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STABILITY OF CERTAIN ENGEL-LIKE DISTRIBUTIONS

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Abstract. We introduce a higher dimensional analogue of the Engel structure, motivated by the Cartan prolongation of contact manifolds. We study the stability of such structure, generalizing the Gray-type stability results for Engel manifolds. We also derive local normal forms defining such a distribution.

Keywords: Engel structure; Cartan prolongation; global stability; nonholonomic distribution; normal form

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1. Introduction

In [7] Montgomery proved that a generic rank r distribution on a manifold of dimension n is not stable if $r(n-r) > n$. Among the cases that are excluded by this inequality are line fields (when $r = 1$), contact and even contact structures (when $r = n-1$) and lastly Engel structures (when $r = 2$, $n = 4$). An Engel structure is a rank 2 distribution $\mathcal{D}$ on a 4-manifold M such that $\mathcal{D}^2$ is a rank 3 distribution and $\mathcal{D}^3 = TM$. Like contact structures, any Engel structure is locally given as the common kernel of two 1-forms (see [8]),

$$\mathrm{d}z - y\,\mathrm{d}x, \quad \mathrm{d}y - w\,\mathrm{d}x.$$

But unlike contact structures, Engel structures are not stable under arbitrary isotopy. In fact, any Engel structure $\mathcal{D}$ defines a complete flag

$$\mathcal{L} \subset \mathcal{D} \subset \mathcal{E},$$

where the line field $\mathcal{L}$, called the *characteristic line field*, is usually not stable under isotopy. Golubev proved a modified version of the Gray-type theorem for Engel structures in [5].

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Engel manifolds are closely related to contact 3-manifolds. Starting with a 3-dimensional manifold with a contact structure ξ, one obtains a circle bundle by the Cartan prolongation of ξ, where the total space of the bundle carries an Engel structure $\mathcal{D}$ with its characteristic line field tangent to fibers, see [8]. The prolongation on an arbitrary contact manifold (N^{2n+1}, ξ) gives rise to fiber bundles $M \to N$ with the fiber $\mathbb{R}P^{n-1}$. The total space of the bundle supports a flag $\mathcal{L} \subset \mathcal{D} \subset \mathcal{E} \subset TM$ on M with the rank vector $(2n-1, 2n, 4n-1, 4n)$, where

$$\mathcal{D}^2 = \mathcal{E}, \quad \mathcal{D}^3 = TM,$$

and $\mathcal{L}$ is the Cauchy characteristic distribution (see Definition 2.1) of $\mathcal{E}$. Motivated by this, we introduce generalized Engel structures on manifolds M as distributions $\mathcal{D}$ of even co-rank such that $\mathcal{E} = \mathcal{D}^2$ is a co-rank 1 distribution, $\mathcal{D}^3 = TM$, and the Cauchy characteristic distribution $\mathcal{L}$ of $\mathcal{E}$ is contained in $\mathcal{D}$ and has co-rank 1 in $\mathcal{D}$. The distribution $\mathcal{L}$ is referred to as the characteristic distribution of $\mathcal{D}$. In general, if we have a flag $\mathcal{D} \subset \mathcal{E} \subset TM$ satisfying $\mathcal{D}^2 = \mathcal{E}$ and $\mathcal{D}^3 = TM$, then it does not necessarily follow that the Cauchy characteristic distribution $\mathcal{L}$ is contained in $\mathcal{D}$ unless dimension M is 4 (see Example 3.2).

The generalized Engel distributions are not generic. However, they are similar to Engel 2-distributions in several ways. The main goal of this article is to demonstrate a Gray-type stability property of these distributions.

Theorem 1.1. *Let $\mathcal{D}_t$, $0 \leqslant t \leqslant 1$, be a smooth one-parameter family of generalized Engel distributions on a closed manifold M. Assume that the characteristic distribution $\mathcal{L}_t$ of $\mathcal{D}_t$ is independent of t and put $\mathcal{L} = \mathcal{L}_t$ for all t. Then there exists an isotopy φ_t of M such that*

$$\varphi_{t*}\mathcal{D}_t = \mathcal{D}_0, \quad \varphi_{t*}\mathcal{L} = \mathcal{L}.$$

We also obtain a local normal form for a set of generators of the annihilating ideal of a generalized Engel distribution $\mathcal{D}$.

It should be mentioned that global stability theorems for other types of (multi)flags have been discussed in literature before, namely:

- ▷ In [9], the authors proved it for a co-rank 1 distribution containing a characteristic distribution of arbitrary co-rank.
- ▷ In [1], Adachi improved upon this result by considering a distribution of arbitrary co-rank containing a co-rank 1 characteristic distribution.
- ▷ Later in [2], Adachi proved a similar result for the *special multiflag*, which can be considered as a direct generalization of the Goursat flag.

The article is organized as follows: In Section 2 we recall some basic notions about distributions. In Section 3 we introduce generalized Engel structures and describe the Pfaffian system defining them. In Sections 4 and 5 we prove the main results of this article.

2. Basic notions and examples

Given any distribution $\mathcal{A}$ on a manifold M, we can think of it as a sheaf of local sections of the sub-bundle $\mathcal{A} \subset TM$. The notation $X \in \mathcal{A}$ would mean some local section X of the distribution $\mathcal{A}$. Given two distributions $\mathcal{A}$, $\mathcal{B}$, define $[\mathcal{A}, \mathcal{B}]$ as the sheaf of vector fields obtained by taking Lie brackets of local sections. Using this notation, recursively define

$$\mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i], \quad \mathcal{D}^1 = \mathcal{D}.$$

At every $x \in M$ we have the integer $q_i(x) = \dim \mathcal{D}^i_x$, where $\mathcal{D}^i_x$ is the stalk at the point x. Note that $\mathcal{D}^i$ defines a distribution if the integer $q_i(x)$ is locally constant. The integer sequence $(q_i(x))_i$ is called the *growth vector* for the distribution $\mathcal{D}$ at x. A distribution is *regular* if the growth vector is independent of the point x. A regular distribution $\mathcal{D}$ is called *nonholonomic* if there is an integer k such that $TM = \mathcal{D}^k$. In this article, we consider only nonholonomic distributions in the above sense.

Before moving onto some examples, we recall the definition of Cauchy characteristic distribution (see [3]), as it will play an important role in understanding generalized Engel distributions.

Definition 2.1. Given a co-rank 1 distribution $\mathcal{E}$ on a manifold M, consider the collection

$$\mathcal{L} = \{X \in \mathcal{E} \colon [X, Y] \in \mathcal{E} \text{ for all } Y \in \mathcal{E}\}.$$

If $\mathcal{L}$ has constant rank everywhere it is called the *Cauchy characteristic distribution* of $\mathcal{E}$.

We can locally define $\mathcal{L}$ as follows. Suppose $\mathcal{E} \underset{\text{loc}}{=} \ker \theta$. Then

$$\mathcal{L} \underset{\text{loc}}{=} \ker \mathrm{d}\theta\big|_{\mathcal{E}} = \{X \in \mathcal{E} \colon \mathrm{d}\theta(X, Y) = 0 \text{ for all } Y \in \mathcal{E}\}.$$

It is easy to see that the Cauchy characteristic distribution is integrable. Indeed, if $X, Y \in \mathcal{L}$ and $Z \in \mathcal{E}$, then we have

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

Now, $[X, Z], [Y, Z] \in \mathcal{E}$ and hence $[[X, Y], Z] \in \mathcal{E}$. Thus $[X, Y] \in \mathcal{L}$. But then $\mathcal{L}$ is integrable by the Frobenius theorem.

Example 2.2.

(a) A contact distribution on an odd-dimensional manifold M is a co-rank 1 distribution ξ such that $\xi^2 = TM$ and the Cauchy characteristic distribution of ξ is trivial.

(b) Similarly, an even contact structure on an even-dimensional manifold M^{2n+2} is a co-rank 1 distribution $\mathcal{E}$ such that $\mathcal{E}^2 = TM$ and the Cauchy characteristic distribution of $\mathcal{E}$ is a line field. Like contact structures, an even contact structure $\mathcal{E}$ is locally given as kernel of some 1-form α satisfying $\alpha \wedge \mathrm{d}\alpha^n \neq 0$.

(c) An Engel structure $\mathcal{D}$ is a co-rank 2 nonholonomic distribution on a 4-dimensional manifold M, such that $\mathcal{D}^2$ is an even contact structure and $\mathcal{D}^3 = TM$. The characteristic line field $\mathcal{L}$ of $\mathcal{D}^2$ turns out to be contained in $\mathcal{D}$, see [8]. Thus the Engel structure completely defines the flag $\mathcal{L} \subset \mathcal{D} \subset \mathcal{D}^2 \subset TM$. Any Engel structure $\mathcal{D}$ can be locally realized as kernel of two 1-forms,

$$\mathrm{d}z - y\,\mathrm{d}x, \quad \mathrm{d}y - w\,\mathrm{d}x.$$

We are particularly interested in distributions in higher dimensions, which exhibit properties similar to Engel structures.

2.1. Cartan prolongation. A prime example of the Engel manifold appears as the Cartan prolongation (see [8], [10]) of contact 3-manifolds (M, ξ). We first describe the prolongation of a contact structure below.

Consider an odd-dimensional manifold N^{2n+1} with a contact structure ξ. On N we construct the Grassmann bundle

$$\mathbb{RP}^{2n-1} \hookrightarrow \mathbb{P}\xi \xrightarrow{\pi} N,$$

where the fiber over a point $x \in N$ is the projective space of lines in the vector space ξ_x. The total space $Q = \mathbb{P}\xi$ is of dimension $4n$. The inverse image of ξ under $\mathrm{d}\pi$ defines a co-rank 1 distribution $\mathcal{E}$ on Q, i.e., $\mathcal{E} = \mathrm{d}\pi^{-1}(\xi)$. On the other hand, there is a distribution $\mathcal{D}$ which is obtained as follows: at a point $[l] \in Q$, where l is a line in ξ_p for $p \in N$, put $\mathcal{D}_{[l]} = \mathrm{d}\pi|_{[l]}^{-1}(l)$. Since π is a submersion, $\mathcal{D}$ is a co-rank $2n$ distribution on Q. Clearly, $\mathcal{D} \subset \mathcal{E}$. Set $\mathcal{L}$ as the vertical sub-bundle of TQ over N, i.e., $\mathcal{L}$ is tangent along the fibers. Thus, we have a flag, $\mathcal{L} \subset \mathcal{D} \subset \mathcal{E}$. The distribution $\mathcal{D}$ is called the *prolongation* of ξ. In particular, if $n = 1$ and N is a contact 3 manifold, then $\dim Q = 4$ and $\mathcal{D}$ is an Engel structure on Q.

We now observe a few general properties of this flag. Since ξ is a contact structure on M it can be locally expressed as $\ker\Big(\mathrm{d}z - \sum\limits_{i=1}^{n} y_i\,\mathrm{d}x_i\Big)$. Therefore,

$$\xi \underset{\mathrm{loc}}{=} \langle \partial_{y_i},\, P_i = \partial_{x_i} + y_i \partial_z \colon\ i = 1, \ldots, n \rangle.$$

Any line $l \subset \xi_p$ is represented by a nontrivial linear combination of these vectors. Hence, on Q we can introduce homogeneous coordinates

$$\{a_1, \dots, a_n, b_1, \dots, b_n\}$$

along the fiber of $\pi\colon Q \to N$. If $q \in Q$ with $\pi(q) = (x_1, \dots, x_n, y_1, \dots, y_n, z)$ then there is a unique 1-dimensional subspace in $T_{\pi(q)}N$ given by

$$Z = \left\langle \sum_{i=1}^{n} a_i \partial_{y_i} + \sum_{i=1}^{n} b_i P_i \right\rangle,$$

where a_i, b_i are homogeneous coordinates of q along fiber. We can describe the flag $\mathcal{L} \subset \mathcal{D} \subset \mathcal{E}$ locally as follows,

$$\begin{aligned} \mathcal{L}_q &= \text{the vertical tangent space of } Q \text{ at } q, & \mathcal{D}_q &= \mathcal{L}_q \oplus \langle Z \rangle, \\ \mathcal{E}_q &= \mathcal{L}_q \oplus \langle \partial_{y_1}, \dots, \partial_{y_n}, P_1, \dots, P_n \rangle, & T_qQ &= \mathcal{E}_q \oplus \langle \partial_z \rangle. \end{aligned}$$

From this description we observe that

▷ co-rank $\mathcal{D}$ is even, co-rank $\mathcal{E}$ is 1 and co-rank of $\mathcal{L}$ in $\mathcal{D}$ is 1,
▷ $\mathcal{D}^2 = \mathcal{E}$, $\mathcal{D}^3 = TM$,
▷ $\mathcal{L}$ is the Cauchy characteristic distribution of $\mathcal{E}$, i.e., $[\mathcal{L}, \mathcal{E}] \subset \mathcal{E}$.

3. Generalized Engel structure

Motivated by the Cartan prolongation of a contact structure, we define a generalized Engel structure.

Definition 3.1. A *generalized Engel structure* or an Engel-like distribution on a manifold M is a distribution $\mathcal{D}$ of even co-rank, such that
(1) $\mathcal{E} = \mathcal{D}^2$ is a co-rank 1 distribution,
(2) $\mathcal{D}^3 = TM$,
(3) $\mathcal{L}$, the Cauchy characteristic distribution of $\mathcal{E}$, is contained in $\mathcal{D}$,
(4) $\mathcal{L}$ has co-rank 1 in $\mathcal{D}$.
Thus, we have the flag

$$\underbrace{\mathcal{L} \subset \mathcal{D}}_{\text{co-rank=1}} \subset \overbrace{\phantom{\mathcal{D}}\!\!\!\!\!\!\!\!\!\!\!\!\mathcal{D}\, \subset \underbrace{\mathcal{E} \subset TM}_{\text{co-rank=1}}}^{\text{even co-rank}}.$$

The distribution $\mathcal{L}$ will be called the *characteristic distribution* of the generalized Engel distribution $\mathcal{D}$.

We observe that, for a generalized Engel distribution $\mathcal{D}$, we must have that $\operatorname{rank} \mathcal{L} \geqslant \operatorname{rank} \mathcal{E}/\mathcal{D}$. The equality is achieved, for example, in the situation of Cartan prolongation of a contact $(2n+1)$-fold, where we get the flag

$$\mathcal{L}^{(2n-1)} \subset \mathcal{D}^{(2n)} \subset \mathcal{E}^{(4n-1)} \subset TQ^{(4n)}$$

and $\operatorname{rank} \mathcal{L} = 2n - 1 = \operatorname{rank} \mathcal{E}/\mathcal{D}$.

3.1. A remark on the definition. When $\dim M = 4$ and $\mathcal{D}$ is of co-rank 2, we have an Engel structure. As mentioned earlier, in this case the Cauchy characteristic distribution $\mathcal{L}$ of $\mathcal{E} = \mathcal{D}^2$ is completely determined by $\mathcal{D}$ and it is contained in $\mathcal{D}$. For a higher co-rank we can not expect this to happen in general as can be seen from the examples below. In the first two examples $\mathcal{L} \not\subset \mathcal{D}$ and in the third one the co-rank of $\mathcal{L}$ in $\mathcal{D}$ is not 1. All the examples are constructed over $\mathbb{R}^8$, where the coordinates are understood from the context.

Example 3.2.

(a) Suppose $\mathcal{D} = \langle \partial_x, \partial_y, \partial_z, \partial_w + x\partial_{x_1} + y\partial_{y_1} + z\partial_{z_1} + z_1\partial_t \rangle$. Then $[\mathcal{D}, \mathcal{D}] = \langle \partial_{x_1}, \partial_{y_1}, \partial_{z_1} \rangle$ and hence,

$$\mathcal{E} = \mathcal{D}^2 = \langle \partial_x, \partial_y, \partial_z, \partial_{x_1}, \partial_{y_1}, \partial_{z_1}, \partial_w + z_1\partial_t \rangle.$$

Lastly, $[\mathcal{D}, \mathcal{D}^2] = \langle \partial_t \rangle$ and so $\mathcal{D}^3 = TM$. The Cauchy characteristic distribution of $\mathcal{E}$ is $\mathcal{L} = \langle \partial_x, \partial_y, \partial_z, \partial_{x_1}, \partial_{y_1} \rangle$ which is a rank 5 distribution. Clearly in this case we have $\mathcal{L} \not\subset \mathcal{D}$.

(b) Consider, $\mathcal{D} = \langle \partial_w, \partial_{x_1} + w\partial_{y_1} + y_1\partial_z, \partial_{x_2} + w\partial_{y_2} + y_2\partial_z, \partial_{x_3} + w\partial_{y_3} \rangle$. Then $[\mathcal{D}, \mathcal{D}] = \langle \partial_{y_1}, \partial_{y_2}, \partial_{y_3} \rangle$ and so

$$\mathcal{E} = \mathcal{D}^2 = \langle \partial_w, \partial_{y_1}, \partial_{y_2}, \partial_{y_3}, \partial_{x_1} + y_1\partial_z, \partial_{x_2} + y_2\partial_z, \partial_{x_3} \rangle.$$

Clearly, $\mathcal{D}^3 = TM$ and the Cauchy characteristic distribution of $\mathcal{E}$ is $\mathcal{L} = \langle \partial_w, \partial_{y_3}, \partial_{x_3} \rangle$. Since $\partial_{y_3} \notin \mathcal{D}$, we have $\mathcal{L} \not\subset \mathcal{D}$.

(c) Let $v_i = \partial_{x_i} + w\partial_{y_i} + y_i\partial_z$ for $i = 1, 2, 3$ and $\mathcal{D} = \langle \partial_w, v_1, v_2, v_3 \rangle$. Then $[\mathcal{D}, \mathcal{D}] = \langle \partial_{y_1}, \partial_{y_2}, \partial_{y_3} \rangle$, and so $\mathcal{E} = \mathcal{D}^2$ is a co-rank 1 distribution and $\mathcal{D}^3 = TM$. Also, $\mathcal{L} = \langle \partial_w \rangle$. In this case, we have the flag $\mathcal{L} \subset \mathcal{D} \subset \mathcal{E}$, where $\mathcal{E}$ is an even contact structure. Further, note that there exists a co-rank 1 integrable distribution $\langle v_1, v_2, v_3 \rangle$ contained in $\mathcal{D}$.

The above examples justify the conditions (3) and (4) in the definition of the generalized Engel structure.

3.2. Pfaffian system. A *Pfaffian system* is a sub-bundle of the cotangent bundle T^*M. Given a distribution $\mathcal{D} \subset TM$, we have an associated Pfaffian system $\mathcal{S}(\mathcal{D})$ defined as the collection of 1-forms which vanish on $\mathcal{D}$. In this section we would like to find out the Pfaffian system for a generalized Engel distribution.

We start with a co-rank $k+1$ generalized Engel distribution $\mathcal{D}$, where $k = 2l+1$ is odd. Suppose locally that

$$\mathcal{E} = \{\theta = 0\}$$

and

$$\mathcal{D} = \{\omega^1 = \ldots = \omega^k = 0 = \theta\}$$

for 1-forms $\theta, \omega^1, \ldots, \omega^k$. Set $\eta^i = \omega^1 \wedge \ldots \wedge \omega^k \wedge \theta \wedge \mathrm{d}\omega^i$.

Proposition 3.3. *We have the following statements.*

(1) $\{\eta^1, \ldots, \eta^k\}$ *is point-wise linearly independent.*

(2) $\omega^i \wedge \theta \wedge \mathrm{d}\theta^{l+1} = 0$ *for all* $i = 1, \ldots, k$.

(3) $\theta \wedge \mathrm{d}\theta^{l+1} \neq 0$.

(4) $\theta \wedge \mathrm{d}\theta^{l+2} = 0$.

Proof. Choose local vector fields D and R such that $\mathcal{D}/\mathcal{L} = \langle D \bmod \mathcal{L}\rangle$ and $TM/\mathcal{E} = \langle R \bmod \mathcal{E}\rangle$. Since $\mathcal{L}$ is integrable and $\mathcal{E} = \mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$, we have that the map

$$\mathcal{L} \to \mathcal{E}/\mathcal{D}, \quad L \mapsto [D, L] \bmod \mathcal{D}$$

is a surjective bundle map. Since $\{\omega^i\}$ are linearly independent and $\mathcal{D}$ is their common kernel in $\mathcal{E}$, we can choose dual vectors $V^i \in \mathcal{E}/\mathcal{D}$. Also from the surjectivity, there exists $L^i \in \mathcal{L}$ such that $V^i = [D, L^i] \bmod \mathcal{D}$. Then $\eta^i \neq 0$ for all i, since we have

$$\eta^i(V^1, \ldots, V^k, R, D, L^i) \neq 0.$$

If possible, let $\{\eta^i\}$ be linearly dependent at the point p. Then without loss of generality we may assume that $\eta^1 = \sum\limits_{i=2}^{k} f_i \eta^i$ at p for some functions f_i. Set $\widetilde{\omega}^1 = \omega^1 - \sum\limits_{i>1} f_i \omega^i$. Then clearly, $\mathcal{D}$ is also defined as $\{\widetilde{\omega}^1 = \omega^2 = \ldots = \omega^k = 0 = \theta\}$. But then we must have that

$$\widetilde{\omega}^1 \wedge \omega^2 \wedge \ldots \omega^k \wedge \theta \wedge \mathrm{d}\widetilde{\omega}^1 \neq 0.$$

On the other hand,

$$
\begin{aligned}
&\widetilde{\omega}^1 \wedge \omega^2 \wedge \ldots \omega^k \wedge \theta \wedge \mathrm{d}\widetilde{\omega}^1 \\
&\qquad = \Bigl(\omega^1 - \sum_{i>1} f_i \omega^i\Bigr) \wedge \omega^2 \wedge \ldots \wedge \omega^k \wedge \theta \wedge \Bigl(\mathrm{d}\omega^1 - \sum_{i>1} \mathrm{d}(f_i\omega^i)\Bigr) \\
&\qquad = \omega^1 \wedge \ldots \omega^k \wedge \theta \wedge \Bigl(\mathrm{d}\omega^1 - \sum_{i>1} \mathrm{d}f_i \wedge \omega^i - \sum_{i>1} f_i \,\mathrm{d}\omega^i\Bigr) \\
&\qquad = \omega^1 \wedge \ldots \omega^k \wedge \theta \wedge \Bigl(\mathrm{d}\omega^1 - \sum_{i>1} f_i \,\mathrm{d}\omega^i\Bigr) = \eta^1 - \sum_{i>1} f_i \eta^i = 0 \text{ at the point } p.
\end{aligned}
$$

This is a contradiction. Hence we have that the set of $(k+3)$-forms $\{\eta^i\}$ are point-wise linearly independent. This proves statement (1).

Now observe that $\mathcal{L} = \ker(\mathrm{d}\theta|_{\ker\theta})$. So, on $\mathcal{E}/\mathcal{L}$ we have that $\mathrm{d}\theta$ has full rank. Since $\operatorname{co-rank}\mathcal{L} = k+2 = 2l+3 \Rightarrow \operatorname{rank}\mathcal{E}/\mathcal{L} = 2(l+1)$, we have

$$\theta \wedge \mathrm{d}\theta^{l+1} \neq 0, \quad \theta \wedge \mathrm{d}\theta^{l+2} = 0,$$

which proves statements (3) and (4).

Next, consider the $(2l+3)$-form $\omega^i \wedge \mathrm{d}\theta^{l+1}|_{\mathcal{E}}$ on $\mathcal{E}$. For any $L \in \mathcal{L}$ we have that $\iota_L\omega^i \wedge \mathrm{d}\theta^{l+1}|_{\mathcal{E}}$ is identically zero, since $\omega^i(L) = 0$ and $\iota_L \mathrm{d}\theta|_{\mathcal{E}} = 0$. Thus, $\mathcal{L}$ is in the kernel of $\omega^i \wedge \mathrm{d}\theta^{l+1}|_{\mathcal{E}}$. But $\mathcal{L}$ has co-rank $2l+2$ in $\mathcal{E}$ and then by a simple rank counting argument, $\omega^i \wedge \mathrm{d}\theta^{l+1}|_{\mathcal{E}}$ is identically zero. Now $\mathcal{E} = \ker\theta$ and hence $\omega^i \wedge \theta \wedge \mathrm{d}\theta^{l+1} = 0$, proving statement (2). □

The converse of Proposition 3.3 is also true. Suppose we are given some co-rank $k+1$ distribution $\mathcal{D}$ on a manifold M, where $k = 2l+1$, such that $\mathcal{D}$ is locally the common kernel of 1-forms $\{\theta, \omega^1, \ldots, \omega^k\}$, satisfying

▷ $\{\eta^1, \ldots, \eta^k\}$ is point-wise linearly independent, where $\eta^i = \omega^1 \wedge \ldots \omega^k \wedge \theta \wedge \mathrm{d}\omega^i$,
▷ $\omega^i \wedge \theta \wedge \mathrm{d}\theta^{l+1} = 0$ for all $i = 1, \ldots, k$,
▷ $\theta \wedge \mathrm{d}\theta^{l+1} \neq 0$,
▷ $\theta \wedge \mathrm{d}\theta^{l+2} = 0$.

Proposition 3.4. *Under the above hypotheses, $\mathcal{D}$ is a generalized Engel structure.*

P r o o f. Set $\mathcal{E} = \ker\theta$ and $\mathcal{L} = \ker \mathrm{d}\theta|_{\mathcal{E}}$. These are locally defined distributions of co-rank 1 and $2l+3$, respectively. We can get a local framing of $TM/\mathcal{L}$ as $\{R, X_i, Y_j \colon i,j = 1, \ldots, l+1\}$ such that $\theta \wedge \mathrm{d}\theta^{l+1}(R, X_1, Y_1, \ldots, X_{l+1}, Y_{l+1}) \neq 0$. Consider $L \in \mathcal{L}$. Then we have

$$0 = \omega^i \wedge \theta \wedge \mathrm{d}\theta^{l+1}(L, R, X_1, \ldots, Y_{l+1}) = \omega^i(L)\theta \wedge \mathrm{d}\theta^{l+1}(R, X_1, \ldots, Y_{l+1})$$

since all other terms vanish. But then $\omega^i(L) = 0$ for all i. Thus, $L \in \mathcal{D}$. So we have the flag

$$\mathcal{L} \subset \mathcal{D} \subset \mathcal{E}.$$

Next we show $\mathcal{E} = \mathcal{D}^2$. First note that $\mathcal{D} \underset{\text{loc}}{=} \mathcal{L} \oplus \langle Z \rangle$ for some choice of a vector field Z. Since $\mathcal{L}$ is the Cauchy characteristic distribution of $\mathcal{E}$, we have $[\mathcal{L}, \mathcal{E}] \subset \mathcal{E}$. In particular, $[\mathcal{L}, Z] \subset \mathcal{E}$. Also $\mathcal{L}$ being integrable, we have $[\mathcal{L}, \mathcal{L}] \subset \mathcal{L}$ by the Frobenius theorem. Then clearly, $[\mathcal{D}, \mathcal{D}] \subset \mathcal{E}$. Thus $\mathcal{D}^2 \subset \mathcal{E}$. For the equality, consider the map

$$\begin{aligned} \Phi\colon\ \mathcal{L} &\to \mathcal{E}/\mathcal{D}, \\ L &\mapsto [Z, L] \bmod \mathcal{D}, \end{aligned}$$

where Φ is a bundle map: $[Z, fL] = Z(f)L + f[Z, L] \equiv f[Z, L] \bmod \mathcal{D}$, as $\mathcal{L} \subset \mathcal{D}$. We show that Φ is of full rank, which implies that $\mathcal{E} = \mathcal{D}^2$. Equivalently this happens if Φ^* is injective. Dualizing Φ we get

$$\begin{aligned} \Phi^*\colon\ \big(\mathcal{E}/\mathcal{D}\big)^* &\to \mathcal{L}^*, \\ [\alpha] &\mapsto -\,\iota_Z\, \mathrm{d}\alpha|_{\mathcal{L}}, \end{aligned}$$

where $(\mathcal{E}/\mathcal{D})^*$ consists of classes of 1-forms α defined on $\mathcal{E}$, which annihilate $\mathcal{D}$. Consider the 1-forms

$$\tau^i := -\iota_Z\, \mathrm{d}\omega^i|_{\mathcal{L}}$$

defined on $\mathcal{L}$. Since ω^i induces a basis for $(\mathcal{E}/\mathcal{D})^*$, it is enough to show that the maps τ^i are point-wise linearly independent for Φ^* to be injective. If not, then without loss of generality assume $\tau^1 = \sum_{i>1} f_i \tau^i$ at some point p for some functions f_i. Get dual vectors $\{R, V_1, \ldots, V_k\}$ in $TM/\mathcal{D}$ of $\{\theta, \omega^1, \ldots, \omega^k\}$, respectively. Now, for any $L \in \mathcal{L}$, we have $\eta^i(V_1, \ldots, V_k, R, Z, L) = \mathrm{d}\omega^i(Z, L) = -\tau^i(L)$. Thus,

$$\eta^1(V_1, \ldots, V_k, R, Z, L) = -\sum_{i>1} f_i \eta^i(V_1, \ldots, V_k, R, Z, L)$$

at the point p. But then $\eta^1 = -\sum_{i>1} f_i \eta^i$ at p, contradicting the point-wise linear independence of $\{\eta^i\}$. Hence, $\{\tau^i\}$ has to be linearly independent point-wise. Thus we get $\mathcal{E} = \mathcal{D}^2$; $\mathcal{E}$ and consequently $\mathcal{L}$ are now globally defined distributions. Also observe that Φ being full rank, we have $\operatorname{rank} \mathcal{L} \geqslant \operatorname{rank} \mathcal{E}/\mathcal{D}$.

Lastly, to verify $\mathcal{D}^3 = TM$, note that $\mathrm{d}\theta$ is non-degenerate on $\mathcal{E}/\mathcal{L}$. In particular, for $Z \in \mathcal{D}$ satisfying $\mathcal{D} = \mathcal{L} \oplus \langle Z \rangle$, we have $\iota_Z\, \mathrm{d}\theta \neq 0$. So, $\mathrm{d}\theta(Z, V) \neq 0$ for some $V \in \mathcal{E}/\mathcal{D}$. Then $V \in [\mathcal{D}, \mathcal{D}]$ and $0 \neq \mathrm{d}\theta(Z, V) = -\theta[Z, V]$. Thus, $TM = \mathcal{E} \oplus \langle [Z, V] \rangle$. So, $TM = \mathcal{D}^3$. $\square$

3.3. Orientability. Suppose M is a manifold of dimension $4n$ and $\mathcal{D}$ is a rank $2n$ generalized Engel structure on it with the associated flag

$$\mathcal{L}^{(2n-1)} \subset \mathcal{D}^{(2n)} \subset \mathcal{E}^{(4n-1)} \subset TM^{(4n)}.$$

Then we have a bundle map

$$\begin{aligned} \Phi\colon\ & \mathcal{L} \otimes \mathcal{D}/\mathcal{L} \to \mathcal{E}/\mathcal{D}, \\ & L \otimes (D \bmod \mathcal{L}) \mapsto [L, D] \bmod \mathcal{D} \end{aligned}$$

which is surjective. Since $\operatorname{rank} \mathcal{L} \otimes \mathcal{D}/\mathcal{L} = 2n-1 = \operatorname{rank} \mathcal{E}/\mathcal{D}$, Φ must be a bundle isomorphism. Thus we get a splitting

$$TM \cong \mathcal{E} \oplus TM/\mathcal{E} \cong \mathcal{D} \oplus \mathcal{E}/\mathcal{D} \oplus TM/\mathcal{E} \cong \mathcal{D} \oplus (\mathcal{L} \otimes \mathcal{D}/\mathcal{L}) \oplus TM/\mathcal{E}.$$

Proposition 3.5. *The distribution $\mathcal{E}$ is orientable.*

P r o o f. We calculate the first Stiefel-Whitney class $\omega_1(\mathcal{E})$. Since

$$\mathcal{E} \cong \mathcal{D} \oplus (\mathcal{L} \otimes \mathcal{D}/\mathcal{L}),$$

we have

$$\omega_1(\mathcal{E}) = \omega_1(\mathcal{D} \oplus (\mathcal{L} \otimes \mathcal{D}/\mathcal{L})) = \omega_1(\mathcal{D}) + \omega_1(\mathcal{L} \otimes \mathcal{D}/\mathcal{L}).$$

We have the formula (see [6]) for the total Stiefel-Whitney class,

$$\omega(\mathcal{L} \otimes \mathcal{D}/\mathcal{L}) = P(\omega_1(\mathcal{L}), \ldots, \omega_{2n-1}(\mathcal{L}), \omega_1(\mathcal{D}/\mathcal{L})) \bmod 2.$$

Here, P is the polynomial of $2n$ variables, given by the identity

$$P(\sigma_1, \ldots, \sigma_{2n-1}, T) = \prod_{i=1}^{2n-1} (1 + X_i + T),$$

where σ_i is the ith degree elementary symmetric polynomial in indeterminates $X_1, \ldots, X_{2n-1}$. Explicitly,

$$P(\sigma_1, \ldots, \sigma_{2n-1}, T) = (1+T)^{2n-1} + \sigma_1(1+T)^{2n-2} + \ldots + \sigma_{2n-2}T + \sigma_{2n-1}.$$

So we get

$$\omega(\mathcal{L} \otimes \mathcal{D}/\mathcal{L}) = (1 + \omega_1(\mathcal{D}/\mathcal{L}))^{2n-1} + \omega_1(\mathcal{L})(1 + \omega_1(\mathcal{D}/\mathcal{L}))^{2n-2} + \ldots + \omega_{2n-1}(\mathcal{L}).$$

Comparing both sides,

$$\omega_1(\mathcal{L} \otimes \mathcal{D}/\mathcal{L}) = (2n-1)\omega_1(\mathcal{D}/\mathcal{L}) + \omega_1(\mathcal{L}) = \omega_1(\mathcal{D}/\mathcal{L}) + \omega_1(\mathcal{L}) = \omega_1(\mathcal{D}/\mathcal{L} \oplus \mathcal{L}) = \omega_1(\mathcal{D}).$$

But then $\omega_1(\mathcal{E}) = \omega_1(\mathcal{D}) + \omega_1(\mathcal{D}) = 0$ and hence $\mathcal{E}$ must be orientable. $\square$

The orientability of $\mathcal{E}$ may also be understood in the following manner. Suppose $L_1, \dots, L_{2n-1}$ is a local frame of $\mathcal{L}$ and X is a local section of $\mathcal{D}$ which is transverse to $\mathcal{L}$. Then, $L_1, \dots, L_{2n-1}, X, [L_1, X], \dots, [L_{2n-1}, X]$ defines a framing of $\mathcal{E}$. It may be shown that this framing uniquely determines an orientation on $\mathcal{E}$ independent of the choices.

4. Stability of generalized Engel structure

Engel structures are not globally stable due to the presence of an integrable subbundle, though they have the local stability property. Golubev proved the following Gray-type theorem for the Engel structure which shows that a homotopy $\mathcal{D}_t$, $0 \leqslant t \leqslant 1$, of Engel structures is obtained by an isotopy provided the characteristics distribution of $\mathcal{D}_t$ is independent of t.

Theorem 4.1 ([5]). *Let $\mathcal{D}_t$, $0 \leqslant t \leqslant 1$, be a one-parameter family of oriented Engel structures on an oriented closed 4-dimensional manifold M such that the characteristic line field $\mathcal{L}(\mathcal{D}_t) = \mathcal{L}$ for all t. Then there exists an isotopy φ_t, $0 \leqslant t \leqslant 1$, of M such that*

$$\varphi_{t*}(\mathcal{D}_t) = \mathcal{D}_0, \quad \varphi_{t*}(\mathcal{L}) = \mathcal{L}.$$

Theorem 1.1 is a direct generalization of Theorem 4.1 for generalized Engel structure. We shall first prove a special case of this theorem.

Theorem 4.2. *Suppose $\mathcal{D}_t$, $0 \leqslant t \leqslant 1$, is a one-parameter family of generalized Engel structures on a closed manifold M such that $\mathcal{D}_t^2$ is independent of t and equals $\mathcal{E}$. If $\mathcal{L}$ is the Cauchy characteristics distribution of $\mathcal{E}$ then there exists an isotopy φ_t of M such that*

$$\varphi_{t*}\mathcal{D}_0 = \mathcal{D}_t, \quad \varphi_{t*}\mathcal{E} = \mathcal{E}, \quad \varphi_{t*}\mathcal{L} = \mathcal{L}.$$

4.1. Proof of Theorem 4.2. The approach of the proof is very similar to that of Adachi in [1]. The proof follows through a sequence of lemmas. Throughout this section, we assume that $\mathcal{D}_t$ is a smooth one-parameter family of co-rank $k+1$, where $k = 2l + 1$, the generalized Engel distribution on a closed manifold M such that $\mathcal{E} = \mathcal{D}_t^2$ is independent of t and the Cauchy characteristic distribution of $\mathcal{E}$ is $\mathcal{L}$.

We first identify a time-dependent vector field X_t, whose flow has the desired property. Then, we prove the existence of such vector fields on M.

Proposition 4.3. *Suppose there exists a time dependent vector field* X_t, $0 \leqslant t \leqslant 1$, *on* M *which satisfies the conditions*

$$\iota_{X_t}\,\mathrm{d}\omega_t^i|_{\mathcal{D}_t} + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big|_{\mathcal{D}_t} = 0 \quad \forall\, i = 1, \ldots, k;\ X_t \in \mathcal{L}, \tag{1}$$

where θ, ω_t^i, $0 \leqslant t \leqslant 1$, $i = 1, 2, \ldots, k$, *is a smooth family of (local)* 1*-forms such that*

$$\mathcal{E} \underset{\mathrm{loc}}{=} \ker\theta \quad \textit{and} \quad \mathcal{D}_t \underset{\mathrm{loc}}{=} \{\omega_t^1 = \ldots = \omega_t^k = 0 = \theta\}.$$

Then the flow φ_t *obtained by integrating the time-dependent vector field* X_t *satisfies,*

$$\varphi_{t*}\mathcal{D}_0 = \mathcal{D}_t, \quad \varphi_{t*}\mathcal{L} = \mathcal{L}.$$

Proof. Since $X_t \in \mathcal{L}$ and $[\mathcal{L}, \mathcal{E}] \subset \mathcal{E}$, we have that $\varphi_{t*}\mathcal{E} = \mathcal{E}$ and hence $\varphi_{t*}\mathcal{L} = \mathcal{L}$, as $\mathcal{L}$ is completely defined by $\mathcal{E}$. So we have $\varphi_t^*\theta = F_t\theta$ for some family of non-vanishing functions F_t.

In order to verify that $\varphi_{t*}\mathcal{D}_0 = \mathcal{D}_t$, we would show the existence of smooth families of functions G_t^{ij} and F_t^i satisfying

$$\varphi_t^*\omega_t^i = \sum_j G_t^{ij}\omega_0^j + F_t^i\theta \quad \forall\, i, \tag{$*$}$$

where the matrix $(G_t^{ij})_{k\times k}$ is non-singular for every t. Furthermore, since φ_0 is the identity map, $G_0^{ij} = \delta_{ij}$ and $F_0^i = 0$ for all i, j.

Suppose we have a family of functions G_t^{ij} and F_t^i which satisfy the relation $(*)$. Differentiating both sides of $(*)$ with respect to t we get

$$\sum_j \frac{\mathrm{d}G_t^{ij}}{\mathrm{d}t}\omega_0^j + \frac{\mathrm{d}F_t^i}{\mathrm{d}t}\theta = \frac{\mathrm{d}}{\mathrm{dt}}\varphi_t^*\omega_t^i = \varphi_t^*\Big(L_{X_t}\omega_t^i + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big) = \varphi_t^*\Big(\iota_{X_t}\,\mathrm{d}\omega_t^i + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big).$$

Now from the hypothesis, $\iota_{X_t}\mathrm{d}\omega_t^i|_{\mathcal{D}_t} + (\mathrm{d}/\mathrm{dt})\omega_t^i|_{\mathcal{D}_t} = 0$ and hence there exists a family of functions g_t^{ij} and f_t^i such that

$$\iota_{X_t}\,\mathrm{d}\omega_t^i + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i = \sum_j g_t^{ij}\omega_t^j + f_t^i\theta.$$

Pulling back by φ_t we have

$$\begin{aligned}
\varphi_t^*\Big(\iota_{X_t}\mathrm{d}\omega_t^i + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big) &= \sum_j (g_t^{ij}\circ\varphi_t)\varphi_t^*\omega_t^j + (f_t^i\circ\varphi_t)\varphi_t^*\theta \\
&= \sum_j (g_t^{ij}\circ\varphi_t)\Big(\sum_p G_t^{jp}\omega_0^p + F_t^j\theta\Big) + (f_t^i\circ\varphi_t)F_t\theta \\
&= \sum_p\Big(\sum_j (g_t^{ij}\circ\varphi_t)G_t^{jp}\Big)\omega_0^p + \Big(\sum_j (g^i j_t\circ\varphi_t)F_t^j + (f_t^i\circ\varphi_t)F_t\Big)\theta.
\end{aligned}$$

Comparing the coefficients of ω_0^i and θ in the last two expressions, we get a system of first order differential equations

$$
\begin{aligned}
\frac{\mathrm{d}G_t^{ip}}{\mathrm{d}t} &= \sum_j (g_t^{ij} \circ \varphi_t) G_t^{jp} \quad \forall\, i, p, \\
\frac{\mathrm{d}F_t^{i}}{\mathrm{d}t} &= \sum_j (g_t^{ij} \circ \varphi_t) F_t^{j} + (f_t^i \circ \varphi_t) F_t \quad \forall\, i
\end{aligned} \tag{2}
$$

with the initial conditions

$$
G_0^{ij} = \partial_{ij}, \quad F_0^i = 0. \tag{3}
$$

Thus it follows that G^{ij} and F^i must be solutions to the initial value problem (2) and (3). Conversely, if G^{ij} and F^i are solutions to the initial value problem (2) and (3), then they satisfy the relation $(*)$.

As the system (2) is affine, its solution exists for all $t \in [0,1]$. Since the initial value matrix $(G_0^{ij}) = I_k$ is non-singular, we have that the solution (G_t^{ij}) is non-singular for every t (see [4]). This completes the proof of the proposition. $\square$

Next we observe that the time-dependent vector field X_t satisfying the hypothesis of Proposition 4.3 depends only on the distribution $\mathcal{D}_t$ and not on the choice of a basis for the Pfaffian system defining the distribution.

Proposition 4.4. *Suppose $\mathcal{E} = \ker\eta$ and $\mathcal{D}_t = \{\mu_t^1 = \ldots = \mu_t^j = 0 = \eta\}$ for a smooth family of (local) 1-forms $\{\mu_t^i, \eta\}$. Then the vector field X_t, $0 \leqslant t \leqslant 1$, in Proposition 4.3 also satisfies the relation*

$$
\iota_{X_t} \mathrm{d}\mu_t^i|_{\mathcal{D}_t} + \frac{\mathrm{d}}{\mathrm{d}t}\mu_t^i\Big|_{\mathcal{D}_t} = 0.
$$

In other words, X_t depends on the distributions $\mathcal{D}_t$, not on the choice of local 1-forms defining the distributions.

P r o o f. Suppose $\{\mu_t^i, \eta\}$ and $\{\omega_t^i, \theta\}$ be as above. Then we must have that $\eta = f\theta$ for some nonzero function f and

$$
\begin{pmatrix} \mu_t^1 \\ \vdots \\ \mu_t^k \end{pmatrix} = A_t \begin{pmatrix} \omega_t^1 \\ \vdots \\ \omega_t^k \end{pmatrix}
$$

for a family of non-singular $k \times k$ matrices $A_t = (A_t^{ij})$. So, $\mu_t^i = \sum\limits_j A_t^{ij} \omega_t^j$. Then $\mathrm{d}\mu_t^i = \sum\limits_j \mathrm{d}A_t^{ij} \wedge \omega_t^j + A_t^{ij}\, \mathrm{d}\omega_t^j$. So,

$$\begin{aligned}\iota_{X_t}\, \mathrm{d}\mu_t^i|_{\mathcal{D}_t} &= \sum_j (\iota_{X_t}\, \mathrm{d}A_t^{ij})\omega_t^j|_{\mathcal{D}_t} - (\iota_{X_t}\omega_t^j)\, \mathrm{d}A_t^{ij}|_{\mathcal{D}_t} + A_t^{ij}\iota_{X_t}\, \mathrm{d}\omega_t^j|_{\mathcal{D}_t} \\ &= \sum_j A_t^{ij}\iota_{X_t}\, \mathrm{d}\omega_t^j|_{\mathcal{D}_t} = -\sum_j A_t^{ij}\frac{\mathrm{d}}{\mathrm{dt}}\omega_t^j\Big|_{\mathcal{D}_t}\end{aligned}$$

as $\omega_t^j(X_t) = 0$ and $\omega_t^j|_{\mathcal{D}_t} = 0$. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{dt}}\mu_t^i\Big|_{\mathcal{D}_t} = \sum_j \frac{\mathrm{d}A_t^{ij}}{\mathrm{dt}}\omega_t^j\Big|_{\mathcal{D}_t} + A_t^{ij}\frac{\mathrm{d}}{\mathrm{dt}}\omega_t^j\Big|_{\mathcal{D}_t} = \sum_j A_t^{ij}\frac{\mathrm{d}}{\mathrm{dt}}\omega_t^j\Big|_{\mathcal{D}_t}.$$

Hence, we have $\iota_{X_t}\mathrm{d}\mu_t^i|_{\mathcal{D}_t} + (\mathrm{d}/\mathrm{dt})\mu_t^i|_{\mathcal{D}_t} = 0$. Thus X_t is the solution for every family of local forms defining $\mathcal{D}_t$. $\square$

4.1.1. Obtaining the time-dependent field. Now, in order to prove Theorem 4.2, we need to find a time-dependent vector field $X_t \in \mathcal{L}$ which locally satisfies the relations

$$\iota_{X_t}\mathrm{d}\omega_t^i|_{\mathcal{D}_t} + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i|_{\mathcal{D}_t} = 0 \quad \forall\, i = 1, \ldots, k$$

on some open subset U, where ω_t^i, θ are as in Proposition 4.3.

To begin with, we introduce a few notations. Put

$$\begin{aligned}\mathcal{K}_t^i &= \ker \mathrm{d}\omega_t^i|_{\mathcal{D}_t} = \{X \in \mathcal{D}_t \colon\ \mathrm{d}\omega_t^i(X, Y) = 0 \text{ for all } Y \in \mathcal{D}_t\} \\ &= \{X \in \mathcal{D}_t \colon\ \omega_t^i([X, Y]) = 0 \text{ for all } Y \in \mathcal{D}_t\}, \\ \mathcal{J}_t^i &= \bigcap_{j \neq i} \mathcal{K}_t^j = \{X \in \mathcal{D}_t \colon\ \omega_t^j([X, Y]) = 0 \text{ for all } Y \in \mathcal{D}_t \text{ for all } j \neq i\}, \\ \mathcal{W}_t &= \bigcap_j \mathcal{K}_t^j = \{X \in \mathcal{D}_t \colon\ [X, Y] \in \mathcal{D}_t \text{ for all } Y \in \mathcal{D}_t\}.\end{aligned}$$

Lemma 4.5. *For each $i = 1, \ldots, k$ we have $\mathcal{K}_t^i \subset \mathcal{L}$ for all t.*

P r o o f. Locally, we have a family of 1-forms α_t such that $\mathcal{L} = \mathcal{D}_t \cap \ker \alpha_t$. Since $\mathcal{L}$ is integrable, from the Frobenius theorem we have, in particular, $\alpha_t \wedge \omega_t^1 \wedge \ldots \wedge \omega_t^k \wedge \mathrm{d}\omega_t^i = 0$, $i = 1, \ldots, k$. Pick some $K \in \mathcal{K}_t^i$. Then we have

$$0 = \iota_K(\alpha_t \wedge \omega_t^1 \wedge \ldots \wedge \omega_t^k \wedge \mathrm{d}\omega_t^i) = \alpha_t(K)\omega_t^1 \wedge \ldots \wedge \omega_t^k \wedge \mathrm{d}\omega_t^i$$

as the other terms vanish. But $\omega_t^1 \wedge \ldots \wedge \omega_t^k \wedge \mathrm{d}\omega_t^i \neq 0$. Hence we have $\alpha_t(K) = 0$ and so $K \in \mathcal{L}$. Thus we have $\mathcal{K}_t^i \subset \mathcal{L}$ for all i and for all t. $\square$

In particular, we have that $\mathcal{J}_t^i \subset \mathcal{L}$ for each $i = 1, \ldots, k$ and $\mathcal{W}_t \subset \mathcal{L}$. Also observe that for any i, $\mathcal{W}_t = \mathcal{J}_t^i \cap \mathcal{K}_t^i$.

Lemma 4.6. *For any $0 \leqslant t \leqslant 1$, $\mathcal{K}_t^i$, $\mathcal{W}_t$ and $\mathcal{J}_t^i$ have constant ranks. Furthermore,* $\operatorname{rank} \mathcal{J}_t^i = \operatorname{rank} \mathcal{W}_t + 1$ *and co-rank of $\mathcal{K}_t^i$ in $\mathcal{L}$ is* 1.

P r o o f. Fix $0 \leqslant t \leqslant 1$. Since $\mathcal{L}$ has co-rank 1 in $\mathcal{D}_t$, choose a vector field Z such that $\mathcal{D}_t = \mathcal{L} \oplus \langle Z \rangle$. Then consider the map

$$\begin{aligned} \Psi \colon\ & \mathcal{L} \to \mathcal{E}/\mathcal{D}_t, \\ & L \mapsto [Z, L] \bmod \mathcal{D}_t, \end{aligned}$$

where Ψ is a bundle map and has full rank, since $\mathcal{D}_t^2 = \mathcal{E}$ and $\mathcal{L}$ is integrable. Clearly $\mathcal{W}_t$ is contained in the kernel of Ψ. Also for $\Psi(L) = 0$, i.e., $[Z, L] \in \mathcal{D}_t$, we have that $[L, X] \in \mathcal{D}_t$ for any $X \in \mathcal{D}_t$, since $\mathcal{L}$ is integrable. Thus $L \in \mathcal{W}_t$ and so $\mathcal{W}_t = \ker \Psi$. Hence $\mathcal{W}_t$ is a constant rank distribution.

Since the induced forms $\omega_t^i|_{\mathcal{E}/\mathcal{D}_t}$ are (point-wise) linearly independent, choose some vector fields $\{V_i\}$ from $\mathcal{E}_t$ such that $\{\overline{V}_i = V_i \bmod \mathcal{D}_t\}$ is the corresponding dual basis. Consider the map

$$\begin{aligned} \Psi_i \colon\ & \mathcal{L} \to \mathcal{E}/(\mathcal{D}_t \oplus \langle V_i \rangle), \\ & L \mapsto [Z, L] \bmod (\mathcal{D}_t \oplus \langle V_i \rangle). \end{aligned}$$

Again Ψ_i is a full rank bundle map. As $\overline{V}_i$ is dual to $\omega_t^i|_{\mathcal{E}/\mathcal{D}_t}$ for any $L \in \mathcal{J}_t^i$, we have that

$$[Z, L] \bmod \mathcal{D}_t = f_i \overline{V}_i$$

for some function f_i and thus $\mathcal{J}_t^i \subset \ker \Psi_i$. Conversely, suppose $\Psi_i(L) = 0$, i.e., $[Z, L] \in \mathcal{D}_t \oplus \langle V_i \rangle$. But then for $j \neq i$, $\omega_t^j[Z, L] = 0$ which implies $\omega_t^j[L, X] = 0$ for any $X \in \mathcal{D}_t$ and for all $j \neq i$. Thus $L \in \mathcal{J}_t^i$ and hence $\mathcal{J}_t^i = \ker \Psi_i$, proving that $\mathcal{J}_t^i$ is of constant rank.

Similarly define the map

$$\begin{aligned} \Phi_i \colon\ & \mathcal{L} \to \mathcal{E}/(\mathcal{D}_t \oplus \langle V_1, \ldots, \widehat{V}_i, \ldots, V_k \rangle), \\ & L \mapsto [Z, L] \bmod (\mathcal{D}_t \oplus \langle V_1, \ldots, \widehat{V}_i, \ldots, V_k \rangle) \end{aligned}$$

and observe that $\ker \Phi_i = \mathcal{K}_t^i$. Since Φ_i is again a full rank bundle map, $\mathcal{K}_t^i$ is of constant rank.

Clearly we have that $\operatorname{rank} \mathcal{J}_t^i = \operatorname{rank} \mathcal{W}_t + 1$ and co-rank of $\mathcal{K}_t^i$ in $\mathcal{L}$ is 1 for any $0 \leqslant t \leqslant 1$. □

Now we find the local field.

Lemma 4.7. *For each i there is a local field $X_t^i \in \mathcal{J}_t^i$ such that*

$$\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big|_{\mathcal{D}_t} = 0.$$

P r o o f. From Lemma 4.6 we have that $\mathcal{J}_t^i = \mathcal{W}_t \oplus \mathcal{U}_t^i$ for some line field $\mathcal{U}_t^i \subset \mathcal{J}_t^i$. Clearly $\mathcal{U}_t^i \not\subset \mathcal{K}_t^i$, as that would imply $\mathcal{J}_t^i \subset \mathcal{K}_t^u$. Now we can get $\mathcal{D}_t = \mathcal{K}_t^i \oplus \mathcal{V}_t^i$ such that $\mathcal{U}_t^i \subset \mathcal{V}_t^i$ for all t. As the co-rank of $\mathcal{K}_t^i$ in $\mathcal{L}$ is 1 we have that $\mathcal{V}_t^i$ is of constant rank with $\operatorname{rank} \mathcal{V}_t^i = 2$. Since by definition $\mathcal{K}_t^i = \ker \mathrm{d}\omega_t^i|_{\mathcal{D}_t}$, we have that $\mathrm{d}\omega_t^i$ is non-degenerate over $\mathcal{V}_t^i$. Hence we have a solution $X_t^i \in \mathcal{V}_t^i$ such that

$$\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{V}_t^i} + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big|_{\mathcal{V}_t^i} = 0.$$

Now, $\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{K}_t^i} = 0$. Also since $\mathcal{K}_t^i \subset \mathcal{L} \subset \mathcal{D}_s$ for every parameter s, we have that $(\mathrm{d}/\mathrm{dt})\omega_t^i|_{\mathcal{K}_t^i} = 0$, since $\mathcal{D}_s \subset \ker \omega_s^i$. Thus we also have that $\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{K}_t^i} + (\mathrm{d}/\mathrm{dt})\omega_t^i|_{\mathcal{K}_t^i} = 0$. Combining this we get that

$$\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big|_{\mathcal{D}_t} = 0$$

as required. We now show that $X_t^i \in \mathcal{U}_t^i$, which will yield that $X_i^i \in \mathcal{J}_t^i$.

Restricting to $\mathcal{U}_t^i$ we see that $\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{U}_t^i} = -(\mathrm{d}/\mathrm{dt})\omega_t^i|_{\mathcal{U}_t^i}$. But $\mathcal{U}_t^i \subset \mathcal{J}_t^i \subset \mathcal{L} \subset \mathcal{D}_s \subset \ker \omega_s^i$ for all s and so $(\mathrm{d}/\mathrm{dt})\omega_t^i|_{\mathcal{U}_t^i} = 0$. Thus we have that

$$\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{U}_t^i} = 0.$$

Now $X_t^i \in \mathcal{V}_t^i$ and $\mathcal{U}_t^i \subset \mathcal{V}_t^i$ is a line field. But $\mathrm{d}\omega_t^i$ is non-degenerate on $\mathcal{V}_t^i$ with $\operatorname{rank} \mathcal{V}_t^i = 2$. Thus $\iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{U}_t^i} = 0$ is possible only if $X_t^i \in \mathcal{U}_t^i$. This completes the proof. □

Set

$$X_t = \sum_i X_t^i.$$

Since each $X_t^i \in \mathcal{J}_t^i \subset \mathcal{L}$, we have that $X_t \in \mathcal{L}$.

Lemma 4.8. *We have $\iota_{X_t} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} + (\mathrm{d}/\mathrm{dt})\omega_t^i|_{\mathcal{D}_t} = 0$ for all $i = 1, \ldots, k$.*

P r o o f. Since $X_t^i \in \mathcal{J}_t^i$, we have that $\iota_{X_t^j} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} = 0$ for all $j \neq i$. Thus,

$$\iota_{X_t} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} = \sum_j \iota_{X_t^j} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} = \iota_{X_t^i} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} = -\frac{\mathrm{d}}{\mathrm{dt}}\omega_t^i\Big|_{\mathcal{D}_t},$$

i.e., $\iota_{X_t} \mathrm{d}\omega_t^i|_{\mathcal{D}_t} + (\mathrm{d}/\mathrm{dt})\omega_t^i|_{\mathcal{D}_t} = 0$ for every $i = 1, \ldots, k$. □

To complete the proof of Theorem 4.2 we have to obtain a global time dependent vector field X_t satisfying the hypothesis of Proposition 4.3. First, suppose that we have a (locally) finite open cover $\{U^\lambda\}$ of M and local fields $X_t^\lambda \in \mathcal{L}|_{U_\lambda}$ on U_λ, which satisfy the relations

$$\iota_{X_t^\lambda} \mathrm{d}\omega_t^{i,\lambda}|_{\mathcal{D}_t} + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^{i,\lambda}\Big|_{\mathcal{D}_t} = 0 \quad \forall\, i = 1, \dots, k$$

on U_λ, where $\mathcal{D}_t|_{U_\lambda} = \{\omega_t^{i,\lambda} = 0 = \theta^\lambda\}$ and $\mathcal{E}|_{U_\lambda} = \ker \theta^\lambda$. Consider a partition of unity $\{\varrho_\lambda\}$ subordinate to the covering. Set

$$X_t = \sum_\lambda \varrho_\lambda X_t^\lambda.$$

Then X_t is a global field and $X_t \in \mathcal{L}$ since every $X_t^\lambda \in \mathcal{L}$. By Proposition 4.4, the local field X_t^λ satisfies

$$\iota_{X_t^\lambda} \mathrm{d}\omega_t^{i,\mu}|_{\mathcal{D}_t} + \frac{\mathrm{d}}{\mathrm{dt}}\omega_t^{i,\mu}\Big|_{\mathcal{D}_t} = 0 \quad \forall\, i = 1, \dots, k$$

on $U_\lambda \cap U_\mu$, whenever the set is nonempty. Then the global field X_t satisfies the hypothesis of Proposition 4.3 over each U_λ as X_t is a convex linear combination of the local fields.

4.2. Proof of Theorem 1.1. The proof of Theorem 1.1, as we will see, follows from Theorem 4.2 by using the following lemma, proof of which can be found in [9].

Lemma 4.9 ([9])**.** *Suppose we are given a one-parameter family of co-rank 1 distributions $\mathcal{E}_t$ on a closed manifold M such that the Cauchy characteristic distribution $\mathcal{L}_t$ of $\mathcal{E}_t$ is independent of t, say $\mathcal{L}_t = \mathcal{L}$ and $\mathcal{E}_t^2 = TM$. Then there exists an isotopy φ_t of M such that*

$$\varphi_{t*}\mathcal{E}_0 = \mathcal{E}_t, \quad \varphi_{t*}\mathcal{L} = \mathcal{L}.$$

P r o o f of Theorem 1.1. Since $\mathcal{D}_t^3 = TM$, in particular we have that $\mathcal{E}_t^2 = TM$. Hence, using Lemma 4.9 we get an isotopy φ_t that fixes $\mathcal{L}$ and $\varphi_{t*}\mathcal{E}_0 = \mathcal{E}_t$. Since φ_t is a diffeomorphism, we get $\varphi_{t*}^{-1}\mathcal{E}_t = \mathcal{E}_0$. Set $\mathcal{D}'_t = \varphi_{t*}^{-1}\mathcal{D}_t$. Clearly we have the flag, $\mathcal{L} \subset \mathcal{D}'_t \subset \mathcal{E}_0$, where $\mathcal{L}$ is the Cauchy characteristic distribution of $\mathcal{E}_0$. Since the Lie brackets are preserved under push-forwards by diffeomorphisms, we have that $\mathcal{D}_t'^2 = \mathcal{E}_0$ and $\mathcal{D}_t'^3 = TM$.

Now using Theorem 4.2, we get another isotopy ψ_t that fixes both $\mathcal{L}$ and $\mathcal{E}_0$, and $\psi_{t*}\mathcal{D}_0 = \mathcal{D}'_t$. Setting $\Phi_t = \varphi_t \circ \psi_t$, we get the desired isotopy since

$$\Phi_{t*}\mathcal{D}_0 = \mathcal{D}_t, \quad \Phi_{t*}\mathcal{L} = \mathcal{L}.$$

□

5. Normal forms

In this section we obtain the normal form of generators of the Pfaffian system defining a generalized Engel structure $\mathcal{D}$. Suppose that $\mathcal{D}$ is of co-rank $k+1$, where $k = 2l+1$, and $\mathcal{L} \subset \mathcal{D} \subset \mathcal{E} \subset TM$ is associated with the canonical flag on M. Suppose $\mathcal{S}_0$ and $\mathcal{S}_1$ are the Pfaffian systems annihilating $\mathcal{D}$ and $\mathcal{E}$, respectively. Since $\mathcal{D} \subset \mathcal{E}$, we have that $\mathcal{S}_0 \supset \mathcal{S}_1$. Suppose $\mathcal{S}_1 = \langle \theta \rangle$ and $\mathcal{S}_0 = \langle \theta, \omega^1, \dots, \omega^k \rangle$ locally. Then by Proposition 3.3 we get certain relations among these forms. We want to get standard normal forms for some bases of $\mathcal{S}_0$ and $\mathcal{S}_1$.

Since $\theta \wedge \mathrm{d}\theta^{l+1} \neq 0$ and $\theta \wedge \mathrm{d}\theta^{l+2} = 0$, around a point $p \in M$ we have some coordinate system (see [3], Theorem 3.1) such that $\mathcal{E}$ is the kernel of

$$\Theta = \mathrm{d}z - \sum_{i=1}^{l+1} x_{i+l+1} \mathrm{d}x_i.$$

Clearly $\mathcal{S}_1 = \langle \Theta \rangle$ and $\{\Theta, \omega^i\}$ is a Pfaffian system associated to the given generalized Engel structure. From $\omega^i \wedge \Theta \wedge \mathrm{d}\Theta^{l+1} = 0$ for all i, we have that $0 = \omega^i \wedge \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{2l+2} \wedge \mathrm{d}z$. Hence, there exist functions a^{ij}, b^i such that

$$\omega^i = \sum_{j=1}^{2l+2} a^{ij}\, \mathrm{d}x_j + b^i\, \mathrm{d}z.$$

Now we have, in particular, $\omega^1 \wedge \dots \wedge \omega^k \wedge \Theta \neq 0$. Therefore, the matrix

$$\begin{pmatrix} a^{11} & \dots & a^{2l+1,1} & -x_{l+2} \\ \vdots & \vdots & \vdots & \vdots \\ a^{1,l+1} & \dots & a^{2l+1,l+1} & -x_{2l+2} \\ a^{1,l+2} & \dots & a^{2l+1,l+2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a^{1,2l+2} & \dots & a^{2l+1,2l+2} & 0 \\ b^1 & \dots & b^{2l+1} & 1 \end{pmatrix}_{(2l+3)\times(2l+2)}$$

has the full rank $2l+2$ everywhere. Evaluating this at the point p, we observe that the last column is $\begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^t$ and hence the last row cannot be linearly dependent on the rest. Hence, without loss of generality we may assume that the $(2l+2)$th row is linearly dependent and the rest of rows are linearly independent about p. Thus we can transform the matrix into

$$\begin{pmatrix} I_{(2l+1)\times(2l+1)} & 0 \\ C & 0 \\ 0 & 1 \end{pmatrix},$$

where $C = (\, c_1 \quad \ldots \quad c_{2l+1}\,)$ is a row vector of functions. Further we can transform ω^i into

$$\Omega^i = \mathrm{d}x_i + c_i \mathrm{d}x_{2l+2} \quad \forall\, i = 1, \ldots, k = 2l+1.$$

Clearly, $\{\Theta, \Omega^i\}$ still forms a basis of $\mathcal{S}_0$. Hence we have $\mu^i = \Omega^1 \wedge \ldots \wedge \Omega^k \wedge \Theta \wedge \mathrm{d}\Omega^i \neq 0$ which gives,

$$\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{2l+2} \wedge \mathrm{d}z \wedge \mathrm{d}c_i \neq 0 \quad \forall\, i.$$

But $\{\mu^i\}$ are also point-wise linearly independent. Hence we have that $\{\mathrm{d}c_i\}$ are point-wise linearly independent as well, i.e., $\{c_i\}$ are coordinate functions around p. Set $y_i = x_{i+l+1}$. Thus we have $2k+2$ coordinate functions $(x_1, \ldots, x_{l+1}, y_1, \ldots, y_{l+1}, z, c_1, \ldots, c_k)$ around the point p. Say, $r = \dim M - (2k+2)$. Clearly, $r \geqslant 0$. Then we obtain a coordinate system

$$(x_1, \ldots, x_{l+1}, y_1, \ldots, y_{l+1}, z, c_1, \ldots, c_k, q_1, \ldots, q_r)$$

around p, where the coordinates $q_1, \ldots, q_r$ are chosen arbitrarily. In this system, the 1-forms $\{\Theta, \Omega^i\}$ can be expressed as

$$\Theta = \mathrm{d}z - \sum_{i=1}^{l+1} y_i \, \mathrm{d}x_i,$$
$$\Omega^i = \begin{cases} \mathrm{d}x_i + c_i \, \mathrm{d}y_{l+1}, & 1 \leqslant i \leqslant l+1, \\ \mathrm{d}y_{i-l-1} + c_i \, \mathrm{d}y_{l+1}, & l+2 \leqslant i \leqslant k = 2l+1, \end{cases}$$

where r is such that $\dim M = r + 2k + 2$, $\mathcal{E} = \ker \Theta$ and $\mathcal{D} = \{\Omega^1 = \ldots \Omega^k = 0 = \Theta\}$.

Note that for $\dim M = 4$, taking $l = 0$ in the above normal form, we obtain the normal form of the Pfaffian system defining an Engel structure on M, namely

$$\{\mathrm{d}z - y_1 \, \mathrm{d}x_1,\ \mathrm{d}x_1 + c_1 \, \mathrm{d}y_1\},$$

i.e., $\{\mathrm{d}z - y \, \mathrm{d}x,\ \mathrm{d}x - w \, \mathrm{d}y\}$ after relabeling $x = x_1$, $y = y_1$, $w = -c_1$. Under the coordinate change $(x, y, z, w) \mapsto (y, -x, z - xy, -w)$ the pair transforms into the standard Engel pair $\{\mathrm{d}z - y \, \mathrm{d}x,\ \mathrm{d}y - w \, \mathrm{d}x\}$.

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