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TRAVELING WAVE SOLUTIONS IN A CLASS OF HIGHER DIMENSIONAL LATTICE DIFFERENTIAL SYSTEMS WITH DELAYS AND APPLICATIONS

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Abstract. In this paper, we are concerned with the existence of traveling waves in a class of delayed higher dimensional lattice differential systems with competitive interactions. Due to the lack of quasimonotonicity for reaction terms, we use the cross iterative and Schauder's fixed-point theorem to prove the existence of traveling wave solutions. We apply our results to delayed higher-dimensional lattice reaction-diffusion competitive system.

Keywords: higher dimensional lattice; traveling wave solution; delay; upper and lower solutions

MSC 2020: 37L60, 34K10, 39A10

1. Introduction

We are concerned with the existence of traveling waves of n dimensional spatially discrete delayed systems

$$\begin{cases} \dfrac{\mathrm{d}u_{1\eta}(t)}{\mathrm{d}t} = d_1(\Delta_n g_1(u_1))_\eta(t) + f_1((u_{1\eta})_t, (u_{2\eta})_t), \\ \dfrac{\mathrm{d}u_{2\eta}(t)}{\mathrm{d}t} = d_2(\Delta_n g_2(u_2))_\eta(t) + f_2((u_{1\eta})_t, (u_{2\eta})_t), \end{cases} \tag{1.1}$$

where $t > 0$, $d_1, d_2 > 0$, $(\Delta_n g_i(w))_\eta = \sum\limits_{|\xi-\eta|=1,\, \xi\in\mathbb{Z}^n} g_i(w_\xi) - 2ng_i(w_\eta)$, $\eta \in \mathbb{Z}^n$, $n \in \mathbb{Z}^+$, $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$, $\tau > 0$ is the maximal delay involved

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in (1.1), $g_i\colon \mathbb{R}\to\mathbb{R}$, $f_i\colon C([-\tau,0];\mathbb{R})\to\mathbb{R}$, $i=1,2$, $(w_\eta)_t\in C([-\tau,0];\mathbb{R})$ with $(w_\eta)_t(\theta)=w_\eta(t+\theta)$ for $\theta\in[-\tau,0]$, and the following conditions hold:

(P1) $f_i(\mathbf{0})=f_i(\mathbf{K})=0$, where $\mathbf{0}=(0,0)$, $\mathbf{K}=(k_1,k_2)$ are constant functions, $k_i>0$, $i=1,2$;

(P2) there exists $L_i>0$ such that

$$|f_i(\Phi_1)-f_i(\Phi_2)|\leqslant L_i\|\Phi_1-\Phi_2\|$$

for $\Phi_i=(\varphi_{1i},\varphi_{2i})\in C([-\tau,0],\mathbb{R}^2)$ with $0\leqslant\varphi_{1i}\leqslant M_1$, $0\leqslant\varphi_{2i}\leqslant M_2$ on $[-\tau,0]$, $M_i>k_i$, $\|\cdot\|$ is the supremum norm in $C([-\tau,0],\mathbb{R}^2)$, $i=1,2$;

(P3) $g_i\colon [0,M_i]\to\mathbb{R}$, $i=1,2$, is Lipschitz continuous and increasing.

We are interested in the traveling waves of (1.1) with nonlinear types (WQM) and (WQM*), see Section 2.

Two typical examples are delayed lattice diffusion-competition systems with two species:

$$(1.2)\qquad \begin{cases} \dfrac{\mathrm{d}u_{1\eta}(t)}{\mathrm{d}t}=d_1(\Delta_n u_1)_\eta+r_1u_{1\eta}[1-a_1u_{1\eta}-b_1u_{2\eta}(t-\tau_1)],\\ \dfrac{\mathrm{d}u_{2\eta}(t)}{\mathrm{d}t}=d_2(\Delta_n u_2)_\eta+r_2u_{2\eta}[1-b_2u_{1\eta}(t-\tau_2)-a_2u_{2\eta}],\end{cases}$$

and

$$(1.3)\qquad \begin{cases} \dfrac{\mathrm{d}u_{1\eta}(t)}{\mathrm{d}t}=d_1(\Delta_n u_1)_\eta+r_1u_{1\eta}[1-a_1u_{1\eta}(t-\tau_1)-b_1u_{2\eta}(t-\tau_2)],\\ \dfrac{\mathrm{d}u_{2\eta}(t)}{\mathrm{d}t}=d_2(\Delta_n u_2)_\eta+r_2u_{2\eta}[1-b_2u_{1\eta}(t-\tau_3)-a_2u_{2\eta}(t-\tau_4)],\end{cases}$$

where $r_i,a_i,b_i>0$, $i=1,2$, $\tau_j>0$, $j=1,2,3,4$.

Now we recall some conclusions about the traveling waves of different dimensional lattice equations with or without delays. In past few years, great progress has been made in the traveling wave solutions for a single equation, see [1], [2], [3], [4], [5], [6], [8], [10], [11], [16], [17], [18], [25], [20], [21], [22], [24], [26], [27], [29] for 1 or 2 dimensional lattices and [19], [23], [28] for higher dimensional lattices. Recently, many authors also paid their attention to the traveling waves for systems with two equations. For example, for $n=1$, Huang, Lu and Ruan [9] investigated the existence of traveling waves of (1.1) with $g_1=g_2$ and partial monotonicity; Li and Li [13] studied the existence of traveling wave solutions of competition-cooperation system as well as asymptotic behavior; Lin and Li [15] investigated the traveling waves of a class of systems including (1.2) and (1.3), which is not applied to higher dimensional lattice systems; Guo et al. [7] and Li et al. [12], respectively, studied the existence, asymptotic behavior and uniqueness of invasive waves for systems (1.2) and (1.3) without delays and with delays. The continuous systems (1.2) and (1.3) with $n = 1$ were also studied by Li et al. [14]. The above existence results of traveling wave solutions depended on the existence of upper and lower solutions. However, the existence of traveling wave solutions for higher dimensional lattice systems (1.2) and (1.3) remains open.

Inspired by the method in [9], [10], [14], [15], we adopt Schauder's fixed-point theorem and upper and lower solutions technique to obtain the existence of traveling waves of (1.1) connecting $\mathbf{0}$ with $\mathbf{K}$. We required that the weak upper solution is larger than coexistence equilibrium and is not necessarily monotone, which can be constructed easier. As applications, we will study the traveling waves of (1.2) and (1.3).

The rest of this paper is organized as follows. Some notations and preliminaries are given in Section 2. In Sections 3 and 4, we prove the existence of traveling waves for the cases (WQM) and (WQM*), respectively. In Section 5, our conclusions are applied to (1.2) and (1.3).

2. EXISTENCE

We first give some notations in $\mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, $x \leqslant y$ is defined by $x_i \leqslant y_i$, $i = 1, 2$, and $x < y$ is defined by $x \leqslant y$ but $x \neq y$, $x \ll y$ is defined by $x \leqslant y$ but $x_i \neq y_i$, $i = 1, 2$. When $x \leqslant y$, denote $(x, y] = \{u \in \mathbb{R}^2;\ x < u \leqslant y\}$, $[x, y) = \{u \in \mathbb{R}^2;\ x \leqslant u < y\}$, $[x, y] = \{u \in \mathbb{R}^2;\ x \leqslant u \leqslant y\}$.

Definition 2.1. The traveling wave solution of (1.1) has the form $u_{1\eta}(t) = \varphi_1(\sigma \cdot \eta + ct)$, $u_{2\eta}(t) = \varphi_2(\sigma \cdot \eta + ct)$, where $\varphi_1(\pm\infty)$ and $\varphi_2(\pm\infty)$ both exist, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n$ with $|\sigma| = 1$, the wave speed $c > 0$, the wave profile $(\varphi_1, \varphi_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$.

Substituting $u_{i\eta}(t) = \varphi_i(\sigma \cdot \eta + ct)$ into (1.1) and denoting $\varphi_{it}(\theta) = \varphi_i(t + \theta)$, $i = 1, 2$, and $\sigma \cdot \eta + ct$ by t, our problem reduces to the existence of solution of system

$$(2.1)\quad \begin{cases} c\varphi_1'(t) = \displaystyle\sum_{j=1}^{n} d_1[g_1(\varphi_1(t + \sigma_k)) - 2g_1(\varphi_1(t)) + g_1(\varphi_1(t - \sigma_k))] + f_1^c(\varphi_{1t}, \varphi_{2t}), \\ c\varphi_2'(t) = \displaystyle\sum_{j=1}^{n} d_2[g_2(\varphi_2(t + \sigma_k)) - 2g_2(\varphi_2(t)) + g_2(\varphi_2(t - \sigma_k))] + f_2^c(\varphi_{1t}, \varphi_{2t}) \end{cases}$$

with

$$(2.2)\qquad \lim_{t\to-\infty}(\varphi_1(t), \varphi_2(t)) = \mathbf{0}, \quad \lim_{t\to\infty}(\varphi_1(t), \varphi_2(t)) = \mathbf{K},$$

where $f_i^c(\varphi_1,\varphi_2)=f_i(\varphi_1^c,\varphi_2^c)$, $(\varphi_1^c(\theta),\varphi_2^c(\theta))=(\varphi_1(c\theta),\varphi_2(c\theta))$, $\theta\in[-\tau,0]$, $i=1,2$. Denote

$$C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^2)=\{(\varphi_1,\varphi_2)\in C(\mathbb{R},\mathbb{R}^2);\ \mathbf{0}\leqslant(\varphi_1(t),\varphi_2(t))\leqslant\mathbf{M},\ t\in\mathbb{R}\},$$

where $\mathbf{M}=(M_1,M_2)$.

The reaction terms $f=(f_1,f_2)$ satisfy weak quasimonotone condition:

(WQM) there exist $\beta_i>0$ such that

$$\begin{aligned}
&f_1(\varphi_1(\theta),\varphi_2(\theta))-f_1(\psi_1(\theta),\varphi_2(\theta))+\beta_1[\varphi_1(0)-\psi_1(0)]\\
&\qquad\qquad\geqslant 2nd_1[g_1(\varphi_1(0))-g_1(\psi_1(0))],\\
&f_1(\varphi_1(\theta),\varphi_2(\theta))-f_1(\varphi_1(\theta),\psi_2(\theta))\leqslant 0,\\
&f_2(\varphi_1(\theta),\varphi_2(\theta))-f_2(\varphi_1(\theta),\psi_2(\theta))+\beta_2[\varphi_2(0)-\psi_2(0)]\\
&\qquad\qquad\geqslant 2nd_2[g_2(\varphi_2(0))-g_2(\psi_2(0))],\\
&f_2(\varphi_1(\theta),\varphi_2(\theta))-f_2(\psi_1(\theta),\varphi_2(\theta))\leqslant 0
\end{aligned}$$

for $\varphi_i(\theta),\psi_i(\theta)\in C([-c\tau,0],\mathbb{R})$, $i=1,2$, with $\mathbf{0}\leqslant(\psi_1(\theta),\psi_2(\theta))\leqslant(\varphi_1(\theta),\varphi_2(\theta))\leqslant\mathbf{M}$ for $\theta\in[-c\tau,0]$,

or weak nonquasimonotone condition:

(WQM*) there exist $\beta_i>0$ such that

$$\begin{aligned}
&f_1(\varphi_1(\theta),\varphi_2(\theta))-f_1(\psi_1(\theta),\varphi_2(\theta))+\beta_1[\varphi_1(0)-\psi_1(0)]\\
&\qquad\qquad\geqslant 2nd_1[g_1(\varphi_1(0))-g_1(\psi_1(0))],\\
&f_1(\varphi_1(\theta),\varphi_2(\theta))-f_1(\varphi_1(\theta),\psi_2(\theta))\leqslant 0,\\
&f_2(\varphi_1(\theta),\varphi_2(\theta))-f_2(\varphi_1(\theta),\psi_2(\theta))+\beta_2[\varphi_2(0)-\psi_2(0)]\\
&\qquad\qquad\geqslant 2nd_2[g_2(\varphi_2(0))-g_2(\psi_2(0))],\\
&f_2(\varphi_1(\theta),\varphi_2(\theta))-f_2(\psi_1(\theta),\varphi_2(\theta))\leqslant 0
\end{aligned}$$

for $\varphi_i(\theta),\psi_i(\theta)\in C([-c\tau,0],\mathbb{R})$, $i=1,2$, with (i) $\mathbf{0}\leqslant(\psi_1(\theta),\psi_2(\theta))\leqslant(\varphi_1(\theta),\varphi_2(\theta))\leqslant\mathbf{M}$ for $\theta\in[-c\tau,0]$, and (ii) $\mathrm{e}^{\beta_1\theta/c}[\varphi_1(\theta)-\psi_1(\theta)]$ and $\mathrm{e}^{\beta_2\theta/c}[\varphi_2(\theta)-\psi_2(\theta)]$ are nondecreasing in $\theta\in[-c\tau,0]$.

Define $H=(H_1,H_2)\colon C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^2)\to C(\mathbb{R},\mathbb{R}^2)$ by

$$\begin{cases}
H_1(\varphi_1,\varphi_2)(t)=f_1^c(\varphi_{1t},\varphi_{2t})+\beta_1\varphi_1(t)\\
\qquad\qquad+d_1\displaystyle\sum_{j=1}^{n}[g_1(\varphi_1(t+\sigma_k))-2g_1(\varphi_1(t))+g_1(\varphi_1(t-\sigma_k))],\\
H_2(\varphi_1,\varphi_2)(t)=f_2^c(\varphi_{1t},\varphi_{2t})+\beta_2\varphi_2(t)\\
\qquad\qquad+d_2\displaystyle\sum_{j=1}^{n}[g_2(\varphi_2(t+\sigma_k))-2g_2(\varphi_2(t))+g_2(\varphi_2(t-\sigma_k))].
\end{cases}$$

Then (2.1) becomes

$$\begin{cases} c\varphi_1' = -\beta_1\varphi_1 + H_1(\varphi_1, \varphi_2), \\ c\varphi_2' = -\beta_2\varphi_2 + H_2(\varphi_1, \varphi_2). \end{cases} \tag{2.3}$$

From (2.3), we can define $F = (F_1, F_2)\colon C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$F_i(\varphi_1, \varphi_2)(t) = \frac{1}{c}\mathrm{e}^{-\beta_i t/c} \int_{-\infty}^{t} \mathrm{e}^{\beta_i s/c} H_i(\varphi_1, \varphi_2)(s)\,\mathrm{d}s, \quad i = 1, 2.$$

It is clear that F is well defined. Furthermore, if $(\varphi_1, \varphi_2) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$, then it solves equations

$$c(F_i(\varphi_1, \varphi_2))'(t) = -\beta_i F_i(\varphi_1, \varphi_2)(t) + H_i(\varphi_1, \varphi_2)(t), \quad i = 1, 2. \tag{2.4}$$

Then it is easy to see that a fixed point of (2.4) solves (2.3).

Let $(B_\nu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\nu)$ be a Banach space of continuous vector valued function from $\mathbb{R}$ to $\mathbb{R}^2$, where the norm $|\cdot|_\nu$ is defined by

$$|\varrho|_\nu = \sup_{\xi \in \mathbb{R}} |\varrho(\xi)|\mathrm{e}^{-\nu|\xi|} \quad \text{and} \quad B_\nu(\mathbb{R}, \mathbb{R}^2) = \Big\{\varrho \in C(\mathbb{R}, \mathbb{R}^2);\ \sup_{\xi \in \mathbb{R}} |\varrho(\xi)|\mathrm{e}^{-\nu|\xi|} < \infty\Big\}$$

for $\nu \in (0, \min\{\beta_1/c, \beta_2/c\})$.

3. THE CASE (WQM)

We first show that $f = (f_1, f_2)$ satisfies (WQM).

Definition 3.1. The continuous functions $\overline{\Phi} = (\overline{\varphi}_1, \overline{\varphi}_2)$, $\underline{\Phi} = (\underline{\varphi}_1, \underline{\varphi}_2) \in C(\mathbb{R}, \mathbb{R}^2)$ is called an *upper (a lower) solution* of (2.1) provided that they satisfy respectively

$$\begin{cases} c\overline{\varphi}_1'(t) \geqslant d_1 \displaystyle\sum_{j=1}^{n} [g_1(\overline{\varphi}_1(t + \sigma_k)) - 2g_1(\overline{\varphi}_1(t)) + g_1(\overline{\varphi}_1(t - \sigma_k))] \\ \qquad\qquad\qquad\qquad\qquad\qquad + f_1^c(\overline{\varphi}_{1t}, \underline{\varphi}_{2t}) \quad \text{in } \mathbb{R}, \\ c\overline{\varphi}_2'(t) \geqslant d_2 \displaystyle\sum_{j=1}^{n} [g_2(\overline{\varphi}_2(t + \sigma_k)) - 2g_2(\overline{\varphi}_2(t)) + g_2(\overline{\varphi}_2(t - \sigma_k))] \\ \qquad\qquad\qquad\qquad\qquad\qquad + f_2^c(\underline{\varphi}_{1t}, \overline{\varphi}_{2t}) \quad \text{in } \mathbb{R}, \end{cases} \tag{3.1}$$

and

$$
\tag{3.2}
\begin{cases}
c\underline{\varphi}_1'(t) \leqslant d_1 \displaystyle\sum_{j=1}^{n}[g_1(\underline{\varphi}_1(t+\sigma_k)) - 2g_1(\underline{\varphi}_1(t)) + g_1(\underline{\varphi}_1(t-\sigma_k))] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad + f_1^c(\underline{\varphi}_{1t}, \overline{\varphi}_{2t}) \quad \text{in } \mathbb{R}, \\
c\underline{\varphi}_2'(t) \leqslant d_2 \displaystyle\sum_{j=1}^{n}[g_2(\underline{\varphi}_2(t+\sigma_k)) - 2g_2(\underline{\varphi}_2(t)) + g_2(\underline{\varphi}_2(t-\sigma_k))] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad + f_2^c(\overline{\varphi}_{1t}, \underline{\varphi}_{2t}) \quad \text{in } \mathbb{R}.
\end{cases}
$$

We assume that $\overline{\Phi} = (\overline{\varphi}_1, \overline{\varphi}_2)$ and $\underline{\Phi} = (\underline{\varphi}_1, \underline{\varphi}_2)$ of (2.1) satisfy

(A1) $\mathbf{0} \leqslant \underline{\Phi}(t) \leqslant \overline{\Phi}(t) \leqslant \mathbf{M}$, $t \in \mathbb{R}$;

(A2) $\lim\limits_{t\to-\infty} \overline{\Phi}(t) = \mathbf{0}$, $\lim\limits_{t\to\infty} \underline{\Phi}(t) = \lim\limits_{t\to\infty} \overline{\Phi}(t) = \mathbf{K}$.

By the definition of H, the following conclusion holds.

Lemma 3.1. *If* (P1)–(P3) *and* (WQM) *are satisfied, then*

$$
H_1(\psi_1, \varphi_2)(t) \leqslant H_1(\varphi_1, \psi_2)(t), \quad H_2(\varphi_1, \psi_2)(t) \leqslant H_2(\psi_1, \varphi_2)(t),
$$

furthermore,

$$
F_1(\psi_1, \varphi_2)(t) \leqslant F_1(\varphi_1, \psi_2)(t), \quad F_2(\varphi_1, \psi_2)(t) \leqslant F_2(\psi_1, \varphi_2)(t)
$$

for $t \in \mathbb{R}$ *if* $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$ *with*

$$
\mathbf{0} \leqslant (\psi_1(t), \psi_2(t)) \leqslant (\varphi_1(t), \varphi_2(t)) \leqslant \mathbf{M}
$$

for $t \in \mathbb{R}$, $i = 1, 2$.

Proof. From (P3) and (WQM), for all $t \in \mathbb{R}$ we have

$$
\begin{aligned}
H_1(\varphi_1, \psi_2)(t) - H_1(\psi_1, \varphi_2)(t) &\geqslant 2nd_1[g_1(\varphi_1(t)) - g_1(\psi_1(t))] \\
&\quad + d_1 \sum_{j=1}^{n}\{[g_1(\varphi_1(t+\sigma_k)) - g_1(\psi_1(t+\sigma_k))] - 2[g_1(\varphi_1(t)) - g_1(\psi_1(t))] \\
&\qquad + [g_1(\varphi_1(t-\sigma_k)) - g_1(\psi_1(t-\sigma_k))]\} \geqslant 0.
\end{aligned}
$$

The inequality for H_2 is obtained by using a similar argument. We also obtain related properties by the relation between F and H. The proof is completed. □

Let

$$\Gamma(\underline{\Phi},\overline{\Phi}) = \{\Phi \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^2);\ \underline{\Phi}(t) \leqslant \Phi(t) \leqslant \overline{\Phi}(t),\ t \in \mathbb{R}\}.$$

Obviously, $\Gamma(\underline{\Phi},\overline{\Phi})$ is nonempty since $\overline{\Phi},\underline{\Phi} \in \Gamma(\underline{\Phi},\overline{\Phi})$ by (A1) and (A2).

The following lemma can be proved by using a similar proof of Lemma 3.3 in [15].

Lemma 3.2. *If* (P1)–(P3) *and* (WQM) *are satisfied, then F is continuous according to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R},\mathbb{R}^2)$.*

Lemma 3.3. *If* (P1)–(P3) *and* (WQM) *are satisfied, then* $F(\Gamma(\underline{\Phi},\overline{\Phi})) \subset \Gamma(\underline{\Phi},\overline{\Phi})$.

Proof. When $(\varphi_1,\varphi_2) \in \Gamma(\underline{\Phi},\overline{\Phi})$, it easily follows from Lemma 3.1 that

$$F_1(\underline{\varphi}_1,\overline{\varphi}_2) \leqslant F_1(\varphi_1,\varphi_2) \leqslant F_1(\overline{\varphi}_1,\underline{\varphi}_2), \quad F_2(\overline{\varphi}_1,\underline{\varphi}_2) \leqslant F_2(\varphi_1,\varphi_2) \leqslant F_2(\underline{\varphi}_1,\overline{\varphi}_2).$$

It is enough to show

$$\underline{\varphi}_1 \leqslant F_1(\underline{\varphi}_1,\overline{\varphi}_2) \leqslant F_1(\overline{\varphi}_1,\underline{\varphi}_2) \leqslant \overline{\varphi}_1, \quad \underline{\varphi}_2 \leqslant F_2(\overline{\varphi}_1,\underline{\varphi}_2) \leqslant F_2(\underline{\varphi}_1,\overline{\varphi}_2) \leqslant \overline{\varphi}_2,$$

which hold by using a similar argument in Lemma 3.5 of [15]. The proof is completed. □

Modifying slightly those arguments in Lemma 3.5 of [10] and Lemma 3.7 of [9], the following conclusion holds.

Lemma 3.4. *If* (P1)–(P3) *and* (WQM) *are satisfied, then* $F\colon \Gamma(\underline{\Phi},\overline{\Phi}) \to \Gamma(\underline{\Phi},\overline{\Phi})$ *is compact according to the norm* $|\cdot|_\nu$.

Theorem 3.1. *If* (P1)–(P3) *and* (WQM) *are satisfied and* (2.1) *has a pair of upper solution* $\overline{\Phi}$ *and lower solution* $\underline{\Phi}$ *in* $C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^2)$ *satisfying* (A1) *and* (A2), *then* (2.1) *and* (2.2) *have a solution.*

Proof. The existence of solution $(\varphi_1^*,\varphi_2^*) \in \Gamma(\underline{\Phi},\overline{\Phi})$ is easily obtained from Schauder's fixed-point theorem. From (A1), $\mathbf{0} \leqslant \underline{\Phi}(t) \leqslant (\varphi_1^*(t),\varphi_2^*(t)) \leqslant \overline{\Phi}(t) \leqslant \mathbf{M}$. The asymptotic boundary conditions are obvious by (A1) and (A2). The proof is completed. □

Remark 3.1. Motivated by the results in [9], [10], [15], the upper and lower solutions defined by Definition 3.1 do not require the smoothness at all points. We only assume that (3.1) and (3.2) are satisfied except for the finite point set because of the continuity of $\underline{\Phi}(t)$ and $\underline{\Phi}(t)$. Then Theorem 3.1 is still valid. We call such upper and lower solutions weak upper and lower solutions.

4. THE CASE (WQM*)

Now we study that $f = (f_1, f_2)$ satisfies (WQM*).

We give another condition on $\overline{\Phi}(t)$ and $\underline{\Phi}(t) \in C(\mathbb{R}, \mathbb{R}^2)$ besides (A1) and (A2).

(A3) $\mathrm{e}^{\beta_i t/c}[\overline{\varphi}_i(t) - \underline{\varphi}_i(t)]$ is nondecreasing in $t \in \mathbb{R}$, $i = 1, 2$.

Let

$$\Gamma^*(\underline{\Phi}, \overline{\Phi}) = \left\{ \Phi = (\varphi_1, \varphi_2) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2);\ \begin{array}{l} \text{(i) } \underline{\Phi}(t) \leqslant \Phi(t) \leqslant \overline{\Phi}(t),\ t \in \mathbb{R}, \\ \text{(ii) } \mathrm{e}^{\beta_i t/c}[\overline{\varphi}_i(t) - \varphi_i(t)], \\ \quad \mathrm{e}^{\beta_i t/c}[\varphi_i(t) - \underline{\varphi}_i(t)],\ i = 1, 2, \\ \quad \text{are nondecreasing in } t \in \mathbb{R}, \end{array} \right\}$$

If $\overline{\Phi}$ and $\underline{\Phi} \in \Gamma^*$ by (A1)–(A3), then they belong to Γ^*.

The following two lemmas are very similar to Lemmas 3.1 and 3.2.

Lemma 4.1. *If* (P1)–(P3) *and* (WQM*) *are satisfied, then*

$$H_1(\psi_1, \varphi_2)(t) \leqslant H_1(\varphi_1, \psi_2)(t), \quad H_2(\varphi_1, \psi_2)(t) \leqslant H_2(\psi_1, \varphi_2)(t),$$

furthermore,

$$F_1(\psi_1, \varphi_2)(t) \leqslant F_1(\varphi_1, \psi_2)(t), \quad F_2(\varphi_1, \psi_2)(t) \leqslant F_2(\psi_1, \varphi_2)(t)$$

for $t \in \mathbb{R}$ *if* $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$ *with* (i) $\mathbf{0} \leqslant (\psi_1(t), \psi_2(t)) \leqslant (\varphi_1(t), \varphi_2(t)) \leqslant \mathbf{M}$ *for* $t \in \mathbb{R}$, (ii) $\mathrm{e}^{\beta_i t/c}[\varphi_i(t) - \psi_i(t)]$ *is nondecreasing in* $t \in \mathbb{R}$, $i = 1, 2$.

Lemma 4.2. *If* (P1)–(P3) *and* (WQM*) *are satisfied, then F is continuous according to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^2)$.*

From (A3), we get the properties of $\Gamma^*(\underline{\Phi}, \overline{\Phi})$.

Lemma 4.3. $\Gamma^*(\underline{\Phi}, \overline{\Phi}) \subset B_\nu(\mathbb{R}, \mathbb{R}^2)$ *is closed, bounded and convex.*

Modifying slightly arguments of Lemmas 3.3, 3.4, it yields two lemmas as follows.

Lemma 4.4. *If* (P1)–(P3) *and* (WQM*) *are satisfied, then* $F(\Gamma^*(\underline{\Phi}, \overline{\Phi})) \subset \Gamma^*(\underline{\Phi}, \overline{\Phi})$.

Lemma 4.5. *If* (P1)–(P3) *and* (WQM*) *are satisfied, then* $F\colon \Gamma^*(\underline{\Phi}, \overline{\Phi}) \to \Gamma^*(\underline{\Phi}, \overline{\Phi})$ *is compact according to the norm* $|\cdot|_\nu$.

Theorem 4.1. *If* (P1)–(P3), (WQM*) *are satisfied and* (2.1) *has a pair of upper solution* $\overline{\Phi}$ *and lower solution* $\underline{\Phi}$ *in* $C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^2)$ *satisfying* (A1)–(A3), *then* (2.1) *and* (2.2) *has a solution.*

Remark 4.1. If $\underline{\Phi}(t)$ and $\underline{\Phi}(t)$ are weak upper and lower solutions stated in Remark 3.1, then Theorem 4.1 still holds.

5. APPLICATIONS

As mentioned in the introduction, we apply the above results to prove the existence of traveling waves of (1.2) and (1.3). Equations (1.2) and (1.3) have the same equilibria

$$\mathbf{0}=(0,0),\quad \Big(\frac{1}{a_1},0\Big),\quad \Big(0,\frac{1}{a_2}\Big),$$
$$\mathbf{K}=(k_1,k_2):=\Big(\frac{a_2-b_1}{a_1a_2-b_1b_2},\frac{a_1-b_2}{a_1a_2-b_1b_2}\Big),\quad k_1>0,\ k_2>0$$

provided that

$$a_1>b_2,\quad a_2>b_1. \tag{5.1}$$

Example 5.1. We study the traveling waves of (1.2) which connects $\mathbf{0}$ with $\mathbf{K}$.

Consider the existence of the solution for system

$$\begin{cases} c\varphi_1'(t)=d_1\displaystyle\sum_{j=1}^{n}[\varphi_1(t+\sigma_k)-2\varphi_1(t)+\varphi_1(t-\sigma_k)]\\ \qquad\qquad\qquad +r_1\varphi_1(t)[1-a_1\varphi_1(t)-b_1\varphi_2(t-c\tau_1)],\\ c\varphi_2'(t)=d_2\displaystyle\sum_{j=1}^{n}[\varphi_2(t+\sigma_k)-2\varphi_2(t)+\varphi_2(t-\sigma_k)]\\ \qquad\qquad\qquad +r_2\varphi_2(t)[1-b_2\varphi_1(t-c\tau_2)-a_2\varphi_1(t)]\end{cases} \tag{5.2}$$

satisfying

$$\lim_{t\to-\infty}(\varphi_1(t),\varphi_2(t))=\mathbf{0},\quad \lim_{t\to\infty}(\varphi_1(t),\varphi_2(t))=\mathbf{K}.$$

For $\varphi_1,\varphi_2\in C([-c\tau,0],\mathbb{R})$, $\tau=\max\{\tau_1,\tau_2\}$, let

$$f_1(\varphi_1,\varphi_2)=r_1\varphi_1(0)[1-a_1\varphi_1(0)-b_1\varphi_2(-c\tau_1)],$$
$$f_2(\varphi_1,\varphi_2)=r_2\varphi_2(0)[1-b_2\varphi_1(-c\tau_2)-a_2\varphi_2(0)].$$

It is easy to see that $f=(f_1,f_2)$ satisfies (P1)–(P3). By using an analogous argument as in [14], [15], the following lemma holds.

Lemma 5.1. *The functional* f *satisfies* (WQM).

Similarly to [9], [10], [15], we can prove the following conclusion.

Lemma 5.2. *Let*

$$\Delta_i(\lambda, c) := d_i \sum_{j=1}^{n} (\mathrm{e}^{\lambda\sigma_k} + \mathrm{e}^{-\lambda\sigma_k} - 2) - c\lambda + r_i, \quad i = 1, 2.$$

Then there exist two positive constants c_1^* *and* c_2^* *such that* $\Delta_1(\lambda, c) = 0$ *and* $\Delta_2(\lambda, c) = 0$ *have only two real roots* $0 < \lambda_1 < \lambda_2$ *and* $0 < \lambda_3 < \lambda_4$*, respectively, and*

$$\Delta_1(\lambda, c) \begin{cases} < 0, & \lambda_1 < \lambda < \lambda_2, \\ > 0, & \text{other } \lambda, \end{cases} \quad \text{and} \quad \Delta_2(\lambda, c) \begin{cases} < 0, & \lambda_3 < \lambda < \lambda_4, \\ > 0, & \text{other } \lambda, \end{cases}$$

but $\Delta_i(\lambda, c) = 0$ *has no real roots for* $0 < c < c_i^*$, $i = 1, 2$.

Now we construct weak upper and lower solutions when $c > c^* := \max\{c_1^*, c_2^*\}$.

Take

$$\upsilon \in \Bigl(1, \min\Bigl\{2, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_4}{\lambda_3}, \frac{\lambda_1 + \lambda_3}{\lambda_1}, \frac{\lambda_1 + \lambda_3}{\lambda_3}\Bigr\}\Bigr),$$

consider functions $h_1(t) = \mathrm{e}^{\lambda_1 t} - q\mathrm{e}^{\upsilon\lambda_1 t}$ and $h_2(t) = \mathrm{e}^{\lambda_3 t} - q\mathrm{e}^{\upsilon\lambda_3 t}$, where $q > 1$ is sufficiently large. One can calculate that the unique global maximum $\varrho_i = \varrho_i(q) > 0$ of $h_i(t)$ is attained at

$$t_i^* = t_i^*(q) = -\frac{1}{(\upsilon - 1)\lambda_i} \ln q\upsilon < 0,$$

furthermore,

$$\begin{aligned} \lim_{q\to\infty} \varrho_1(q) = \lim_{q\to\infty} \varrho_2(q) = 0, \lim_{q\to\infty} \mathrm{e}^{\lambda_1 t_1^*(q)} \\ = \lim_{q\to\infty} q\mathrm{e}^{\upsilon\lambda_1 t_1^*(q)} = \lim_{q\to\infty} \mathrm{e}^{\lambda_3 t_2^*(q)} = \lim_{q\to\infty} q\mathrm{e}^{\upsilon\lambda_3 t_2^*(q)} = 0. \end{aligned}$$

The properties of $h_i(t)$ imply that it is strictly increasing on $(-\infty, t_i^*]$ and strictly decreasing on $[t_i^*, \infty)$. Then

$$\begin{cases} h_1(t) = h_1(t_1^* - 1) \text{ has only two real roots } t_{1*} \text{ and } t_1, \\ \qquad\qquad \text{with } t_{1*} < t_1^* < t_1 \text{ and } t_1 - t_{1*} > 1, \\ h_2(t) = h_2(t_2^* - 1) \text{ has only two real roots } t_{3*} \text{ and } t_3 \\ \qquad\qquad \text{with } t_{3*} < t_2^* < t_3 \text{ and } t_3 - t_{3*} > 1. \end{cases} \tag{5.3}$$

So for any $\lambda > 0$ there exist two positive constants ε_2 and ε_4 satisfying

$$h_1(t_1) = k_1 - \varepsilon_2\mathrm{e}^{-\lambda t_1} \text{ and } h_2(t_3) = k_2 - \varepsilon_4\mathrm{e}^{-\lambda t_3}.$$

By (5.1), we can choose three positive constants $\varepsilon_0, \varepsilon_1$ and ε_3 satisfying

$$\begin{cases} a_1\varepsilon_1 - b_1\varepsilon_4 > \varepsilon_0, & a_2\varepsilon_3 - b_2\varepsilon_2 > \varepsilon_0, \\ a_1\varepsilon_2 - b_1\varepsilon_3 > \varepsilon_0, & a_2\varepsilon_4 - b_2\varepsilon_1 > \varepsilon_0. \end{cases} \tag{5.4}$$

For $\lambda > 0$ and $q > 1$, define the continuous functions

$$\overline{\varphi}_1(t) = \begin{cases} \mathrm{e}^{\lambda_1 t}, & t \leqslant t_2, \\ k_1 + \varepsilon_1 \mathrm{e}^{-\lambda t}, & t > t_2, \end{cases} \qquad \overline{\varphi}_2(t) = \begin{cases} \mathrm{e}^{\lambda_3 t}, & t \leqslant t_4, \\ k_2 + \varepsilon_3 \mathrm{e}^{-\lambda t}, & t > t_4, \end{cases}$$

and

$$\underline{\varphi}_1(t) = \begin{cases} \mathrm{e}^{\lambda_1 t} - q\mathrm{e}^{\upsilon\lambda_1 t}, & t \leqslant t_1, \\ k_1 - \varepsilon_2 \mathrm{e}^{-\lambda t}, & t > t_1, \end{cases} \qquad \underline{\varphi}_2(t) = \begin{cases} \mathrm{e}^{\lambda_3 t} - q\mathrm{e}^{\upsilon\lambda_3 t}, & t \leqslant t_3, \\ k_2 - \varepsilon_4 \mathrm{e}^{-\lambda t}, & t > t_3. \end{cases}$$

Obviously, $(M_1, M_2) := (\max\limits_{t\in\mathbb{R}} \overline{\varphi}_1(t), \max\limits_{t\in\mathbb{R}} \overline{\varphi}_2(t)) \gg (k_1, k_2), \overline{\varphi}_i(t)$ and $\underline{\varphi}_i(t)$, $i = 1, 2$, satisfy (A1) and (A2) and

$$\min\{t_2, t_4\} - \max\{c\tau_1, c\tau_2\} \geqslant \{t_1, t_3\}$$

for sufficiently small λ and sufficiently large q. From the definitions of υ we have

$$\Delta_1(\upsilon\lambda_1, c) < 0 \quad \text{and} \quad \Delta_2(\upsilon\lambda_3, c) < 0.$$

Lemma 5.3. *If* (5.1) *holds, then* $(\overline{\varphi}_1(t), \overline{\varphi}_2(t))$ *and* $(\underline{\varphi}_1(t), \underline{\varphi}_2(t))$*, respectively, are a pair of weak upper and lower solutions of* (5.2).

Proof. We can assume $\sigma_k > 0$. We only need to show $\overline{\varphi}_1$ and $\underline{\varphi}_1$ since the others can use a similar argument. Define

$$\begin{aligned} P(\varphi_1, \varphi_2)(t) := c\varphi_1'(t) - d_1 \sum_{j=1}^{n} [\varphi_1(t + \sigma_k) - 2\varphi_1(t) + \varphi_1(t - \sigma_k)] \\ - r_1\varphi_1(t)[1 - a_1\varphi_1(t) - b_1\varphi_2(t - c\tau_1)]. \end{aligned}$$

For $\overline{\varphi}_1(t)$ there are two cases to discuss.
(i) If $t < t_2$, in view of $\overline{\varphi}_1(t \pm \sigma_k) \leqslant \mathrm{e}^{\lambda_1(t\pm\sigma_k)}$, then

$$\begin{aligned} P(\overline{\varphi}_1, \underline{\varphi}_2)(t) &\geqslant c\overline{\varphi}_1'(t) - d_1 \sum_{j=1}^{n} [\overline{\varphi}_1(t + \sigma_k) - 2\overline{\varphi}_1(t) + \overline{\varphi}_1(t - \sigma_k)] - r_1\overline{\varphi}_1(t) \\ &\geqslant -\mathrm{e}^{\lambda_1 t}\Delta_1(\lambda_1, c) = 0. \end{aligned}$$

(ii) If $t > t_2$, since $\overline{\varphi}_1(t \pm \sigma_k) \leqslant k_1 + \varepsilon_1 \mathrm{e}^{-\lambda(t \pm \sigma_k)}$ and $t_2 \geqslant t_3 + c\tau_1$, we can get

$$\begin{aligned} P(\overline{\varphi}_1, \underline{\varphi}_2)(t) \geqslant \mathrm{e}^{-\lambda t}\Big\{ &\varepsilon_1 \Big[-c\lambda - d_1 \sum_{j=1}^{n} (\mathrm{e}^{\lambda\sigma_k} + \mathrm{e}^{-\lambda\sigma_k} - 2) \Big] \\ &+ r_1(k_1 + \varepsilon_1 \mathrm{e}^{-\lambda t})(a_1\varepsilon_1 - b_1\varepsilon_4 \mathrm{e}^{\lambda c\tau_1}) \Big\} := \mathrm{e}^{-\lambda t} I_1(\lambda). \end{aligned}$$

$I_1(\lambda) > 0$ for λ small enough, because $I_1(0) = r_1(k_1 + \varepsilon_1)(a_1\varepsilon_1 - b_1\varepsilon_4) > 0$ by (5.4).

Now we verify $\underline{\varphi}_1(t)$.

(i) If $t < t_1 < 0$, in view of $t_1 \to -\infty$ as $q \to \infty$, we have

$$J(q) := \frac{a_1}{q} \mathrm{e}^{(2-\upsilon)\lambda_1 t} + \frac{b_1}{q} \mathrm{e}^{((\lambda_1+\lambda_2)/\lambda_1 - \upsilon)\lambda_1 t} \to 0 \quad \text{as } q \to \infty.$$

Since $\underline{\varphi}_1(t \pm \sigma_k) \geqslant \mathrm{e}^{\lambda_1(t \pm \sigma_k)} - q\mathrm{e}^{\upsilon\lambda_1(t \pm \sigma_k)}$, $\underline{\varphi}_1(t) \leqslant \mathrm{e}^{\lambda_1 t}$ and $\overline{\varphi}_2(t - c\tau_1) \leqslant \mathrm{e}^{\lambda_3(t - c\tau_1)} \leqslant \mathrm{e}^{\lambda_3 t}$, then

$$\begin{aligned} P(\underline{\varphi}_1, \overline{\varphi}_2)(t) &\leqslant q\mathrm{e}^{\upsilon\lambda_1 t}\Delta_1(\upsilon\lambda_1, c) + r_1(a_1\mathrm{e}^{2\lambda_1 t} + b_1\mathrm{e}^{(\lambda_1+\lambda_3)t}) \\ &\leqslant q\mathrm{e}^{\upsilon\lambda_1 t}[\Delta_1(\upsilon\lambda_1, c) + r_1 J(q)] \leqslant 0 \end{aligned}$$

for $q > 1$ large enough.

(ii) If $t > t_1$, we have $\overline{\varphi}_2(t - c\tau_1) \leqslant k_2 + \varepsilon_3 \mathrm{e}^{-\lambda(t - c\tau_1)}$ and $\underline{\varphi}_1(t \pm \sigma_k) \geqslant k_1 - \varepsilon_2 \mathrm{e}^{-\lambda(t \pm \sigma_k)}$ by (5.3), we have

$$\begin{aligned} P(\underline{\varphi}_1, \overline{\varphi}_2)(t) \leqslant \mathrm{e}^{-\lambda t}\Big\{ &\varepsilon_2 \Big[d_1 \sum_{j=1}^{n} (\mathrm{e}^{\lambda\sigma_k} + \mathrm{e}^{-\lambda\sigma_k} - 2) + c\lambda \Big] \\ &+ r_1(k_1 - \varepsilon_2 \mathrm{e}^{-\lambda t})(b_1\varepsilon_3 \mathrm{e}^{\lambda c\tau_1} - a_1\varepsilon_2) \Big\} := \mathrm{e}^{-\lambda t} I_2(\lambda). \end{aligned}$$

$I_2(\lambda) > 0$ for λ small enough, because $I_2(0) = r_1(k_1 - \varepsilon_2)(b_1\varepsilon_3 - a_1\varepsilon_2) < 0$ by (5.4). This completes the proof. □

Theorem 5.1. *For any* $c > c^*$, (1.2) *has a traveling wave solution* $(\varphi_1(\xi), \varphi_2(\xi))$ *connecting* $\mathbf{0}$ *with* $\mathbf{K}$ *if* (5.1) *holds. Furthermore,*

$$\begin{aligned} &\lim_{\xi \to -\infty} (\varphi_1(\xi)\mathrm{e}^{-\gamma_1\xi}, \varphi_2(\xi)\mathrm{e}^{-\gamma_2\xi}) = (1, 1), \\ &\lim_{\xi \to -\infty} (\varphi_1'(\xi)\mathrm{e}^{-\gamma_1\xi}, \varphi_2'(\xi)\mathrm{e}^{-\gamma_2\xi}) = (\gamma_1, \gamma_2), \end{aligned} \tag{5.5}$$

where $\gamma_1 = \lambda_1$, $\gamma_2 = \lambda_3$, $\xi = \sigma \cdot \eta + ct$. *But for* $0 < c < c^*$ *there are no traveling wave solutions of* (1.2) *satisfying* (5.5) *connecting* $\mathbf{0}$ *with* $\mathbf{K}$.

Proof. By Theorem 3.1 and Remark 3.1, we get the existence conclusion. The asymptotic behavior $\lim\limits_{\xi\to-\infty}(\varphi_1(\xi)\mathrm{e}^{-\lambda_1\xi},\varphi_2(\xi)\mathrm{e}^{-\lambda_3\xi})=(1,1)$ is obvious by $\overline{\varphi}_i(t)$, $\underline{\varphi}_i(t)$, $i=1,2$, and also

$$\lim_{\xi\to-\infty}\varphi_1'(\xi)\mathrm{e}^{-\lambda_1\xi}=\frac{1}{c}\Bigl[d_1\sum_{j=1}^{n}(\mathrm{e}^{\lambda_1\sigma_k}+\mathrm{e}^{-\lambda_1\sigma_k}-2)+r_1\Bigr]=\lambda_1.$$

The second part is similar.

When $c_1^*\geqslant c_2^*$, we have $c^*=c_1^*$. Assume that there is a traveling wave solution $(\varphi(\sigma\cdot\eta+ct),\psi(\sigma\cdot\eta+ct))$ of (1.1) and (5.5) connecting $\mathbf{0}$ and $\mathbf{K}$ for $0<c<c^*$. Then the asymptotic behavior of $(\varphi(\xi),\psi(\xi))$ leads to $\Delta_1(\gamma_1,c)=0$, which is impossible. The case $c_2^*\geqslant c_1^*$ is similar. The proof is completed. $\square$

Example 5.2. We study the traveling waves of (1.3) which connects $\mathbf{0}$ with $\mathbf{K}$.

Consider the existence of the solution for system

$$\tag{5.6}\begin{cases}c\varphi_1'(t)=d_1\sum\limits_{j=1}^{n}[\varphi_1(t+\sigma_k)-2\varphi_1(t)+\varphi_1(t-\sigma_k)]\\ \qquad\qquad\qquad+r_1\varphi_1(t)[1-a_1\varphi_1(t-c\tau_1)-b_1\varphi_2(t-c\tau_2)],\\ c\varphi_2'(t)=d_2\sum\limits_{j=1}^{n}[\varphi_2(t+\sigma_k)-2\varphi_2(t)+\varphi_2(t-\sigma_k)]\\ \qquad\qquad\qquad+r_2\varphi_2(t)[1-b_2\varphi_1(t-c\tau_3)-a_2\varphi_2(t-c\tau_4)]\end{cases}$$

satisfying

$$\lim_{t\to-\infty}(\varphi_1(t),\varphi_2(t))=\mathbf{0},\quad \lim_{t\to\infty}(\varphi_1(t),\varphi_2(t))=\mathbf{K}.$$

For $\varphi_1,\varphi_2\in C([-c\tau,0],\mathbb{R}),\tau=\max\limits_{1\leqslant i\leqslant 4}\tau_i$, let

$$\begin{aligned}f_1(\varphi_1,\varphi_2)&=r_1\varphi_1(0)[1-a_1\varphi_1(-c\tau_1)-b_1\varphi_2(-c\tau_2)],\\ f_2(\varphi_1,\varphi_2)&=r_2\varphi_2(0)[1-b_2\varphi_1(-c\tau_3)-a_2\varphi_2(-c\tau_4)].\end{aligned}$$

One can check that $f=(f_1,f_2)$ satisfies (P1)–(P3).

By using analogous argument as in [14], [15], the following lemma holds.

Lemma 5.4. *For τ_1,τ_4 small enough, the functional f satisfies* (PQM*).

Let $(\overline{\varphi}_1(t),\overline{\varphi}_2(t))$ and $(\underline{\varphi}_1(t),\underline{\varphi}_2(t))$ be as described above. Obviously, (A3) is satisfied and

$$\min\{t_2,t_4\}-\max\{c\tau_1,c\tau_2,c\tau_3,c\tau_4\}\geqslant\{t_1,t_3\}$$

for sufficiently small $\lambda>0$ and sufficiently large $q>1$.

Lemma 5.5. *If* (5.1) *holds, then* $(\overline{\varphi}_1(t),\overline{\varphi}_2(t))$ *and* $(\underline{\varphi}_1(t),\underline{\varphi}_2(t))$, *respectively, are a pair of weak upper and lower solutions of* (5.6) *for sufficiently small* τ_1,τ_4.

Proof. We verify $\overline{\varphi}_1(t)$. For $t<t_2$ and $t>t_2+c\tau_1$ we can use a similar argument as in Lemma 5.3. For the case $t_2<t<t_2+c\tau_1$, $I_1(\lambda)$ becomes

$$\tilde{I}_1(\lambda)=\varepsilon_1\Biggl[-c\lambda-d_1\sum_{j=1}^{n}(\mathrm{e}^{\lambda\sigma_k}+\mathrm{e}^{-\lambda\sigma_k}-2)\Biggr]+r_1(k_1+\varepsilon_1\mathrm{e}^{-\lambda t})(a_1\varepsilon_1\mathrm{e}^{\lambda c\tau_1}-b_1\varepsilon_4\mathrm{e}^{\lambda c\tau_2}),$$

and from Lemma 5.3, $\tilde{I}_1(0)<0$ when $t=t_2+c\tau_1$. Then $P(\overline{\varphi}_1,\underline{\varphi}_2)(t)\geqslant 0$ for $t_2<t<t_2+c\tau_1$ with sufficiently small τ_1 because of uniform boundedness and continuity of $\overline{\varphi}_1'(t),\overline{\varphi}_1(t)$ and $\underline{\varphi}_2(t)$ for $t\in\mathbb{R}\setminus\{t_2,t_3\}$ as well as of independency of τ_1. The cases $t_4<t<t_4+c\tau_4$ for $\overline{\varphi}_2(t)$, $t_1<t<t_1+c\tau_1$ for $\underline{\varphi}(t)$, and $t_3<t<t_3+c\tau_4$ for $\underline{\varphi}_2(t)$ are very similar. This completes the proof. $\square$

From Theorem 4.1 and Remark 4.1, the existence result follows.

Theorem 5.2. *For any* $c>c^*$, (1.3) *has a traveling wave solution* $(\varphi_1(\xi),\varphi_2(\xi))$ *connecting* $\mathbf{0}$ *with* $\mathbf{K}$ *for sufficiently small* τ_1,τ_4 *if* (5.1) *holds. Furthermore,*

$$\begin{aligned}\lim_{\xi\to-\infty}(\varphi_1(\xi)\mathrm{e}^{-\gamma_1\xi},\varphi_2(\xi)\mathrm{e}^{-\gamma_2\xi})&=(1,1),\\ \lim_{\xi\to-\infty}(\varphi_1'(\xi)\mathrm{e}^{-\gamma_1\xi},\varphi_2'(\xi)\mathrm{e}^{-\gamma_2\xi})&=(\gamma_1,\gamma_2),\end{aligned}\tag{5.7}$$

where $\gamma_1=\lambda_1$, $\gamma_2=\lambda_3$, $\xi=\sigma\cdot\eta+ct$. *But for* $0<c<c^*$ *there are no traveling wave solutions of* (1.2) *satisfying* (5.5) *connecting* $\mathbf{0}$ *with* $\mathbf{K}$.

Remark 5.1. The results of Theorems 5.1 and 5.2 show that the interspecific delays have no effect on the existence of traveling waves and monotonicity of the system. But the intraspecific delays τ_1,τ_4 in (1.3) do.

References

[1] *J. W. Cahn, J. Mallet-Paret, E. S. Van Vleck*: Traveling wave solutions for systems of ODEs on a two-dimensional spatial lattice. SIAM J. Appl. Math. *59* (1999), 455–493. zbl MR doi

[2] *X. Chen, J.-S. Guo*: Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations. J. Differ. Equations *184* (2002), 549–569. zbl MR doi

[3] *X. Chen, J.-S. Guo*: Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. Math. Ann. *326* (2003), 123–146. zbl MR doi

[4] *C.-P. Cheng, W.-T. Li, Z.-C. Wang, S. Zheng*: Traveling waves connecting equilibrium and periodic orbit for a delayed population model on a two-dimensional spatial lattice. Int. J. Bifurcation Chaos Appl. Sci. Eng. *26* (2016), Article ID 1650049, 13 pages. zbl MR doi

[5] *S.-N. Chow, J. Mallet-Paret, W. Shen*: Traveling waves in lattice dynamical systems. J. Differ. Equations *149* (1998), 248–291. zbl MR doi

[6] *J.-S. Guo, C.-H. Wu*: Existence and uniqueness of traveling waves for a monostable 2-D lattice dynamical system. Osaka J. Math. *45* (2008), 327–346. zbl MR

[7] *J.-S. Guo, C.-H. Wu*: Traveling wave front for a two-component lattice dynamical system arising in competition models. J. Differ. Equations *252* (2012), 4357–4391. zbl MR doi

[8] *D. Hankerson, B. Zinner*: Wavefronts for a cooperative tridiagonal system of differential equations. J. Dyn. Differ. Equations *5* (1993), 359–373. zbl MR doi

[9] *J. Huang, G. Lu, S. Ruan*: Traveling wave solutions in delayed lattice differential equations with partial monotonicity. Nonlinear Anal., Theory Methods Appl., Ser. A *60* (2005), 1331–1350. zbl MR doi

[10] *J. Huang, G. Lu, X. Zou*: Existence of traveling wave fronts of delayed lattice differential equations. J. Math. Anal. Appl. *298* (2004), 538–558. zbl MR doi

[11] *J. P. Keener*: Propagation and its failure in coupled systems of discrete excitable cells. SIAM J. Appl. Math. *47* (1987), 556–572. zbl MR doi

[12] *K. Li, J. Huang, X. Li, Y. He*: Traveling wave fronts in a delayed lattice competitive system. Appl. Anal. *97* (2018), 982–999. zbl MR doi

[13] *K. Li, X. Li*: Traveling wave solutions in a delayed lattice competition-cooperation system. J. Difference Equ. Appl. *24* (2018), 391–408. zbl MR doi

[14] *W.-T. Li, G. Lin, S. Ruan*: Existence of travelling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition systems. Nonlinearity *19* (2006), 1253–1273. zbl MR doi

[15] *G. Lin, W.-T. Li*: Traveling waves in delayed lattice dynamical systems with competition interactions. Nonlinear Anal., Real World Appl. *11* (2010), 3666–3679. zbl MR doi

[16] *J. Mallet-Paret*: The Fredholm alternative for functional-differential equations of mixed type. J. Dyn. Differ. Equations *11* (1999), 1–47. zbl MR doi

[17] *J. Mallet-Paret*: The global structure of traveling waves in spatially discrete dynamical systems. J. Dyn. Differ. Equations *11* (1999), 49–127. zbl MR doi

[18] *P. Weng*: Spreading speed and traveling wavefront of an age-structured population diffusing in a 2D lattice strip. Discrete Contin. Dyn. Syst., Ser. B *12* (2009), 883–904. zbl MR doi

[19] *S.-L. Wu, S.-Y. Liu*: Travelling waves in delayed reaction-diffusion equations on higher dimensional lattices. J. Difference Equ. Appl. *19* (2013), 384–401. zbl MR doi

[20] *J. Wu, X. Zou*: Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations. J. Differ. Equations *135* (1997), 315–357. zbl MR doi

[21] *Z.-X. Yu, R. Yuan*: Nonlinear stability of wavefronts for a delayed stage-structured population model on a 2-D lattice. Osaka J. Math. *50* (2013), 963–976. zbl MR

[22] *Z.-X. Yu, W. Zhang, X. Wang*: Spreading speeds and travelling waves for non-monotone time-delayed 2D lattice systems. Math. Comput. Modelling *58* (2013), 1510–1521. MR doi

[23] *H.-Q. Zhao*: Asymptotic stability of traveling fronts in delayed reaction-diffusion monostable equations on higher-dimensional lattices. Electron. J. Differ. Equ. *2013* (2013), Article ID 119, 15 pages. zbl MR

[24] *H.-Q. Zhao, S.-L. Wu*: Wave propagation for a reaction-diffusion model with a quiescent stage on a 2D spatial lattice. Nonlinear Anal., Real World Appl. *12* (2011), 1178–1191. zbl MR doi

[25] *B. Zinner*: Stability of traveling wavefronts for the discrete Nagumo equation. SIAM J. Math. Anal. *22* (1991), 1016–1020. zbl MR doi

[26] *B. Zinner*: Existence of traveling wavefront solutions for the discrete Nagumo equation. J. Differ. Equations *96* (1992), 1–27. zbl MR doi

[27] *B. Zinner, G. Harris, W. Hudson*: Traveling wavefronts for the discrete Fisher's equation. J. Differ. Equations *105* (1993), 46–62. zbl MR doi

[28] *X. Zou*: Traveling wave fronts in spatially discrete reaction-diffusion equations on higher-dimensional lattices. Electron. J. Differ. Equ. *1997* (1997), 211–222. zbl MR

[29] *X. Zou, J. Wu*: Local existence and stability of periodic traveling waves of lattice functional-differential equations. Can. Appl. Math. Q. *6* (1998), 397–418. zbl MR

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