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Pre-derivations and description of non-strongly nilpotent filiform Leibniz algebras

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Abstract. In this paper we give the description of some non-strongly nilpotent Leibniz algebras. We pay our attention to the subclass of nilpotent Leibniz algebras, which is called filiform. Note that the set of filiform Leibniz algebras of fixed dimension can be decomposed into three disjoint families. We describe the pre-derivations of filiform Leibniz algebras for the first and second families and determine those algebras in the first two classes of filiform Leibniz algebras that are non-strongly nilpotent.

1 Introduction

It is well-known that a Lie algebra over a field of characteristic zero admitting a non-singular (invertible) derivation is nilpotent [13]. The first example of a nilpotent Lie algebra, whose derivations are nilpotent (and hence, singular) was constructed in [8]. Further, such Lie algebras got the name characteristically nilpotent and various papers are devoted to the investigation of characteristically nilpotent Lie algebras [2], [5], [7], [16], [17], [19].

The study of derivations of Lie algebras led to the appearance of a natural generalization: pre-derivations of Lie algebras [23]. The set of pre-derivations of a Lie algebra L is the Lie algebra of the Lie group of pre-automorphisms of L (see [4]). A pre-derivation of L is just a derivation of the Lie triple system induced by L.

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Research on pre-derivations has been related to Lie algebra degenerations, Lie triple systems and bi-invariant semi-Riemannian metrics on Lie groups [6]. In [4] it was proved that the analogue of Jacobson's result for pre-derivations is also true. Similar to the example of Dixmier and Lister, several examples of nilpotent Lie algebras whose pre-derivations are nilpotent were presented in [6]. Such Lie algebras are called strongly nilpotent.

In [22], a generalized notion of derivations and pre-derivations of Lie algebras is given. These derivations are called Leibniz-derivations of order k, and it is proved that a finite-dimensional Lie algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible Leibniz-derivation. Furthermore, an analogue of this result was shown for alternative, Jordan, Zinbiel, Malcev and Leibniz algebras [9], [14], [15].

The notion of Leibniz algebra was introduced in [21] as a non-antisymmetric generalization of Lie algebras. Since the study of derivations and automorphisms of a Lie algebra plays an essential role in the structure theory of algebras, it is a natural question whether the corresponding results for Lie algebras can be extended to more general objects. An analogue of Jacobson's result for Leibniz algebras was proved in [20]. Moreover, it was shown that, similarly to the case of Lie algebras, the converse of this statement does not hold and the notion of a characteristically nilpotent Lie algebra can be extended to Leibniz algebras [18], [24].

Since a Leibniz-derivation of order 3 of a Leibniz algebra is a pre-derivation, it is natural to define the notion of strongly nilpotent Leibniz algebras. Note that the every strongly nilpotent Leibniz algebra is characteristically nilpotent. Thus, one of the approaches to the classification of nilpotent Leibniz algebras considers a subclass of Leibniz algebras, in which any Leibniz derivation of order k is nilpotent and any algebra admits a non-nilpotent Leibniz-derivation of order $k+1$. In the case of $k=1$ we have the class of non-characteristically nilpotent Leibniz algebras. The filiform Leibniz algebras in this class were determined in [18]. Some classes of finite-dimensional filiform Leibniz algebras up to dimension less than 10 were classified in [1], [10], [25], [26].

This paper is devoted to the study of algebras for the case $k=2$, i.e., the class of characteristically nilpotent filiform Leibniz algebras which are non-strongly nilpotent. It is known that the class of all filiform Leibniz algebras can be divided into three disjoint families [3], [11], where one of the families contains filiform Lie algebras and the other two families arise from naturally graded non-Lie filiform Leibniz algebras. We determine those algebras in the first two classes of filiform Leibniz algebras that are non-strongly nilpotent for any finite dimension. For the third class this questions is reduced to the same question about Lie algebras. Note that the classification of non-strongly nilpotent filiform Lie algebras is known only up to dimension 11 in [6].

In order to achieve our goal, we have organized the paper as follows. In Section 2, we present necessary definitions and results that will be used in the rest of the paper. In Section 3, we describe pre-derivations of filiform Leibniz algebras of the first and second families. Finally, in Section 4, we give a description of characteristically nilpotent filiform Leibniz algebras which are non-strongly nilpotent.

Throughout the paper, all the spaces and algebras are assumed to be finite--dimensional.

2 Preliminaries

In this section we give necessary definitions and preliminary results.

Definition 1. An algebra $(L, [-,-])$ over a field F is called a (*right*) *Leibniz algebra* if for any $x, y, z \in L$, the so-called *Leibniz identity*

$$\big[[x,y],z\big] = \big[[x,z],y\big] + \big[x,[y,z]\big]$$

holds.

Note that a derivation of a Leibniz algebra L is a linear transformation, such that

$$d([x,y]) = [d(x),y] + [x,d(y)],$$

for any $x, y \in L$.

Pre-derivations of Leibniz algebras are generalization of derivations which are defined as follows.

Definition 2. A linear transformation P of the Leibniz algebra L is called a *pre--derivation* if for any $x, y, z \in L$,

$$P([[x,y],z]) = [[P(x),y],z] + [[x,P(y)],z] + [[x,y],P(z)].$$

For a given Leibniz algebra L we consider the following central lower series:

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1], \qquad k \geq 1.$$

Definition 3. A Leibniz algebra L is called *nilpotent* if there exists $s \in \mathbb{N}$ such that $L^s = 0$.

A nilpotent Leibniz algebra is called *characteristically nilpotent* if all its derivations are nilpotent. We say that a Leibniz algebra is *strongly nilpotent* if any pre-derivation is nilpotent.

Since any derivation of a Leibniz algebra is a pre-derivation, every strongly nilpotent Leibniz algebra is characteristically nilpotent. An example of a characteristically nilpotent, but non-strongly nilpotent Leibniz algebra could be found in [9], [18], [24].

Definition 4. A Leibniz algebra L is said to be *filiform* if $\dim L^i = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.

The following theorem divides all n-dimensional filiform Leibniz algebras into three families.

Theorem 1 ([3], [11]). *Any n-dimensional complex filiform Leibniz algebra admits a basis $\{e_1, e_2, \ldots, e_n\}$ such that the table of multiplication of the algebra has one of the following forms:*

$$F_1(\alpha_4,\ldots,\alpha_n,\theta) = \begin{cases} [e_1,e_1] = e_3, & \\ [e_i,e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1,e_2] = \sum\limits_{t=4}^{n-1} \alpha_t e_t + \theta e_n, & \\ [e_j,e_2] = \sum\limits_{t=j+2}^{n} \alpha_{t-j+2} e_t, & 2 \le j \le n-2, \end{cases}$$

$$F_2(\beta_4,\ldots,\beta_n,\gamma) = \begin{cases} [e_1,e_1] = e_3, & \\ [e_i,e_1] = e_{i+1}, & 3 \le i \le n-1, \\ [e_1,e_2] = \sum\limits_{t=4}^{n} \beta_t e_t, & \\ [e_2,e_2] = \gamma e_n, & \\ [e_j,e_2] = \sum\limits_{t=j+2}^{n} \beta_{t-j+2} e_t, & 3 \le j \le n-2, \end{cases}$$

$$F_3(\theta_1,\theta_2,\theta_3) = \begin{cases} [e_i,e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1,e_i] = -e_{i+1}, & 3 \le i \le n-1, \\ [e_1,e_1] = \theta_1 e_n, & \\ [e_1,e_2] = -e_3 + \theta_2 e_n, & \\ [e_2,e_2] = \theta_3 e_n, & \\ [e_i,e_j] = -[e_j,e_i] \in \\ \quad \in \langle e_{i+j+1}, e_{i+j+2}, \ldots, e_n \rangle, & 2 \le i < j \le n-1, \\ [e_i,e_{n+1-i}] = -[e_{n+1-i},e_i] = \\ \quad = \alpha(-1)^{i+1} e_n, & 2 \le i \le n-1, \end{cases}$$

where all omitted products are equal to zero and $\alpha \in \{0,1\}$ for even n and $\alpha = 0$ for odd n.

It is easy to see that algebras of the first and the second families are non-Lie algebras. Note that if $(\theta_1, \theta_2, \theta_3) = (0,0,0)$, then an algebra of the third class is a Lie algebra.

Further we shall need the notion of Catalan numbers. The p-th Catalan numbers were defined in [12] by the formula

$$C_n^p = \frac{1}{(p-1)n+1} \binom{pn}{n}.$$

It should be noted that for the p-th Catalan numbers the following identity holds:

$$\sum_{k=1}^{n} C_k^p C_{n-k}^p = \frac{2n}{(p-1)n+p+1} C_{n+1}^p . \tag{1}$$

3 Pre-derivations of filiform Leibniz algebras

In this section we give the a description of pre-derivations of filiform Leibniz algebras. First, we consider the filiform Leibniz algebras from the first family.

Proposition 1. *The pre-derivations of the filiform Leibniz algebras from the family $F_1(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta)$ have the following form:*

$P(e_1) = \sum\limits_{t=1}^{n} a_t e_t,$	
$P(e_2) = (a_1 + a_2)e_2 + \sum\limits_{t=3}^{n-2} a_t e_t + b_{n-1}e_{n-1} + b_n e_n,$	
$P(e_3) = \sum\limits_{t=2}^{n} c_t e_t,$	
$P(e_{2i}) = ((2i-1)a_1 + a_2)e_{2i} + \sum\limits_{t=2i+1}^{n} (a_{t-2i+2} + (2i-2)a_2\alpha_{t-2i+3})e_t,$	$2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$
$P(e_{2i+1}) = c_2 e_{2i} + ((2i-2)a_1 + c_3)e_{2i+1} + \sum\limits_{t=2i+2}^{n} (c_{t-2i+2} + (2i-2)a_2\alpha_{t-2i+2})e_t,$	$2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$

where $\lfloor a \rfloor$ is the integer part of the real number a and

$(1 + (-1)^n)c_2 = 0, \quad c_2\alpha_t = 0,$	$4 \leq t \leq n-1,$
$(a_1 - a_2)\alpha_4 = 0, \quad (3a_1 - c_3)\alpha_4 = 0,$	
$\sum\limits_{t=3}^{k} (a_{2k-2t+3} - c_{2k-2t+4} + a_2\alpha_{2k-2t+4})\alpha_{2t-2} = 0,$	$3 \leq k \leq \lfloor \frac{n-1}{2} \rfloor,$
$(2a_1 + a_2 - c_3)\alpha_{2k} + \sum\limits_{t=3}^{k} (a_{2k-2t+4} - c_{2k-2t+5} + a_2\alpha_{2k-2t+5})\alpha_{2t-2} = 0,$	$3 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1,$
$(a_2 - (k-3)a_1)\alpha_k = \frac{k-1}{2} a_2 \sum\limits_{t=5}^{k} \alpha_{t-1}\alpha_{k-t+4},$	$5 \leq k \leq n-2,$

$(a_2-(n-4)a_1)\alpha_{n-1}=a_2\sum\limits_{t=3}^{\frac{n-2}{2}}(2t-3)\alpha_{n-2t+3}\alpha_{2t-1}+$ $+\sum\limits_{t=2}^{\frac{n-2}{2}}(c_{n-2t+2}-a_{n-2t+1}+(2t-3)a_2\alpha_{n-2t+2})\alpha_{2t},$	n *is even,*
$(2a_2-c_3-(n-6)a_1)\alpha_{n-1}=$ $=a_2\sum\limits_{t=3}^{\frac{n-1}{2}}(2t-3)\alpha_{n-2t+3}\alpha_{2t-1}+$ $+\sum\limits_{t=2}^{\frac{n-3}{2}}(c_{n-2t+2}-a_{n-2t+1}+(2t-3)a_2\alpha_{n-2t+2})\alpha_{2t},$	n *is odd.*

(2)

Proof. Let L be a filiform Leibniz algebra from the family $F_1(\alpha_4,\alpha_5,\dots,\alpha_n,\theta)$ and let $P\colon L\to L$ be a pre-derivation of L. Put

$$P(e_1)=\sum_{t=1}^{n}a_te_t,\quad P(e_2)=\sum_{t=1}^{n}b_te_t,\quad P(e_3)=\sum_{t=1}^{n}c_te_t.$$

From

$$0=P([[e_1,e_1],e_3])=[[P(e_1),e_1],e_3]+[[e_1,P(e_1)],e_3]+[[e_1,e_1],P(e_3)]$$
$$=[e_3,\sum_{t=1}^{n}c_te_t]=c_1e_4+c_2\sum_{t=4}^{n-1}\alpha_te_{t+1},$$

we have

$c_1=0$	
$c_2\alpha_t=0,$	$4\le t\le n-1.$

By the definition of a pre-derivation, we have

$$P(e_4)=P([[e_1,e_1],e_1])=[[P(e_1),e_1],e_1]+[[e_1,P(e_1)],e_1]+[[e_1,e_1],P(e_1)]$$
$$=[(a_1+a_2)e_3+\sum_{t=4}^{n}a_{t-1}e_t,e_1]+[a_1e_3+a_2\big(\sum_{t=4}^{n-1}\alpha_te_t+\theta e_n\big),e_1]$$
$$+a_1e_4+a_2\sum_{t=5}^{n}\alpha_{t-1}e_t$$
$$=(3a_1+a_2)e_4+\sum_{t=5}^{n}(a_{t-2}+2a_2\alpha_{t-1})e_t.$$

On the other hand,

$$P(e_4) = P([[e_2, e_1], e_1]) = [[P(e_2), e_1], e_1] + [[e_2, P(e_1)], e_1] + [[e_2, e_1], P(e_1)]$$
$$= [(b_1 + b_2)e_3 + \sum_{t=4}^{n} b_{t-1}e_t, e_1] + [a_1e_3 + a_2 \sum_{t=4}^{n} \alpha_t e_t, e_1]$$
$$+ a_1e_4 + a_2 \sum_{t=5}^{n} \alpha_{t-1}e_t$$
$$= (2a_1 + b_1 + b_2)e_4 + \sum_{t=5}^{n}(b_{t-2} + 2a_2\alpha_{t-1})e_t.$$

By comparing the coefficients of the basis elements we have

$b_1 + b_2 = a_1 + a_2,$	
$b_t = a_t,$	$3 \leq t \leq n-2.$

Using the property of pre-derivation, we get

$$P(e_5) = P([[e_3, e_1], e_1]) = [[P(e_3), e_1], e_1] + [[e_3, P(e_1)], e_1] + [[e_3, e_1], P(e_1)]$$
$$= [c_2e_3 + \sum_{t=4}^{n} c_{t-1}e_t, e_1] + [a_1e_4 + a_2 \sum_{t=5}^{n} \alpha_{t-1}e_t, e_1]$$
$$+ a_1e_5 + a_2 \sum_{t=6}^{n} \alpha_{t-2}e_t$$
$$= c_2e_4 + (2a_1 + c_3)e_5 + \sum_{t=6}^{n}(c_{t-2} + 2a_2\alpha_{t-2})e_t.$$

Similarly, from the identity

$P(e_{j+2}) = P([[e_j, e_1], e_1]) =$ $= [[P(e_j), e_1], e_1] + [[e_j, P(e_1)], e_1] + [[e_j, e_1], P(e_1)],$	$2 \leq j \leq n-2,$

inductively, we derive

$P(e_{2i}) = ((2i-1)a_1 + a_2)e_{2i} + \sum_{t=2i+1}^{n} (a_{t-2i+2} + (2i-2)a_2\alpha_{t-2i+3})e_t,$	$2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$
$P(e_{2i+1}) = c_2e_{2i} + ((2i-2)a_1 + c_3)e_{2i+1} + \sum_{t=2i+2}^{n} (c_{t-2i+2} + (2i-2)a_2\alpha_{t-2i+2})e_t,$	$2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$

Moreover, in the case of n being even we deduce from the definition of a pre-derivation applied to the triple $\{e_{n-1}, e_1, e_1\}$ that $c_2 = 0$. Thus we get

$$(1 + (-1)^n)c_2 = 0.$$

Consider

$$P([[e_1,e_1],e_2]) = P([e_3,e_2]) = P\Big(\sum_{t=5}^{n}\alpha_{t-1}e_t\Big)$$

$$= \sum_{t=2}^{\lfloor\frac{n-1}{2}\rfloor}\Big[c_2e_{2t} + ((2t-2)a_1+c_3)e_{2t+1}$$

$$+ \sum_{k=2t+2}^{n}(c_{k-2t+2}+(2t-2)a_2\alpha_{k-2t+2})e_k\Big]\alpha_{2t}$$

$$+\sum_{t=3}^{\lfloor\frac{n}{2}\rfloor}\Big[((2t-1)a_1+a_2)e_{2t} + \sum_{k=2t+1}^{n}(a_{k-2t+2}+(2t-2)a_2\alpha_{k-2t+3})e_k\Big]\alpha_{2t-1}$$

$$= \sum_{k=2}^{\lfloor\frac{n-1}{2}\rfloor}((2k-2)a_1+c_3)\alpha_{2k}e_{2k+1} + \sum_{k=6}^{n}\sum_{t=2}^{\lfloor\frac{k-2}{2}\rfloor}(c_{k-2t+2}+(2t-2)a_2\alpha_{k-2t+2})\alpha_{2t}e_k$$

$$+\sum_{k=3}^{\lfloor\frac{n}{2}\rfloor}((2k-1)a_1+a_2)\alpha_{2k-1}e_{2k}$$

$$+\sum_{k=7}^{n}\sum_{t=3}^{\lfloor\frac{k-1}{2}\rfloor}(a_{k-2t+2}+(2t-2)a_2\alpha_{k-2t+3})\alpha_{2t-1}e_k$$

$$= (2a_1+c_3)\alpha_4e_5$$

$$+\sum_{k=3}^{\lfloor\frac{n}{2}\rfloor}\Big[((2k-1)a_1+a_2)\alpha_{2k-1} + \sum_{t=2}^{k-1}(c_{2k-2t+2}+(2t-2)a_2\alpha_{2k-2t+2})\alpha_{2t}$$

$$+\sum_{t=3}^{k-1}(a_{2k-2t+2}+(2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t-1}\Big]e_{2k}$$

$$+\sum_{k=3}^{\lfloor\frac{n-1}{2}\rfloor}\Big[((2k-2)a_1+c_3)\alpha_{2k} + \sum_{t=2}^{k-1}(c_{2k-2t+3}+(2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t}$$

$$+\sum_{t=3}^{k}(a_{2k-2t+3}+(2t-2)a_2\alpha_{2k-2t+4})\alpha_{2t-1}\Big]e_{2k+1}.$$

On the other hand, using the property of pre-derivation, we get

$$P([[e_1,e_1],e_2]) = [[P(e_1),e_1],e_2] + [[e_1,P(e_1)],e_2] + [[e_1,e_1],P(e_2)]$$

$$= [(a_1+a_2)e_3 + \sum_{t=4}^{n}a_{t-1}e_t, e_2] + [a_1e_3 + a_2\Big(\sum_{t=4}^{n-1}\alpha_te_t+\theta e_n\Big), e_2]$$

$$+ b_1e_4 + b_2\sum_{t=5}^{n}\alpha_{t-1}e_t$$

$$= (a_1 + a_2)\sum_{t=5}^{n}\alpha_{t-1}e_t + \sum_{t=4}^{n-2}a_{t-1}\sum_{k=t+2}^{n}\alpha_{k-t+2}e_k$$
$$+ a_1\sum_{t=5}^{n}\alpha_{t-1}e_t + a_2\sum_{t=4}^{n-2}\alpha_t\sum_{k=t+2}^{n}\alpha_{k-t+2}e_k + b_1e_4 + b_2\sum_{t=5}^{n}\alpha_{t-1}e_t$$
$$= b_1e_4 + (2a_1 + a_2 + b_2)\alpha_4e_5$$
$$+ \sum_{k=3}^{\lfloor \frac{n}{2}\rfloor}\Big[(2a_1 + a_2 + b_2)\alpha_{2k-1} + \sum_{t=4}^{2k-2}(a_{t-1} + a_2\alpha_t)\alpha_{2k-t+2}\Big]e_{2k}$$
$$+ \sum_{k=3}^{\lfloor \frac{n-1}{2}\rfloor}\Big[(2a_1 + a_2 + b_2)\alpha_{2k} + \sum_{t=4}^{2k-1}(a_{t-1} + a_2\alpha_t)\alpha_{2k-t+3}\Big]e_{2k+1}.$$

By comparing the coefficients of the basis elements we have

$b_1 = 0,$ $(2a_1 + a_2 + b_2)\alpha_4 = (2a_1 + c_3)\alpha_4,$		(3)
$(2a_1 + a_2 + b_2)\alpha_{2k-1} + \sum\limits_{t=4}^{2k-2}(a_{t-1} + a_2\alpha_t)\alpha_{2k-t+2} =$ $= ((2k-1)a_1 + a_2)\alpha_{2k-1} +$ $+\sum\limits_{t=2}^{k-1}(c_{2k-2t+2} + (2t-2)a_2\alpha_{2k-2t+2})\alpha_{2t} +$ $++\sum\limits_{t=3}^{k-1}(a_{2k-2t+2} + (2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t-1},$	$3 \le k \le \left\lfloor\frac{n}{2}\right\rfloor$	(4)
$(2a_1 + a_2 + b_2)\alpha_{2k} + \sum\limits_{t=4}^{2k-1}(a_{t-1} + a_2\alpha_t)\alpha_{2k-t+3} =$ $= ((2k-2)a_1 + c_3)\alpha_{2k} +$ $+\sum\limits_{t=2}^{k-1}(c_{2k-2t+3} + (2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t} +$ $+\sum\limits_{t=3}^{k}(a_{2k-2t+3} + (2t-2)a_2\alpha_{2k-2t+4})\alpha_{2t-1}.$	$3 \le k \le \left\lfloor\frac{n-1}{2}\right\rfloor$	(5)

Now, we consider

$$P([[e_3, e_1], e_2]) = P([e_4, e_2]) = P\Big(\sum_{t=6}^{n}\alpha_{t-2}e_t\Big)$$
$$= \sum_{t=3}^{\lfloor \frac{n-1}{2}\rfloor}\Big[((2t-2)a_1 + c_3)e_{2t+1} + \sum_{k=2t+2}^{n}(c_{k-2t+2} + (2t-2)a_2\alpha_{k-2t+2})e_k\Big]\alpha_{2t-1}$$
$$+ \sum_{t=3}^{\lfloor \frac{n}{2}\rfloor}\Big[((2t-1)a_1 + a_2)e_{2t} + \sum_{k=2t+1}^{n}(a_{k-2t+2} + (2t-2)a_2\alpha_{k-2t+3})e_k\Big]\alpha_{2t-2}$$

$$= \sum_{k=3}^{\lfloor \frac{n-1}{2} \rfloor} ((2k-2)a_1 + c_3)\alpha_{2k-1}e_{2k+1}$$

$$+ \sum_{k=8}^{n} \sum_{t=3}^{\lfloor \frac{k-2}{2} \rfloor} (c_{k-2t+2} + (2t-2)a_2\alpha_{k-2t+2})\alpha_{2t-1}e_k$$

$$+ \sum_{k=3}^{\lfloor \frac{n}{2} \rfloor} ((2k-1)a_1 + a_2)\alpha_{2k-2}e_{2k}$$

$$+ \sum_{k=7}^{n} \sum_{t=3}^{\lfloor \frac{k-1}{2} \rfloor} (a_{k-2t+2} + (2t-2)a_2\alpha_{k-2t+3})\alpha_{2t-2}e_k$$

$$= (5a_1 + a_2)\alpha_4 e_6 + \sum_{k=3}^{\lfloor \frac{n-1}{2} \rfloor} \Big[((2k-2)a_1 + c_3)\alpha_{2k-1}$$

$$+ \sum_{t=3}^{k-1} (c_{2k-2t+3} + (2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t-1}$$

$$+ \sum_{t=3}^{k} (a_{2k-2t+3} + (2t-2)a_2\alpha_{2k-2t+4})\alpha_{2t-2} \Big] e_{2k+1}$$

$$+ \sum_{k=4}^{\lfloor \frac{n}{2} \rfloor} \Big[((2k-1)a_1 + a_2)\alpha_{2k-2}$$

$$+ \sum_{t=3}^{k-1} (c_{2k-2t+2} + (2t-2)a_2\alpha_{2k-2t+2})\alpha_{2t-1}$$

$$+ \sum_{t=3}^{k-1} (a_{2k-2t+2} + (2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t-2} \Big] e_{2k}.$$

On the other hand,

$$P([[e_3, e_1], e_2]) = [[P(e_3), e_1], e_2] + [[e_3, P(e_1)], e_2] + [[e_3, e_1], P(e_2)]$$

$$= [\sum_{t=3}^{n} c_{t-1}e_t, e_2] + [a_1 e_4 + a_2 \sum_{t=5}^{n} \alpha_{t-1}e_t, e_2] + b_2 \sum_{t=6}^{n} \alpha_{t-2}e_t$$

$$= \sum_{t=4}^{n-2} c_{t-1} \sum_{k=t+2}^{n} \alpha_{k-t+2}e_k + a_1 \sum_{t=6}^{n} \alpha_{t-2}e_t$$

$$+ a_2 \sum_{t=5}^{n-2} \alpha_{t-1} \sum_{k=t+2}^{n} \alpha_{k-t+2}e_k + b_2 \sum_{t=6}^{n} \alpha_{t-2}e_t$$

$$= (2a_1 + a_2 + c_3)\alpha_4 e_6$$

$$+ \sum_{k=7}^{n} \Big[(2a_1 + a_2 + c_3)\alpha_{k-2} + \sum_{t=5}^{k-2} (c_{t-1} + a_2\alpha_{t-1})\alpha_{k-t+2} \Big] e_k$$

$$= (2a_1 + a_2 + c_3)\alpha_4 e_6 + \sum_{k=3}^{\lfloor \frac{n-1}{2} \rfloor} \Big[(2a_1 + a_2 + c_3)\alpha_{2k-1} + \sum_{t=5}^{2k-1} (c_{t-1} + a_2\alpha_{t-1})\alpha_{2k-t+3}\Big] e_{2k+1} + \sum_{k=4}^{\lfloor \frac{n}{2} \rfloor} \Big[(2a_1 + a_2 + c_3)\alpha_{2k-2} + \sum_{t=5}^{2k-2} (c_{t-1} + a_2\alpha_{t-1})\alpha_{2k-t+2}\Big] e_{2k}.$$

By comparing the coefficients of the basis elements we get

$(2a_1 + a_2 + c_3)\alpha_4 = (5a_1 + a_2)\alpha_4$		(6)
$(2a_1 + a_2 + c_3)\alpha_{2k-1} + \sum_{t=5}^{2k-1} (c_{t-1} + a_2\alpha_{t-1})\alpha_{2k-t+3} = ((2k-2)a_1 + c_3)\alpha_{2k-1} + \sum_{t=3}^{k-1}(c_{2k-2t+3} + (2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t-1} + \sum_{t=3}^{k}(a_{2k-2t+3} + (2t-2)a_2\alpha_{2k-2t+4})\alpha_{2t-2},$	$3 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$	(7)
$(2a_1 + a_2 + c_3)\alpha_{2k-2} + \sum_{t=5}^{2k-2} (c_{t-1} + a_2\alpha_{t-1})\alpha_{2k-t+2} = ((2k-1)a_1 + a_2)\alpha_{2k-2} + \sum_{t=3}^{k-1}(c_{2k-2t+2} + (2t-2)a_2\alpha_{2k-2t+2})\alpha_{2t-1} + \sum_{t=3}^{k-1}(a_{2k-2t+2} + (2t-2)a_2\alpha_{2k-2t+3})\alpha_{2t-2}.$	$4 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$	(8)

According to $b_1 + b_2 = a_1 + a_2$, from equalities (3) and (6) we obtain

$$(a_1 - a_2)\alpha_4 = 0, \quad (3a_1 - c_3)\alpha_4 = 0.$$

By subtracting of identities (4) and (7) we obtain

$\sum_{t=3}^{k}(a_{2k-2t+3} - c_{2k-2t+4} + a_2\alpha_{2k-2t+4})\alpha_{2t-2} = 0,$	$3 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor.$

By summarizing of identities (4) and (7) we get

$\alpha_k(a_2 - (k-3)a_1) = \frac{k-1}{2} a_2 \sum_{t=5}^{k} \alpha_{t-1}\alpha_{k-t+4}$	$5 \le k \le n-2$, k is odd.

Similarly, we obtain from identities (5) and (8)

$(2a_1 + a_2 - c_3)\alpha_{2k} + \sum_{t=3}^{k}(a_{2k-2t+4} - c_{2k-2t+5} + a_2\alpha_{2k-2t+5})\alpha_{2t-2} = 0,$	$3 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$

and

$\alpha_k(a_2 - (k-3)a_1) = \frac{k-1}{2}a_2 \sum_{t=5}^{k} \alpha_{t-1}\alpha_{k-t+4}$	$5 \leq k \leq n-2$, k is even.

From equalities (4) and (5) in the case of $2k = n$ and $2k = n-1$, respectively, we obtain last the two restrictions of equality (2).

Considering the properties of the pre-derivation for $P([[e_i, e_1], e_2])$, $4 \leq i \leq n$ and $P([[e_i, e_2], e_2])$, $3 \leq i \leq n$, we have identical equalities. □

In the following proposition we give descriptions of the pre-derivations of algebras from the second class of filiform Leibniz algebras.

Proposition 2. *The pre-derivations of the filiform Leibniz algebras from the family $F_2(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$ have the following form:*

$P(e_1) = \sum_{t=1}^{n} a_t e_t,$	
$P(e_2) = b_2e_2 + b_{n-1}e_{n-1} + b_ne_n,$	
$P(e_3) = \sum_{t=2}^{n} c_t e_t,$	
$P(e_{2i}) = (2i-1)a_1e_{2i} + \sum_{t=2i+1}^{n}(a_{t-2i+2} + (2i-2)a_2\beta_{t-2i+3})e_t,$	$2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$
$P(e_{2i+1}) = ((2i-2)a_1 + c_3)e_{2i+1} + \sum_{t=2i+2}^{n}(c_{t-2i+2} + (2i-2)a_2\beta_{t-2i+2})e_t,$	$2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$

where

$(c_3 - 2a_1)\beta_4 = 0,$ $(b_2 - 2a_1)\beta_4 = 0,$ $c_2\beta_t = 0,$	$4 \leq t \leq n-1,$
$\sum_{t=3}^{k}(a_{2k-2t+3} - c_{2k-2t+4} + a_2\beta_{2k-2t+4})\beta_{2t-2} = 0,$	$3 \leq k \leq \lfloor \frac{n-1}{2} \rfloor,$

$(c_3-2a_1)\beta_{2k}=\sum\limits_{t=3}^{k}(a_{2k-2t+4}-c_{2k-2t+5}+a_2\beta_{2k-2t+5})\beta_{2t-2},$	$3\leq k\leq \lfloor\frac{n}{2}\rfloor-1,$
$(b_2-(k-2)a_1)\beta_k=\frac{k-1}{2}a_2\sum\limits_{t=5}^{k}\beta_{t-1}\beta_{k-t+4},$	$5\leq k\leq n-2,$
$(b_2-c_3-(n-5)a_1)\beta_{n-1}=a_2\sum\limits_{t=3}^{\frac{n-1}{2}}(2t-3)\beta_{n-2t+3}\beta_{2t-1}+$ $+\sum\limits_{t=2}^{\frac{n-3}{2}}(c_{n-2t+2}-a_{n-2t+1}+(2t-3)a_2\beta_{n-2t+2})\beta_{2t},$	*n is odd,*
$(b_2-(n-3)a_1)\beta_{n-1}=a_2\sum\limits_{t=3}^{\frac{n-2}{2}}(2t-3)\beta_{n-2t+3}\beta_{2t-1}+$ $+\sum\limits_{t=2}^{\frac{n-2}{2}}(c_{n-2t+2}-a_{n-2t+1}+(2t-3)a_2\beta_{n-2t+2})\beta_{2t},$	*n is even.*

(9)

Proof. Let P be a pre-derivation of a filiform Leibniz algebra L from the second class. Put

$$P(e_1)=\sum_{t=1}^{n}a_te_t,\quad P(e_2)=\sum_{t=1}^{n}b_te_t,\quad P(e_3)=\sum_{t=1}^{n}c_te_t.$$

From the definition of a pre-derivation we obtain that

$$\begin{aligned}P([[e_2,e_1],e_1])&=[[P(e_2),e_1],e_1]+[[e_2,P(e_1)],e_1]+[[e_2,e_1],P(e_1)]\\&=[[\sum_{t=1}^{n}b_te_t,e_1],e_1]+[[e_2,\sum_{t=1}^{n}a_te_t],e_1]\\&=[b_1e_3+\sum_{t=3}^{n-1}b_te_{t+1},e_1]=b_1e_4+\sum_{t=5}^{n}b_{t-2}e_t.\end{aligned}$$

On the other hand, $P([[e_2,e_1],e_1])=0$, since $[e_2,e_1]=0$. Thus, we have

$b_1=0,$	
$b_t=0,$	$3\leq t\leq n-2.$

Hence, $P(e_2)=b_2e_2+b_{n-1}e_{n-1}+b_ne_n$.

From

$$\begin{aligned}0=P([[e_1,e_1],e_3])&=[[P(e_1),e_1],e_3]+[[e_1,P(e_1)],e_3]+[[e_1,e_1],P(e_3)]\\=[e_3,\sum_{t=1}^{n}c_te_t]&=c_1e_4+c_2\sum_{t=5}^{n}\beta_{t-1}e_t,\end{aligned}$$

we get

$$c_1 = 0, \quad c_2\beta_t = 0, \quad 4 \leq t \leq n-1.$$

Inductively, we obtain from the definition of a pre-derivation applied to the triples $\{e_1, e_1, e_1\}$ and $\{e_i, e_1, e_1\}$ for $3 \leq i \leq n-2$,

$P(e_{2i}) = (2i-1)a_1e_{2i} + \sum\limits_{t=2i+1}^{n}(a_{t-2i+2} + (2i-2)a_2\beta_{t-2i+3})e_t,$	$2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$
$P(e_{2i+1}) = ((2i-2)a_1 + c_3)e_{2i+1} + \sum\limits_{t=2i+2}^{n}(c_{t-2i+2} + (2i-2)a_2\beta_{t-2i+2})e_t,$	$2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$

Now, we consider

$$\begin{aligned} P([[e_1, e_1], e_2]) &= [[P(e_1), e_1], e_2] + [[e_1, P(e_1)], e_2] + [[e_1, e_1], P(e_2)] \\ &= [[\sum_{t=1}^{n} a_te_t, e_1], e_2] + [[e_1, \sum_{t=1}^{n} a_te_t], e_2] + [e_3, b_2e_2 + b_{n-1} + b_te_t] \\ &= (2a_1 + b_2)\beta_4e_5 \\ &\quad + \sum_{k=3}^{\lfloor \frac{n}{2} \rfloor}\Big[(2a_1 + b_2)\beta_{2k-1}\sum_{t=4}^{2k-2}(a_{t-1} + a_2\beta_t)\beta_{2k-t+2}\Big]e_{2k} \\ &\quad + \sum_{k=3}^{\lfloor \frac{n-1}{2} \rfloor}\Big[(2a_1 + b_2)\beta_{2k} + \sum_{t=4}^{2k-1}(a_{t-1} + a_2\beta_t)\beta_{2k-t+3}\Big]e_{2k+1}. \end{aligned}$$

On the other hand,

$$P([[e_1, e_1], e_2]) = P([e_3, e_2]) = P\Big(\sum_{t=5}^{n}\beta_{t-1}e_t\Big)$$

$$\begin{aligned} &= \beta_4(2a_1 + c_3)e_5 + \sum_{k=3}^{\lfloor \frac{n}{2} \rfloor}\Big[\beta_{2k-1}(2k-1)a_1 \\ &\quad + \sum_{t=2}^{k-1}\beta_{2t}(c_{2k-2t+2} + (2t-2)a_2\beta_{2k-2t+2}) \\ &\quad + \sum_{t=3}^{k-1}\beta_{2t-1}(a_{2k-2t+2} + (2t-2)a_2\beta_{2k-2t+3})\Big]e_{2k} \\ &+ \sum_{k=3}^{\lfloor \frac{n-1}{2} \rfloor}\Big[\beta_{2k}((2k-2)a_1 + c_3) + \sum_{t=2}^{k-1}\beta_{2t}(c_{2k-2t+3} + (2t-2)a_2\beta_{2k-2t+3}) \\ &\quad + \sum_{t=3}^{k}\beta_{2t-1}(a_{2k-2t+3} + (2t-2)a_2\beta_{2k-2t+4})\Big]e_{2k+1}. \end{aligned}$$

By comparing the coefficients of the basis elements, we have

$(2a_1+b_2)\beta_4=(2a_1+c_3)\beta_4,$		(10)
$(2a_1+b_2)\beta_{2k-1}+\sum\limits_{t=4}^{2k-2}(a_{t-1}+a_2\beta_t)\beta_{2k-t+2}= = \beta_{2k-1}(2k-1)a_1+ +\sum\limits_{t=2}^{k-1}\beta_{2t}(c_{2k-2t+2}+(2t-2)a_2\beta_{2k-2t+2})+ +\sum\limits_{t=3}^{k-1}\beta_{2t-1}(a_{2k-2t+2}+(2t-2)a_2\beta_{2k-2t+3}),$	$3\le k\le\left\lfloor\frac{n}{2}\right\rfloor$	(11)
$(2a_1+b_2)\beta_{2k}+\sum\limits_{t=4}^{2k-1}(a_{t-1}+a_2\beta_t)\beta_{2k-t+3}= = \beta_{2k}((2k-2)a_1+c_3)+ +\sum\limits_{t=2}^{k-1}\beta_{2t}(c_{2k-2t+3}+(2t-2)a_2\beta_{2k-2t+3})+ +\sum\limits_{t=3}^{k}\beta_{2t-1}(a_{2k-2t+3}+(2t-2)a_2\beta_{2k-2t+4}).$	$3\le k\le\left\lfloor\frac{n-1}{2}\right\rfloor$	(12)

Analogously, from the identity

$$P([e_4,e_2])=P([[e_3,e_1],e_2])=[[P(e_3),e_1],e_2]+[[e_3,P(e_1)],e_2]+[[e_3,e_1],P(e_2)],$$

we obtain the following conditions for the coefficients:

$(a_1+b_2+c_3)\beta_4=5a_1\beta_4,$		(13)
$(a_1+b_2+c_3)\beta_{2k-1}+\sum\limits_{t=5}^{2k-1}(c_{t-1}+a_2\beta_{t-1})\beta_{2k-t+3}= = \beta_{2k-1}((2k-2)a_1+c_3)+ +\sum\limits_{t=3}^{k-1}\beta_{2t-1}(c_{2k-2t+3}+(2t-2)a_2\beta_{2k-2t+3})+ +\sum\limits_{t=3}^{k}\beta_{2t-2}(a_{2k-2t+3}+(2t-2)a_2\beta_{2k-2t+4}),$	$3\le k\le\left\lfloor\frac{n-1}{2}\right\rfloor$	(14)

$(a_1+b_2+c_3)\beta_{2k-2}+\sum\limits_{t=5}^{2k-2}(c_{t-1}+a_2\beta_{t-1})\beta_{2k-t+2}= =\beta_{2k-2}(2k-1)a_1+ +\sum\limits_{t=3}^{k-1}\beta_{2t-1}(c_{2k-2t+2}+(2t-2)a_2\beta_{2k-2t+2})+ +\sum\limits_{t=3}^{k-1}\beta_{2t-2}(a_{2k-2t+2}+(2t-2)a_2\beta_{2k-2t+3}).$	$4\leq k\leq\left\lfloor\frac{n}{2}\right\rfloor$	(15)

It is not difficult to see from (10) and (13) that we have

$$(c_3-2a_1)\beta_4=0,\quad (b_2-2a_1)\beta_4=0.$$

Similarly to the proof of Proposition 1, by summarizing and subtracting equalities (11) and (14), we obtain

$\sum\limits_{t=3}^{k}(a_{2k-2t+3}-c_{2k-2t+4}+a_2\beta_{2k-2t+4})\beta_{2t-2}=0,$	$3\leq k\leq\left\lfloor\frac{n-1}{2}\right\rfloor$

and

$(b_2-(k-2)a_1)\beta_k=\frac{k-1}{2}a_2\sum\limits_{t=5}^{k}\beta_{t-1}\beta_{k-t+4}$	$5\leq k\leq n-2,$ k is odd.

From equalities (12) and (15) we have

$(c_3-2a_1)\beta_{2k}=\sum\limits_{t=3}^{k}(a_{2k-2t+4}-c_{2k-2t+5}+a_2\beta_{2k-2t+5})\beta_{2t-2},$	$3\leq k\leq\left\lfloor\frac{n}{2}\right\rfloor-1,$

and

$(b_2-(k-2)a_1)\beta_k=\frac{k-1}{2}a_2\sum\limits_{t=5}^{k}\beta_{t-1}\beta_{k-t+4}$	$5\leq k\leq n-2,$ k is even.

From equalities (11) and (12) in the case of $2k=n-1$ and $2k=n$, respectively, we obtain the last two restrictions of equalities (9).

Considering the properties of the pre-derivation for $P([[e_i,e_1],e_2])$ for $4\leq i\leq n$ and $P([[e_i,e_2],e_2])$ for $3\leq i\leq n$, we have the identical equalities. $\square$

4 Strongly nilpotent filiform Leibniz algebras

In this section we determine those algebras in the first two classes of filiform Leibniz algebras that are non-strongly nilpotent. For the third class this question is reduced to the same question about Lie algebras.

First, we consider the case of filiform Leibniz algebras from the first class. From Proposition 1 it is obvious that if there exist the parameters a_1, a_2, c_2, c_3 such that $(a_1, a_2, c_2, c_3) \neq (0,0,0,0)$ and the conditions (2) hold, then a filiform Leibniz algebra of the first family is non-strongly nilpotent, otherwise is strongly nilpotent.

Proposition 3. *Let $L(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$ be a filiform Leibniz algebra of the first family. If $\alpha_4 = \alpha_5 = \dots = \alpha_{n-1} = 0$, then L is non-strongly nilpotent.*

Proof. It is immediate that, if $\alpha_4 = \alpha_5 = \dots = \alpha_{n-1} = 0$, then the restriction (2) holds for any values of a_1, a_2, c_3. Thus, we have that L has a non-nilpotent pre-derivation, which implies that L is non-strongly nilpotent. □

It is obvious that any algebra from the family $F_1(0, \dots, 0, \alpha_n, \theta)$ is isomorphic to one of the following four algebras:

$$F_1(0,\dots,0,0,0), \quad F_1(0,\dots,0,0,1), \quad F_1(0,\dots,0,1,0), \quad F_1(0,\dots,0,1,1).$$

Note that the algebras $F_1(0,\dots,0,0,0)$, $F_1(0,\dots,0,0,1)$ and $F_1(0,\dots,0,1,1)$ are non-characteristically nilpotent (see [18]). The algebra $F_1(0,\dots,0,1,0)$ is characteristically nilpotent, but non-strongly nilpotent.

Now we consider the case of $\alpha_i \neq 0$ for some i $(4 \leq i \leq n-1)$. Then from (2) we have that $c_2 = 0$.

Theorem 2. *Let L be a filiform Leibniz algebra from the family $F_1(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$ and let n be even. Then L is non-strongly nilpotent if and only if the parameters $(\alpha_4, \alpha_5, \alpha_6, \dots, \alpha_{n-1}, \alpha_n, \theta)$ have one of the following values:*

i)	$\alpha_4 \neq 0$	
	$\alpha_k = (-1)^k C_{k-3}^2 \alpha_4^{k-3}$,	$5 \leq k \leq n-2$;

ii)	$\alpha_{(2s-3)t+3} = (-1)^{t+1} C_t^{2s-2} \alpha_{2s}^t$,	$3 \leq s \leq \frac{n-2}{2}$, $1 \leq t \leq \left\lfloor \frac{n-5}{2s-3} \right\rfloor$,
	$\alpha_j = 0$,	$4 \leq j \leq n-2$, $j \neq (2s-3)t+3$;

iii)	$\alpha_{2i} = 0$,	$2 \leq i \leq \frac{n-2}{2}$;

where $C_n^p = \frac{1}{(p-1)n+1}\binom{pn}{n}$ is the p-th Catalan number.

Proof. From Proposition 1 we have

$(a_1-a_2)\alpha_4=0, \quad (3a_1-c_3)\alpha_4=0,$	
$\sum_{t=3}^{k}(a_{2k-2t+3}-c_{2k-2t+4}+a_2\alpha_{2k-2t+4})\alpha_{2t-2}=0,$	$3\leq k\leq \frac{n-2}{2},$
$(2a_1+a_2-c_3)\alpha_{2k}+ + \sum_{t=3}^{k}(a_{2k-2t+4}-c_{2k-2t+5}+a_2\alpha_{2k-2t+5})\alpha_{2t-2}=0,$	$3\leq k\leq \frac{n-2}{2},$
$(a_2-(k-3)a_1)\alpha_k=\frac{k-1}{2}a_2\sum_{t=5}^{k}\alpha_{t-1}\alpha_{k-t+4},$	$5\leq k\leq n-2,$
$(a_2-(n-4)a_1)\alpha_{n-1}=a_2\sum_{t=3}^{\frac{n-2}{2}}(2t-3)\alpha_{n-2t+3}\alpha_{2t-1}+ + \sum_{t=2}^{\frac{n-2}{2}}(c_{n-2t+2}-a_{n-2t+1}+(2t-3)a_2\alpha_{n-2t+2})\alpha_{2t}.$	

(16)

Case 1. If $\alpha_4\neq 0$, then from the first two equalities of (16) we get $a_2=a_1$, $c_3=3a_1$ and from the next two equalities of (16) we obtain

$c_i=a_{i-1}+a_1\alpha_i,$	$4\leq i\leq n-3.$

Since $a_2=a_1$, $c_3=3a_1$, we get that L is non-strongly nilpotent if and only if $a_1\neq 0$. Therefore we have

$\alpha_k=\frac{k-1}{2(4-k)}\sum_{t=5}^{k}\alpha_{t-1}\alpha_{k-t+4},$	$5\leq k\leq n-2,$
$(5-n)a_1\alpha_{n-1}= = (c_{n-2}-a_{n-3}+a_1\alpha_{n-2})\alpha_4+a_1\sum_{t=5}^{n-2}(t-2)\alpha_{n-t+2}\alpha_t.$	

Using identity (1) we get that

$\alpha_k=(-1)^kC_{k-3}^2\alpha_4^{k-3},$	$5\leq k\leq n-2$

and

$$c_{n-2}=\frac{1}{\alpha_4}\Big((5-n)a_1\alpha_{n-1}-a_1\sum_{t=5}^{n-2}(t-2)\alpha_{n-t+2}\alpha_t\Big)+a_{n-3}-a_1\alpha_{n-2}.$$

Note that the parameters α_{n-1},α_n and θ are free and we have the case i).

Case 2. Let $\alpha_{2s} \neq 0$ for some s $(3 \leq s \leq \frac{n-2}{2})$ and $\alpha_{2i} = 0$ for $2 \leq i \leq s-1$. Then similarly to the previous case from identity (16) we get

$$(2a_1 + a_2 - c_3)\alpha_{2s} = 0, \qquad (a_2 - (2s-3)a_1)\alpha_{2s} = 0,$$

which derive $a_2 = (2s-3)a_1$, $c_3 = (2s-1)a_1$ and

$c_i = a_{i-1} + a_2\alpha_i,$	$4 \leq i \leq n-2s+1.$

If L is non-strongly nilpotent then $a_1 \neq 0$. Consequently, we have

$(2s-k)\alpha_k = \dfrac{(k-1)(2s-3)}{2}\displaystyle\sum_{t=5}^{k}\alpha_{t-1}\alpha_{k-t+4},$	$5 \leq k \leq n-2.$

Inductively, we obtain by using $\alpha_{2i} = 0$ for $2 \leq i \leq s-1$ and applying (1) that

$\alpha_j = 0,$	$4 \leq j \leq n-2,$ $j \neq (2s-3)t+3,$
$\alpha_{(2s-3)t+3} = (-1)^{t+1}C_t^{2s-2}\alpha_{2s}^t,$	$1 \leq t \leq \left\lfloor \dfrac{n-5}{2s-3} \right\rfloor.$

From the last identity of (16) we have

$$c_{n-2s+2} = \frac{1}{\alpha_{2s}}\Big((2s+1-n)a_1\alpha_{n-1} - (2s-3)a_1\sum_{t=2s+1}^{n-2s+2}(t-2)\alpha_{n-t+2}\alpha_t\Big) + a_{n-2s+1} - (2s-3)^2a_1\alpha_{n-2s+2}.$$

The parameters α_{n-1}, α_n and θ may take any values and we obtain case ii).

Case 3. Let $\alpha_{2i} = 0$ for all i $(2 \leq i \leq \frac{n-2}{2})$. Then the first four equalities of (16) hold and from the last equalities we have

$\alpha_5(a_2 - 2a_1) = 0,$	
$\alpha_{2s+1}(a_2 - (2s-2)a_1) = sa_2\displaystyle\sum_{t=3}^{s}\alpha_{2t-1}\alpha_{2s+5-2t},$	$3 \leq s \leq \dfrac{n-2}{2}.$
$(a_2 - (n-4)a_1)\alpha_{n-1} = a_2\displaystyle\sum_{t=3}^{\frac{n-2}{2}}(2t-3)\alpha_{n-2t+3}\alpha_{2t-1}.$	

Taking $a_1 = a_2 = 0$ and $c_3 \neq 0$, we have that previous equalities hold for any values of α_{2s+1}. Since $c_3 \neq 0$, this algebra is non-strongly nilpotent and we obtain the case iii). □

Theorem 3. *Let L be a filiform Leibniz algebra from the family $F_1(\alpha_4, \alpha_5, \dots, \alpha_n, \theta)$ and let n be odd. L is non-strongly nilpotent if and only if the parameters $(\alpha_4, \alpha_5, \alpha_6, \dots, \alpha_{n-1}, \alpha_n, \theta)$ have one of the following values:*

i)	$\alpha_4 \neq 0$	
	$(-1)^k C_{k-3}^2 \alpha_4^{k-3}$,	$5 \leq k \leq n-2$;
ii)	$\alpha_{(2s-3)t+3} = (-1)^{t+1} C_t^{2s-2} \alpha_{2s}^t$,	$3 \leq s \leq \frac{n-3}{2}$, $1 \leq t \leq \lfloor \frac{n-5}{2s-3} \rfloor$,
	$\alpha_j = 0$,	$4 \leq j \leq n-2$, $j \neq (2s-3)t+2$;
iii)	$\alpha_{2i} = 0$,	$2 \leq i \leq \frac{n-1}{2}$;
iv)	$\alpha_{(2s-2)t+3} = (-1)^{t+1} C_t^{2s-1} \alpha_{2s+1}^t$,	$2 \leq s \leq \frac{n-3}{2}$, $1 \leq t \leq \lfloor \frac{n-5}{2s-2} \rfloor$,
	$\alpha_j = 0$,	$4 \leq j \leq n-2$ $j \neq (2s-2)t+3$.

Proof. From Proposition 1 we have

$(a_1 - a_2)\alpha_4 = 0$,	
$(3a_1 - c_3)\alpha_4 = 0$,	
$\sum\limits_{t=3}^{k} (a_{2k-2t+3} - c_{2k-2t+4} + a_2\alpha_{2k-2t+4})\alpha_{2t-2} = 0$,	$3 \leq k \leq \frac{n-1}{2}$,
$(2a_1 + a_2 - c_3)\alpha_{2k} + \sum\limits_{t=3}^{k} (a_{2k-2t+4} - c_{2k-2t+5} + a_2\alpha_{2k-2t+5})\alpha_{2t-2} = 0$,	$3 \leq k \leq \frac{n-3}{2}$,
$(a_2 - (k-3)a_1)\alpha_k = \frac{k-1}{2} a_2 \sum\limits_{t=5}^{k} \alpha_{t-1}\alpha_{k-t+4}$,	$5 \leq k \leq n-2$,
$(2a_2 - c_3 - (n-6)a_1)\alpha_{n-1} = a_2 \sum\limits_{t=3}^{\frac{n-1}{2}} (2t-3)\alpha_{n-2t+3}\alpha_{2t-1} + \sum\limits_{t=2}^{\frac{n-3}{2}} (c_{n-2t+2} - a_{n-2t+1} + (2t-3)a_2\alpha_{n-2t+2})\alpha_{2t}$	

(17)

Case 1. $\alpha_{2s} \neq 0$ for some s $(2 \leq s \leq \frac{n-3}{2})$ and $\alpha_{2i} = 0$ for $2 \leq i \leq s-1$. Then similarly to the proof of Theorem 2 we get

$a_2 = (2s-3)a_1,$	
$c_3 = (2s-1)a_1,$	
$c_i = a_{i-1} + a_2\alpha_i,$	$4 \leq i \leq n-2s+1.$

Consequently we have

$\alpha_k = \dfrac{(k-1)(2s-3)}{2(2s-k)} \sum\limits_{t=5}^{k} \alpha_{t-1}\alpha_{k-t+4},$	$5 \leq k \leq n-2.$

Inductively, we obtain by applying (1) that

$\alpha_{(2s-3)t+3} = (-1)^{t+1} C_t^{2s-2} \alpha_{2s}^t,$	$1 \leq t \leq \left\lfloor \dfrac{n-5}{2s-3} \right\rfloor,$
$\alpha_j = 0,$	$4 \leq j \leq n-2,$ $j \neq (2s-3)t+3.$

From the last identity of (17) we have

$$c_{n-2s+2} = \frac{1}{\alpha_{2s}}\Big((2s+1-n)a_1\alpha_{n-1} - (2s-3)a_1 \sum_{t=2s+1}^{n-2s+2} (t-2)\alpha_{n-t+2}\alpha_t\Big)$$
$$+ a_{n-2s+1} - (2s-3)^2 a_1 \alpha_{n-2s+2}.$$

Thus, we have the cases i) and ii).

Case 2. Let $\alpha_{2i} = 0$ for all i $(2 \leq i \leq \frac{n-3}{2})$. Then the first four equalities of (17) hold and from the last two equalities we have

$\alpha_5(a_2 - 2a_1) = 0,$	
$\alpha_{2s+1}(a_2 - (2s-2)a_1) = sa_2 \sum\limits_{t=3}^{s} \alpha_{2t-1}\alpha_{2s+5-2t},$	$3 \leq s \leq \dfrac{n-3}{2},$
$(2a_2 - c_3 - (n-6)a_1)\alpha_{n-1} = 0.$	

(18)

If $\alpha_{n-1} = 0$, then taking $a_1 = a_2 = 0$ and $c_3 \neq 0$, we have a non-nilpotent pre-derivation for any values of α_{2i+1}, and so we have the case iii).

If $\alpha_{n-1} \neq 0$, then $c_3 = 2a_2 + (n-6)a_1$ and we obtain that non-nilpotency of pre-derivations depends of the parameters α_{2i+1}.

Let α_{2s+1} be the first non-vanishing parameter among $\{\alpha_5, \alpha_7, \ldots, \alpha_{n-4}, \alpha_{n-2}\}$. Then we get

$$a_2 = (2s-2)a_1.$$

Inductively, we obtain by applying (1) and (18) that

$\alpha_{(2s-2)t+3} = (-1)^{t+1} C_t^{2s-1} \alpha_{2s+1}^t,$	$1 \le t \le \left\lfloor \dfrac{n-5}{2s-3} \right\rfloor$

and

$\alpha_j = 0,$	$4 \le j \le n-2,$ $j \neq (2s-2)t+3.$

Therefore, we have the case iv). □

Now we give the classification of non-strongly nilpotent filiform Leibniz algebras from the second class.

Proposition 4. *Let $L(\beta_4, \beta_5, \dots, \beta_n, \gamma)$ be a filiform Leibniz algebra of the second family. If $\beta_4 = \beta_5 = \dots = \beta_{n-1} = 0$, then L is non-strongly nilpotent.*

Proof. Analogously to the proof of Proposition 3. □

It is obvious that any algebra from the family $F_2(0, \dots, 0, \beta_n, \gamma)$ is isomorphic to one of the algebras $F_2(0, \dots, 0, 0, 0)$, $F_2(0, \dots, 0, 1, 0)$, $F_2(0, \dots, 0, 0, 1)$. Note that these algebras are non-characteristically nilpotent (see [18]).

Now we consider the case of $\beta_i \neq 0$ for some i $(4 \le i \le n-1)$. Then from (9) we have that $c_2 = 0$.

Theorem 4. *Let L be a n-dimensional complex non-strongly nilpotent filiform Leibniz algebra from the family $F_2(\beta_4, \dots, \beta_n, \gamma)$ and n even. Then L is isomorphic to one of the following algebras:*

$F_2^{2s}(0, \dots, 0, \beta_{2s}, 0 \dots, 0, \beta_{n-1}, \beta_n, \gamma),$	$\beta_{2s} = 1,$	$2 \le s \le \dfrac{n-2}{2},$
$F_2(0, \beta_5, 0, \beta_7, 0, \dots, 0, \beta_{n-1}, \beta_n, \gamma).$		

Proof. From Proposition 2 we have:

$(c_3 - 2a_1)\beta_4 = 0,$	
$(b_2 - 2a_1)\beta_4 = 0,$	
$\displaystyle\sum_{t=3}^{k} (a_{2k-2t+3} - c_{2k-2t+4} + a_2\beta_{2k-2t+4})\beta_{2t-2} = 0,$	$3 \le k \le \dfrac{n-2}{2},$
$\displaystyle(c_3 - 2a_1)\beta_{2k} = \sum_{t=3}^{k} (a_{2k-2t+4} - c_{2k-2t+5} + a_2\beta_{2k-2t+5})\beta_{2t-2},$	$3 \le k \le \dfrac{n-2}{2},$
$\displaystyle(b_2 - (k-2)a_1)\beta_k = \frac{k-1}{2} a_2 \sum_{t=5}^{k} \beta_{t-1}\beta_{k-t+4},$	$5 \le k \le n-2,$

$(b_2-(n-3)a_1)\beta_{n-1}=a_2\sum\limits_{t=3}^{\frac{n-2}{2}}(2t-3)\beta_{n-2t+3}\beta_{2t-1}+$ $+\sum\limits_{t=2}^{\frac{n-2}{2}}(c_{n-2t+2}-a_{n-2t+1}+(2t-3)a_2\beta_{n-2t+2})\beta_{2t}.$	

(19)

Case 1. Let $\beta_4\neq 0$, then from (19) we have

$c_3=b_2=2a_1,$	
$c_i=a_{i-1}+a_2\beta_i,$	$4\leq i\leq n-3,$
$(4-k)a_1\beta_k=\dfrac{k-1}{2}a_2\sum\limits_{t=5}^{k}\beta_{t-1}\beta_{k-t+4},$	$5\leq k\leq n-2,$
$(5-n)a_1\beta_{n-1}=(c_{n-2}-a_{n-3}-a_2\beta_{n-2})\beta_4+$ $+a_2\sum\limits_{t=5}^{n-2}(t-2)\beta_{n-t+2}\beta_t.$	

Since L is non-strongly nilpotent, we get $a_1\neq 0$ and

$\beta_5=-\dfrac{2a_2}{a_1}\beta_4^2,$	
$\beta_k=\dfrac{(k-1)a_2}{2(4-k)a_1}\sum\limits_{t=5}^{k}\beta_{t-1}\beta_{k-t+4},$	$6\leq k\leq n-2.$

From the isomorphism criterion in [10, Theorem 4.4] we have that if two algebras from the family $F_2(\beta_4,\ldots,\beta_n,\gamma)$ are isomorphic, then there exist $A,B,D\in\mathbb{C}$, such that

$$\beta_4'=\frac{D}{A^2}\beta_4,\qquad \beta_5'=\frac{D}{A^3}(\beta_5-\frac{2B}{A}\beta_4^2),$$

where β_i and β_i' are parameters of the first and second algebras, respectively.

Putting $D=\frac{A^2}{\beta_4}$ and $B=\frac{A\beta_5}{2\beta_4^2}$ we obtain $\beta_4'=1$, $\beta_5'=0$. Therefore, we have shown that, if L is a non-strongly nilpotent algebra from the family $F_2(\beta_4,\ldots,\beta_n,\gamma)$, with $\beta_4\neq 0$, then we may always suppose

$$\beta_4=1,\qquad \beta_5=0.$$

Moreover, from $\beta_5=-\frac{2a_2}{a_1}\beta_4^2$, we obtain $a_2=0$, which implies $\beta_k=0$ for $6\leq k\leq n-2$ and

$$c_{n-2}=a_{n-3}+\frac{(5-n)a_1\beta_{n-1}}{\beta_4}.$$

Thus, L is isomorphic to the algebra $F_2(1,0,\ldots,0,\beta_{n-1},\beta_n,\gamma)$.

Case 2. Let $\beta_{2s} \neq 0$ for some s ($3 \leq s \leq \frac{n-2}{2}$) and $\beta_{2i} = 0$ for $2 \leq i \leq s-1$. Then we have

$b_2 = (2s-2)a_1,$	
$c_3 = 2a_1,$	
$c_i = a_{i-1} + a_2\beta_i,$	$4 \leq i \leq n-2s+1,$
$(2s-k)a_1\beta_k = \frac{k-1}{2}a_2\sum\limits_{t=5}^{k}\beta_{t-1}\beta_{k-t+4},$	$5 \leq k \leq n-2,$
$(2s+1-n)a_1\beta_{n-1} = (c_{n-2s+2} - a_{n-2s+1} - a_2\beta_{n-2s+2})\beta_{2s} + a_2\sum\limits_{t=5}^{n-2}(t-2)\beta_{n-t+2}\beta_t.$	

Since L is non-strongly nilpotent, we get $a_1 \neq 0$, which implies

$$\beta_5 = \cdots = \beta_{2s-1} = 0.$$

Moreover, we have

$$\beta_{4s-3} = -\frac{(2s-2)a_2}{(2s-3)a_1}\beta_{2s}^2.$$

From the isomorphism criterion for filiform Leibniz algebras of the second class in [10, Theorem 4.4], we obtain

$$\beta'_{2s} = \frac{D}{A^{2s-2}}\beta_{2s},$$
$$\beta'_{4s-3} = \frac{D}{A^{4s-2}}\left(\beta_{4s-3} - \frac{(2s-2)B}{A}\beta_{2s}^2\right).$$

Putting $D = \frac{A^{2s-2}}{\beta_{2s}}$ and $B = \frac{A\beta_{4s-3}}{(2s-2)\beta_{2s}^2}$, we have $\beta'_{2s} = 1$, $\beta'_{4s-3} = 0$. Therefore, we have shown that, if L is non-strongly nilpotent algebra from the family $F_2(\beta_4, \ldots, \beta_n, \gamma)$, with $\beta_{2s} \neq 0$ and $\beta_{2i} = 0$ for $2 \leq i \leq s-1$, then we may always suppose

$$\beta_{2s} = 1, \quad \beta_{4s-3} = 0.$$

Moreover, from $\beta_{4s-3} = -\frac{(2s-2)a_2}{(2s-3)a_1}\beta_{2s}^2$, we obtain $a_2 = 0$, which implies $\beta_k = 0$ for $2s+1 \leq k \leq n-2$ and

$$c_{n-2s+2} = a_{n-2s+1} + \frac{(2s+1-n)a_1\beta_{n-1}}{\beta_{2s}}.$$

Thus, L isomorphic to the algebra

$$F_2^{2s}(0, \ldots, 0, \beta_{2s}, 0 \ldots, 0, \beta_{n-1}, \beta_n, \gamma), \quad \beta_{2s} = 1.$$

Case 3. Let $\beta_{2s} = 0$ for $2 \leq s \leq \frac{n-2}{2}$, then we have

$(b_2 - (k-2)a_1)\beta_k = \frac{k-1}{2} a_2 \sum_{t=5}^{k} \beta_{t-1}\beta_{k-t+4},$	$5 \leq k \leq n-2,$
$(b_2 - (n-3)a_1)\beta_{n-1} = a_2 \sum_{t=3}^{\frac{n-2}{2}} (2t-3)\beta_{n-2t+3}\beta_{2t-1}.$	

Taking $a_1 = b_2 = 0$ and $c_3 \neq 0$, we have that the previous identities hold for any values of β_{2s+1}. Since $c_3 \neq 0$, this algebra is non-strongly nilpotent. Therefore, we obtain the algebra $F_2(0, \beta_5, 0, \beta_7, 0, \dots, 0, \beta_{n-1}, \beta_n, \gamma)$. □

Theorem 5. *Let L be a n-dimensional complex non-strongly nilpotent filiform Leibniz algebra from the family $F_2(\beta_4, \dots, \beta_n, \gamma)$ and n odd. Then L is isomorphic to one of the following algebras:*

$F_2^j(0, \dots, 0, \beta_j, 0 \dots, 0, \beta_{n-1}, \beta_n, \gamma),$	$\beta_j = 1,$	$4 \leq j \leq n-2,$
$F_2(0, \beta_5, 0, \beta_7, 0, \dots, 0, \beta_{n-2}, 0, \beta_n, \gamma).$		

Proof. Analogously to the proofs of Theorems 3 and 4. □

Now, we consider a Leibniz algebra L from the third family $F_3(\theta_1, \theta_2, \theta_3)$. Note that L is a parametric algebra with parameters $\theta_1, \theta_2, \theta_3$ and $\alpha_{i,j}^k$. The parameters $\alpha_{i,j}^k$ appear from the multiplications $[e_i, e_j]$ for $\leq i < j \leq n-1$. In the case of $\theta_1 = \theta_2 = \theta_3 = 0$, the algebra L is a Lie algebra.

In the next proposition we assert that the strongly nilpotency of $L(\theta_1, \theta_2, \theta_3)$ is equivalent to the strongly nilpotency of $L(0, 0, 0)$.

Proposition 5. *An algebra $L(\theta_1, \theta_2, \theta_3)$ from the family $F_3(\theta_1, \theta_2, \theta_3)$ is strongly nilpotent if and only if the algebra $L(0, 0, 0)$ is strongly nilpotent.*

Proof. Note that the parameters $\theta_1, \theta_2, \theta_3$ appear only in the multiplications $[e_1, e_1]$, $[e_1, e_2]$, $[e_2, e_2]$. Since $[P(x), y],\ [x, P(y)],\ [x, y] \in L^2$ and $e_1, e_2 \notin L^2$ the parameters θ_i do not appear in the identity

$$P([[x, y], z]) = [[P(x), y], z] + [[x, P(y)], z] + [[x, y], P(z)], \quad x, y, z \in L.$$

Therefore, the spaces of pre-derivations of $L(\theta_1, \theta_2, \theta_3)$ and $L(0, 0, 0)$ coincide. □

It should be noted that family $F_3(0, 0, 0)$ give us the class of filiform Lie algebras. Filiform Lie algebras admitting a non-singular pre-derivation but no non-singular derivation are classified up to dimension 11 in [6]. From this result we have the classification of non-strongly nilpotent filiform Lie algebras with dimension $n \leq 11$, but it is not known the description of strongly nilpotent filiform Lie algebras for arbitrary dimension.

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