Mohamed Mourad Lhannafi Ait Yahia; Hamid Haddadou

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A GENERAL HOMOGENIZATION RESULT OF SPECTRAL PROBLEM FOR LINEARIZED ELASTICITY IN PERFORATED DOMAINS

Mohamed Mourad Lhannafi Ait Yahia, Hamid Haddadou, Algiers

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Abstract. The goal of this paper is to establish a general homogenization result for linearized elasticity of an eigenvalue problem defined over perforated domains, beyond the periodic setting, within the framework of the H^0-convergence theory. Our main homogenization result states that the knowledge of the fourth-order tensor A^0, the H^0-limit of A^ε, is sufficient to determine the homogenized eigenvalue problem and preserve the structure of the spectrum. This theorem is proved essentially by using Tartar's method of test functions, and some general arguments of spectral analysis used in the literature on the homogenization of eigenvalue problems. Moreover, we give a result on a particular case of a simple eigenvalue of the homogenized problem. We conclude our work by some comments and perspectives.

Keywords: homogenization; H-convergence; perforated domain; linear elasticity; eigenvalue problem

MSC 2020: 35B40, 35B27, 74B05, 47A75

1. Introduction

Homogenization of spectral problems has been of great importance in various branches of engineering sciences like material science, porous media, structural optimization and so on. It was extensively explored in the literature, especially for the periodic setting, because of its numerous industrial applications.

Periodic homogenization of the second-order elliptic eigenvalue problem with homogeneous Dirichlet conditions on a fixed domain goes back to Kesaven [14]. The case of a periodically perforated domain was first studied by Vanninathan [23] dealing with some eigenvalue problems for the Laplace operator with different boundary conditions on holes. Briane et al. [2] have generalized this result to the case of ad-

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missible holes with Neumann condition. During the last twenty years, considerable progress has been achieved in homogenization techniques, namely the multi-scale convergence method and periodic unfolding method, which were initially developed for the classical periodic setting, and then generalized to variant cases. This allowed the appearance of numerous results for the elliptic scalar case dealing with the homogenization of spectral problems (with different boundary conditions, such as Robin conditions etc.) defined on periodic, quasi-periodic or locally periodic structures, let us cite for example [1], [3], [6], [20].

For the case of linearized elasticity, several works exist in literature. In fixed domains, homogenization of spectral problems has been developed using the G-strong convergence method in general context. In perforated domains, spectral problems have been considered in the classical periodic setting. For more details see [13], [17], [18]. To the best of our knowledge, all works which have studied homogenization of the spectral problem for linearized elasticity with Neumann condition on holes have been considered only for the classical periodic perforated domains and for rapidly oscillating periodic coefficients.

The purpose of our work is to establish a homogenization result of the eigenvalue problem for linearized elasticity defined over perforated domains, beyond periodic setting, within the framework of the H^0-convergence. More precisely, we are interested in the homogenization of spectral problems for linearized elasticity in perforated domains, not necessary periodic, with Neumann conditions on holes. The classical periodic hypothesis on holes is replaced by a more general one, namely the e-admissibility hypothesis. This hypothesis assumes the existence of some extension operator P_ε from the perforated domain to the entire domain. In addition, the tensor A^ε associated to our spectral problem is only assumed to be H^0-convergent to some fourth-order tensors A^0.

The paper is organized as follows: In Section 2, we provide the principal definitions and results of the H_e^0-convergence given in [8] to be used in the proof of our main results. In Section 3 we set our spectral problem given by

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} -\operatorname{div}(A^\varepsilon e(u^\varepsilon)) = \lambda^\varepsilon u^\varepsilon & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(u^\varepsilon))\vartheta = 0 & \text{on } \partial S_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ is the perforated domain, with S_ε a sequence of compact subsets of Ω e-admissible in the sense of [8], and A^ε is a sequence of fourth-order tensors of $M_e(\alpha, \beta, \Omega)$. We assume that for the whole sequence (ε) the pair $(A^\varepsilon, S_\varepsilon)$ H_e^0-converges to $A^0 \in M_e(\alpha/C^2, \beta^2/\alpha, \Omega)$ and the sequence χ_ε, of the characteristic function of Ω_ε, converges weakly $\star$ in $L^\infty(\Omega)$ to a function χ_0 such that $\chi_0 \geqslant \delta > 0$ almost everywhere in Ω, with δ a positive constant.

Section 4 is devoted to establishing two preliminary results. First, we give necessary bounds on the eigenvalues (Proposition 4.1) by using the minimax principles [4], [14], [17], [23]. Second, we describe (Lemma 4.1) the behavior of the sequences λ_s^ε and the associated normalized eigenfunctions u_s^ε, for a subsequence of (ε), using the standard diagonal process.

In Section 5, we give our main results. The first one (Theorem 5.1) brings out the relationship between the limits found in Lemma 4.1 and the problem

$$(\mathcal{P}_0) \qquad \begin{cases} -\operatorname{div}(A^0 e(u)) = \lambda \chi_0 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More precisely, it states that every limit point of the sequence $(\lambda_s^\varepsilon, u_s^\varepsilon)_\varepsilon$, for each integer $s \geqslant 1$, is necessarily a pair of an eigenvalue and an associated normalized eigenfunction of $(\mathcal{P}_0)$. The proof is based essentially on Tartar's method of test functions [21] and on the different properties of the H_e^0-convergence. In the second result (Theorem 5.2), the process of homogenization is finally completed. It establishes that $(\mathcal{P}_0)$ represents the homogenized eigenvalue problem by showing that every eigenvalue λ_s of $(\mathcal{P}_0)$ is a limit of the sequence of eigenvalues λ_s^ε of $(\mathcal{P}_\varepsilon)$, for each integer $s \geqslant 1$, and (up to subsequence)

$$\begin{cases} P_\varepsilon u_s^\varepsilon \rightharpoonup u_s & \text{in } H_0^1(\Omega) \text{ weakly}, \\ \widetilde{u_s^\varepsilon} \rightharpoonup \chi_0 u_s & \text{in } L^2(\Omega) \text{ weakly}, \end{cases}$$

where u_s^ε, u_s are normalized eigenfunctions associated to λ_s^ε and λ_s, respectively; and $\widetilde{\cdot}$ denotes the extension by 0 outside Ω_ε. The third result (Theorem 5.3) deals with a particular case of a simple eigenvalue of the homogenized problem $(\mathcal{P}_0)$. Section 6 closes our work, gives some comments and perspectives.

Throughout the paper we use the convention on the summation over repeated indices and the following notations:

- ▷ (ε) denotes a decreasing sequence converging to zero.
- ▷ $\chi_{\mathcal{O}}$ denotes the characteristic function of a subset $\mathcal{O}$ of $\mathbb{R}^n$.
- ▷ $\tilde{f}$ denotes the extension by 0 outside $\mathcal{O}$ of a function f defined on $\mathcal{O}$.
- ▷ If $\zeta = (\zeta_{ij})_{1 \leqslant i,j \leqslant n}$ and $\xi = (\xi_{ij})_{1 \leqslant i,j \leqslant n}$ are two square matrices, we set

$$\zeta \cdot \xi = \sum_{i,j=1}^n \zeta_{ij} \xi_{ij} \quad \text{and} \quad |\xi| = \left(\sum_{i,j=1}^n \xi_{ij}^2 \right)^{1/2}.$$

▷ If $v=(v_1,\dots,v_n)$ is a vector-valued function and $\zeta=(\zeta_{ij})_{1\leqslant i,j\leqslant n}$ is a second-order tensor of the variable $x=(x_1,\dots,x_n)$, we set

$$\begin{cases} (\nabla v)_{ij}=\dfrac{\partial v_i}{\partial x_j},\\ e(v)=\dfrac{1}{2}(\nabla v+(\nabla v)^\top),\\ (\operatorname{div}\xi)_i=\dfrac{\partial \xi_{ij}}{\partial x_j}.\end{cases}$$

▷ For any $\alpha,\beta\in\mathbb{R}$ such that $0<\alpha<\beta$, and an open set $\mathcal{O}$, we denote by $M_e(\alpha,\beta,\mathcal{O})$ the set of fourth-order tensors $A=(A_{ijkl})_{1\leqslant i,j,k,l\leqslant n}$ defined on $\mathcal{O}$ such that a.e. on $\mathcal{O}$, we have

$$\begin{aligned} &\text{(i)} && A_{ijkl}\in L^\infty(\mathcal{O}) \text{ for any } i,j,l,k=1,\dots,n,\\ &\text{(ii)} && A_{ijkl}=A_{jikl}=A_{klij} \text{ for any } i,j,l,k=1,\dots,n,\\ &\text{(iii)} && \alpha|\eta|^2\leqslant A\eta\eta \text{ for any symmetric matrix } \eta,\\ &\text{(iv)} && |A\eta|\leqslant\beta|\eta| \text{ for any matrix } \eta.\end{aligned}\tag{1.1}$$

▷ If $A=(A_{ijkl})_{1\leqslant i,j,k,l\leqslant n}$ is a fourth-order tensor and $\Upsilon=(\Upsilon_{ij})_{1\leqslant i,j\leqslant n}$ is a square matrix, we have

$$A\Upsilon=((A\Upsilon)_{ij})_{1\leqslant i,j\leqslant n}=(A_{ijkl}\Upsilon_{kl})_{1\leqslant i,j\leqslant n}.$$

▷ For any $i,j=1,2,\dots,n$, $\delta_{ij}=1$ if $i=j$ and 0 otherwise, represents the Kronecker symbol.

2. Preliminary results on the H_e^0-convergence

As mentioned above, the limit problem is obtained within the framework of the H^0-convergence theory for linearized elasticity. The notion of H_e^0-convergence was introduced by El Hajji and Donato [8], it extends the H^0-convergence given by Briane, Damlamian and Donato [2] to linearized elasticity. The paper [2] generalizes the G-convergence introduced by Spagnolo [19] and the H-convergence given by Murat and Tartar [16] to the case of a perforated domain, not necessarily periodic. Furthermore, the H-convergence was extended to linearized elasticity by Francfort and Murat [9], and denoted in [12] by H_e-convergence.

In this section we recall the definition of the H_e^0-convergence as well as its main properties used in the proof of our main results.

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geqslant 2$, with boundary $\partial\Omega$. We consider the perforated domain Ω_ε defined by

$$\Omega_\varepsilon = \Omega \setminus S_\varepsilon,$$

where S_ε is a sequence of compact subsets of Ω. We denote by χ_ε the characteristic function $\chi_{\Omega_\varepsilon}$, by ϑ the outward normal unit vector on the boundary of Ω_ε.

Consider the Hilbert space

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon)^n ;\ v_{|\partial\Omega} = 0\}$$

equipped with the H^1-norm.

In what follows, we assume that S_ε satisfies the definition of e-admissibility given in [8].

Definition 2.1 ([8]). The set S_ε is said to be admissible in Ω for linearized elasticity, or e-admissible in Ω, if and only if:

- ▷ Every $L^\infty(\Omega)$ weak $\star$ limit point of χ_ε is positive almost everywhere in Ω.
- ▷ There exists a positive real constant C and for each ε an extension operator P_ε from V_ε to $H_0^1(\Omega)^n$ such that

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)^n), \\ (P_\varepsilon v)_{|\Omega_\varepsilon} = v \quad \forall\, v \in V_\varepsilon, \\ \|e(P_\varepsilon v)\|_{L^2(\Omega)^{n\times n}} \leqslant C\|e(v)\|_{L^2(\Omega_\varepsilon)^{n\times n}} \quad \forall\, v \in V_\varepsilon. \end{cases} \tag{2.1}$$

R e m a r k 2.1. Conditions (2.1) of e-admissibility give an information on the regularity of the holes S_ε and the way in which they approach the boundary $\partial\Omega$. As an example of e-admissible holes, we cite the case of perforated domain with holes of size ε or δ_ε (where $0 < \delta_\varepsilon << \varepsilon$) treated in [15], [10] respectively. We can cite also the case of a perforated domain with double periodicity [7] and the case of finitely many periodic scales introduced in [11].

We denote by P_ε^* the adjoint operator of P_ε defined from $H^{-1}(\Omega)^n$ to V_ε', the dual of V_ε, given by

$$P_\varepsilon^* \colon\ f \in H^{-1}(\Omega)^n \to P_\varepsilon^* f \in V_\varepsilon'$$

with

$$\langle P_\varepsilon^* f, v\rangle_{V_\varepsilon', V_\varepsilon} = \langle f, P_\varepsilon v\rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n}.$$

R e m a r k 2.2 ([8]). Korn's inequality remains valid in V_ε with a constant independent of ε, i.e., there exists a positive constant $\mathcal{C}$ independent of ε such that

$$\|v\|_{H^1(\Omega_\varepsilon)^n} \leqslant \mathcal{C}\|e(v)\|_{L^2(\Omega_\varepsilon)^{n\times n}} \quad \forall\, v \in V_{\varepsilon*}.$$

Then, the quantity

$$\|e(\cdot)\|^2_{L^2(\Omega_\varepsilon)^{n\times n}} \doteq \sum_{i,j=1}^{n} \int_{\Omega_\varepsilon} |e_{ij}(\cdot)|^2 \,\mathrm{d}x$$

defines a norm on V_ε equivalent to the natural one of $H^1(\Omega_\varepsilon)^n$ with constants independent of ε.

Remark 2.3. The definition of V_ε ensures that

$$\begin{cases} V_\varepsilon \subset L^2(\Omega_\varepsilon)^n \text{ with compact injection,} \\ V_\varepsilon \text{ is dense in } L^2(\Omega_\varepsilon)^n. \end{cases}$$

Now, we give the definition of the H-convergence for linearized elasticity in the case of a perforated domain and its principal properties.

Definition 2.2 ([8], [12]). Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ and S_ε be a sequence of compact subsets e-admissible in Ω. We say that the pair $(A^\varepsilon, S_\varepsilon)$ *H^0_e-converges to* $A^0 \in M_e(\alpha',\beta',\Omega)$ and we write $(A^\varepsilon, S_\varepsilon) \overset{H^0_e}{\rightharpoonup} A^0$, if for each function f of $H^{-1}(\Omega)^n$ the solution u^ε of

$$\begin{cases} -\mathrm{div}(A^\varepsilon e(u^\varepsilon)) = P^*_\varepsilon f & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(u^\varepsilon))\vartheta = 0 & \text{on } \partial S_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies the weak convergence

$$\begin{cases} P_\varepsilon u^\varepsilon \rightharpoonup u & \text{weakly in } H^1_0(\Omega)^n, \\ \widetilde{A^\varepsilon e(u^\varepsilon)} \rightharpoonup A^0 e(u) & \text{weakly in } L^2(\Omega)^{n\times n}, \end{cases}$$

where u is the unique solution of the problem

$$\begin{cases} -\mathrm{div}(A^0 e(u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 2.4. In view of [8], Definition 2.2 can be rewritten as follows: For every function $g \in L^2(\Omega)$ and any sequence (ε) such that

$$\chi_\varepsilon \rightharpoonup \chi_0 \text{ weakly} \star \text{ in } L^\infty(\Omega),$$

the solution v^ε of the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon e(v^\varepsilon)) = g & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(v^\varepsilon))\vartheta = 0 & \text{on } \partial S_\varepsilon, \\ v^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$\begin{cases} P_\varepsilon v^\varepsilon \rightharpoonup v^0 & \text{weakly in } H_0^1(\Omega)^n, \\ \widetilde{A^\varepsilon e(v^\varepsilon)} \rightharpoonup A^0 e(v^0) & \text{weakly in } L^2(\Omega)^{n\times n}, \end{cases}$$

where v^0 is the unique solution of the problem

$$\begin{cases} -\operatorname{div}(A^0 e(v^0)) = \chi_0 g & \text{in } \Omega, \\ v^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 2.5 ([8]). The notion of H_e^0-convergence is independent of the choice of the extension operator P_ε. Furthermore, if $v^\varepsilon \in V_\varepsilon$ such that $P_\varepsilon v^\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)^n$, then for all $\phi \in D(\Omega)$, $P_\varepsilon(\phi_{|\Omega_\varepsilon} v^\varepsilon) \rightharpoonup \phi v$ weakly in $H_0^1(\Omega)^n$.

The notion of H_e^0-convergence makes sense in view of the following result:

Theorem 2.1 (Compactness Theorem [8], [11]). *Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ and S_ε be e-admissible in Ω. Then there exists a subsequence of (ε) (still denoted by (ε)) and a tensor $A^0 \in M_e(\alpha/C^2, \beta^2/\alpha, \Omega)$ such that the sequence $(A^\varepsilon, S_\varepsilon)$ H_e^0-converges to A^0.*

3. Setting of the problem

In the rest of the paper, we consider a bounded open domain Ω of $\mathbb{R}^n$, the perforated domain $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ with e-admissible holes S_ε and a sequence A^ε of fourth-order tensors of $M_e(\alpha,\beta,\Omega)$.

For the whole sequence (ε), the following assumptions are made on A^ε and χ_ε:

(H1) The pair $(A^\varepsilon, S_\varepsilon)$ H_e^0-converges to $A^0 \in M_e(\alpha/C^2, \beta^2/\alpha, \Omega)$.

(H2) The sequence χ_ε converges weakly $\star$ in $L^\infty(\Omega)$ to a function χ_0 satisfying the condition that

$$\text{there exists a positive constant } \delta \text{ such that } \chi_0 \geqslant \delta > 0 \text{ a.e. in } \Omega.$$

Assumptions (H1) and (H2) are reasonable hypotheses. Assumption (H1) draws its validity from the compactness theorem (Theorem 2.1), while assumption (H2) is verified in more general situations, see [22].

Now, we are ready to present our problem. Consider the eigenvalue problem of finding a real value λ^ε and a vector-valued function u^ε such that

$$\begin{cases} -\operatorname{div}(A^\varepsilon e(u^\varepsilon)) = \lambda^\varepsilon u^\varepsilon & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(u^\varepsilon))\vartheta = 0 & \text{on } \partial S_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Its weak formulation reads:

Find $\lambda^\varepsilon \in \mathbb{R}$ (eigenvalues) and $u^\varepsilon \in V_\varepsilon - \{0\}$ (eigenfunctions) such that

$$a_\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon (u^\varepsilon, v)_\varepsilon \quad \text{for any } v \in V_\varepsilon,$$

where the bilinear form a_ε is

$$\forall\, u, v \in V_\varepsilon;\ a_\varepsilon(u, v) = \int_{\Omega_\varepsilon} A^\varepsilon e(u) \cdot e(v)\,\mathrm{d}x,$$

and $(u, v)_\varepsilon = \int_{\Omega_\varepsilon} uv\,\mathrm{d}x$ denotes the inner product of $L^2(\Omega_\varepsilon)^n$.

It is easy to see from (1.1) and Remark 2.2 that the bilinear form a_ε is symmetric and coercive. According to a well-known result in spectral theory, Remark 2.3 ensures the existence of a sequence of eigenvalues $\{\lambda_s^\varepsilon\}_{s=1}^\infty$ and a sequence of normalized eigenfunctions $\{u_s^\varepsilon\}_{s=1}^\infty$ satisfying problem (3.1), such that

$$\text{(3.2)} \qquad \begin{cases} 0 < \lambda_1^\varepsilon \leqslant \lambda_2^\varepsilon \leqslant \lambda_3^\varepsilon \leqslant \ldots \leqslant \lambda_s^\varepsilon \to \infty, \\ \lambda_s^\varepsilon \text{ is of finite multiplicity for each } s, \\ \text{and } \{u_s^\varepsilon\}_{s=1}^\infty \subset V_\varepsilon \text{ forms an orthogonal basis in } L^2(\Omega_\varepsilon)^n \\ \qquad\qquad\qquad\qquad \text{equipped with its natural norm.} \end{cases}$$

We characterize each eigenvalue λ_s^ε with the help of the Rayleigh quotient

$$\forall\, v \in V_\varepsilon\colon\ v \neq 0;\ \mathcal{R}_\varepsilon(v) = \frac{a_\varepsilon(v, v)}{(v, v)_\varepsilon}.$$

Then, the minimax principle states that

$$\text{(3.3)} \qquad \lambda_s^\varepsilon = \min_{\substack{\mathcal{W}_\varepsilon^s \subset V_\varepsilon \\ \dim \mathcal{W}_\varepsilon^s = s}} \max_{v \in \mathcal{W}_\varepsilon^s} \mathcal{R}_\varepsilon(v) = \max_{v \in E_\varepsilon(s)} \mathcal{R}_\varepsilon(v) = \max_{\substack{v \in V_\varepsilon \\ v \perp E_\varepsilon(s-1)}} \mathcal{R}_\varepsilon(v),$$

where $E_\varepsilon(s)$ is the subspace of V_ε spanned by $\{u_1^\varepsilon, u_2^\varepsilon, \ldots, u_s^\varepsilon\}$.

4. Estimates on eigenvalues and convergence

In this section, we give two primordial preliminary results to the homogenization process. For this, let us denote by $L^2(\Omega, \chi_0)^n$ the space $L^2(\Omega)^n$ equipped with the inner product

$$(u, v)_0 = \int_\Omega \chi_0 uv\,\mathrm{d}x \quad \text{for any } u, v \text{ in } L^2(\Omega)^n,$$

where the corresponding norm

$$\|u\|_0 = \left(\int_\Omega \chi_0 u^2 \, \mathrm{d}x \right)^{1/2} \quad \text{for any } u \text{ in } L^2(\Omega)^n,$$

is equivalent to the natural one.

In order to present the first result, which provides necessary bounds on the eigenvalues λ_s^ε, we introduce the following intermediate eigenvalue problem for the Lamé tensor L,

$$\begin{cases} -\mathrm{div}(Le(\tau)) = -\mathrm{div}(e(\tau)) = \chi_0 \mu \tau & \text{in } \Omega, \\ \tau = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where L is the fourth-order tensor defined by $L_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ for all $i, j, k, l \in \{1, 2, \ldots, n\}$.

The weak formulation of problem (4.1) is: Find $\mu \in \mathbb{R}$ (eigenvalues) and $\tau \in H_0^1(\Omega)^n - \{0\}$ (eigenfunctions) such that

$$b_0(\tau, v) = \mu(\tau, v)_0 \quad \text{for any } v \in H_0^1(\Omega)^n,$$

where the bilinear form b_0 is

$$\forall\, u, v \in H_0^1(\Omega)^n;\; b_0(u, v) = \int_\Omega e(u) \cdot e(v) \, \mathrm{d}x.$$

Since $L \in M_e(1, 2, \Omega)$, problem (4.1) has an increasing sequence of eigenvalues $\{\mu_s\}_{s=1}^\infty$ of finite multiplicity such that $\mu_s \to \infty$, as $s \to \infty$, and a sequence of normalized eigenfunctions $\{\tau_s\}_{s=1}^\infty \subset H_0^1(\Omega)^n$ forming an orthogonal basis in $L^2(\Omega, \chi_0)^n$. These eigenvalues μ_s can be characterized by

$$\mu_s = \min_{\substack{\mathcal{W}^s \subset H_0^1(\Omega)^n \\ \dim \mathcal{W}^s = s}} \max_{v \in \mathcal{W}^s} \mathcal{R}(v) = \max_{v \in E(s)} \mathcal{R}(v) = \max_{\substack{v \in H_0^1(\Omega)^n \\ v \perp E(s-1)}} \mathcal{R}(v), \tag{4.2}$$

where $E(s)$ is the subspace of $H_0^1(\Omega)^n$ spanned by $\{\tau_1, \tau_2, \ldots, \tau_s\}$ and for any $v \in H_0^1(\Omega)^n$, $v \neq 0$, $\mathcal{R}(v) = b_0(v, v)/(v, v)_0$.

The following proposition gives an estimate of eigenvalues λ_s^ε by eigenvalues μ_s of the spectral problem (4.1), which are independent of ε. The proof is essentially based on the comparison of the eigenvalues λ_s^ε and μ_s using the minimax principles.

Proposition 4.1. *For each $s \geqslant 1$ there exists a real constant $c > 0$ independent of ε such that for all $\varepsilon > 0$*

$$\frac{\alpha\delta}{C^2} \mu_s \leqslant \lambda_s^\varepsilon \leqslant c\beta\mu_s, \tag{4.3}$$

where α, β, C and δ are given in assumptions (H1) *and* (H2).

P r o o f. The proof is based on a general argument of spectral analysis, see [4], [14], [17], [23] involving problem (4.1). Let us prove first the left-hand side of (4.3), for this let $\tau_1, \tau_2, \ldots, \tau_s$ be the first s normalized eigenfunctions of problem (4.1) and consider the set

$$B_s^\varepsilon = \{\tau_{1|\Omega_\varepsilon}, \tau_{2|\Omega_\varepsilon}, \ldots, \tau_{s|\Omega_\varepsilon}\} \subset V_\varepsilon.$$

We claim that B_s^ε is an independent family. Indeed, on the contrary we would have sequences of reals $c_1^\varepsilon, c_2^\varepsilon, \ldots, c_s^\varepsilon$ not all equal to zero such that

$$\sum_{i=1}^{s} c_i^\varepsilon \tau_{i|\Omega_\varepsilon} = 0 \quad \text{in } \Omega_\varepsilon,$$

so, for any $j \in \{1, 2, \ldots, s\}$

$$\begin{aligned}\Bigl(\sum_{i=1}^{s} c_i^\varepsilon \tau_i|_{\Omega_\varepsilon}, \tau_j|_{\Omega_\varepsilon}\Bigr)_\varepsilon &= \sum_{i=1}^{s} c_i^\varepsilon (\tau_i|_{\Omega_\varepsilon}, \tau_j|_{\Omega_\varepsilon})_\varepsilon = \sum_{i=1}^{s} c_i^\varepsilon \int_{\Omega_\varepsilon} \tau_i|_{\Omega_\varepsilon} \tau_j|_{\Omega_\varepsilon}\,\mathrm{d}x \\ &= \sum_{i=1}^{s} c_i^\varepsilon \int_\Omega \chi_\varepsilon \tau_i \cdot \tau_j\,\mathrm{d}x = 0.\end{aligned}$$

Without loss of generality, we can assume that for a subsequence of (ε) (still denoted by (ε)) there exists an integer $j_0 \in \{1, 2, \ldots, s\}$ such that $c_{j_0}^\varepsilon = 1$ and $|c_i^\varepsilon| \leqslant 1$ for all $i = 1, 2, \ldots, s$, and real constants $c_1, c_2, \ldots, c_s$ such that $c_i^\varepsilon \to c_i$ as $\varepsilon \to 0$ with $c_{j_0} = 1$. Then, by passing to the limit, as $\varepsilon \to 0$, in the last equation with $j = j_0$ we get

$$\sum_{i=1}^{s} c_i (\tau_i, \tau_{j_0})_0 = 0.$$

Thus $c_{j_0} = 0$, which contradicts our assumption, hence the desired claim is valid.

Now, let us choose in the first equality of (3.3) the subspace of V_ε, denoted by B_s^ε, spanned by the family B_s^ε and by taking into account (1.1), we get

$$\lambda_s^\varepsilon \leqslant \max_{v \in B_s^\varepsilon} \frac{a_\varepsilon(v, v)}{(v, v)_\varepsilon} \leqslant \beta \max_{v \in B_s^\varepsilon} \frac{\|e(v)\|^2_{L^2(\Omega_\varepsilon)^{n\times n}}}{(v, v)_\varepsilon}.$$

From the fact that the subspace B_s^ε represents the restriction to Ω_ε of the elements of $E(s)$ defined in (4.2),

$$\lambda_s^\varepsilon \leqslant \beta \max_{v \in E(s)} \frac{b_0(v, v)}{(v, v)_\varepsilon},$$

so

$$\lambda_s^\varepsilon \leqslant \beta \max_{v \in E(s)} \frac{(v, v)_0}{(v, v)_\varepsilon} \max_{v \in E(s)} \frac{b_0(v, v)}{(v, v)_0}.$$

The characterization (4.2) of eigenvalues μ_s leads to

$$\lambda_s^\varepsilon \leqslant \beta\mu_s \max_{v\in E(s)} \frac{(v,v)_0}{(v,v)_\varepsilon}. \tag{4.4}$$

The right-hand side of inequality (4.4) is bounded above by a constant independent of ε. Otherwise, we would have for a subsequence $\varepsilon_m \to 0$ of ε a sequence of normalized vectors $(v_m)_m$ in $E(s)$, such that

$$\frac{1}{m}\int_\Omega \chi_0 v_m^2 \,\mathrm{d}x > \int_{\Omega_m} v_m^2 \,\mathrm{d}x \tag{4.5}$$

with $\Omega_m = \Omega_{\varepsilon_m}$.

Since $E(s)$ is of finite dimension, we have (up to a subsequence)

$$v_m \to v_0 \quad \text{strongly in } L^2(\Omega)^n \text{ with } \|v_0\|_0 = 1,$$

by passing to the limits, for this subsequence, in (4.5) we get that $\|v_0\|_0 = 0$. This contradiction implies that there exists a constant c independent of ε such that

$$\lambda_s^\varepsilon \leqslant c\beta\mu_s.$$

Now, to prove the left-hand side of (4.3) let $u_1^\varepsilon, u_2^\varepsilon, \dots, u_s^\varepsilon$ be the first s normalized eigenfunctions of problem (3.1) and consider the family

$$D_s^\varepsilon = \{P_\varepsilon u_1^\varepsilon, P_\varepsilon u_2^\varepsilon, \dots, P_\varepsilon u_s^\varepsilon\} \subset H_0^1(\Omega)^n,$$

it is easy to see from properties (2.1) that D_s^ε is an independent family.

Let us choose in (4.2) the subspace of $H_0^1(\Omega)^n$, denoted by D_s^ε, spanned by the family D_s^ε, and from the fact that D_s^ε represents the extension of all functions of $E_\varepsilon(s)$ by P_ε, we have

$$\mu_s \leqslant \max_{v\in D_s^\varepsilon} \frac{b_0(v,v)}{(v,v)_0} = \max_{v\in E_\varepsilon(s)} \frac{b_0(P_\varepsilon v, P_\varepsilon v)}{(P_\varepsilon v, P_\varepsilon v)_0},$$

from properties (2.1) and (1.1) we get

$$\mu_s \leqslant C^2 \max_{v\in E_\varepsilon(s)} \frac{\|e(v)\|^2_{L^2(\Omega_\varepsilon)^{n\times n}}}{(P_\varepsilon v, P_\varepsilon v)_0} \leqslant \frac{C^2}{\alpha} \max_{v\in E_\varepsilon(s)} \frac{a_\varepsilon(v,v)}{(P_\varepsilon v, P_\varepsilon v)_0},$$

so

$$\mu_s \leqslant \frac{C^2}{\alpha} \max_{v\in E_\varepsilon(s)} \frac{(v,v)_\varepsilon}{(P_\varepsilon v, P_\varepsilon v)_0} \max_{v\in E_\varepsilon(s)} \frac{a_\varepsilon(v,v)}{(v,v)_\varepsilon}.$$

The characterization of the eigenvalue λ_s^ε given by (3.3) yields

$$\mu_s \leqslant \frac{C^2}{\alpha}\lambda_s^\varepsilon \max_{v\in E_\varepsilon(s)} \frac{(v,v)_\varepsilon}{(P_\varepsilon v, P_\varepsilon v)_0}.$$

Finally, assumption (H2) leads to

$$\mu_s \leqslant \frac{C^2}{\alpha\delta}\lambda_s^\varepsilon.$$

This completes the proof. □

As a natural consequence of Proposition 4.1, we establish the following result, which describes the behavior of the sequences λ_s^ε and the associated normalized eigenfunctions u_s^ε, for a subsequence of (ε), using essentially the standard diagonal process:

Lemma 4.1. *There exists a subsequence of (ε) (still denoted by (ε)) such that for any integer $s \geqslant 1$ there exists a real $\bar\lambda_s > 0$, a vector-valued function $\bar u_s \in H_0^1(\Omega)^n$ and a symmetric matrix $\bar\xi_s \in L^2(\Omega)^{n\times n}$ satisfying*

$$\begin{cases} \lambda_s^\varepsilon \to \bar\lambda_s, & \\ P_\varepsilon u_s^\varepsilon \rightharpoonup \bar u_s & \text{weakly in } H_0^1(\Omega)^n, \\ \widetilde{\xi_s^\varepsilon} \rightharpoonup \bar\xi_s & \text{weakly in } L^2(\Omega)^{n\times n}, \end{cases} \tag{4.6}$$

where ξ_s^ε is the symmetric matrix defined by $\xi_s^\varepsilon = A^\varepsilon e(u_s^\varepsilon)$.

Furthermore,

$$0 < \bar\lambda_1 \leqslant \bar\lambda_2 \leqslant \bar\lambda_3 \leqslant \ldots \leqslant \bar\lambda_s \leqslant \ldots \to \infty, \tag{4.7}$$

$$(\bar u_s, \bar u_p)_0 = \delta_{sp}. \tag{4.8}$$

Proof. For any integer $s \geqslant 1$, by taking u_s^ε as a test function in the weak formulation of problem (3.1), since the normalized eigenfunctions satisfy $\|u_s^\varepsilon\|_{L^2(\Omega_\varepsilon)^n} = 1$, we have

$$\int_{\Omega_\varepsilon} A^\varepsilon e(u_s^\varepsilon) e(u_s^\varepsilon)\,\mathrm{d}x = \lambda_s^\varepsilon.$$

Due to properties (1.1) of the set $M_e(\alpha,\beta,\Omega)$, Proposition 4.1 and properties (2.1) of the extension operator P_ε, it easy to see

$$\|e(P_\varepsilon u_s^\varepsilon)\|^2_{L^2(\Omega)^{n\times n}} < cC^2\Big(\frac{\beta}{\alpha}\Big)\mu_s. \tag{4.9}$$

Moreover, by considering the symmetric matrix $\xi^\varepsilon = A^\varepsilon e(u^\varepsilon)$, properties (1.1) and inequality (4.9) lead to

$$\|\widetilde{\xi_s^\varepsilon}\|^2_{L^2(\Omega)^{n\times n}} < cC^2\Big(\frac{\beta^3}{\alpha}\Big)\mu_s. \tag{4.10}$$

Consequently, inequalities (4.3), (4.9), (4.10) combined with the standard diagonal process allow us to deduce that a subsequence of (ε), still denoted by (ε), can be found such that for any integer $s \geqslant 1$ there exists a real $\bar\lambda_s$, a vector-valued function $\bar u_s \in H_0^1(\Omega)^n$ and a symmetric matrix $\bar\xi_s \in L^2(\Omega)^{n\times n}$ satisfying (4.6).

Furthermore, by passing to the limit for this subsequence in (4.3) and (3.2), we get

$$0 < \bar\lambda_1 \leqslant \bar\lambda_2 \leqslant \bar\lambda_3 \leqslant \ldots \leqslant \bar\lambda_s \leqslant \ldots,$$

and due to $\tau_s \to \infty$, by passing to the limit in (4.3) twice, firstly as $\varepsilon \to 0$, secondly as $s \to \infty$, we deduce that $\bar\lambda_s \to \infty$ as $s \to \infty$.

On the other hand, for any two positive integers $s, p \geqslant 1$ we have

$$\delta_{sp} = (u_s^\varepsilon, u_p^\varepsilon)_\varepsilon = \int_{\Omega_\varepsilon} u_s^\varepsilon u_p^\varepsilon \,\mathrm{d}x = \int_\Omega \widetilde{u_s^\varepsilon}\widetilde{u_p^\varepsilon}\,\mathrm{d}x = \int_\Omega \chi_\varepsilon (P_\varepsilon u_s^\varepsilon)(P_\varepsilon u_p^\varepsilon)\,\mathrm{d}x,$$

then, according to (4.6) and (H2), we get

$$\lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon (P_\varepsilon u_s^\varepsilon)(P_\varepsilon u_p^\varepsilon)\,\mathrm{d}x = \int_\Omega \chi_0 \bar u_s \bar u_p \,\mathrm{d}x = (\bar u_s, \bar u_p)_0 = \delta_{sp}.$$

$\square$

5. Main homogenization theorems

The aim of this section is to show that the homogenized spectral problem associated to problem (3.1) is given by

$$\begin{cases} -\mathrm{div}(A^0 e(u)) = \lambda\chi_0 u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

This problem is similar to problem (4.1) with the fourth-order tensor A^0 instead of L. Since $A^0 \in M_e(\alpha/C^2, \beta^2/\alpha, \Omega)$, problem (5.1) admits a sequence of eigenvalues $\{\lambda_s\}_{s=1}^\infty$ and a sequence of normalized eigenfunctions $\{u_s\}_{s=1}^\infty \subset H_0^1(\Omega)^n$, such that

$$\begin{cases} 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \to \infty,\\ \lambda_s \text{ is of finite multiplicity for each } s,\\ \{u_s\}_{s=1}^\infty \text{ forms an orthogonal basis for } L^2(\Omega,\chi_0)^n. \end{cases}$$

To do this, we start by stating and proving an important result to bring out the relationship between the limits found in Lemma 4.1 and problem (5.1).

Theorem 5.1. *For any integer $s \geqslant 1$, let λ_s^ε be an eigenvalue and u_s^ε an associated normalized eigenfunction of problem* (3.1). *Then, under assumptions* (H1) *and* (H2), *the limits $\bar{\lambda}_s$ and $\bar{u}_s$ given in Lemma* 4.1 *represent, respectively, an eigenvalue and an associated normalized eigenfunction of problem* (5.1).

Proof. Let us consider any $w \in H_0^1(\Omega)^n$. By choosing $w_{|\Omega_\varepsilon}$ as a test function in the weak formulation of problem (3.1), we get for any integer $s \geqslant 1$

$$\int_{\Omega_\varepsilon} \xi_s^\varepsilon \cdot e(w_{|\Omega_\varepsilon})\,\mathrm{d}x = \int_{\Omega_\varepsilon} A^\varepsilon e(u_s^\varepsilon) \cdot e(w_{|\Omega_\varepsilon})\,\mathrm{d}x = \lambda_s^\varepsilon \int_{\Omega_\varepsilon} u_s^\varepsilon w_{|\Omega_\varepsilon}\,\mathrm{d}x,$$

then

$$\int_\Omega \widetilde{\xi_s^\varepsilon} \cdot e(w)\,\mathrm{d}x = \lambda_s^\varepsilon \int_\Omega \chi_\varepsilon (P_\varepsilon u_s^\varepsilon) w\,\mathrm{d}x.$$

Hence, (H2) and (4.6) imply

$$\int_\Omega \bar{\xi}_s \cdot e(w)\,\mathrm{d}x = \bar{\lambda}_s \int_\Omega \chi_0 \bar{u}_s w\,\mathrm{d}x \quad \text{for any } w \in H_0^1(\Omega)^n,$$

so

$$-\operatorname{div}\bar{\xi}_s = \bar{\lambda}_s \chi_0 \bar{u}_s \quad \text{a.e. in} \quad \Omega.$$

Therefore, the proof will be complete if we show that

$$\bar{\xi}_s = A^0 e(\bar{u}_s).$$

Consider for any function $\varphi \in H_0^1(\Omega)$ and any symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$ the unique solution $\theta_\Lambda^\varepsilon \in V_\varepsilon$ of the problem

$$\begin{cases} -\operatorname{div}(A^\varepsilon e(\theta_\Lambda^\varepsilon)) = P_\varepsilon^*(-\operatorname{div}(A^0 e(\Lambda x\varphi))) & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(\theta_\Lambda^\varepsilon))\cdot\vartheta = 0 & \text{on } \partial S_\varepsilon, \\ \theta_\Lambda^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Taking into account that $(A^\varepsilon, S_\varepsilon) \xrightharpoonup{H_\varepsilon^0} A^0$, we deduce the existence of a function $\theta_\Lambda \in H_0^1(\Omega)^n$ such that

$$\begin{cases} P_\varepsilon\theta_\Lambda^\varepsilon \rightharpoonup \theta_\Lambda \text{ weakly in } H_0^1(\Omega)^n, \\ \widetilde{A^\varepsilon e(\theta_\Lambda^\varepsilon)} \rightharpoonup A^0 e(\theta_\Lambda) \text{ weakly in } L^2(\Omega)^{n\times n}, \end{cases}$$

where θ_Λ satisfies the problem

$$\begin{cases} -\operatorname{div}(A^0 e(\theta_\Lambda - \Lambda x\varphi)) = 0 & \text{in } \Omega, \\ \theta_\Lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $A^0 \in M_e(\alpha/C^2, \beta^2/\alpha, \Omega)$, the last problem has a unique solution, which is the null function, then $\theta_\Lambda = \Lambda x\varphi$.

Now, set $\eta^\varepsilon = A^\varepsilon e(\theta^\varepsilon_\Lambda)$ and for any $\psi \in D(\Omega)$ choose the function $u^\varepsilon_s\psi$ as a test function in problem (5.2), then we get due to the property of symmetry given in (1.1)

$$\begin{aligned}(5.3)\qquad \int_{\Omega_\varepsilon} \eta^\varepsilon \cdot e(u^\varepsilon_s\psi)\,\mathrm{d}x &= \langle -\operatorname{div}(A^0 e(\Lambda x\varphi)), P_\varepsilon(u^\varepsilon_s\psi)\rangle_{H^{-1}(\Omega)^n, H^1_0(\Omega)^n} \\ &= \int_\Omega A^0 e(\Lambda x\varphi)\cdot e(P_\varepsilon(u^\varepsilon_s\psi))\,\mathrm{d}x.\end{aligned}$$

Since $e(u^\varepsilon_s\psi) = (e_{ij}(u^\varepsilon_s\psi))_{1\leqslant i,j\leqslant n}$ with

$$e_{ij}(u^\varepsilon_s\psi) = \frac{1}{2}\Big[\frac{\partial(u^\varepsilon_s\psi)_j}{\partial x_i} + \frac{\partial(u^\varepsilon_s\psi)_i}{\partial x_j}\Big] = e_{ij}(u^\varepsilon_s)\psi + \frac{1}{2}\Big[(u^\varepsilon_s)_j\frac{\partial\psi}{\partial x_i} + (u^\varepsilon_s)_i\frac{\partial\psi}{\partial x_j}\Big],$$

we get

$$(5.4)\quad \int_{\Omega_\varepsilon} \eta^\varepsilon \cdot e(u^\varepsilon_s\psi)\,\mathrm{d}x = \int_{\Omega_\varepsilon} \eta^\varepsilon \cdot e(u^\varepsilon_s)\psi\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_\varepsilon} \eta^\varepsilon_{ij}\Big[(u^\varepsilon_s)_j\frac{\partial\psi}{\partial x_i} + (u^\varepsilon_s)_i\frac{\partial\psi}{\partial x_j}\Big]\,\mathrm{d}x.$$

Notice that from the property of symmetry given in (1.1),

$$(5.5)\qquad \int_{\Omega_\varepsilon} \eta^\varepsilon \cdot e(u^\varepsilon_s)\psi\,\mathrm{d}x = \int_{\Omega_\varepsilon} \xi^\varepsilon_s \cdot e(\theta^\varepsilon_\Lambda)\psi\,\mathrm{d}x,$$

then from (5.3), (5.4) and (5.5), we have

$$\int_{\Omega_\varepsilon} \xi^\varepsilon_s\cdot e(\theta^\varepsilon_\Lambda)\psi\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_\varepsilon} \eta^\varepsilon_{ij}\Big[(u^\varepsilon_s)_j\frac{\partial\psi}{\partial x_i} + (u^\varepsilon_s)_i\frac{\partial\psi}{\partial x_j}\Big]\,\mathrm{d}x = \int_\Omega A^0 e(\Lambda x\varphi)\cdot e(P_\varepsilon(u^\varepsilon_s\psi))\,\mathrm{d}x.$$

By passing to the limit, as $\varepsilon \to 0$, in each term of this last equation:

First term. We have

$$\int_{\Omega_\varepsilon} \xi^\varepsilon_s \cdot e(\theta^\varepsilon_\Lambda)\psi\,\mathrm{d}x = \int_\Omega \widetilde{\xi^\varepsilon_s} \cdot e(P_\varepsilon\theta^\varepsilon_\Lambda)\psi\,\mathrm{d}x.$$

Thanks to the weak convergences of $\widetilde{\xi^\varepsilon_s}$ to $\bar{\xi}_s$ in $L^2(\Omega)^{n\times n}$, $P_\varepsilon\theta^\varepsilon_\Lambda$ to $\Lambda x\varphi$ in $H^1_0(\Omega)^n$, and due to (3.1) and the div-curl argument (see [22]), we have

$$\int_{\Omega_\varepsilon} \xi^\varepsilon_s \cdot e(\theta^\varepsilon_\Lambda)\psi\,\mathrm{d}x \to \int_\Omega \bar{\xi}_s \cdot e(\Lambda x\varphi)\psi\,\mathrm{d}x.$$

S e c o n d t e r m. We have

$$\int_{\Omega_\varepsilon} \eta^\varepsilon_{ij} \Big[(u^\varepsilon_s)_j \frac{\partial \psi}{\partial x_i} + (u^\varepsilon_s)_i \frac{\partial \psi}{\partial x_j} \Big] \,\mathrm{d}x = \int_\Omega \widetilde{\eta^\varepsilon_{ij}} \Big[(P_\varepsilon u^\varepsilon_s)_j \frac{\partial \psi}{\partial x_i} + (P_\varepsilon u^\varepsilon_s)_i \frac{\partial \psi}{\partial x_j} \Big] \,\mathrm{d}x,$$

from the weak convergences of η^ε to $A^0 e(\Lambda x \varphi)$ in $L^2(\Omega)^{n\times n}$ and of $P_\varepsilon u^\varepsilon_s$ to $\bar{u}_s$ in $H^1_0(\Omega)^n$, we get

$$\int_{\Omega_\varepsilon} \eta^\varepsilon_{ij} \Big[(u^\varepsilon_s)_j \frac{\partial \psi}{\partial x_i} + (u^\varepsilon_s)_i \frac{\partial \psi}{\partial x_j} \Big] \,\mathrm{d}x \to \int_\Omega (A^0 e(\Lambda x \varphi))_{ij} \Big[(\bar{u}_s)_j \frac{\partial \psi}{\partial x_i} + (\bar{u}_s)_i \frac{\partial \psi}{\partial x_j} \Big] \,\mathrm{d}x.$$

T h i r d t e r m. From the weak convergence of $P_\varepsilon u^\varepsilon_s$ to $\bar{u}_s$ in $H^1_0(\Omega)^n$ and Remark 2.5 we obtain

$$\int_\Omega A^0 e(\Lambda x \varphi) \cdot e(P_\varepsilon(u^\varepsilon_s \psi)) \,\mathrm{d}x \to \int_\Omega A^0 e(\Lambda x \varphi) \cdot e(\bar{u}_s \psi) \,\mathrm{d}x.$$

However, due to

$$e_{ij}(\bar{u}_s \psi) = e_{ij}(\bar{u}_s)\psi + \frac{1}{2} \Big[(\bar{u}_s)_j \frac{\partial \psi}{\partial x_i} + (\bar{u}_s)_i \frac{\partial \psi}{\partial x_j} \Big],$$

we get

$$\begin{aligned} \int_\Omega A^0 e(\Lambda x \varphi) \cdot e(P_\varepsilon(u^\varepsilon_s \psi)) \,\mathrm{d}x \to & \int_\Omega A^0 e(\Lambda x \varphi) \cdot e(\bar{u}_s)\psi \,\mathrm{d}x \\ & + \frac{1}{2} \int_\Omega (A^0 e(\Lambda x \varphi))_{ij} \Big[(\bar{u}_s)_j \frac{\partial \psi}{\partial x_i} + (\bar{u}_s)_i \frac{\partial \psi}{\partial x_j} \Big] \,\mathrm{d}x. \end{aligned}$$

Then, we conclude that

$$\int_\Omega \bar{\xi}_s \cdot e(\Lambda x \varphi)\psi \,\mathrm{d}x = \int_\Omega A^0 e(\Lambda x \varphi) \cdot e(\bar{u}_s)\psi \,\mathrm{d}x \quad \text{for any } \psi \in D(\Omega),$$

and so

$$\bar{\xi}_s \cdot e(\Lambda x \varphi) = A^0 e(\Lambda x \varphi) \cdot e(\bar{u}_s) \quad \text{in } \Omega.$$

Furthermore, for any compact set $\omega \subset\subset \Omega$ by choosing the function φ such that $\varphi \equiv 1$ in ω

$$\bar{\xi}_s \cdot \Lambda = A^0 \Lambda \cdot e(\bar{u}_s) \quad \text{in } \omega,$$

from property of symmetry of the fourth-order tensor A^0

$$\bar{\xi}_s \cdot \Lambda = A^0 e(\bar{u}_s) \cdot \Lambda.$$

Thanks to the symmetry property of A^0 and $\bar{\xi}_s$, we deduce that $\bar{\xi}_s = A^0 e(\bar{u}_s)$ and finally

$$-\mathrm{div}(A^0 e(\bar{u}_s)) = \bar{\lambda}_s \chi_0 \bar{u}_s \quad \text{a.e. in } \Omega.$$

□

We are now able to complete our homogenization process by establishing our second main theorem, where we prove that problem (5.1) represents the homogenized spectral problem. The proof is based on the result established by Theorem (5.1) and the minimax principles of the eigenvalues [4], [13], [14].

Theorem 5.2. *Let* $\{\lambda_s^\varepsilon\}_{s\geqslant 1}$, *be the eigenvalues of problem* (3.1) *and let* $\{u_s^\varepsilon\}_{s\geqslant 1}$ *be the corresponding normalized eigenfunctions. Then under assumptions* (H1) *and* (H2), *we have*

(1) *for each integer* $s \geqslant 1$

$$\lambda_s^\varepsilon \to \lambda_s \quad \text{as } \varepsilon \to 0,$$

(2) *there exists a subsequence of* (ε) (*still denoted by* (ε)) *such that for each integer* $s \geqslant 1$

$\triangleright$ $P_\varepsilon u_s^\varepsilon \rightharpoonup u_s$ *weakly in* $H_0^1(\Omega)^n$,

$\triangleright$ $\widetilde{A^\varepsilon e(u_s^\varepsilon)} \rightharpoonup A^0 e(u_s)$ *weakly in* $L^2(\Omega)^{n\times n}$,

where λ_s *represents an eigenvalue and* u_s *an associated normalized eigenfunction of problem* (5.1).

P r o o f. In virtue of Lemma 4.1 and Theorem 5.1, to prove the first assertion of the theorem it suffices to prove that

$$\bar{\lambda}_s = \lambda_s \quad \forall\, s \geqslant 1,$$

where $\{\lambda_s\}_{s=1}^\infty$ represents the spectrum of problem (5.1).

For this, first we show that the whole spectrum $\{\lambda_s\}_{s=1}^\infty$ of problem (5.1) is included in the sequence $\{\bar{\lambda}_s\}_{s=1}^\infty$.

Let us argue by contradiction, assume that there exists a real $\lambda \in \mathbb{R}_+^*$, an eigenvalue of problem (5.1), such that for every $s \geqslant 1$ $\lambda \neq \bar{\lambda}_s$, and let u be an associated normalized eigenfunction satisfying

$$(u, \bar{u}_s)_0 = 0 \quad \text{for any } s \geqslant 1. \tag{5.6}$$

From (4.7) we have the existence of a positive integer s_0, such that $\lambda < \bar{\lambda}_{s_0}$ with $\bar{\lambda}_{s_0}$ an eigenvalue of problem (5.1) of finite multiplicity k.

Let U^ε be the unique solution in V_ε of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon e(U^\varepsilon)) = -P_\varepsilon^*[\operatorname{div}(A^0 e(u))] & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(U^\varepsilon))\vartheta = 0 & \text{on } \partial S_\varepsilon, \\ U^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Since $(A^\varepsilon, S_\varepsilon) \xrightarrow{H_e^0} A^0$, there exists a vector-valued function $U^0 \in H_0^1(\Omega)^n$ such that

$$(5.8) \qquad \begin{cases} P_\varepsilon U^\varepsilon \rightharpoonup U^0 & \text{weakly in } H_0^1(\Omega)^n, \\ \widetilde{A^\varepsilon e(U^\varepsilon)} \rightharpoonup A^0 e(U^0) & \text{weakly in } L^2(\Omega)^{n\times n} \end{cases}$$

with

$$\begin{cases} -\operatorname{div}(A^0 e(U^0)) = -\operatorname{div}(A^0 e(u)) & \text{in } \Omega, \\ U^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Because $A^0 \in M_e(\alpha/C^2, \beta^2/\alpha, \Omega)$, the last problem admits the vector-valued function u as a unique solution, then $U^0 = u$.

Now consider the vector

$$v^\varepsilon = U^\varepsilon - \sum_{i=1}^{s_0+k-1} (U^\varepsilon, u_i^\varepsilon)_\varepsilon u_i^\varepsilon.$$

Obviously, by construction we have

$$v^\varepsilon \in V_\varepsilon, \quad v^\varepsilon \neq 0, \quad v^\varepsilon \perp E_\varepsilon(s_0 + k - 1),$$

where $E_\varepsilon(s_0+k-1)$ is the finite-dimensional subspace of V_ε spanned by $\{u_1^\varepsilon, u_2^\varepsilon, \dots, u_{s_0+k-1}^\varepsilon\}$.

The characterization (3.3) of eigenvalue $\lambda_{s_0+k}^\varepsilon$ leads to

$$\lambda_{s_0+k}^\varepsilon = \min_{\substack{v \in V_\varepsilon \\ v \perp E_\varepsilon(s_0+k-1)}} \mathcal{R}_\varepsilon(v) \leqslant \mathcal{R}_\varepsilon(v^\varepsilon) = \frac{a_\varepsilon(v^\varepsilon, v^\varepsilon)}{(v^\varepsilon, v^\varepsilon)_\varepsilon},$$

so

$$(5.9) \qquad a_\varepsilon(v^\varepsilon, v^\varepsilon) \geqslant \lambda_{s_0+k}^\varepsilon (v^\varepsilon, v^\varepsilon)_\varepsilon,$$

due to (3.2), by developing each member of (5.9) we obtain

$$(v^\varepsilon, v^\varepsilon)_\varepsilon = (U^\varepsilon, U^\varepsilon)_\varepsilon - \sum_{j=1}^{s_0+k-1} (U^\varepsilon, u_j^\varepsilon)_\varepsilon^2.$$

Likewise, from the weak formulation of problem (5.7) and problem (3.1) we get

$$\begin{aligned} a_\varepsilon(v^\varepsilon, v^\varepsilon) = {} & \langle -\operatorname{div}(A^0 e(u)), P_\varepsilon U^\varepsilon \rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n} \\ & - 2 \sum_{j=1}^{s_0+k-1} (U^\varepsilon, u_j^\varepsilon)_\varepsilon \langle -\operatorname{div}(A^0 e(u)), P_\varepsilon u_j^\varepsilon \rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n} \\ & + \sum_{j=1}^{s_0+k-1} \lambda_j^\varepsilon (U^\varepsilon, u_j^\varepsilon)_\varepsilon^2. \end{aligned}$$

Moreover, since u is a normalized eigenfunction of problem (5.1) associated to the eigenvalue λ, we obtain

$$\langle -\mathrm{div}(A^0 e(u)), P_\varepsilon U^\varepsilon \rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n} = \lambda \int_\Omega \chi_0 u P_\varepsilon U^\varepsilon \, \mathrm{d}x,$$

and

$$\langle -\mathrm{div}(A^0 e(u)), P_\varepsilon u_j^\varepsilon \rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n} = \lambda \int_\Omega \chi_0 u P_\varepsilon u_j^\varepsilon \, \mathrm{d}x.$$

Under assumption (5.6) and due to (5.8), it is easy to see that

$$\begin{cases} (U^\varepsilon, U^\varepsilon)_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 1, \\ (U^\varepsilon, u_j^\varepsilon)_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0, \\ \langle -\mathrm{div}(A^0 e(u)), P_\varepsilon U^\varepsilon \rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n} \underset{\varepsilon \to 0}{\longrightarrow} \lambda, \\ \langle -\mathrm{div}(A^0 e(u)), P_\varepsilon u_j^\varepsilon \rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n} \underset{\varepsilon \to 0}{\longrightarrow} 0. \end{cases}$$

Thus, passing to the limit, as $\varepsilon \to 0$, in (5.9) we obtain

$$\lambda \geqslant \bar{\lambda}_{s_0+k},$$

which contradicts the existence of s_0. Then necessarily there exists some $p \geqslant 1$ such that $\lambda = \bar{\lambda}_p$ and so the sequence $\{\bar{\lambda}_s, s = 1, 2, 3, \ldots\}$ represents the entire spectrum of problem (5.1).

Second, we have to show that eigenvalues λ_s and $\bar{\lambda}_s$ have the same multiplicity. For this purpose, it suffices to prove, by using the same arguments as before, that the family $\{\bar{u}_s, s \geqslant 1\}$ is a complete orthogonal basis of $L^2(\Omega, \chi_0)^n$.

To complete the proof of the first assertion, consider any subsequence (ε') of (ε) such that for each $s \geqslant 1$, $\lambda_s^{\varepsilon'}$ converges to some $\varpi_s \in \mathbb{R}_+^*$ and using the same steps as before we conclude naturally that $\lambda_s = \varpi_s$. Then, for each $s \geqslant 1$ the whole sequence λ_s^ε converges to λ_s.

The second assertion is a direct consequence of the previous one and Lemma 4.1.

□

Remark 5.1. It easy to deduce from assumption (H2) and the second assertion of Theorem 5.2 that $\widetilde{u_s^\varepsilon}$ converges weakly to $\chi_0 u_s$ in $L^2(\Omega)$.

In the particular case of a simple eigenvalue of problem (5.1), we give the following result concerning the weak convergence of the eigenfunction and a particular error estimate:

Theorem 5.3. *Under assumptions* (H1) *and* (H2), *let for some* $p \geqslant 1$, λ_p *be a simple eigenvalue of problem* (5.1). *Then the eigenvalue* λ_p^ε *of problem* (3.1) *which satisfies* $\lambda_p^\varepsilon \to \lambda_p$, *as* $\varepsilon \to 0$, *is a simple eigenvalue too.*

Let u *be any normalized eigenfunction corresponding to* λ_p, *then a normalized eigenfunction* u_p^ε *associated to* λ_p^ε *can be found such that the whole sequence* $P_\varepsilon u_p^\varepsilon$ *converges weakly in* $H_0^1(\Omega)^n$ *to* u.

Furthermore, for any real κ *such that* $0 < \kappa < 1$, *we get for a sufficiently small* ε

$$|\lambda_p^\varepsilon - \lambda_p| < \Big(\frac{1+\kappa}{1-\kappa}\Big)\lambda_p\Big(\Big\|\frac{\chi_\varepsilon - \chi_0}{\chi_0}\Big\|_{L^\infty(\Omega)} + \Big\|\frac{\chi_\varepsilon}{\chi_0}\Big\|_{L^\infty(\Omega)}\|P_\varepsilon w_p^\varepsilon - u\|_0\Big),$$

where w_p^ε *is the unique solution in* V_ε *of the problem*

$$\text{(5.10)}\qquad \begin{cases} -\operatorname{div}(A^\varepsilon e(w_p^\varepsilon)) = -P_\varepsilon^*[\operatorname{div}(A^0 e(u))] & \text{in } \Omega_\varepsilon,\\ (A^\varepsilon e(w_p^\varepsilon))\vartheta = 0 & \text{on } \partial S_\varepsilon,\\ w_p^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

P r o o f. First, due to the convergence of the sequence λ_p^ε to λ_p, as ε goes to 0, one can easily see from the proof of the previous theorem that the eigenvalue λ_p^ε is of simple multiplicity too. We stress that in general the multiplicity of λ_p^ε may be less than or equal to that of λ_p.

Now, let u_p^ε be a normalized eigenfunction of problem (3.1) corresponding to the eigenvalue λ_p^ε such that

$$\text{(5.11)}\qquad (u_p^\varepsilon, u)_\varepsilon > 0.$$

As a consequence of Theorem 5.2, for a subsequence ε' of ε, we have

$$P_{\varepsilon'} u_p^{\varepsilon'} \rightharpoonup u_p \quad \text{weakly in } H_0^1(\Omega)^n,$$

where u_p is a normalized eigenfunction of problem (5.1) associated to the eigenvalue λ_p. Then, by passing to the limit in (5.11) we get

$$(u_p, u)_0 > 0.$$

We conclude that u_p and u are two normalized eigenfunctions corresponding to a simple eigenvalue with the same orientation, so $u_p = u$. Then, the whole sequence $P_\varepsilon u_p^\varepsilon$ converges to u, as $\varepsilon \to 0$.

Now, consider w_p^ε, the unique solution in V_ε of problem (5.10). Because of the H_e^0-convergence of $(A^\varepsilon, S_\varepsilon)$ to $A^0 \in M_e(\alpha/C^2, \beta^2/\alpha, \Omega)$ we obtain

$$\begin{cases} P_\varepsilon w_p^\varepsilon \rightharpoonup u & \text{weakly in } H_0^1(\Omega)^n,\\ \widetilde{A^\varepsilon e(w_p^\varepsilon)} \rightharpoonup A^0 e(u) & \text{weakly in } L^2(\Omega)^{n\times n}. \end{cases}$$

By choosing test functions u_p^ε and w_p^ε in the weak formulation of problems (5.10) and (3.1), respectively, we get

$$\int_{\Omega_\varepsilon} A^\varepsilon e(u_p^\varepsilon)e(w_p^\varepsilon)\,\mathrm{d}x = \lambda_p^\varepsilon \int_{\Omega_\varepsilon} u_p^\varepsilon w_p^\varepsilon\,\mathrm{d}x = \lambda_p^\varepsilon \int_\Omega \chi_\varepsilon (P_\varepsilon u_p^\varepsilon)(P_\varepsilon w_p^\varepsilon)\,\mathrm{d}x$$

and

$$\int_{\Omega_\varepsilon} A^\varepsilon e(w_p^\varepsilon)e(u_p^\varepsilon)\,\mathrm{d}x = \int_\Omega A^0 e(u)e(P_\varepsilon u_p^\varepsilon)\,\mathrm{d}x = \lambda_p \int_\Omega \chi_0 u(P_\varepsilon u_p^\varepsilon)\,\mathrm{d}x.$$

Since $A^\varepsilon e(u_p^\varepsilon)e(w_p^\varepsilon) = A^\varepsilon e(w_p^\varepsilon)e(u_p^\varepsilon)$, we have

$$\lambda_p^\varepsilon \int_\Omega \chi_\varepsilon P_\varepsilon u_p^\varepsilon P_\varepsilon w_p^\varepsilon\,\mathrm{d}x = \lambda_p \int_\Omega \chi_0 u P_\varepsilon u_p^\varepsilon\,\mathrm{d}x,$$

so

$$(\lambda_p^\varepsilon - \lambda_p)\int_\Omega \chi_\varepsilon P_\varepsilon u_p^\varepsilon P_\varepsilon w_p^\varepsilon\,\mathrm{d}x = \lambda_p \int_\Omega (\chi_0 u - \chi_\varepsilon P_\varepsilon w_p^\varepsilon)P_\varepsilon u_p^\varepsilon\,\mathrm{d}x.$$

From the previous limits $\int_\Omega \chi_\varepsilon P_\varepsilon u_p^\varepsilon P_\varepsilon w_p^\varepsilon\,\mathrm{d}x \to \int_\Omega \chi_0 u^2\,\mathrm{d}x = 1$, so for ε sufficiently small we have $\int_\Omega \chi_\varepsilon P_\varepsilon u_p^\varepsilon P_\varepsilon w_p^\varepsilon\,\mathrm{d}x > 1-\kappa$ with $0<\kappa<1$ and then

$$\Big(\frac{1-\kappa}{\lambda_p}\Big)|\lambda_p^\varepsilon - \lambda_p| < \Big|\int_\Omega (\chi_\varepsilon - \chi_0)uP_\varepsilon u_p^\varepsilon\,\mathrm{d}x\Big| + \Big|\int_\Omega \chi_\varepsilon(P_\varepsilon w_p^\varepsilon - u)P_\varepsilon u_p^\varepsilon\,\mathrm{d}x\Big|. \tag{5.12}$$

From assumption (H2), the two terms on the left-hand side of (5.12) can be rewritten as follows:

$$\Big|\int_\Omega (\chi_\varepsilon - \chi_0)uP_\varepsilon u_p^\varepsilon\,\mathrm{d}x\Big| = \Big|\int_\Omega \frac{\chi_\varepsilon - \chi_0}{\chi_0}(\sqrt{\chi_0}u)(\sqrt{\chi_0}P_\varepsilon u_p^\varepsilon)\,\mathrm{d}x\Big|,$$
$$\Big|\int_\Omega \chi_\varepsilon(P_\varepsilon w_p^\varepsilon - u)P_\varepsilon u_p^\varepsilon\,\mathrm{d}x\Big| = \Big|\int_\Omega \frac{\chi_\varepsilon}{\chi_0}(\sqrt{\chi_0}(P_\varepsilon w_p^\varepsilon - u))(\sqrt{\chi_0}P_\varepsilon u_p^\varepsilon)\,\mathrm{d}x\Big|.$$

Observe that from the definition of $L^2(\Omega,\chi_0)$ and its norm $\|\cdot\|_0$, we have for any function $v\in L^2(\Omega)$ $\|\sqrt{\chi_0}v\|_{L^2(\Omega)} = \|v\|_0$, so we get

$$\Big|\int_\Omega (\chi_\varepsilon - \chi_0)uP_\varepsilon u_p^\varepsilon\,\mathrm{d}x\Big| \leqslant \Big\|\frac{\chi_\varepsilon - \chi_0}{\chi_0}\Big\|_{L^\infty(\Omega)}\|u\|_0\|P_\varepsilon u_p^\varepsilon\|_0,$$
$$\Big|\int_\Omega \chi_\varepsilon(P_\varepsilon w_p^\varepsilon - u)P_\varepsilon u_p^\varepsilon\,\mathrm{d}x\Big| \leqslant \Big\|\frac{\chi_\varepsilon}{\chi_0}\Big\|_{L^\infty(\Omega)}\|P_\varepsilon w_p^\varepsilon - u\|_0\|P_\varepsilon u_p^\varepsilon\|_0.$$

Since $\|u\|_0 = 1$ and $P_\varepsilon u_p^\varepsilon \to u$ strongly in $L^2(\Omega)^n$, for a sufficiently small ε

$$\|P_\varepsilon u_p^\varepsilon\|_0 < 1+\kappa,$$

so we obtain

$$|\lambda_p^\varepsilon - \lambda_p| < \Big(\frac{1+\kappa}{1-\kappa}\Big)\lambda_p\Big(\Big\|\frac{\chi_\varepsilon - \chi_0}{\chi_0}\Big\|_{L^\infty(\Omega)} + \Big\|\frac{\chi_\varepsilon}{\chi_0}\Big\|_{L^\infty(\Omega)}\|P_\varepsilon w_p^\varepsilon - u\|_0\Big).$$

□

6. Comments and perspectives

Our main result states that to homogenize a spectral problem for linearized elasticity in perforated domains with e-admissible holes (beyond periodic setting), it suffices to determine the H_e^0-limit of the associated fourth-order tensor. Thanks to Theorem 2.1 the existence of this H_e^0-limit is guaranteed at least for a subsequence of (ε).

As an application, we can consider a spectral problem for linearized elasticity in perforated domains with two types of holes, the thick ones are periodically distributed in Ω and the thin ones are assumed to be only e-admissible as introduced in [11]. Let Y be a reference cell with paving properties. Set $Y^\star \equiv Y \setminus \mathcal{T}^\star$, where $\mathcal{T}^\star$ is a compact subset of Y, consider a sequence $\mathfrak{S}_\varepsilon$ of compact subsets of $Y^\star$, $H^1(Y^\star)$-admissible in the sense of [11]. Then, the set $T_\varepsilon = T_\varepsilon^\star \cup S_\varepsilon$, where

$$\begin{cases} T_\varepsilon^\star = \Big\{ \bigcup \varepsilon\{\mathcal{T}^\star + k_l b_l\} \text{ s.t. } k \in \mathbb{Z}^n \text{ and } \varepsilon\{\mathcal{T}^\star + k_l b_l\} \subset \Omega \Big\}, \\ S_\varepsilon = \Big\{ \bigcup \varepsilon\{\mathfrak{S}_\varepsilon + k_l b_l\} \text{ s.t. } k \in \mathbb{Z}^n \text{ and } \varepsilon\{\mathfrak{S}_\varepsilon + k_l b_l\} \subset \Omega \Big\}, \end{cases}$$

represents the e-admissible holes and the homogenized spectral problem associated to this problem can be deduced directly from the results given in [11].

As a consequence, we will obtain a homogenization result of a large class of spectral problems for elasticity in perforated domains and obviously more information about the limit of the sequence λ_s^ε. We cite the two following situations:

- ▷ holes with finitely many periodic scales,
- ▷ two types of holes will be considered, the thick ones are periodic and the thin ones are locally periodic in $Y^\star$.

These ideas are the subject of a paper in preparation.

To conclude, we note that our result would not be valid if the holes were not admissible, which leads to a question similar to the one asked by Damlamian and Donato [5] for the scalar case: which holes are admissible for linearized elasticity? This is an open problem.

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Authors' addresses: *Mohamed Mourad Lhannafi Ait Yahia* (corresponding author), AMNEDP Laboratory, Faculty of Mathematics, University of Sciences and Technology Houari Boumedienne, POB 32 El Alia, Algiers, Algeria, e-mail: `maityahia@usthb.dz`; *Hamid Haddadou*, LCSI Laboratory, Ecole Nationale Supérieure d'Informatique (ESI), Oued Smar, Algiers, Algeria, e-mail: `h_haddadou@esi.dz`.