $E_f(S_1,2) = E_g(S_1,2)$ and $E_f(S_2,2) = E_g(S_2,2)$. Suppose F, G be given by (2.1). Then F and G share $(1,2)$. We consider the following cases.

Case 1. Suppose that $\Psi \not\equiv 0$.

Subcase 1.1. Let $H \not\equiv 0$. Then for $n = 2m+3$ using Lemma 2.6 for $p=2$, $t=2$, Lemma 2.5 for $p=2$, $p=0$, Lemma 2.3 and Lemma 2.8 we obtain

$$\begin{aligned}&(n+m)(T(r,f)+T(r,g))\\ &\quad\leqslant 2\left(\overline{N}(r,0;f) + \sum_{i=1}^{m}\overline{N}(r,c_i;f)\right) + \frac{1}{8}(\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G))\\ &\qquad + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + \frac{1}{2}(N(r,1;F) + N(r,1;G))\\ &\qquad - \frac{1}{2}\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\ &\quad\leqslant \frac{25}{8}(\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)) + \frac{n}{2}(T(r,f)+T(r,g))\\ &\qquad + \frac{5}{8}\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\ &\quad\leqslant \left(\frac{n}{2} + \frac{25}{8}\right)(T(r,f)+T(r,g)) + \frac{5}{24}(\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g))\\ &\qquad + S(r,f) + S(r,g)\\ &\quad\leqslant \left(\frac{n}{2} + \frac{10}{3}\right)(T(r,f)+T(r,g)) + S(r,f) + S(r,g),\end{aligned}$$

which is a contradiction.

For $n > 2m+3$, in a similar way as above we get

$$(n+m)(T(r,f)+T(r,g)) \leqslant \left(\frac{n}{2} + \frac{7}{2} + \frac{1}{3}\right)(T(r,f)+T(r,g)) + S(r,f) + S(r,g),$$

which is again a contradiction.

The rest of the proof can be dealt the same as the proof of part (i) of this theorem. □

P r o o f of Theorem 1.2. Let f and g be nonconstant meromorphic functions such that $E_f(S_1,0) = E_g(S_1,0)$ and $E_f(S_2,4) = E_g(S_2,4)$. Suppose F, G be given by (2.1). Then F and G share $(1,4)$. We consider the following cases.

Case 1. Suppose that $\Psi \not\equiv 0$.

Subcase 1.1. Let $H \not\equiv 0$. Then using Lemmas 2.6, 2.8 for $t = 4$, Lemma 2.5 for $p = 0$, Lemma 2.3 and proceeding similarly as in Subcase 1.1 of part (i) of Theorem 1.1 we obtain

$$(n+2m)(T(r,f)+T(r,g)) \leqslant \Big(\frac{n}{2}+5+\frac{1}{6}\Big)(T(r,f)+T(r,g)) + S(r,f) + S(r,g),$$

which is a contradiction for $n \geqslant 4m+3$.

Subcase 1.2. Let $H \equiv 0$. Then from (2.2) we get

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B, \tag{3.14}$$

where A $(\neq 0)$ and B are two constants. So in view of Lemma 2.3, from (3.14) we get

$$T(r,f) = T(r,g) + O(1). \tag{3.15}$$

Subcase 1.2.1. Suppose $B \neq 0$. Then from (3.14) we get

$$F-1 \equiv \frac{G-1}{BG+A-B}. \tag{3.16}$$

Subcase 1.2.1.1. If $A-B \neq 0$, then noting that $(B-A)/B \neq 1$, from (3.16) we get

$$\overline{N}\Big(r,\frac{B-A}{B};G\Big) = \overline{N}(r,\infty;F).$$

Now let us consider the following subcases.

Subcase 1.2.1.1.1. Suppose that $(B-A)/B \neq \beta_i/c$ for all $i = 1,2,\dots m$. Therefore in view of equation (3.15) using the second fundamental theorem we have

$$\begin{aligned}(n+2m)T(r,g) &\leqslant \overline{N}(r,0;g) + \sum_{i=1}^{2m}\overline{N}(r,c_i;g) + \overline{N}(r,\infty;g) + \overline{N}\Big(r,\frac{B-A}{B};G\Big) + S(r,g)\\ &\leqslant (2m+2)T(r,g) + \overline{N}(r,\infty;f) + S(r,g),\end{aligned}$$

which is a contradiction for $n \geqslant 4$.

Subcase 1.2.1.1.2. Suppose that $(B-A)/B = \beta_i/c$ for one $i \in \{1,2,\dots m\}$. Since $a^2/4b \neq n(n-2m)/(n-m)^2$, then from Lemma 2.2 we get that

$$G' = n\frac{g^{n-2m-1}(ng^{2m}+(n-m)ag^m+b(n-2m))}{-c}g'. \tag{3.17}$$

Also from Remark 1.1 of [8], for $a^2/4b \neq 1,\, n(n-2m)/(n-m)^2$, it is uncertain whether $P(z)$ is critically injective or not and at the same time we have $\beta_i \neq 0$ by Lemma 2.11. Therefore

$$z^n + az^{n-m} + bz^{n-2m} + \beta_i \tag{3.18}$$

may have more than one multiple zero which are nothing but c_i's for $i = 1, 2, \ldots, 2m$. But it is certain that if (3.18) has r multiple zeros, say $c_1, c_2, \ldots, c_r$; then each of them is of multiplicity 2 because they are simple zeros of $nz^{2m} + (n-m)az^m + b(n-2m)$. Hence

$$g^n + ag^{n-m} + bg^{n-2m} + \beta_i = (g-c_1)^2(g-c_2)^2 \ldots (g-c_r)^2 \prod_{i=1}^{n-2r}(g-\zeta_i),$$

where ζ_i's are $(n-2r)$ distinct zeros of (3.18) such that $\zeta_i \neq c_i, 0$ for $i = 1, 2, \ldots, 2m$. Then from (3.16) we have

$$B(F-1) \equiv \frac{-c(G-1)}{(g-c_1)^2(g-c_2)^2 \ldots (g-c_r)^2 \prod\limits_{i=1}^{n-2r}(g-\zeta_i)}. \tag{3.19}$$

Since $E_f(S_1, 0) = E_g(S_1, 0)$, c_i-points of g are not poles of F and hence c_i-points are e.v.P. of g. Now if $r \geqslant 3$, then g is constant which is a contradiction. If $r \leqslant 2$, then observe that each ζ_i-point of g of multiplicity p is a pole of F of multiplicity q (say). Therefore $p = nq \geqslant n$. So by the second fundamental theorem we get

$$\begin{aligned}(n-r)T(r,g) &\leqslant \overline{N}(r,0;g) + \sum_{i=1}^{r}\overline{N}(r,c_i;g) + \overline{N}(r,\infty;g) + \sum_{i=1}^{n-2r}\overline{N}(r,\zeta_j;g) + S(r,g)\\ &\leqslant \Big(2 + \frac{n-2r}{n}\Big)T(r,g) + S(r,g).\end{aligned}$$

Since $r \leqslant 2 \leqslant 2m$, therefore we arrive at a contradiction for $n \geqslant 2m+3$, i.e. for $n \geqslant 4m+3$.

Subcase 1.2.1.2. If $A - B = 0$, then as here $a^2 \neq 4b$, this case can be dealt the same as in the proof of Subcase 1.2.1.2. of Theorem 1.1.

Subcase 1.2.2. Suppose $B = 0$. Then from (3.14) we get that

$$G - 1 = A(F-1),$$

i.e.

$$G' = AF',$$

which implies $\Psi \equiv 0$, a contradiction.

Case 2. Let $\Psi \equiv 0$. Then on integration we get

$$G - 1 = A(F - 1),$$

i.e.

$$g^n + ag^{n-m} + bg^{n-2m} \equiv A\Big(f^n + af^{n-m} + bf^{n-2m} + c\frac{A-1}{A}\Big) \tag{3.20}$$

and

$$g^n + ag^{n-m} + bg^{n-2m} + c(1 - A) \equiv A(f^n + af^{n-m} + bf^{n-2m}). \tag{3.21}$$

Subcase 2.1. Let $A \neq 1$. Then this case also can be resorted the same as Subcase 2.1. of Theorem 1.1.

Subcase 2.2. Let $A = 1$. Then we get $P(g) \equiv P(f)$, i.e.

$$g^{n-2m}(g^{2m} + ag^m + b) \equiv f^{n-2m}(f^{2m} + af^m + b), \tag{3.22}$$

which implies f, g share ∞ CM. Since $E_f(S_1, 0) = E_g(S_1, 0)$, then equation (3.22) also implies f, g share 0 CM. Now suppose $h = g/f$. Then clearly h does not have any zero or pole. Substituting $g = fh$ into $P(g) \equiv P(f)$ we have

$$f^{2m}(h^n - 1) + af^m(h^{n-m} - 1) + b(h^{n-2m} - 1) = 0. \tag{3.23}$$

Subcase 2.2.1. If h is constant, then as g is nonconstant, $h^n = h^{n-m} = h^{n-2m} = 1$, which implies $h = 1$ as $\gcd(m, n) = 1$. Therefore $f \equiv g$.

Subcase 2.2.2. If h is nonconstant, then from (3.23) we get

$$\Big(f^m + \frac{a}{2}\frac{h^{n-m} - 1}{h^n - 1}\Big)^2 = \frac{\varphi(h)}{4(h^n - 1)^2}, \tag{3.24}$$

where $\varphi(h) = a^2(h^{n-m} - 1)^2 - 4b(h^{n-2m} - 1)(h^n - 1)$. Now in view of part (ii) and (i) of Lemma 2.9 we get that each multiple zero of $\varphi(z)$ is of multiplicity 2 and those zeros are of the form e^{t_0} such that t_0 satisfies $\cosh mt_0 = 1$ or $\cosh mt_0 = a^2(n - m)^2/2bn(n - 2m) - 1$, i.e. at most $m + 2m = 3m$ multiple zeros are there. So $\varphi(z)$ can have at least $2n - 2m - 6m$ distinct simple zeros, say ν_i for $i = 1, 2, \ldots, 2n - 8m$. From (3.24) it is clear that each ν_i-point of h is of multiplicity at least 2.

Therefore by the second fundamental theorem we get

$$\begin{aligned}(2n - 8m)T(r, h) &\leqslant \sum_{i=1}^{2n-8m} \overline{N}(r, \nu_i; h) + \overline{N}(r, 0; h) + \overline{N}(r, \infty; h) + S(r, h)\\ &\leqslant (n - 4m)T(r, h) + S(r, h),\end{aligned}$$

which is a contradiction for $n \geqslant 4m + 3$. $\square$

P r o o f of Theorem 1.3. Proceeding similarly as in the proof of Theorem 1.2. for $a=0$ and $\gcd(n,2m)=1$, we can obtain the result. $\square$

P r o o f of Theorem 1.4. Here, proceeding in a similar fashion like in the proof of Theorem 1.2 and using the fact that

$$\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\leqslant\frac{1}{2}(N(r,\infty;f)+N(r,\infty;g)),$$

we can obtain the result. $\square$

P r o o f of Theorem 1.5. Similarly as in the proof of Theorem 1.4 we use the inequality

$$\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\leqslant\frac{1}{2}(N(r,\infty;f)+N(r,\infty;g))$$

and obtain the result. $\square$

It is to be noted that using the same method as adopted in this paper, one can easily show that for any meromorphic functions having no simple poles, the sets $S_1=\{0,c_1,c_2,\ldots,c_m\}$, $S_2=\{z\colon\ z^n+az^{n-m}+c=0\}$, where $\gcd(m,n)=1$, $a\in\mathbb{C}^*$ and $c\neq 0,\beta_i$, are BURSM0, 3 for $n\geqslant 2m+3$, BURSM1, 2 for $n\geqslant 2m+4$ and BURSM1, 3 for $n\geqslant 2m+1$ under the additional supposition that $c\neq\beta_i\beta_j/(\beta_i+\beta_j)$.

4. Application

Application of Theorem 1.1. Let us consider the sets defined in Theorem C. Then we have

$$\begin{aligned}S_2&=\bigl\{z\colon\ \tfrac{1}{2}(n-1)(n-2)z^n-n(n-2)z^{n-1}+\tfrac{1}{2}n(n-1)z^{n-2}-d=0\bigr\}\\&=\Bigl\{z\colon\ z^n+\frac{-2n}{n-1}z^{n-1}+\frac{n}{n-2}z^{n-2}-\frac{2d}{(n-1)(n-2)}=0\Bigr\}\\&=\{z\colon\ z^n+az^{n-m}+bz^{n-2m}+c=0\},\end{aligned}$$

where $a=-2n/(n-1)$, $b=n/(n-2)$, $c=-2d/(n-1)(n-2)$ and $m=1$. Observe that here $a^2/4b=n(n-2m)/(n-m)^2$, $\gcd(m,n)=1$. Hence, the roots of

$$nz^{2m}+a(n-m)z^m+b(n-2m)=0$$

are $c_i=c_1=a(1-n)/2n=1$ and

$$\beta_i=-(c_i^n+ac_i^{(n-m)}+bc_i^{(n-2m)})=-\Bigl(1-\frac{2n}{n-1}+\frac{n}{n-2}\Bigr)=\frac{-2}{(n-1)(n-2)}=\beta_j.$$

Therefore $\beta_i\beta_j/(\beta_i+\beta_j) = -1/(n-1)(n-2)$. Also we have $S_1 = \{0,1\} = \{0,c_1\}$ and $n \geqslant 5 = 2\cdot 1+3 = 2m+3$. Therefore all the conditions of Theorem 1.1 are satisfied and hence S_i's as used in Theorem C are BURSM$1,3$, BURSM$2,2$ for $c \neq 0, \beta_i, \beta_i\beta_j/(\beta_i+\beta_j)$, i.e. $-2d/(n-1)(n-2) \neq 0, -2/(n-1)(n-2), -(n-1)^{-1}\times (n-2)^{-1}$, i.e. $d \neq 0, 1, \frac{1}{2}$.

Remark 4.1. The above result significantly improves Theorem C by removing the condition $d^2-d+1 \neq 0$ as well as relaxing the nature of sharing from $(3,2)$ to $(2,2)$.

Remark 4.2. Example 1.1 shows that whenever $m = 1$ and $n = 5$, then $c \neq \beta_i\beta_j/(\beta_i+\beta_j)$ is a must for Theorem 1.1. Consequently, $d \neq \frac{1}{2}$ in Theorem C is a must whenever $n = 5$. In this case we would have any two nonconstant meromorphic functions f and g such that $f+g = 1$ and they share S_1 and S_2 CM but $f \not\equiv g$.

5. Some relevant issues

To get the best possible answer of Question A, Yi (see [19]) also introduced the following polynomial in the literature:

$$P_1(w) = a_1w^n - n(n-1)w^2 + 2n(n-2)b_1w - (n-1)(n-2)b_1^2, \tag{5.1}$$

where $n \geqslant 3$ is an integer and a_1 and b_1 are two nonzero complex numbers satisfying $a_1b_1^{n-2} \neq 2$. It has also been proved that $P_1(w)$ has only simple zeros.

A huge number of researchers (see [19], [11], [2], [3], [7]) devoted themselves to the best possible solution of Question A under the ambit of this polynomial. In all these theorems, authors resorted to the same technique so as to reduce the cardinality of one set containing n elements, as small as possible, as the other set, namely the set of poles, is always fixed. One can easily point out that the least possible value of n devoid of any deficiency conditions have so far been obtained is 8. In the sequel we will show that to further reduce the value of n without any deficiency conditions the notion of bi-unique range sets plays a vital role if we slightly manipulate the initial definition in [4]. By adopting this new notion we will also be able to execute our second motivation as stated earlier. Hence, we initiate the following definition.

Definition 5.1. Suppose $S_1 = S^* \cup \{\infty\}$, where $S^* \subset \mathbb{C}$. Further suppose $S_2 \subset \mathbb{C}$. Then S_1 and S_2 are called extended bi-unique range sets for meromorphic (entire) functions with weights m, k if for any two nonconstant meromorphic (entire) functions f and g, $E_f(S_1,m) = E_g(S_1,m)$, $E_f(S_2,k) = E_g(S_2,k)$ imply $f \equiv g$. We say S_i's, $i = 1,2$, are EBURSMm,k (EBURSEm,k) in short. As usual if both $m = k = \infty$, we say S_i's, $i = 1,2$, are EBURSM (EBURSE).

In connection to this new definition, we are now going to provide the following results which will improve, supplement and generalize all the results obtained so far for $P_1(w)$ as far as the possible answer of Question A is concerned.

Let $Q(w) = cw^n + bw^{2m} + aw^m + 1$, where $n, m \in \mathbb{N}$, and $a, b, c \in \mathbb{C}^*$ be such that $n > 2m$, $\gcd(n, m) = 1$, $a^2/4b = n(n-2m)/(n-m)^2$ and $c \neq -(2mbe_i^{2m} + ame_i^m)/ne_i^n$ $(= \gamma_i)$ with e_i being the roots of the equation

$$w^m = -\frac{2n}{(n-m)a}. \tag{5.2}$$

Now, $Q'(w) = ncw^{n-1} + 2mbw^{2m-1} + amw^{m-1}$. Therefore the zeros of $Q'(w)$ are the roots of $ncw^{n-1} + 2mbw^{2m-1} + amw^{m-1} = 0$. Clearly, for any zero 's' of $Q'(w)$ we have $ncs^{n-1} + 2mbs^{2m-1} + ams^{m-1} = 0$, i.e. $ncs^n + 2mbs^{2m} + ams^m = 0$, i.e. $cs^n = -(2mbs^{2m} + ams^m)/n$.

Now for $s = 0$

$$Q(0) = 1 \neq 0$$

and for $s \neq 0$

$$\begin{aligned} Q(s) &= -\frac{2mbs^{2m} + ams^m}{n} + bs^{2m} + as^m + 1 \\ &= \frac{(n-2m)bs^{2m} + (n-m)as^m + n}{n} \\ &= \frac{a^2(n-m)^2 s^{2m} + 4a(n-m)s^m + 4n^2}{4n^2} \\ &= \frac{(a(n-m)s^m + 2n)^2}{4n^2}. \end{aligned}$$

So, 's' is a zero of $Q(w)$ if $s^m = -2n/((n-m)a)$, i.e. if $s \in \{e_1, e_2, \ldots, e_m\}$. But then we would have $ce_i^n = -(2mbe_i^{2m} + ame_i^m)/n$ for $i \in \{1, 2, \ldots, m\}$, which is a contradiction as $c \neq \gamma_i = -(2mbe_i^{2m} + ame_i^m)/ne_i^n$. Hence, $Q(w)$ has only simple zeros.

Theorem 5.1. *Let $S_1^* = \{\infty, e_1, e_2, \ldots e_m\}$ and $S_2^* = \{w\colon cw^n + bw^{2m} + aw^m + 1 = 0\}$, where $n \geqslant 2m + 3$, $\gcd(n, m) = 1$, $a^2/4b = n(n-2m)/(n-m)^2$, and $a, b, c \in \mathbb{C}^*$ be such that $c \notin \{0, \gamma_i, \gamma_i\gamma_j/(\gamma_i + \gamma_j)\}$. Then*

(i) *S_i^*'s, $i = 1, 2$, are EBURSM1, 3.*

(ii) *S_i^*'s, $i = 1, 2$, are EBURSM2, 2.*

Proof. Let f and g be two nonconstant meromorphic functions such that $E_f(S_1^*, p) = E_g(S_1^*, p)$ and $E_f(S_2^*, t) = E_g(S_2^*, t)$, where $(p, t) = (1, 3), (2, 2)$.

We have $S_2^* = \{w\colon\ cw^n + bw^{2m} + aw^m + 1 = 0\}$ and suppose that $S_2 = \{z\colon z^n + az^{n-m} + bz^{n-2m} + c = 0\}$ with the same condition on a, b, c, n, m as given in the theorem.

Observe that

$$\begin{aligned} S_2^* &= \{w\colon\ cw^n + bw^{2m} + aw^m + 1 = 0\} \\ &= \left\{\frac{1}{z}\colon\ \frac{z^n + az^{n-m} + bz^{n-2m} + c}{z^n} = 0\right\} \\ &= \left\{\frac{1}{z}\colon\ z^n + az^{n-m} + bz^{n-2m} + c = 0\right\}. \end{aligned}$$

Suppose $w_1, w_2, \dots, w_n$ are distinct roots of $cw^n + bw^{2m} + aw^m + 1 = 0$ and $z_1, z_2, \dots, z_n$ are distinct roots of $z^n + az^{n-m} + bz^{n-2m} + c = 0$. Clearly the elements of S_2 and S_2^* are reciprocals, so after suitable arrangement of the elements of these sets we can write $w_i = 1/z_i$. Further suppose $f_1 = 1/f$, $g_1 = 1/g$.

Let a_{ij} be any w_i-point of f. Then a_{ij} is $1/w_i$ point of $1/f$, i.e. a_{ij} is z_i-point of f_1 and vice-versa.

Now $E_f(S_2^*, t) = \bigcup_{w_i \in S_2^*} E_t(w_i, f) = \bigcup_{z_i \in S_2} E_t(z_i, f_1) = E_{f_1}(S_2, t)$. So, $E_f(S_2^*, t) = E_g(S_2^*, t)$ implies $E_{f_1}(S_2, t) = E_{g_1}(S_2, t)$.

We recall that $S_1^* = \{\infty, e_1, e_2, \dots e_m\}$, where e_i's, $i \in \{1, 2, \dots, m\}$, are the distinct mth roots of the equation

$$z^m = -\frac{2n}{(n-m)a}. \tag{5.3}$$

Suppose that $S_1 = \{0, c_1, c_2, \dots, c_m\}$, where c_i's, $i \in \{1, 2, \dots, m\}$, are the distinct mth roots of the equation

$$w^m = -\frac{(n-m)a}{2n}. \tag{5.4}$$

Now putting $w = 1/z$ in (5.4), we get (5.3). Then by similar argument as deployed to find the relation between w_i and z_i we can write $c_i = 1/e_i$.

So for $f = 1/f_1$ and $g = 1/g_1$, using the similar argument as done above we have that $E_f(S_1^*, p) = E_g(S_1^*, p)$ implies $E_{f_1}(S_1, p) = E_{g_1}(S_1, p)$.

Also from the proof of Lemma 2.11 observe that

$$\gamma_i = -\frac{2mbe_i^{2m} + ame_i^m}{ne_i^n} = -c_i^{n-2m}\frac{m}{n}(ac_i^m + 2b) = \beta_i.$$

Hence, all the conditions of this theorem coincide with all the conditions of Theorem 1.1. Therefore $E_{f_1}(S_1, p) = E_{g_1}(S_1, p)$ and $E_{f_1}(S_2, t) = E_{g_1}(S_2, t)$ imply $f_1 = g_1$ for $(p, t) = (1, 3), (2, 2)$. Hence $1/f = 1/g$, i.e. $f \equiv g$. $\square$

Corollary 5.1. *Let $S_1^* = \{b_1, \infty\}$ and $S_2^* = \{w\colon\ a_1w^n - n(n-1)w^2 + 2n(n-2) \times b_1w - (n-1)(n-2)b_1^2 = 0\}$, where $n \geqslant 5$, and $a_1, b_1 \in \mathbb{C}$ be such that $a_1b_1^{n-2} \notin \{0, 1, 2\}$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S_1^*, p) = E_g(S_1^*, p)$ and $E_f(S_2^*, t) = E_g(S_2^*, t)$. Then $f \equiv g$ for $(p, t) = (1, 3), (2, 2)$.*

P r o o f. Given that

$$\begin{aligned} S_2^* &= \{w : a_1w^n - n(n-1)w^2 + 2n(n-2)b_1w - (n-1)(n-2)b_1^2 = 0\} \\ &= \Big\{w\colon\ \frac{-a_1}{(n-1)(n-2)b_1^2}w^n + \frac{n}{(n-2)b_1^2}w^2 - \frac{2n}{(n-1)b_1}w + 1 = 0\Big\} \\ &= \{w\colon\ cw^n + bw^{2m} + aw^m + 1 = 0\}, \end{aligned}$$

where $c = -a_1/(n-1)(n-2)b_1^2$, $b = n/(n-2)b_1^2$, $a = -2n/(n-1)b_1$ and $m = 1$, observe that here $a^2/4b = n(n-2m)/(n-m)^2$ and $\gcd(n, m) = 1$. Now the roots of $w^m = -2n/(n-m)a$ are $e_i = e_1 = -2n/(n-1)a = b_1$ and hence $\gamma_i = -(2mbe_i^{2m} + ame_i^m)/ne_i^n = -(2bb_1^2 + ab_1)/nb_1^n = -(2n/(n-2))/nb_1^n + (2n/(n-1))/nb_1^n = -2/((n-1)(n-2)b_1^n) = \gamma_j$. Therefore $\gamma_i\gamma_j/(\gamma_i + \gamma_j) = \frac{1}{2}\gamma_i = -1/(n-1)(n-2)b_1^n$. So, we have $S_1^* = \{b_1, \infty\} = \{e_1, \infty\}$ and $n \geqslant 5 = 2 \cdot 1 + 3 = 2m + 3$. Now $a_1b_1^{n-2} \notin \{0, 1, 2\}$ implies $-a_1b_1^n/(n-1)(n-2)b_1^2 \notin \{0, -1/(n-1)(n-2), -2/(n-1)(n-2)\}$, i.e. $c \notin \{0, \gamma_i\gamma_j/(\gamma_i + \gamma_j), \gamma_i\}$. Therefore all the conditions of Theorem 5.1 are satisfied. Hence, the corollary immediately follows from Theorem 5.1. $\square$

R e m a r k 5.1. Clearly Theorem 5.1 and Corollary 5.1 significantly reduces the value of n from 8 to 5 at the cost of forming EBURSM without any deficiency conditions over the functions.

From the last section of the proof of Corollary 5.1 and the discussion just above Theorem 5.1, it is clear that $a_1b_1^{n-2} \notin \{0, 2\}$ is mandatory for all the roots of

$$a_1w^n - n(n-1)w^2 + 2n(n-2)b_1w - (n-1)(n-2)b_1^2 = 0$$

to be simple. Now we exhibit the following example which shows that the condition $a_1b_1^{n-2} \neq 1$ is also sharp for $n = 5$ in Corollary 5.1.

E x a m p l e 5.1. Let $R(w) = a_1w^n/n(n-1)(w-\alpha_1)(w-\alpha_2)$, where α_1 and α_2 are two distinct roots of $n(n-1)w^2 - 2n(n-2)b_1w + (n-1)(n-2)b_1^2 = 0$ with $a_1, b_1 \in \mathbb{C}^*$ being such that $a_1b_1^{n-2} \neq 2$. Suppose f be any nonconstant meromorphic function and $g = b_1f/(f-b_1)$. Also let $F_1 = R(f)$ and $G_1 = R(g)$ for $n = 5$ with

$a_1 b_1^{n-2} = 1$, i.e. $a_1 b_1^3 = 1$. Now

$$
\begin{aligned}
(5.5) \qquad 2G_1 &= \frac{a_1 g^5}{10(g-\alpha_1)(g-\alpha_2)} \\
&= \frac{a_1 b_1^5 f^5}{10(f-b_1)^5 (b_1 f(f-b_1)^{-1}-\alpha_1)(b_1 f(f-b_1)^{-1}-\alpha_2)} \\
&= \frac{b_1^2 f^5}{10(f-b_1)^3 (f(b_1-\alpha_1)+b_1\alpha_1)(f(b_1-\alpha_2)+b_1\alpha_2)} \\
&= \frac{b_1^2 f^5}{10(f-b_1)^3 (f+b_1\alpha_1(b_1-\alpha_1)^{-1})} \\
&\quad \times \frac{1}{(f+b_1\alpha_2(b_1-\alpha_2)^{-1})(b_1-\alpha_1)(b_1-\alpha_2)}.
\end{aligned}
$$

Note that here α_i's are the roots of $10w^2 - 15b_1 w + 6b_1^2 = 0$. Therefore $\alpha_i = \frac{1}{20}(15 \pm \sqrt{15}\mathrm{i})b_1$ and so $(b_1-\alpha_1)(b_1-\alpha_2) = \frac{1}{10}b_1^2$. Putting these values in (5.5) we get

$$
\begin{aligned}
(5.6) \qquad 2G_1 &= \frac{f^5}{(f-b_1)^3 (f+b_1\alpha_1(b_1-\alpha_1)^{-1})(f+b_1\alpha_2(b_1-\alpha_2)^{-1})} \\
&= \frac{a_1 f^5}{(f-b_1)^3 a_1 (f+b_1\alpha_1(b_1-\alpha_1)^{-1})(f+b_1\alpha_2(b_1-\alpha_2)^{-1})}.
\end{aligned}
$$

Here $b_1\alpha_i/(b_1-\alpha_i) = \frac{1}{2}b_1(3 \pm \sqrt{15}\mathrm{i})$. So, (5.6) reduces to

$$
2G_1 = \frac{a_1 f^5}{(f-b_1)^3 (a_1 f^2 + 3a_1 b_1 f + 6a_1 b_1^2)} = \frac{a_1 f^5}{a_1 f^5 - 10f^2 + 15b_1 f - 6b_1^2} = \frac{F_1}{F_1 - \frac{1}{2}},
$$

which implies $G_1 = F_1/(2F_1-1)$, i.e. $G_1 - 1 = (1-F_1)/(2F_1-1)$, hence F_1, G_1 share 1 CM and so $E_f(S_2^*, \infty) = E_g(S_2^*, \infty)$ for $n = 5$. Obviously $E_f(S_1^*, \infty) = E_g(S_1^*, \infty)$ but $f \not\equiv g$.

In [8] it has been shown that S_2 of Theorems 1.1, 1.2, 1.3 forms unique range sets (URSM) with weight 2. Using the techniques of Theorem 5.1 one can easily show that S_2^* of the same theorem is also a URSM with weight 2. Therefore natural questions arise:

Question 5.2. Does there exist BURSM for every URSM? If so, then what is the relation between the cardinalities of URSM and BURSM?

Question 5.3. What happens to Theorems 1.1, 1.2, 1.3 if we use the notion of EBURSM instead of BURSM?

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