Funda Karaçal; Ümit Ertuğrul; M. Nesibe Kesicioğlu
Generating methods for principal topologies on bounded lattices

Kybernetika, Vol. 57 (2021), No. 4, 714–736

Persistent URL: `http://dml.cz/dmlcz/149216`

KYBERNETIKA — VOLUME 57 (2021), NUMBER 4, PAGES 714–736

GENERATING METHODS FOR PRINCIPAL TOPOLOGIES ON BOUNDED LATTICES

Funda Karaçal, Ümit Ertuğrul and M. Nesibe Kesicioğlu

In this paper, some generating methods for principal topology are introduced by means of some logical operators such as uninorms and triangular norms and their properties are investigated. Defining a pre-order obtained from the closure operator, the properties of the pre-order are studied.

Keywords: principal topology, bounded lattice, generating method, uninorm, triangular norm

Classification: 03E72, 03B52, 08A72, 54A10, 06B30, 06F30

1. INTRODUCTION

The notion of principal space introduced by [1], with the name of Diskrete Rắume, is a topological space in which any intersections of open sets is also open. Later on, this kind of topologies has been called as principal spaces [5, 18, 19, 20, 22, 25].

Principal spaces have significant roles in the topological sense. [22] has shown that the lattice of topologies on any set is complemented and each topology has a principal topology complement. This demonstrates the importance of them.

Principal spaces with more applications like geometry, digital topology, diverse branches of computer sciences, natural and social sciences are used in theoretical physics [23].

Finite spaces are special forms of principal spaces [3]. Note that principal topologies are also known as Alexandrov topologies and there exists a one-to-one correspondence between Alexandrov topologies and preorders [18, 25]. The close relationships between the principal spaces and posets raise the question of whether it is possible to obtain a principle topology by means of some operators on bounded lattices. In this paper, we give some generating methods for principal topologies. In this sense, triangular norms and uninorms known existence on bounded lattices provide a very rich resource for generating principal topologies. The paper is organized as follows: In Section 2, we review some basic concepts and notations which will be used in the paper. In Section 3, for a set X and a family of the functions $\{f_i\}_{i\in I}$ with for all $i \in I$, $f_i : X \to X$, we define a topology denoted by $\mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle}$. We show that $(X, \mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle})$ is a quasi-Hausdorff space. We give a relationship between the principal topology given in [13] and $\langle\{U_{x_0}\}_{x_0\in A}\rangle$-topology on a bounded lattice L. We introduce $\langle\{f_i\}_{i\in I}\rangle$-point for

DOI: 10.14736/kyb-2021-4-0714

the topological space $(X, \mathcal{T}_{\langle \{f_i\}_{i \in I} \rangle})$. We determine a relationship between the notions of minimal point and $\langle \{U_{x_0}\}_{x_0 \in A} \rangle$-point. We present some properties of $\langle \{U_{x_0}\}_{x_0 \in A} \rangle$-point when U is a t-norm or t-conorm. Also, we show that the topology $(L, \mathcal{P}(T_{x_0}))$ is a T_0 space. We introduce the notion of fixed point for the topology $(L, \mathcal{T}_{\langle \{f_i\}_{i \in I} \rangle})$. We present a relationship between a $\langle \{U_{x_0}\}_{x_0 \in A} \rangle$-point and a fixed point. In Section 4, we give two topological closure operators $c^*_{U,A}$ and $c^{**}_{U,A}$ by means of a uninorm U and a set $A \subseteq L$ with under some conditions. We define two topologies $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$ obtained from the introduced closure operators. We investigate some properties of the introduced topologies. We show that $\mathfrak{T}_{c^*_{U,A}}$ is a principal topology. Also, we present that $\mathfrak{T}_{c^*_{U,A}}$ is a quasi Hausdorff space but it is not a Hausdorff space. We investigate the relationship between the topologies $\mathfrak{T}_{c_{U,A}}$, $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$.

2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

In this section, we recall some basic notions and results.

A bounded lattice $(L, \leqslant)$ has a top and a bottom element, denoted by 1 and 0, respectively, i. e., there exist two elements $1, 0 \in L$ such that $0 \leqslant x \leqslant 1$, for all $x \in L$.

Definition 2.1. (Karaçal and Mesiar [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

Definition 2.2. (Ma and Wu [21]) An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

Recall that a uninorm U possessing a neutral element $e = 1$ is, in fact, a triangular norm. Similarly, a uninorm U with a neutral element $e = 0$ is, in fact, a triangular conorm.

Definition 2.3. (Grabisch et al. [12], Kesicioğlu et al. [17]) A uninorm U is called conjunctive or disjunctive if
$U(0, 1) = 0$ or $U(0, 1) = 1$ respectively.

Definition 2.4. (Kelley [16]) A map $c : \wp(X) \to \wp(X)$ is called a topological closure operator on non-empty set X if it satisfies following contitions;

C1. $c(\emptyset) = \emptyset$,

C2. $A \subseteq c(A)$ for all $A \subseteq X$,

C3. $c(c(A)) = c(A)$ for all $A \subseteq X$,

C4. $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subseteq X$.

Note that a closure operator c on X is called algebraic if

$$c(A) = \bigcup \{c(F) \ : \ F \text{ is a finite subset of } A\}.$$

The following theorem of Kuratowski shows that these four statements are actually characteristic of closure. The topology defined below is the topology associated with a closure operator.

Theorem 2.5. (Kelley [16]) Let c be a topological closure operator on X, Let $\mathfrak{F}$ be the family of all subsets A of X for which $c(A) = A$, and let $\mathfrak{T}$ be the family of complements of members of $\mathfrak{F}$. That is, $\mathfrak{T} = \{L \setminus A : A \in \mathfrak{F}\}$. Then $\mathfrak{T}$ is a topology for X, and $c(A)$ is the τ_c-closure of A for each subset A of X.

Definition 2.6. (Kelley [16]) Let X be a topological space. X is called $T_0 -$ space iff for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subseteq X$ so that U contains one of x and y but not the other.

Proposition 2.7. (Kelley [16]) Let X be a topological space. Then, X is a $T_0 -$ space if and only if either $x \notin c(\{y\})$ or $y \notin c(\{x\})$ for all $x, y \in X$ such that $x \neq y$.

Definition 2.8. (Dixmier [6]) A topological space X is said to be a Hausdorff space if any two distinct points of X admit disjoint neighborhoods.

Definition 2.9. (Echi [8]) A space X is said to be a quasi-Hausdorff space if for each distinct points $x, y \in X$, either there exists $z \in X$ such that $x, y \in c(\{z\})$ or x and y have disjoint neighborhoods.

Definition 2.10. (Echi [8]) Let X be a set and $f : X \to X$ a function. A subset A of X is called f-invariant if the condition $f(A) \subseteq A$ holds.

Definition 2.11. (Echi [8]) Let X be a set and $f : X \to X$ be a function. The topology $\mathcal{P}(f)$ on X is defined with closed sets exactly those A which are f-invariant, i. e., $f(A) \subseteq A$.

Clearly, $\mathcal{P}(f)$ provides a principal topology on X.

Definition 2.12. (Echi [8]) The pre-order determined by $(X, \mathcal{P}(f))$ is given by: for all $x, y \in X$

$$x \leq_f y \Leftrightarrow f^n(y) = x \text{ for some } n \in \mathbb{N}.$$

Definition 2.13. (Echi [8]) A minimal point of a pre-ordered set $(X, \leq)$ is an $x \in X$ satisfying the property: for each $y \in X$ if $y \leq x$, then $x \leq y$.

Definition 2.14. (Echi [8]) An element x is called a fixed point of the topology $(X, \mathcal{P}(f))$ if $f(x) = x$ holds.

3. THE $\mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle}$-TOPOLOGY

In this section, we define a topology denoted by $\mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle}$ for a family of the functions $\{f_i\}_{i\in I}$ with $f_i : X \to X$ for a set X and all $i \in I$. Some properties of the topology are investigated. Especially, some properties of $(X, \leq_{\langle\{U_{x_0}\}_{x_0\in A}\rangle})$ are investigated.

Let X be a set, the family of the functions $\{f_i\}_{i\in I}$ with for all $i \in I$, $f_i : X \to X$. Define the sets as follows:

$$\langle\{f_i\}_{i\in I}\rangle = \{f_{i_1}^{n_1} \circ f_{i_2}^{n_2} \circ \cdots \circ f_{i_k}^{n_k} |\quad k, n_1, n_2, \ldots, n_k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in I\}$$

and for all $x \in X$

$$\mathcal{O}_{\langle\{f_i\}_{i\in I}\rangle}(x) = \{f_{i_1}^{n_1} \circ f_{i_2}^{n_2} \circ \cdots \circ f_{i_k}^{n_k}(x) |\quad k, n_1, n_2, \ldots, n_k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in I\}.$$

Proposition 3.1. Let X be a set and $A \subseteq X$. The operation $c : \wp(X) \to \wp(X)$ defined by $c(A) = \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} h(A)$ is a closure operator.

Proof. (i) For $j \in I$, consider the function $h^* = f_j \in \langle\{f_i\}_{i\in I}\rangle$. Since $A = id(A) = f_j(A) \subseteq \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} h(A)$, we have that $A \subseteq c(A)$.

(ii) Now, let us show that $c(c(A)) = c(A)$.

$$\begin{aligned} c(c(A)) &= \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} h(c(A)) \\ &= \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} h\Big(\bigcup_{h^*\in\langle\{f_i\}_{i\in I}\rangle} h^*(A)\Big) \\ &= \bigcup_{h'=h\circ h^*\in\langle\{f_i\}_{i\in I}\rangle} h'(A) = c(A). \end{aligned}$$

(iii) Let $A, B \subseteq X$. Then,

$$\begin{aligned} c(A\cup B) &= \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} h(A\cup B) \\ &= \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} (h(A)\cup h(B)) \\ &= \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} h(A) \cup \bigcup_{h\in\langle\{f_i\}_{i\in I}\rangle} h(B) \\ &= c(A)\cup c(B). \end{aligned}$$

□

The topology defined by $\mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle} = \{A' |\quad A \in \mathcal{C}_{\langle\{f_i\}_{i\in I}\rangle}\}$ is the topology with the closure operator c, where $\mathcal{C}_{\langle\{f_i\}_{i\in I}\rangle} = \{A \subseteq X |\quad c(A) = A\}$. The topology $\mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle}$ is called $\langle\{f_i\}_{i\in I}\rangle$-topology.

Definition 3.2. The pre-order determined by $(X, \mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle})$ is given by for any $a, b \in X$

$$a \leq_{\langle \{f_i\}_{i\in I}\rangle} b \Leftrightarrow \exists h \in \langle \{f_i\}_{i\in I}\rangle \text{ such that } h(b) = a.$$

Proposition 3.3. Let X be a set and $x \in X$. Then,

(i) $c(\{x\}) = \mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(x) = (\downarrow x)_{\langle \{f_i\}_{i\in I}\rangle}$, where $(\downarrow x)_{\langle \{f_i\}_{i\in I}\rangle} = \{y \in X|\ y \leq_{\langle \{f_i\}_{i\in I}\rangle} x\}$.

(ii) The smallest open set containing x is the set defined by
$V_{\langle \{f_i\}_{i\in I}\rangle}(x) = \{y \in X \mid \text{there exists } h \in \langle \{f_i\}_{i\in I}\rangle \text{ such that } h(y) = x\} = (x\uparrow)_{\leq_{\langle \{f_i\}_{i\in I}\rangle}}$, where $(x\uparrow)_{\leq_{\langle \{f_i\}_{i\in I}\rangle}} = \{y \in X|\quad x \leq_{\langle \{f_i\}_{i\in I}\rangle} y\}$

(iii) $(X, \mathcal{T}_{\langle \{f_i\}_{i\in I}\rangle})$ is a quasi-Hausdorff space.

Proof. (i) Since $c(\{x\}) = \bigcup_{h\in\langle \{f_i\}_{i\in I}\rangle} h(\{x\})$, we have that

$$\mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(x) = \{h(x)|\quad h \in \langle \{f_i\}_{i\in I}\rangle\} \subseteq c(\{x\}).$$

Let $a \in c(\{x\})$. Then, there exists $h^* \in \langle \{f_i\}_{i\in I}\rangle$ such that $a \in h^*(\{x\})$, whence $a = h^*(x) \in \mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(x)$. Thus, $c(\{x\}) \subseteq \mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(x)$ holds. By the definitions of the sets, it is clear that $\mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(x) = (\downarrow x)_{\langle \{f_i\}_{i\in I}\rangle}$.

(ii) For $j \in I$, consider the function $h = f_j \in \langle \{f_i\}_{i\in I}\rangle$. Since $h(x) = x$, it is clear that $x \in V_{\langle \{f_i\}_{i\in I}\rangle}(x)$. Now, let us show that $V_{\langle \{f_i\}_{i\in I}\rangle}(x) \in \mathcal{T}_{\langle \{f_i\}_{i\in I}\rangle}$. That is, it is sufficient to prove that

$$c(X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x)) = X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x).$$

Let $y \in c(X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x)) = \bigcup_{h\in\langle \{f_i\}_{i\in I}\rangle} h(X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x))$. Then, there exists $h^* \in \langle \{f_i\}_{i\in I}\rangle$ and $t \in X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x)$ such that

$$y = h^*(t).$$

If $y \in V_{\langle \{f_i\}_{i\in I}\rangle}(x)$, it would exist $h^{**} \in \langle \{f_i\}_{i\in I}\rangle$ such that $h^{**}(y) = x$. Then, it would be obtained that

$$\begin{aligned} x &= h^{**}(y) = h^{**}(h^*(t)) \\ &= (h^{**} \circ h^*)(t). \end{aligned}$$

Since $h^{**} \circ h^* \in \langle \{f_i\}_{i\in I}\rangle$, we would have $t \in V_{\langle \{f_i\}_{i\in I}\rangle}(x)$, a contradiction. Then, $c(X \setminus V(x)) \subseteq X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x)$. Thus, the equality

$$c(X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x)) = X \setminus V_{\langle \{f_i\}_{i\in I}\rangle}(x)$$

is satisfied. Let us prove that $V_{\langle \{f_i\}_{i\in I}\rangle}(x)$ is the smallest open set containing the element x. Let $M \in \mathcal{T}_{\langle \{f_i\}_{i\in I}\rangle}$ be a set containing x. Then,

$$X \setminus M = c(X \setminus M) = \bigcup_{h\in\langle \{f_i\}_{i\in I}\rangle} h(X \setminus M).$$

Let $y \in V_{\langle \{f_i\}_{i\in I}\rangle}(x)$. Then, there exists $h \in \langle \{f_i\}_{i\in I}\rangle$ such that $h(y) = x \in M$. Suppose that $y \notin M$. Then, $y \in X \setminus M = \bigcup_{h\in\langle \{f_i\}_{i\in I}\rangle} h(X \setminus M)$. Thus, there exists $h^* \in \langle \{f_i\}_{i\in I}\rangle$ and $k \in X\setminus M$ such that $y = h^*(k)$. Then, $h(y) = h(h^*(k)) = (h\circ h^*)(k) \in \bigcup_{h\in\langle \{f_i\}_{i\in I}\rangle} h(X \setminus M) = X \setminus M$, whence $h(y) \notin M$, contradiction. Thus, it must be $y \in M$. Hence, we have that $V_{\langle \{f_i\}_{i\in I}\rangle}(x) \subseteq M$.

(iii) For the smallest open sets $V_{\langle \{f_i\}_{i\in I}\rangle}(x)$ and $V_{\langle \{f_i\}_{i\in I}\rangle}(y)$ containing x and y, respectively, if $V_{\langle \{f_i\}_{i\in I}\rangle}(x) \cap V_{\langle \{f_i\}_{i\in I}\rangle}(y) = \emptyset$, the proof is clear.

Let $V_{\langle \{f_i\}_{i\in I}\rangle}(x) \cap V_{\langle \{f_i\}_{i\in I}\rangle}(y) \neq \emptyset$. Then, there exists an element z such that $z \in V_{\langle \{f_i\}_{i\in I}\rangle}(x) \cap V_{\langle \{f_i\}_{i\in I}\rangle}(y)$. Thus, there exists $h, h^* \in \langle \{f_i\}_{i\in I}\rangle$ such that $h(z) = x$ and $h^*(z) = y$, whence $x \in c(\{z\})$ and $y \in c(\{z\})$. Thus, $(X, \mathcal{T}_{\langle \{f_i\}_{i\in I}\rangle})$ is a quasi-Hausdorff space. □

Remark 3.4. Let X be a set. Consider the topology $(X, \mathcal{P}(f))$. By Proposition 1.2. [8], we know that for any $x, y \in X$, either $V_f(x) \cap V_f(y) = \emptyset$ or $V_f(x)$ and $V_f(y)$ are comparable.

If we take the topology $(X, \mathcal{T}_{\langle \{f_i\}_{i\in I}\rangle})$, the claim stated above need not be true. Let us look at the following example.

Example 3.5. Consider the lattice $(L, \leq, 0, 1)$ whose lattice diagram is depicted as in Figure 1.

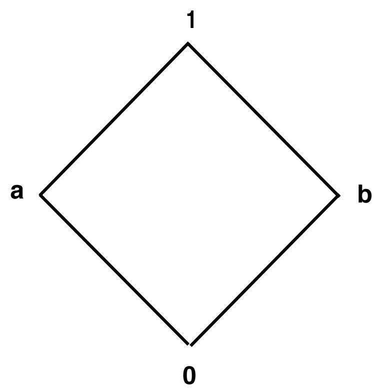

Fig. 1. $(L, \leq, 0, 1)$.

Take the topology $(L, \mathcal{T}_{\langle \{T_{x_0}\}_{x_0\in L}\rangle})$ defined by $T_{x_0}(x) = T(x_0, x)$, where T is a t-norm on L. For $T = T_W = \begin{cases} x \wedge y & x = 1 \text{ or } y = 1 \\ 0 & \text{otherwise,} \end{cases}$, it is clear that $T_0(x) = 0$, $T_1(x) = x$, $T_b(x) = \begin{cases} b & x = 1, \\ 0 & \text{otherwise,} \end{cases}$ and $T_a(x) = \begin{cases} a & x = 1, \\ 0 & \text{otherwise.} \end{cases}$

In this case, $V_{\langle \{T_{x_0}\}_{x_0\in L}\rangle}(a) = \{1, a\}$ an $V_{\langle \{T_{x_0}\}_{x_0\in L}\rangle}(b) = \{1, b\}$. As seen, $V_{\langle \{T_{x_0}\}_{x_0\in L}\rangle}(a) \cap V(b) \neq \emptyset$ and no one of the sets $V_{\langle \{T_{x_0}\}_{x_0\in L}\rangle}(a)$ and $V_{\langle \{T_{x_0}\}_{x_0\in L}\rangle}(b)$ do not contain the other one.

Proposition 3.6. (Karaçal and Köroğlu [13]) Let L be a bounded lattice and U be a uninorm on L with the neutral element e. Let A be a subset of L such that $e \in A$ and $U(A,A) \subseteq A$. The operator $c_{U,A} : \wp(L) \to \wp(L)$ defined by $c_{U,A}(X) = U(X,A)$ is a topological closure operator on L.

Definition 3.7. (Karaçal and Köroğlu [13]) A subset X of L is called a closed subset if $c_{U,A}(X) = X$. The family of all closed subsets of L is denoted by $\mathfrak{F}_{c_{U,A}} = \{X \subseteq L : c_{U,A}(X) = X\}$.

Proposition 3.8. (Karaçal and Köroğlu [13]) Let $c_{U,A}(X)$ be the closure of $X \subseteq L$. Then, $\mathfrak{T}_{c_{U,A}} = \{L \setminus X : X \in \mathfrak{F}_{c_{U,A}}\}$ is a principal topology on L.

Proposition 3.9. The principal topology $\mathfrak{T}_{c_{U,A}}$ given in [13] is a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$-topology, where $U_{x_0} : L \to L$ defined by $U_{x_0}(x) = U(x_0, x),\ x_0 \in A$.

Proof. Let $K \in \mathfrak{F}_{c_{U,A}}$. Then, $c_{U,A}(K) = U(K,A) = K$. Suppose that $K \notin \mathcal{C}_{\langle\{U_{x_0}\}_{x_0 \in A}\rangle}$. Then, $c(K) \neq K$. Thus, $K \subsetneq c(K)$. Then, there exists an element $k \in c(K)$ such that $k \notin K$. Since $k \in c(K) = \bigcup_{\langle\{U_{x_0}\}_{x_0 \in A}\rangle} h(K)$, there exists an $h \in \langle\{U_{x_0}\}_{x_0 \in A}\rangle$ such that $k \in h(K)$. Thus, there exists an element $k^* \in K$ such that for some $x_0 \in A$

$$k = h(k^*) = U_{x_0}(k^*) = U(x_0, k^*) \notin K.$$

For $x_0 \in A$ and $k^* \in K$, since $k = U(x_0, k^*) \in c_{U,A}(K) = K$, we have a contradiction.

Conversely, let $B \in \mathcal{C}_{\langle\{U_{x_0}\}_{x_0 \in A}\rangle}$. Then, $c(B) = B$. Let us show that $U(A,B) = B$. Since $c(B) = \bigcup_{h \in \langle\{U_{x_0}\}_{x_0 \in A}\rangle} h(B)$, $B = \bigcup_{h \in \langle\{U_{x_0}\}_{x_0 \in A}\rangle} h(B)$. Since $e \in A$, it is clear that $B = U(\{e\}, B) \subseteq U(A,B)$. Let $U(x,y) \in U(A,B)$ for $x \in A$ and $y \in B$. Since $U_x \in \langle\{U_{x_0}\}_{x_0 \in A}\rangle$, it is clear that $U(x,y) = U_x(y) \in \bigcup_{h \in \langle\{U_{x_0}\}_{x_0 \in A}\rangle} h(B) = B$, whence $U(A,B) \subseteq B$. Thus, since $c_{U,A}(B) = U(A,B) = B$, we have that $B \in \mathfrak{F}_{c_{U,A}}$. □

Remark 3.10. Let L be a lattice, U be a uninorm on L and $A \subseteq L$. Consider the set $\{U_{x_0}\}_{x_0 \in A}$. Let $U_{x_0} \circ U_{y_0} \in \{U_{x_0}\}_{x_0 \in A}$ for any $x_0, y_0 \in A$. Then, there exists an element $z_0 \in A$ such that

$$U_{x_0} \circ U_{y_0} = U_{z_0}.$$

It is clear that $U_{x_0} \circ U_{y_0} = U_{U(x_0,y_0)}$. Thus,

$$U(z_0, x) = U_{z_0}(x) = U_{U(x_0,y_0)}(x) = U(U(x_0,y_0), x).$$

If we take as $x = e$, it can be easily seen that $U(x_0, y_0) = z_0$ from the last equalities. Thus, for any $x_0, y_0 \in A$, we have that $U(x_0, y_0) \in A$, which means that $U(A,A) \subseteq A$.

Definition 3.11. Let X be a set and A be closed set with respect to $\langle\{f_i\}_{i \in I}\rangle$-topology. The subset A of X is called a minimal set of $\langle\{f_i\}_{i \in I}\rangle$-topology if A is a minimal element in the set of all nonempty closed sets.

Definition 3.12. Consider the topological space $(X, \mathcal{T}_{\langle\{f_i\}_{i \in I}\rangle})$. An element $x \in X$ is said to be a $\langle\{f_i\}_{i \in I}\rangle$- point if there exists some $h \in \langle\{f_i\}_{i \in I}\rangle$ with $h \neq id$ such that $h(x) = x$.

Note that, in Definition 3.12 if we take only one function instead of family, then it is clear that $\langle \{f_i\}_{i\in I}\rangle$- point coincides the periodic point notion.

Proposition 3.13. (i) The minimal sets of the topology $(X, \mathcal{T}_{\langle \{f_i\}_{i\in I}\rangle})$ are exactly the closures of $\langle \{f_i\}_{i\in I}\rangle$- points, where $f_i \neq id$ for all $i \in I$.

(ii) Let L be a lattice and $\emptyset \neq A \subseteq L$. If x is a minimal point of $(X, \leq_{\langle \{U_{x_0}\}_{x_0\in A}\rangle})$, then x is a $\langle \{U_{x_0}\}_{x_0\in A}\rangle$- point, where $U_{x_0} \neq id$ for $x_0 \in A$.

Proof.

(i) Let $A \subseteq X$ be a minimal subset and $x \in A$. Then, $\{x\} \subseteq A$. Since A is a closed set, it is clear that $c(\{x\}) \subseteq c(A) = A$, whence $c(\{x\}) \subseteq A$. Since A is minimal set, we have that

$$c(\{x\}) = A.$$

Now, let us show that for any $x \in A$ and $h \in \langle \{f_i\}_{i\in I}\rangle$, A coincides with the closure of $h(x)$. Since $c(\{x\}) = \bigcup_{h^*\in\langle \{f_i\}_{i\in I}\rangle} h^*(x)$, it is clear that $h(x) \in c(\{x\}) = A$. Then, $\{h(x)\} \subseteq A$. Since A is closed, it is obtained that

$$c(\{h(x)\}) \subseteq A.$$

By the minimality of A again, we have that

$$\mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(h(x)) = c(\{h(x)\}) = A.$$

Thus, we obtain that

$$A = \mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(h(x)) = c(\{x\}) = c(\{h(x)\}).$$

Since $x \in c(\{x\}) = c(\{h(x)\}) = \bigcup_{h^*\in\langle \{f_i\}_{i\in I}\rangle} h^*(h(x))$, there exists an $h^* \in \langle \{f_i\}_{i\in I}\rangle$ such that $x = h^*(h(x)) = (h^* \circ h)(x)$. Thus, x is a $\langle \{f_i\}_{i\in I}\rangle$- point.

Conversely, let x be a $\langle \{f_i\}_{i\in I}\rangle$- point. Let us prove that the closure of x is a minimal set. Since $c(\{x\}) = \mathcal{O}_{\langle \{f_i\}_{i\in I}\rangle}(x)$ and $c(\{x\})$ is the least closed set containing x, it is minimal.

(ii) Let x be a minimal point. By $h(x) = h^1(x)$, it is clear that $h(x) \leq_{\langle \{U_{x_0}\}_{x_0\in A}\rangle} x$. Since x is a minimal point, we have that $x \leq_{\langle \{U_{x_0}\}_{x_0\in A}\rangle} h(x)$. Then, there exists $h^* \in \langle \{U_{x_0}\}_{x_0\in A}\rangle$ such that $h^*(h(x)) = x$. Thus, x is a $\langle \{U_{x_0}\}_{x_0\in A}\rangle$- point. □

Remark 3.14. Let L be a bounded lattice, $\emptyset \neq A \subseteq L$ and T be a t-norm on L. Consider the topology $(L, \mathcal{T}_{\langle \{T_{x_i}\}_{x_i\in A}\rangle})$. If 1 is a $\langle \{T_{x_i}\}_{x_i\in A}\rangle$- point, then $1 \in A$. Indeed, suppose that 1 is a $\langle \{T_{x_i}\}_{x_i\in A}\rangle$- point. Then, there exists $h \in \langle \{T_{x_i}\}_{x_i\in A}\rangle$ such that $h(1) = 1$. Since $h \in \langle \{T_{x_i}\}_{x_i\in A}\rangle$, there exists $n_1, n_2, \ldots, n_k \in \mathbb{N}$, $x_1, x_2, \ldots, x_k \in A$ such that

$$h = T_{x_1}^{n_1} \circ T_{x_2}^{n_2} \circ \cdots \circ T_{x_k}^{n_k}.$$

Thus,

$$1 = h(1) \quad = \quad T_{x_1}^{n_1}(T_{x_2}^{n_2} \cdots T_{x_{k-1}}^{n_{k-1}}(T_{x_k}^{n_k}(1)))$$

$$
\begin{aligned}
&= T_{x_1}^{n_1}(T_{x_2}^{n_2}\cdots T_{x_{k-1}}^{n_{k-1}}(T_{x_k}^{n_k-1}(T_{x_k}(1))))\\
&= T_{x_1}^{n_1}(T_{x_2}^{n_2}\cdots T_{x_{k-1}}^{n_{k-1}}(T_{x_k}^{n_k-1}(T(x_k,1))))\\
&= T_{x_1}^{n_1}(T_{x_2}^{n_2}\cdots T_{x_{k-1}}^{n_{k-1}}(T_{x_k}^{n_k-1}(x_k)))\\
&= T_{x_1}^{n_1}(T_{x_2}^{n_2}\cdots T_{x_{k-1}}^{n_{k-1}}((x_k)_T^{n_k-1}))\\
&\leq T_{x_1}^{n_1}(T_{x_2}^{n_2}\cdots T_{x_{k-2}}^{n_{k-2}}(T_{x_{k-1}}^{n_{k-1}}(1)))\\
&= \cdots = T_{x_1}^{n_1}(T_{x_2}^{n_2}\cdots T_{x_{k-2}}^{n_{k-2}}((x_{k-1})_T^{n_{k-1}-1}))\\
&\leq T_{x_1}^{n_1}(T_{x_2}^{n_2}\cdots T_{x_{k-2}}^{n_{k-2}}(1)) \leq \cdots \leq T_{x_1}^{n_1}(1)\\
&= T_{x_1}^{n_1-1}(T_{x_1}(1)) = T_{x_1}^{n_1-1}(T(x_1,1))\\
&= T_{x_1}^{n_1-1}(x_1) = (x_1)_T^{n_1-1} \leq T(x_1,1) = x_1,
\end{aligned}
$$

whence we have that $1 = x_1 \in A$.

Proposition 3.15. Let L be a lattice and $\emptyset \neq A \subseteq L$. In the topological space $(L, \mathcal{T}_{\langle\{U_{x_0}\}_{x_0\in A}\rangle})$, the following are satisfied.

(i) If all points of A are $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- points, then $A \subseteq U(A,A)$.

(ii) If U is a t-norm (t-conorm, resp.), then 0 (1, resp.) is a $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- point.

(iii) Let x, y be two $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- points. If U is a t-norm (or t-conorm), $U(x,y)$ is also a $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- point.

Proof. (i) Let all points of A be $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- point and $a \in A$. Then, there exists $U_{x_0} \in \langle\{U_{x_0}\}_{x_0\in A}\rangle$ such that $U_{x_0}(a) = a$. Since $U(x_0, a) = a$, it is clear that $A \subseteq U(A,A)$.

(ii) Let U be a t-norm. For any element $x \in A$, since $U_x(0) = U(x,0) = 0$, it is clear that 0 is a $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- point.

Similarly, it can be easily shown that 1 is $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- point when U is a t-conorm.

(iii) Let U be a t-norm. Since x and y are two $\langle\{U_{x_0}\}_{x_0\in A}\rangle$- points,

$$x = (U_{x_0} \circ U_{x_1} \circ \cdots \circ U_{x_n})(x) \text{ and } y = (U_{x_0{}'} \circ U_{x_1{}'} \circ \cdots \circ U_{x_m{}'})(y).$$

Then,

$$
\begin{aligned}
U(x,y) &= U((U_{x_0} \circ U_{x_1} \circ \cdots \circ U_{x_n})(x), (U_{x_0{}'} \circ U_{x_1{}'} \circ \cdots \circ U_{x_m{}'})(y))\\
&= U(U(x_0, U(x_1, U(x_2, \ldots, U(x_n, x)))), U(x_0{}', U(x_1{}', U(x_2{}',\\
&\quad \ldots, U(x_m{}', y)))))\\
&= U(U(x, U(x_1, U(x_2, \ldots, U(x_n, x_0)))), U(y, U(x_1{}', U(x_2{}',\\
&\quad \ldots, U(x_m{}', x_0{}')))))\\
&= U(U(x,a), U(y,b)) = U(U(x, U(a, U(y,b))))\\
&= U(U(x, U(y, U(a,b)))) = U(U(x,y), U(a,b))
\end{aligned}
$$

$$= U_{U_{(a,b)}}(U(x,y)),$$

where $U_{(a,b)} \in \langle \{U_{x_0}\}_{x_0 \in A} \rangle$, $a := U(x_1, U(x_2, \ldots, U(x_n, x_0)))$ and $b := U(y, U(x_1{}', U(x_2{}', \ldots, U(x_m{}', x_0{}'))))$. Thus, $U(x,y)$ is a $\langle \{U_{x_0}\}_{x_0 \in A} \rangle$- point. □

Proposition 3.16. Let $(L, \leq, 0, 1)$ be a bounded lattice and $\emptyset \neq A \subseteq L$. Consider the topology $(L, \mathcal{P}(T_{x_0}))$ such that T is a t-norm and $x_0 \in A$. If x is a $\langle \{T_{x_0}\}_{x_0 \in A} \rangle$- point, then $x \leq x_0$.

Proof. Let x be $\langle \{T_{x_0}\}_{x_0 \in A} \rangle$- point. Then, there exists an element $m \in \mathbb{N}^*$ such that

$$T^m_{x_0}(x) = x,$$

since $x = T^m_{x_0}(x) = T((x_0)^m_T, x) \leq (x_0)^m_T \leq x_0$, $x \leq x_0$. □

Proposition 3.17. Let $(L, \leq, 0, 1)$ be a bounded lattice, $\emptyset \neq A \subseteq L$ and T be a t-norm. Then,

(i) The topology $(L, \mathcal{P}(T_{x_0}))$ is a T_0-space.

(ii) $(L, \mathcal{P}(T_{x_0}))$ does not have any $\langle T_{x_0} \rangle$- point except for fixed points.

Proof. (i) Let $c(\{x\}) = c(\{y\})$ for any $x, y \in L$. Since $x \in c(\{y\})$, there exists $n \in \mathbb{N}^*$ such that $x = T^n_{x_0}(y)$, whence $x = T((x_0)^n_T, y) \leq y$. Similarly, it can be shown that $y \leq x$. Thus, we have that $x = y$.

(ii) Let x be a $\langle T_{x_0} \rangle$- point. Since $c(\{x\}) = \{x, T_{x_0}(x), T^2_{x_0}(x), \ldots, T^{n-1}_{x_0}(x), T^n_{x_0}(x) = x\}$ and $c(\{T_{x_0}(x)\}) = \{T_{x_0}(x), T^2_{x_0}(x), \ldots, T^n_{x_0}(x) = x, T^{n+1}_{x_0}(x) = T_{x_0}(x)\}$, it is clear that $c(\{x\}) = c(\{T_{x_0}(x)\})$. By (i), since $(L, \mathcal{P}(T_{x_0}))$ is a T_0-space, $x = T_{x_0}(x)$, whence x is a fixed point. □

Remark 3.18. Let $L = [0,1]$. For the topology $(L, \mathcal{P}(T_{x_0}))$ with $x_0 \neq 1$, the only fixed point is the point 0. Indeed, for any $n \in \mathbb{N}$, since $x \leq (x_0)^n_T$, it is clear that

$$x \leq \lim (x_0)^n_T = 0,$$

whence $x = 0$.

Proposition 3.19. Let $(L, \leq, 0, 1)$ be a bounded lattice, U be a uninorm with the neutral element $e \in L$. For any $x_0, y_0 \in L$ such that $x_0 \leq e$ and $y_0 > e$, if $\mathcal{P}(U_{x_0}) = \mathcal{P}(U_{y_0})$, then x_0 and y_0 are $\langle \{U_{x_0}\} \rangle$- points.

Proof. It is clear that the closures of the set $\{x_0\}$ in the spaces $(L, \mathcal{P}(U_{x_0}))$ and $(L, \mathcal{P}(U_{y_0}))$ are respectively as follows:

$$c_{\mathcal{P}(U_{x_0})}(\{x_0\}) = \{x_0, U_{x_0}(x_0), \ldots, U^n_{x_0}(x_0), \ldots\}$$

and

$$\begin{aligned} c_{\mathcal{P}(U_{y_0})}(\{x_0\}) &= \{x_0, U_{y_0}(x_0), \ldots, U_{y_0}^n(x_0), \ldots\} \\ &= \{x_0, U(y_0, x_0), \ldots, U((y_0)^n_{U_{y_0}}, x_0), \ldots\}. \end{aligned}$$

Since $U(x_0, y_0) \in c(\{x_0\})$, there exists an $m \in \mathbb{N}^*$ such that

$$U(x_0, y_0) = U_{x_0}^m(x_0) = U((x_0)^m_{U_{x_0}}, x_0).$$

Then, it follows $x_0 \le U(x_0, y_0) = U((x_0)^m_{U_{x_0}}, x_0) = (x_0)^{m+1}_{U_{x_0}} \le x_0$ from $x_0 \le e$. Thus, $(x_0)^{m+1}_{U_{x_0}} = x_0$, that is, x_0 is a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point.

Similarly, it can be easily seen that y_0 is a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point. $\square$

Example 3.20. Take the uninorm [4] defined by

$$U(x, y) = \begin{cases} 2xy & (x, y) \in [0, \frac{1}{2})^2, \\ 1 & x, y \in (\frac{1}{2}, 1]^2, \\ 0 & x = 0 \quad \text{or} \quad y = 0, \\ x & y = \frac{1}{2}, \\ y & x = \frac{1}{2}, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Suppose that there exists $n \in \mathbb{N}$ such that $U_{x_0}^n(x_0) \ge \frac{1}{2}$ for $x_0 < \frac{1}{2}$. Since $U((x_0)^{n-1}_U, x_0) = 2^{n-1}.x_0^n \ge \frac{1}{2}$, we have that $x_0 \ge \frac{1}{2}$, contradiction. Then, for any $x_0 < \frac{1}{2}$, it must be $U(x_0)^n(x_0) < \frac{1}{2}$. Now, let us show that x_0 is not a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point. Suppose that x_0 is a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point. Then, there exists $n \in \mathbb{N}^*$ such that $U_{x_0}^n(x_0) = x_0$. Since $2^{n-1}.x_0^n = x_0$, we have that $x_0^{n-1} = (\frac{1}{2})^{n-1}$, whence $x_0 = \frac{1}{2}$. This is a contradiction. Thus, x_0 is not a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point. That is, x_0 is a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point in the topology $([0, 1], \mathcal{T}_{\langle U_{x_0}\rangle})$. An element $y_0 > \frac{1}{2}$ being $\mathcal{P}(U_{x_0}) = \mathcal{P}(U_{y_0})$ does not exist.

Remark 3.21. Take the function defined by

$$U(x, y) = \begin{cases} 0 & x, y \in [0, \frac{1}{2}], \\ \max(x, y) & x, y \in (\frac{1}{2}, 1], \\ \min(x, y) & \text{otherwise.} \end{cases}$$

The function U is a t-norm [9]. Consider the topology $(L, \mathcal{P}(U_{\frac{2}{3}}))$. For any $x \ge \frac{2}{3}$, since $U_{\frac{2}{3}}(x) = U(\frac{2}{3}, x) = x$, x is a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point. Let $\frac{1}{2} < x < \frac{2}{3}$. Since $U_{\frac{2}{3}}(x) = U(\frac{2}{3}, x) = \frac{2}{3}$, x cannot be a fixed point. For $x \le \frac{1}{2}$, since $U_{\frac{2}{3}}(x) = U(\frac{2}{3}, x) = x$, any element $x \in [0, \frac{1}{2}]$ is a fixed point. Suppose that an element x belongs to the interval $(\frac{1}{2}, \frac{2}{3})$ is a $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- point. There must exist $n \in \mathbb{N}$ such that $U_{\frac{2}{3}^n}(x) = x$. Since $U((\frac{2}{3})_{U^n_{\frac{2}{3}}}, x) = U(\frac{2}{3}, x) = \frac{2}{3} \ne x$, this is a contradiction. Thus, the points in the interval $(\frac{1}{2}, \frac{2}{3})$ cannot be $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- points. Since any fixed points are also $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- points, the set of $\langle\{U_{x_0}\}_{x_0 \in A}\rangle$- points in the topology $(L, \mathcal{P}(U_{\frac{2}{3}}))$ is equal to $[0, \frac{1}{2}] \cup [\frac{2}{3}, 1]$.

Lemma 3.22. Let $(L, \leq, 0, 1)$ be a bounded lattice, $\emptyset \neq A \subseteq L$. Let T be a t-norm and $x_0 \in A$. Every $\langle T_{x_0} \rangle$- point in the topology $(L, \mathcal{P}(T_{x_0}))$ is a fixed point.

Proof. Let x be a $\langle T_{x_0} \rangle$- point in the topology $(L, \mathcal{P}(T_{x_0}))$. There exists an $m \in \mathbb{N}^*$ such that $(x)^m_{T_{x_0}} = x$. Then,

$$T^{m+1}_{x_0}(x) = T_{x_0}(T^m_{x_0}(x)) = T_{x_0}(x).$$

Since $c(\{x\}) = \{T^n_{x_0}(x) | \quad n \in \mathbb{N}\}$, it is clear that $T^{m+1}_{x_0}(x) \in c(\{x\})$. Thus, $\{T_{x_0}(x)\} \subseteq c(\{x\})$, whence we have that

$$c(\{T_{x_0}(x)\}) \subseteq c(\{x\}).$$

Then, $c(\{T_{x_0}(x)\}) = c(\{x\})$. Thus, we have that $T_{x_0}(x) = x$, that is, x is a fixed point. □

The Lemma 3.22 may not be true for any uninorm U. Also, the topological space $(L, \mathcal{P}(U_{x_0}))$ need not be a T_0-space.

Example 3.23. Let $(L, \leq, 0, 1)$ be a bounded lattice whose lattice diagram as in Figure 2.

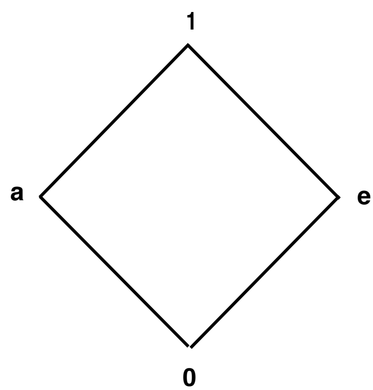

Fig. 2. $(L, \leq, 0, 1)$.

Take the uninorm $U : L^2 \to L$ defined as in Table 1:

U	0	a	e	1
0	0	0	0	1
a	0	e	a	1
e	0	a	e	1
1	1	1	1	1

Tab. 1. The uninorm U.

Consider the topology $(L, \mathcal{T}_{\langle U_a \rangle})$. Since $U_a^2(a) = U_a(U(a,a)) = U_a(e) = a$, a is $\langle U_a \rangle$-point. But, $U_a(a) = U(a,a) = e$, that is, it is not a fixed point.

Corollary 3.24. Consider the topology $(L, \mathcal{P}(U_{x_0}))$ with $x_0 < e$. If any element x with $x < e$ is a $\langle U_{x_0} \rangle$- point, it is a fixed point.

Especially, when L is a chain, Lemma 3.22 is also true for uninorms. Let us investigate the following proposition.

Proposition 3.25. Let $(L, \leq, 0, 1)$ be a chain, $\emptyset \neq A \subseteq L$. Let U be a uninorm with the neutral element e and $x_0 \in A$. $\langle U_{x_0} \rangle$- point in the topology $(L, \mathcal{P}(U_{x_0}))$ is a fixed point.

Proof. If $x_0 = e$, for any element $x \in L$, since $U_e(x) = x$, x is a fixed point, whence it is $\langle \{U_{x_0}\}_{x_0 \in A} \rangle$- point. Now, let $x_0 \neq e$ and x be a $\langle \{U_{x_0}\}_{x_0 \in A} \rangle$- point. Then, there exists an $n \in \mathbb{N}^*$ such that $U_{x_0}^n(x) = x$. Since L is a chain, either $x_0 \leq e$ or $x_0 \geq e$.

Let $x_0 \leq e$. Since $x = U_{x_0}^n(x) = U((x_0)_U^n, x) \leq U(x_0, x) \leq U(e,x) = x$, we have that $U_{x_0}(x) = x$, that is, x is a fixed point.

Let $x_0 \geq e$. Since $x = U_{x_0}^n(x) = U((x_0)_U^n, x) \geq U(x_0, x) \geq U(e,x) = x$, we have that $U_{x_0}(x) = x$, that is, x is a fixed point in this case. □

Corollary 3.26. Let $(L, \leq, 0, 1)$ be a chain, $\emptyset \neq A \subseteq L$. Let U be a uninorm with the neutral element e and $x_0 \in A$. Consider the topology $(L, \mathcal{P}(U_{x_0}))$. x is a $\langle U_{x_0} \rangle$- point iff it is a fixed point.

The topologies $\langle U_{x_0} \rangle$ of different points may be coincident. Let us look at the following example.

Example 3.27. Let $(L, \leq, 0, 1)$ be a bounded lattice whose lattice diagram as in Figure 3.

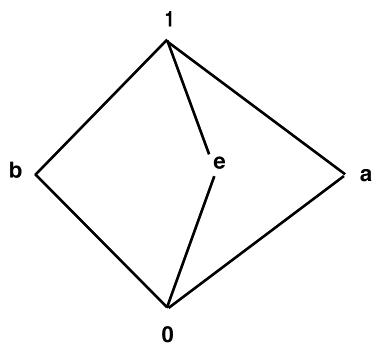

Fig. 3. $(L, \leq, 0, 1)$.

Consider the uninorm given in Table 2.

U	0	a	b	e	1
0	0	0	0	0	0
a	0	1	1	a	1
b	0	1	1	b	1
e	0	a	b	e	1
1	0	1	1	1	1

Tab. 2. The uninorm U.

Although $a \neq b$, $\mathcal{T}_{\langle U_a \rangle} = \{\emptyset^{'}, \{0\}^{'}, \{1\}^{'}, \{0,1\}^{'}, L^{'}\} = \mathcal{T}_{\langle U_b \rangle}$. Indeed,

$$\begin{array}{rl} U_a(\emptyset) \subseteq \emptyset, & (U_b(\emptyset) \subseteq \emptyset), \\ U_a(\{0\}) \subseteq \{0\} & (U_b(\{0\}) \subseteq \{0\}), \\ U_a(\{1\}) \subseteq \{1\}, & (U_b(\{1\}) \subseteq \{1\}), \\ U_a(\{0,1\}) \subseteq \{0,1\}, & (U_b(\{0,1\}) \subseteq \{0,1\}), \\ U_a(L) \subseteq L, & (U_b(L) \subseteq L). \end{array}$$

4. SOME DIFFERENT GENERATING METHODS FOR PRINCIPAL TOPOLOGIES

In this section, two topological closure operators $c^*_{U,A}$ and $c^{**}_{U,A}$ are given by means of a uninorm U and a set $A \subseteq L$ with under some conditions. Also, two topologies $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$ are presented and some properties of the introduced topologies are investigated. Moreover, we search the relationship between the topologies $\mathfrak{T}_{c_{U,A}}$, $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$.

Proposition 4.1. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0,e] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e and $U(A,A) \subseteq A$. The operation $c^*_{U,A} : \wp(L) \to \wp(L)$ defined by

$$c^*_{U,A}(X) = \{y \in L : \text{ there exists } a \in A \text{ and } x \in X \text{ such that } U(a,x) \leq y\}$$

is a topological closure operator.

Proof. (i) Let $x \in X$ be arbitrary element. $U(a,x) \leq U(e,x) = x$ for $a \in [0,e] \cap A$. Thus, it is obtained that $x \in c^*_{U,A}(X)$. Then, $X \subseteq c^*_{U,A}(X)$.

(ii) Now, let us show that $c^*_{U,A}(c^*_{U,A}(X)) = c^*_{U,A}(X)$ for all $X \in \wp(L)$. It is clearly obtained from (i) that $c^*_{U,A}(X) \subseteq c^*_{U,A}(c^*_{U,A}(X))$. Conversely, $k \in c^*_{U,A}(c^*_{U,A}(X))$ be an arbitrary element. Then, there exists $u \in c^*_{U,A}(X)$ and $a \in A$ such that $U(u,a) \leq k$. It is obtained that $U(x',a') \leq u$ such that $a' \in A$ and $x' \in X$ from $u \in c^*_{U,A}(X)$. Then, $k \in c^*_{U,A}(X)$ since $U(x',U(a',a)) = U(U(a',x'),a) \leq U(u,a) \leq k$. Thus, $c^*_{U,A}(c^*_{U,A}(X)) \subseteq c^*_{U,A}(X)$. $c^*_{U,A}(c^*_{U,A}(X)) = c^*_{U,A}(X)$ is obtained.

(iii) Let's first show $c^*_{U,A}(K) \subseteq c^*_{U,A}(L)$ for $K \subseteq L$.
$m \in c^*_{U,A}(K)$ be arbitrary element. Then, there exists $a \in A$ and $k \in K$ such that

$U(a,k) \leq m$. Since $k \in L$, $m \in c^*_{U,A}(L)$. Therefore, it is obtained that $c^*_{U,A}(K) \subseteq c^*_{U,A}(L)$.

Since $X, Y \subseteq X \cup Y$, we have $c^*_{U,A}(X) \cup c^*_{U,A}(Y) \subseteq c^*_{U,A}(X \cup Y)$. Conversely, let $s \in c^*_{U,A}(X \cup Y)$. Then, there exists $a \in A$ and $t \in X \cup Y$ such that $U(a,t) \leq s$. t is either an element of X or an element of Y. It follows from that $s \in c^*_{U,A}(X)$ or $s \in c^*_{U,A}(Y)$, respectively. Thus, $c^*_{U,A}(X \cup Y) \subseteq c^*_{U,A}(X) \cup c^*_{U,A}(Y)$. □

The topology defined by $\mathfrak{T}_{c^*_{U,A}} = \{L \setminus X \mid \quad X \in \mathfrak{F}_{c^*_{U,A}}\}$ is the topology with the closure operator $c^*_{U,A}$, where $\mathfrak{F}_{c^*_{U,A}} = \{X \subseteq L \mid \quad c^*_{U,A}(X) = X\}$.

Remark 4.2. In Proposition 4.1, $[0,e] \cap A \neq \emptyset$ means that $A \neq \emptyset$ if U is a t-norm on L.

Proposition 4.3. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[e,1] \cap A \neq \emptyset$, U be a uninorm on L with neutral element e and $U(A,A) \subseteq A$. The operation $c^{**}_{U,A} : \wp(L) \to \wp(L)$ defined by $c^{**}_{U,A}(X) = \{y \in L : \text{ there exists } a \in A \text{ and } x \in X \text{ such that } U(a,x) \geq y\}$ is a topological closure operator.

Proof. The proof is similar to Proposition 4.1. □

The topology defined by $\mathfrak{T}_{c^{**}_{U,A}} = \{X' \mid \quad X \in \mathfrak{F}_{c^{**}_{U,A}}\}$ is the topology with the closure operator $c^{**}_{U,A}$, where $\mathfrak{F}_{c^{**}_{U,A}} = \{X \subseteq L \mid \quad c^{**}_{U,A}(X) = X\}$.

Proposition 4.4. Let $(L, 0, 1, \leq)$ be a bounded lattice, $A, B \subseteq L$ be nonempty sets such that $1 \in A$ and $0 \in B$, T be a t-norm on L, S be a t-conorm on L and $x, y \in L$. Consider topological closure operators $c^*_{S,B}$, $c^{**}_{T,A}$.

(i) $c^{**}_{T,A}(\{x\}) \cap c^*_{S,B}(\{y\}) \neq \emptyset$ if and only if $y \leq x$.

(ii) $c^*_{S,B}(\{x\}) = x\uparrow$.

(iii) $c^{**}_{T,A}(\{x\}) = x\downarrow$.

(iv) $c^{**}_{T,A}(\{x\}) \cap c^*_{S,B}(\{x\}) = \{x\}$.

Proof. (i) Let $z \in c^{**}_{T,A}(\{x\}) \cap c^*_{S,B}(\{y\})$ be an arbitrary element. Then, there exists $a_1 \in A$ and $a_2 \in B$ such that $z \leq T(x, a_1)$ and $S(a_2, y) \leq z$. Therefore, $z \leq x$ and $y \leq z$. Thus, $y \leq x$. Conversely, let $y \leq x$. Since $y \leq x = T(x,1)$ for $1 \in A_1$, $y \in c^{**}_{T,A}(\{x\})$.Thus, $y \in c^{**}_{T,A}(\{x\}) \cap c^*_{S,B}(\{y\})$.

(ii) Let $t \in c^*_{S,B}(\{x\})$. Then, there exists an element $a_2 \in B$ such that $S(a_2, x) \leq t$. It follows from $x \leq t$ that $t \in x\uparrow$. Conversely, let $k \in x\uparrow$. Thus, $x \leq k$. Since $S(0,x) \leq k$ for $0 \in B$, $k \in c^*_{S,B}(\{x\})$.

(iii) It can be proved in similar ways as done in (ii).

(iv) It is immediately obtained from (ii) and (iii). □

Lemma 4.5. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0, e] \cap A \neq \emptyset$, U be a uninorm on L with neutral element e and $U(A, A) \subseteq A$. The operation $c^*_{U,A} : \wp(L) \to \wp(L)$ defined by $c^*_{U,A}(X) = \{y \in L :$ there exists $a \in A$ and $x \in X$ such that $U(a, x) \leq y\}$ is an algebraic closure operator.

Proof. Let $u \in c^*_{U,A}(Y)$. Then, $U(a, y) \leq u$ such that there exists $a \in A$ and $y \in Y$. Therefore, $u \in c^*_{U,A}(\{y\})$. Thus, $u \in \bigcup_{\substack{X \subseteq Y \\ X \text{ is finite set}}} \{c^*_{U,A}(X) \ : \ X \subseteq Y \text{ is finite set}\}$.

Conversely, it is well known that $c^*_{U,A}(X) \subseteq c^*_{U,A}(Y)$ for $X \subseteq Y$ since $c^*_{U,A}$ topological closure operator. Then,

$$\bigcup_{\substack{X \subseteq Y \\ X \text{ is finite set}}} \{c^*_{U,A}(X) \ : \ X \subseteq Y \text{ is finite set}\} \subseteq c^*_{U,A}(Y).$$

□

Lemma 4.6. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0, e] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e and $U(A, A) \subseteq A$. Consider the topological closure operator $c^*_{U,A}$. Then, $c^*_{U,A}(X) = \bigcup_{x \in X} c^*_{U,A}(\{x\})$.

Proof. $c^*_{U,A}(\{x\}) \subseteq c^*_{U,A}(X)$ since $c^*_{U,A}$ is topological closure operator. Then, $\bigcup_{x \in X} c^*_{U,A}(\{x\}) \subseteq c^*_{U,A}(X)$. Conversely, let $u \in c^*_{U,A}(X)$ be an arbitrary element. It follows that $U(x, a) \leq u$, where $x \in X$ and $a \in A$. Thus, $u \in c^*_{U,A}(\{x\})$. Then, $c^*_{U,A}(X) \subseteq \bigcup_{x \in X} c^*_{U,A}(\{x\})$. □

Remark 4.7. Lemma 4.6 also shows that $c^*_{U,A}$ is an algebraic closure operator.

Proposition 4.8. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0, e] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e and $U(A, A) \subseteq A$. Consider the topological closure operator $c^*_{U,A}$.

(i) If $0 \in X$, $c^*_{U,A}(X) = L$.

(ii) $1 \in c^*_{U,A}(X)$ for every set $\emptyset \neq X \subseteq L$.

Proof. (i) $U(a, 0) \leq U(e, 0) = 0$ for $a \in [0, e] \cap A$ since $[0, e] \cap A \neq \emptyset$ and $0 \in X$. Thus, $U(a, 0) = 0 \leq y$ for all $y \in L$, i.e., $y \in c^*_{U,A}(X)$ for all $y \in L$. Therefore, $c^*_{U,A}(X) = L$.

(ii) Since $U(a, x) \leq 1$ for $a \in A$ and $x \in X$, $1 \in c^*_{U,A}(X)$. □

Throughout the paper, denote by $\overline{X}$ the upper bound of a set X.

Corollary 4.9. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0, e] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e, $U(A, A) \subseteq A$ and $x, y \in L$. Consider the topological closure operator $c^*_{U,A}$.

(i) $\overline{X} \subseteq c^*_{U,A}(X)$ for every $X \subseteq L$.

(ii) If $A=\{e\}$, $c^*_{U,A}(X)=\bigcup\limits_{x\in X} x\uparrow$.

(iii) If $A=\{e\}$, $c^*_{U,A}(\{x\})=c^*_{U,A}(\{y\})\Leftrightarrow x=y$.

(iv) As a clear result of (iii), if $A=\{e\}$, the topology defined by $\mathfrak{T}_{c^*_{U,A}}$ is T_0-space.

Proof. (i) Let $y\in\overline{X}$. Then, $x\leq y$ for every $x\in X$. Since $[0,e]\cap A\neq\emptyset$, there exists $t\in[0,e]\cap A$. Thus, $y\in c^*_{U,A}(X)$ since $U(t,x)\leq U(e,x)=x\leq y$ for every $x\in X$.

(ii) Let $A=\{e\}$. Then, $c^*_{U,A}(X)=\{y\in L$: there exists $x\in X$ such that $U(x,e)\leq y\}=\{y\in L$: there exists $x\in X$ such that $x\leq y\}=\bigcup\limits_{x\in X} x\uparrow$.

(iii) Let $A=\{e\}$ and $c^*_{U,A}(\{x\})=c^*_{U,A}(\{y\})$. $x\in c^*_{U,A}(\{x\})=c^*_{U,A}(\{y\})$, then $y\leq U(y,e)\leq x$. Similarly, $y\in c^*_{U,A}(\{y\})=c^*_{U,A}(\{x\})$, then $x\leq U(x,e)\leq y$. Therefore, $x=y$. If $x=y$, $c^*_{U,A}(\{x\})=c^*_{U,A}(\{y\})$ is clear. □

The converse of Corollary 4.9 (i) may not be true. Let us investigate the following example.

Example 4.10. Consider the lattice $(L,\leq,\{0,a,b,1\})$ whose lattice diagram is depicted as in Figure 1.

Take the t-norm T_W on L and $A=\{1,b\}$. If we consider the set $X=\{b\}$, it is clearly seen that $\overline{X}=\{1,b\}$ but $c^*_{T_W,A}(X)=L$.

If we consider the set $X=\{0\}$, it is clearly seen that $\overline{X}=c^*_{T_W,A}(X)=L$ but $A=\{1,b\}$.

The converse of Corollary 4.9 (iv) may not be true. Let us look at the following proposition.

Proposition 4.11. Let L be a bounded lattice, $e\in L$, $A\subseteq L$ such that $[0,e]\cap A\neq\emptyset$, U be a uninorm on L with a neutral element e and $U(A,A)\subseteq A$. Consider the topological closure operator $c^*_{U,A}$. If there exist the elements a,b satisfying a is smaller than all elements of A, $a\neq b$ and $U(b,a)=a$, then $c^*_{U,A}(\{a\})=c^*_{U,A}(\{b\})=a\uparrow$, i. e., $\mathfrak{T}_{c^*_{U,A}}$ is not T_0-space.

Proof. Since $U(a,e)=a$, $y\in c^*_{U,A}(\{a\})$ for every element y such that $a\leq y$. Suppose that there exists an element $t<a$ such that $t\in c^*_{U,A}(\{a\})$. Then, there exists $a_t\in A$ such that $U(a,a_t)\leq t$. Since a is the lowest element of A, $U(a,a_t)\leq U(a,e)=a$. Considering $U(A,A)\subseteq A$, it is obtained that $U(a,a)=a$. Therefore, we have $a=U(a,a)\leq U(a,x)\leq t<a$, contradiction. Then, $c^*_{U,A}(\{a\})=a\uparrow$.

Since $U(a,b)=a$, $y\in c^*_{U,A}(\{b\})$ for every element y such that $a\leq y$. Suppose that there exists element $s<a$ such that $s\in c^*_{U,A}(\{b\})$. Then, there exists $a_s\in A$ such that $U(a,a_s)\leq s$. Therefore, we have $a=U(a,b)\leq U(a_s,b)\leq t<a$, contradiction. Then, $c^*_{U,A}(\{b\})=a\uparrow$.

Since $c^*_{U,A}(\{a\})=c^*_{U,A}(\{b\})=a\uparrow$ when $a\neq b$, $\mathfrak{T}_{c^*_{U,A}}$ is not T_0-space. □

Theorem 4.12. The topology $\mathfrak{T}_{c^*_{U,A}}$ is a principal topology.

P r o o f. Let us consider the arbitrary family of open sets $\{O_\tau : \tau \in I\}$. Then, $c^*_{U,A}(L\backslash O_\tau) = L\backslash O_\tau$ for all $\tau \in I$. We have to show that $\bigcap_{\tau\in I} O_\tau$ is open set, i. e., $L\backslash(\bigcap_{\tau\in I} O_\tau) = \bigcup_{\tau\in I}(L\backslash O_\tau)$ is closed set. $\bigcup_{\tau\in I}(L\backslash O_\tau) \subseteq c^*_{U,A}(\bigcup_{\tau\in I}(L\backslash O_\tau))$ is trivial since $c^*_{U,A}$ is a closure operator on L. Conversely, $p \in c^*_{U,A}(\bigcup_{\tau\in I}(L\backslash O_\tau))$ be an arbitrary element. Then, there exists $a \in A$ and $m \in \bigcup_{\tau\in I}(L\backslash O_\tau)$ such that $U(a, m) \leq p$. Therefore, there exists $\tau^* \in I$ such that $m \in L\backslash O_{\tau^*}$ since $m \in \bigcup_{\tau\in I}(L\backslash O_\tau)$. Obviously, $p \in c^*_{U,A}(L\backslash O_{\tau^*}) \subseteq \bigcup_{\tau\in I} c^*_{U,A}(L\backslash O_\tau)$ since $U(a, m) \leq p$. Then, $c^*_{U,A}(\bigcup_{\tau\in I}(L\backslash O_\tau)) \subseteq \bigcup_{\tau\in I}(c^*_{U,A}(L\backslash O_\tau))$. Therefore, $c^*_{U,A}(\bigcup_{\tau\in I}(L\backslash O_\tau)) = \bigcup_{\tau\in I}(c^*_{U,A}(L\backslash O_\tau))$,
i. e., $\mathfrak{T}_{c^*_{U,A}}$ is a principal topology. □

Proposition 4.13. (Karaçal and Köroğlu [13]) Consider the uninorms U_1 and U_2 on bounded lattices L_1 and L_2 with the neutral elements e_1, e_2 respectively. Then, the direct product $U_1 \times U_2$ of U_1 and U_2, defined by

$$U_1 \times U_2((x_1, y_1), (x_2, y_2)) = (U_1(x_1, x_2), U_2(y_1, y_2))$$

is a uninorm with the neutral element (e_1, e_2) on the product lattice $L_1 \times L_2$.

Lemma 4.14. Let L be a bounded lattice, $e \in L$, $A_1, A_2 \subseteq L$ such that $[0, e] \cap A_1 \neq \emptyset$, $[0, e] \cap A_2 \neq \emptyset$ and U_1, U_2 be uninorms on L with neutral element e satisfying $U_1(A_1, A_1) \subseteq A_1$ and $U_2(A_2, A_2) \subseteq A_2$, respectively, and consider topological closure operator $c^*_{U_1\times U_2,A_1\times A_2}$. Then, $c^*_{U_1\times U_2,A_1\times A_2}(\{(x, y)\}) = c^*_{U_1,A_1}(\{x\}) \times c^*_{U_2,A_2}(\{y\})$ for $x, y \in L$.

P r o o f.
Let $(m, l) \in c^*_{U_1\times U_2,A_1\times A_2}(\{(x, y)\})$. Then,

$$\begin{aligned}
(m, l) \in c^*_{U_1\times U_2,A_1\times A_2}(\{(x, y)\}) &\Leftrightarrow \text{There exists } (a_1, a_2) \in A_1 \times A_2 \\
&\quad \text{such that } U_1 \times U_2((a_1, a_2), (x, y)) \leq (m, l) \\
&\Leftrightarrow (U_1(a_1, x), U_2(a_2, y)) \leq (m, l) \\
&\quad \text{where } a_1 \in A_1 \text{ and } a_2 \in A_2. \\
&\Leftrightarrow U_1(a_1, x) \leq m \text{ and } U_2(a_2, y) \leq l \\
&\quad \text{where } a_1 \in A_1 \text{ and } a_2 \in A_2. \\
&\Leftrightarrow m \in c^*_{U_1,A_1}(\{x\}) \text{ and } l \in c^*_{U_2,A_2}(\{y\}) \\
&\quad \text{where } a_1 \in A_1 \text{ and } a_2 \in A_2. \\
&\Leftrightarrow (m, l) \in c^*_{U_1,A_1}(\{x\}) \times c^*_{U_2,A_2}(\{y\}).
\end{aligned}$$

□

Lemma 4.15. Let L be a bounded lattice, $e \in L$, $A_1, A_2 \subseteq L$ such that $[0, e] \cap A_1 \neq \emptyset$, $[0, e] \cap A_2 \neq \emptyset$ and U_1, U_2 be uninorms on L with the neutral element e satisfying $U_1(A_1, A_1) \subseteq A_1$ and $U_2(A_2, A_2) \subseteq A_2$, respectively. Then, $\mathfrak{T}_{c^*_{U_1,A_1}} \times \mathfrak{T}_{c^*_{U_2,A_2}} = \mathfrak{T}_{c^*_{U_1\times U_2,A_1\times A_2}}$.

P r o o f. Let $O_1 \times O_2 \in \mathfrak{T}_{c^*_{U_1,A_1}} \times \mathfrak{T}_{c^*_{U_2,A_2}}$. Considering Lemma 4.14,

$$\begin{aligned} c^*_{U_1\times U_2,A_1\times A_2}((L\times L)\backslash(O_1\times O_2)) &= c^*_{U_1\times U_2,A_1\times A_2}(((L\backslash O_1)\times L) \\ &\cup (L\times(L\backslash O_2))) \\ &= c^*_{U_1\times U_2,A_1\times A_2}((L\backslash O_1)\times L) \\ &\cup c^*_{U_1\times U_2,A_1\times A_2}(L\times(L\backslash O_2)) \\ &= c^*_{U_1,A_1}(L\backslash O_1)\times c^*_{U_2,A_2}(L) \\ &\cup c^*_{U_1,A_1}(L)\times c^*_{U_2,A_2}(L\backslash O_2) \\ &= (L\backslash O_1)\times L\cup L\times(L\backslash O_2) \\ &= (L\times L)\backslash(O_1\times O_2). \end{aligned}$$

Thus, $O_1 \times O_2 \in \mathfrak{T}_{c^*_{U_1\times U_2,A_1\times A_2}}$. Then, $\mathfrak{T}_{c^*_{U_1,A_1}} \times \mathfrak{T}_{c^*_{U_2,A_2}} \subseteq \mathfrak{T}_{c^*_{U_1\times U_2,A_1 x A_2}}$. Conversely, let $O \in \mathfrak{T}_{c^*_{U_1\times U_2,A_1\times A_2}}$. Then, $c^*_{U_1\times U_2,A_1\times A_2}(K) = K$ where $(L\times L)\backslash O = K \in \mathfrak{F}_{c^*_{U_1\times U_2,A_1\times A_2}}$. By Lemma 4.6 and $K = \bigcup_{(x,y)\in K}\{(x,y)\}$, we have that

$$\begin{aligned} K = c^*_{U_1\times U_2,A_1\times A_2}(K) &= c^*_{U_1\times U_2,A_1\times A_2}\left(\bigcup_{(x,y)\in K}\{(x,y)\}\right) \\ &= \bigcup_{(x,y)\in K} c^*_{U_1\times U_2,A_1\times A_2}(\{(x,y)\}) \\ &= \bigcup_{(x,y)\in K}\left(c^*_{U_1,A_1}(\{x\})\times c^*_{U_2,A_2}(\{y\})\right). \end{aligned}$$

Since $c^*_{U_1,A_1}(\{x\}) \in \mathfrak{F}_{c^*_{U_1,A_1}}$, $c^*_{U_2,A_2}(\{y\}) \in \mathfrak{F}_{c^*_{U_2,A_2}}$ and the product of the principle topologies is again a principal topology, $c^*_{U_1,A_1}(\{x\}) \times c^*_{U_2,A_2}(\{y\}) \in \mathfrak{F}_{c^*_{U_1,A_1}} \times \mathfrak{F}_{c^*_{U_2,A_2}}$. Moreover, since the union of close sets is close set, $\bigcup_{(x,y)\in K}\left(c^*_{U_1,A_1}(\{x\})\times c^*_{U_2,A_2}(\{y\})\right)$ is close set. Thus, $O \in \mathfrak{T}_{c^*_{U_1,A_1}} \times \mathfrak{F}_{c^*_{U_2,A_2}}$. Then, it is obtained that $\mathfrak{T}_{c^*_{U_1\times U_2,A_1 x A_2}} \subseteq \mathfrak{T}_{c^*_{U_1,A_1}} \times \mathfrak{T}_{c^*_{U_2,A_2}}$. □

Proposition 4.16. Let $(L,\leq,0,1)$ be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0,e]\cap A \neq \emptyset$, U be a uninorm on L with the neutral element e and $U(A,A) \subseteq A$. Consider the topological closure operator $c^*_{U,A}$.

(i) $\mathfrak{T}_{c^*_{U,A}}$ is not Hausdorff space.

(ii) $\mathfrak{T}_{c^*_{U,A}}$ is a quasi-Hausdorff space.

P r o o f. (i) The proof is clear by Proposition 4.8 (ii).

(ii) $x, y \in c^*_{U,A}(\{0\})$ for all $x, y \in L$ such that $x \neq y$ since $c^*_{U,A}(\{0\}) = L$. Then, $\mathfrak{T}_{c^*_{U,A}}$ is a quasi-Hausdorff space. □

We can compare the topologies obtained from the closure operators $c_{U,A}$, $c^*_{U,A}$ and $c^{**}_{U,A}$ as follows.

Proposition 4.17. The topology $\mathfrak{T}_{c_{U,A}}$ is finer than the topology $\mathfrak{T}_{c^*_{U,A}}$.

Proof.
Let $K \in \mathfrak{T}_{c^*_{U,A}}$. Then, $c^*_{U,A}(L\backslash K) = L\backslash K$. We aim to show that $c_{U,A}(L\backslash K) = L\backslash K$. Let $y \in c_{U,A}(L\backslash K)$. Then, there exists $a \in A$ and $x \in L\backslash K$ such that $y = U(a, x)$. Since $U(a, x) \leq y$, it is obtained that $y \in c^*_{U,A}(L\backslash K) = L\backslash K$. Therefore, $c_{U,A}(L\backslash K) \subseteq L\backslash K$. It clearly follows that $c_{U,A}(L \setminus K) = L \setminus K$. Then, $K \in \mathfrak{T}_{c_{U,A}}$. □

Proposition 4.18. The topology $\mathfrak{T}_{c_{U,A}}$ is finer than the topology $\mathfrak{T}_{c^{**}_{U,A}}$.

Proof. The proof is similar to the proof of Proposition 4.17. □

The topologies $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$ may not be compared. Let us investigate the following example.

Example 4.19. Consider the lattice $(L, \leq, \{0, a, b, 1\})$ whose lattice diagram is depicted as in Figure 3, the uninorm given in Table 2 and $A = \{e\}$. If we consider the set $B = \{0, e, b\}$, it is easily seen that $B \in \mathfrak{T}_{c^*_{U,A}}$ and $B \notin \mathfrak{T}_{c^{**}_{U,A}}$. Similarly, it is easily seen that $C \notin \mathfrak{T}_{c^*_{U,A}}$ and $C \in \mathfrak{T}_{c^{**}_{U,A}}$ when we consider the set $C = \{e, b, 1\}$.

5. RELATIONSHIP BETWEEN ORDER TOPOLOGY AND THE THREE NEWLY GENERATED TOPOLOGIES

It is well-known fact that a lattice possesses an order, thus inducing Alexandroff topology. In this section, we investigate the relationship between the order topology on the lattice and newly generated topologies $\mathfrak{T}_{c_{U,A}}$, $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$.

We refer our readers to the Section 2 of [18] for the detailed information of the Alexandrov topology induced by an order and its open sets.

Definition 5.1. (Karaçal and Köroğlu [13]) Let U be a uninorm with the neutral element e on a bounded lattice L and a subset A of L such that $U(A, A) \subseteq A$, $e \in A$. The preorder $x \preceq_{c_{U,A}} y \Leftrightarrow x \in c_{U,A}(\{y\})$ is called a U-preorder for U.

Definition 5.2. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0, e] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e and $U(A, A) \subseteq A$. The preorder $x \preceq_{c^*_{U,A}} y$ is defined with $x \in c^*_{U,A}(\{y\})$.

Definition 5.3. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[e, 1] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e and $U(A, A) \subseteq A$. The preorder $x \preceq_{c^{**}_{U,A}} y$ is defined with $x \in c^{**}_{U,A}(\{y\})$.

The following Propositions 5.4, 5.5 and 5.6 show that the order topology coincides with the topologies generated topologies $\mathfrak{T}_{c_{U,A}}$, $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$ in some particular cases.

Proposition 5.4. Let U be a uninorm with the neutral element e on a bounded lattice L and a subset A of L such that $U(A,A) \subseteq A$, $e \in A$. Consider the topological closure operator $c_{U,A}$, $U = \wedge$ and $A = L$. Then, $\preceq_{c_{U,A}} = \leq$.

Proof. For $x, y \in L$,

$$\begin{aligned} x \preceq_{c_{\wedge,L}} y &\Leftrightarrow x \in c_{\wedge,L}(\{y\}) \\ &\Leftrightarrow y \wedge x = x \text{ for } x \in L. \\ &\Leftrightarrow x \leq y. \end{aligned}$$

□

Proposition 5.5. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[0,e] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e and $U(A,A) \subseteq A$. Consider the topological closure operator $c^*_{U,A}$ and $U = \vee$. Then, $\preceq_{c^*_{U,A}} = \geq$.

Proof. For $x, y \in L$,

$$\begin{aligned} x \preceq_{c^*_{\vee,A}} y &\Leftrightarrow x \in c^*_{\vee,A}(\{y\}) \\ &\Leftrightarrow \text{ there exists an element } a \in A \text{ such that } y \vee a \leq x. \\ &\Leftrightarrow y \leq x. \end{aligned}$$

□

Proposition 5.6. Let L be a bounded lattice, $e \in L$, $A \subseteq L$ such that $[e,1] \cap A \neq \emptyset$, U be a uninorm on L with the neutral element e, $U(A,A) \subseteq A$. Consider the topological closure operator $c^*_{U,A}$ and $U = \wedge$. Then, $\preceq_{c^{**}_{U,A}} = \leq$.

Proof. For $x, y \in L$,

$$\begin{aligned} x \preceq_{c^{**}_{\wedge,A}} y &\Leftrightarrow y \in c^{**}_{\wedge,A}(\{x\}) \\ &\Leftrightarrow \text{ there exists an element } a \in A \text{ such that } x \leq y \wedge a. \\ &\Leftrightarrow x \leq y. \end{aligned}$$

□

With Propositions 5.4, 5.5 and 5.6, it has been shown above that the order topology coincides with the topologies $\mathfrak{T}_{c_{U,A}}$, $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$ in some particular cases. But in general, the order topology is different from the the topologies $\mathfrak{T}_{c_{U,A}}$, $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$.

We note that the Alexandrov topology induced by an order only satisfies T_0 separation axiom (it is not Hausdorff), we refer our readers to the Section 2 of [18] for the detailed information.

6. CONCLUDING REMARKS

It is well-known fact that there exists always a t-norm or uninorm on any bounded lattice. Taking into consideration the fact, some construction methods of principal topologies are presented in this paper. The topology $\mathcal{T}_{\langle\{f_i\}_{i\in I}\rangle}$ is defined and showed that it is a quasi-Hausdorff space. Moreover, $\langle\{f_i\}_{i\in I}\rangle$-point and some related notions are investigated. It is worth noting that $(L, \mathcal{P}(T_{x_0}))$ is a T_0 space. $x_0 \in A\rangle$-point and a fixed point. Also, two topological closure operators $c^*_{U,A}$ and $c^{**}_{U,A}$ are proposed by means of a uninorm U and a set $A \subseteq L$ and some properties of the introduced topologies are investigated. The relationship between the order topology and the topologies $\mathfrak{T}_{c_{U,A}}$, $\mathfrak{T}_{c^*_{U,A}}$ and $\mathfrak{T}_{c^{**}_{U,A}}$ are investigated.

(Received January 14, 2020)

REFERENCES

[1] P. Alexandroff: Diskrete Räume. Mat. Sb. (N.S.) *2* (1937), 501–518.

[2] G. Birkhoff: Lattice Theory. American Mathematical Society, New York 1948.

[3] X. Chen: Cores of Alexandroff spaces. REU (2015), Available at `http://math.uchicago.edu/REUDOCS/Chen,Xi(Cathy)`.

[4] G. D. Çaylı, Ü. Ertuğrul, T. Köroğlu, and F. Karaçal: Notes on locally internal uninorm on bounded lattices. Kybernetika *53* (2017), 911–921. DOI:10.14736/kyb-2017-5-0911

[5] I. Dahane, S. Lazaar, T. Richmond, and T. Turki: On resolvable primal spaces. Quaestiones Mathematicae (2018), 15–35. DOI:10.2989/16073606.2018.1437093

[6] J. Dixmier: General Topology. Springer–Verlag, 1984.

[7] O. Echi: Quasi-homeomorphisms, Goldspectral spaces and Jacspectral spaces. Boll Unione Mat. Ital. Sez. B Artic. Ric. Mat. *8* (2003), 489–507.

[8] O. Echi: The category of flows of Set and Top. Topol. Appl. *159* (2012), 2357–2366. DOI:10.1016/j.topol.2011.11.059

[9] Ü. Ertuğrul, M. N. Kesicioğlu, and F. Karaçal: Ordering based on uninorms. Inform. Sci. *330* (2016), 315–327. DOI:10.1016/j.ins.2015.10.019

[10] Ü. Ertuğrul, F. Karaçal, and R. Mesiar: Modified ordinal sums of triangular norms and triangular conorms on bounded lattices. Int. J. Intell. Systems *30* (2015), 807–817. DOI:10.1002/int.21713

[11] J. Fodor, R. Yager, and A. Rybalov: Structure of uninorms. Int. J. Uncertain. Fuzziness Knowledge-Based Systems *5* (1997), 411–427.

[12] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap: Aggregation Functions. Cambridge University Press, 2009.

[13] F. Karaçal and T. Köroğlu: An Alexandroff Topology Obtained from Uninorms. Submitted (2019).

[14] F. Karaçal and R. Mesiar: Uninorms on bounded lattices. Fuzzy Sets and Systems *261* (2015), 33–43. DOI:10.1016/j.fss.2014.05.001

[15] A. Katsevich and P. Mikusiński: Order of spaces of pseudoquotients. Top. Proc. *44* (2014), 21–31.

[16] J. L. Kelley: General Topology. Springer, New York 1975.

[17] M. N. Kesicioğlu, Ü. Ertuğrul, and F. Karaçal: Some notes on U-partial order. Kybernetika *55* (2019), 3, 518–530. DOI:10.14736/kyb-2019-3-0518

[18] H. Lai and D. Zhang: Fuzzy preorder and fuzzy topology. Fuzzy Sets Systems *157* (2006), 14, 1865–1885. DOI:10.1016/j.fss.2006.02.013

[19] S. Lazaar, T. Richmond, and S. Houssem: The autohomeomorphism group of connected homogeneous functionally Alexandroff spaces. Comm. Algebra (2019). DOI:10.1080/00927872.2019.1570240

[20] S. Lazaar, T. Richmond, and T. Turki: Maps generating the same primal space. Quaest. Math. *40* (2017), 17–28. DOI:10.2989/16073606.2016.1260067

[21] Z. Ma and W. M. Wu: Logical operators on complete lattices. Information Sciences *55* (1991), 77–97. DOI:10.1016/0020-0255(91)90007-H

[22] A. K. Steiner: The lattice of topologies: Structure and complementation. Trans. Amer. Math. Soc. *122* (1969), 379–398.

[23] P. Walden: Effective topology from spacetime tomography. J. Physics: Conference Series *68* (2007), 12–28.

[24] R. R. Yager and A. Rybalov: Uninorm aggregation operators. Fuzzy Sets Systems *80* (1996), 111–120. DOI:10.1016/0165-0114(95)00133-6

[25] H.-P. Zhang, R. Pérez.-Fernández, and B. De Baets: Topologies induced by the representatiton of a betweenness relation as a family of order relations. Topol. Appl. *258* (2019), 100–114. DOI:10.1016/j.topol.2019.02.045

Funda Karaçal, Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080 Trabzon. Turkey.
e-mail: fkaracal@yahoo.com

Ümit Ertuğrul, Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080 Trabzon. Turkey.
e-mail: uertugrul@ktu.edu.tr

M. Nesibe Kesicioğlu, Department of Mathematics, Recep Tayyip Erdogan University, 53100 Rize. Turkey.
e-mail: m.nesibe@gmail.com