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ON THE DISTRIBUTION OF (k,r)-INTEGERS IN PIATETSKI-SHAPIRO SEQUENCES

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Abstract. A natural number n is said to be a (k,r)-integer if $n=a^kb$, where $k>r>1$ and b is not divisible by the rth power of any prime. We study the distribution of such (k,r)-integers in the Piatetski-Shapiro sequence $\{\lfloor n^c\rfloor\}$ with $c>1$. As a corollary, we also obtain similar results for semi-r-free integers.

Keywords: (k,r)-integer; Piatetski-Shapiro sequence

MSC 2020: 11L07, 11N37

1. Introduction and results

Piatetski-Shapiro sequences (PS-sequences) are defined by

$$\mathbb{N}^c=\{\lfloor n^c\rfloor\}_{n\in\mathbb{N}}\quad(c>1,\ c\notin\mathbb{N}),$$

where $\lfloor z\rfloor$ is the integer part of a real z. The PS-sequence was introduced first by Piatetski-Shapiro (see [5]) to study prime numbers in a sequence of the form $\lfloor f(n)\rfloor$, where $f(n)$ is a polynomial. A positive integer is called *r-free* if it is not divisible by the rth power of any prime. In 1978, Rieger in [6] remarked on a paper of Stux (see [7]) concerning square-free integers of the form $\lfloor n^c\rfloor$, and showed that for real $x>1$,

$$\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor \text{ is square-free}}}1=\frac{6}{\pi^2}x+O(x^{(2c+1)/4+\varepsilon})\quad\text{for } 1<c<\frac{3}{2}.$$

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In [1], Cao and Zhai improved Rieger's range by the method of exponential sums and proved that for $\varepsilon > 0$,

$$\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor \text{ is square-free}}} 1 = \frac{6}{\pi^2}x + O(x^{36(c+1)/97+\varepsilon}) \quad \text{for } 1 < c < \frac{61}{36},$$

and in [2] they improved their range to $c < \frac{149}{87}$. In 2017, Zhang and Li in [9] studied in case of cube-free integers and showed that for $\varepsilon > 0$

$$\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor \text{ is cube-free}}} 1 = \frac{x}{\zeta(3)} + O(x^{1-\varepsilon}) \quad \text{for } 1 < c < \frac{11}{6}. \tag{1.1}$$

Very recently, Deshouillers in [3] used the approach of Rieger to improve the error term in (1.1) and showed that

$$\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor \text{ is cube-free}}} 1 = \frac{x}{\zeta(3)} + O(x^{(c+1)/3}\log x) \quad \text{for } 1 < c < 2.$$

It is natural to investigate this problem for other special integers. In this paper, we study the distribution of (k,r)-integers in PS-sequences. Let k and r be fixed integers such that $1 < r < k$. Any positive integer n is called a *(k,r)-integer* if n can be written in the form $a^k b$, where b is an r-free integer. When k tends to infinity, a (∞, r)-integer is the same as an r-free integer, one might consider (k,r)-integers as generalized r-free integers. A positive integer n is called *semi-r-free* if in its canonical factorization no exponent is equal to r. The (k,r)-integers include also the semi-r-free integers when $k = r+1$. The semi-r-free integers in PS-sequences are also obtained in this paper. Let us first recall the notion of exponent pair taken from [4], Chapter 2.

Definition 1.1. Let $A \geqslant 1$, $B \geqslant 1$, and suppose that for all C in $[B, 2B]$,

$$\sum_{B\leqslant n\leqslant C\leqslant 2B} \mathrm{e}^{2\pi \mathrm{i} f(n)} = O(A^{\kappa}B^{\lambda})$$

for some pair (κ, λ) of real numbers satisfying $0 \leqslant \kappa \leqslant 1/2 \leqslant \lambda \leqslant 1$, and for any real function $f \in C^{\infty}[B, 2B]$ satisfying for all $r \geqslant 1$ and for $x \in [B, 2B]$, $AB^{1-r} \ll |f^{(r)}(x)| \ll AB^{1-r}$, where the constants implied by $\ll$ depend only on r. Then we call (κ, λ) an *exponent pair*.

Using the simple one-dimensional exponent pair, we obtain the following results.

Theorem 1.1. *Let k and r be fixed integers such that $1 < r < k$. For any two exponent pairs (κ_1, λ_1) and (κ_2, λ_2) satisfying $r\lambda_1 - \kappa_1 < 1$, $r\lambda_2 - \kappa_2 > 1$, $k\lambda_1 - \kappa_1 > 1$ and $(r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2)) > 1$, we have*

$$\sum_{\substack{n \leqslant N \\ \lfloor n^c \rfloor \text{ is } (k,r)\text{-integer}}} 1 = \frac{\zeta(k)}{\zeta(r)} N + O(N^{(c/r) + \delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2)} \log N)$$

for $1 < c < (r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2))$, where

$$\delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2) = \frac{(\lambda_2 \kappa_1 - \lambda_1 \kappa_2) + r^{-1}(\kappa_2 - \kappa_1)}{\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2)}.$$

For example, if $(\kappa_1, \lambda_1) = (\frac{2}{7}, \frac{4}{7})$ and $(\kappa_2, \lambda_2) = (\frac{1}{6}, \frac{2}{3})$, we obtain the result in Theorem 1.1 for $(k, 2)$-integers as

$$\sum_{\substack{n \leqslant N \\ \lfloor n^c \rfloor \text{ is square-free}}} 1 = \frac{\zeta(k)}{\zeta(2)} N + O(N^{(c/2) + (3/16)} \log N) \quad \text{for } 1 < c < \frac{13}{8}.$$

Since the (k, r)-integers include the semi-r-free integers when $k = r + 1$, we obtain the following corollary.

Corollary 1.1. *With the same assumptions as in Theorem 1.1, we have*

$$\sum_{\substack{n \leqslant N \\ \lfloor n^c \rfloor \text{ is semi-}r\text{-free}}} 1 = \frac{\zeta(r+1)}{\zeta(r)} N + O(N^{(c/r) + \delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2)} \log N)$$

for $1 < c < (r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2))$.

Notation. Throughout this paper, ε denotes a fixed positive constant, not necessarily the same in all occurrenes. As usual, let $\mu(n)$ denote the Möbius function, and $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

2. Lemmas

In this section we state the lemmas which are needed in our proof.

Lemma 2.1 ([8], Lemma 2.6)**.** *If $q_{k,r}$ denotes the characteristic function of the set of (k,r)-integers, then*

$$q_{k,r}(n) = \sum_{a^k b^r c = n} \mu(b).$$

Lemma 2.2 ([8], Theorem 3.1)**.** *For $x \geqslant 3$, we have*

$$Q_{k,r}(x) := \sum_{n \leqslant x} q_{k,r}(n) = \frac{x\zeta(k)}{\zeta(r)} + O(x^{1/r} \exp(-B \log^{3/5}(x)(\log\log x)^{-1/5})),$$

where B is a positive constant depending on r and the O-estimate is uniform in k.

Lemma 2.3 ([1], Lemma)**.** *Let $y > 0$, $X > 1$, $0 \leqslant \sigma < 1$, $g(n) = (n+\sigma)^\gamma$. Then, for any exponent pair (κ, λ),*

$$\sum_{n \sim X} \psi(yg(n)) \ll y^{\kappa/(1+\kappa)} X^{(\lambda+\gamma\kappa)/(1+\kappa)} + y^{-1} X^{1-\lambda}.$$

3. Proof of Theorem 1.1

Let k and r be fixed integers such that $1 < r < k$. For any two exponent pairs (κ_1, λ_1) and (κ_2, λ_2) satisfying $r\lambda_1 - \kappa_1 < 1$, $r\lambda_2 - \kappa_2 > 1$, $k\lambda_1 - \kappa_1 > 1$ and $1 < c < (r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1+\kappa_1) - \lambda_1(1+\kappa_2))$, let $\gamma = 1/c$ and

$$T_c(N) = \sum_{\substack{n \leqslant N \\ \lfloor n^c \rfloor \text{ is } (k,r)\text{-integer}}} 1.$$

We note that, $\lfloor n^c \rfloor$ is a (k,r)-integer, if and only if $m^\gamma \leqslant n < (m+1)^\gamma$, where m is a (k,r)-integer. Therefore

$$T_c(N) = \sum_{\substack{m \leqslant N^c \\ m \text{ is } (k,r)\text{-integer}}} (\lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor) + O(1).$$

In view of Lemma 2.1, we have

$$T_c(N) = \sum_{a^k b^r m \leqslant N^c} \mu(b)(\lfloor -a^{k\gamma} b^{r\gamma} m^\gamma \rfloor - \lfloor -(a^k b^r m + 1)^\gamma \rfloor) + O(1).$$

Thus,

$$T_c(N) = \sum_{a^k b^r m \leqslant N^c} \mu(b)((a^k b^r m + 1)^\gamma - a^{k\gamma} b^{r\gamma} m^\gamma) + E_c(N), \tag{3.1}$$

where

$$E_c(N) = \sum_{a^k b^r m \leqslant N^c} \mu(b)(\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) + O(1). \tag{3.2}$$

The first sum on the right-hand side of (3.1) is

$$\begin{aligned}
&\sum_{a^k b^r m \leqslant N^c} \mu(b) a^{k\gamma} b^{r\gamma} m^\gamma \Big(\frac{\gamma}{a^k b^r m} + O(a^{-2k} b^{-2r} m^{-2}) \Big) \\
&\qquad = \gamma \sum_{a^k b^r m \leqslant N^c} \mu(b) a^{k\gamma - k} b^{r\gamma - r} m^{\gamma - 1} + O\Bigg(\Bigg| \sum_{a^k b^r m \leqslant N^c} a^{k\gamma - 2k} b^{r\gamma - 2r} m^{\gamma - 2} \Bigg| \Bigg) \\
&\qquad = \gamma \sum_{a^k b^r m \leqslant N^c} \mu(b) a^{k\gamma - k} b^{r\gamma - r} m^{\gamma - 1} + O(1) = \gamma \sum_{\substack{n \leqslant N^c \\ n \text{ is } (k,r)\text{-integer}}} n^{\gamma - 1} + O(1).
\end{aligned}$$

We apply Abel's identity and Lemma 2.2, then we have

$$\gamma \sum_{\substack{n \leqslant N^c \\ n \text{ is } (k,r)\text{-integer}}} n^{\gamma - 1} = \frac{\zeta(k)}{\zeta(r)} N + O(N^{(r - cr + c)/r} \exp(-C \log^{3/5}(x)(\log\log x)^{-1/5})), \tag{3.3}$$

where C is a positive constant depending on r and the O-estimate is uniform in k.

Now it remains to bound (3.2). For this we use the simple one dimensional exponent pair in Lemma 2.3. We write the sum in (3.2) as

$$\begin{aligned}
&\sum_{a^k b^r m \leqslant N^c} \mu(b)(\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\
&\qquad = \sum_{a \leqslant N^{c/k}} \sum_{b \leqslant N^{c/r}/a^{k/r}} \mu(b) \sum_{m \leqslant N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)).
\end{aligned}$$

Taking $B = N^{c/r + (\kappa_2 - \kappa_1)/(r(\lambda_2(\kappa_1 + 1) - \lambda_1(\kappa_2 + 1)))}$, we write

$$\begin{aligned}
&\sum_{a \leqslant N^{c/k}} \sum_{b \leqslant N^{c/r}/a^{k/r}} \mu(b) \sum_{m \leqslant N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\
&\qquad = \sum_{a \leqslant N^{c/k}} \sum_{b \leqslant B} \mu(b) \sum_{m \leqslant N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\
&\qquad\quad + \sum_{a \leqslant N^{c/k}} \sum_{B < b \leqslant N^{c/r}/a^{k/r}} \mu(b) \\
&\qquad\quad \times \sum_{m \leqslant N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)).
\end{aligned} \tag{3.4}$$

In view of Lemma 2.3 with the exponent pair $(\kappa,\lambda)=(\kappa_1,\lambda_1)$, we have

$$
\begin{aligned}
(3.5)\quad & \sum_{b\leqslant B}\mu(b)\sum_{m\leqslant N^c/a^kb^r}(\psi(-(a^kb^rm+1)^\gamma)-\psi(-a^{k\gamma}b^{r\gamma}m^\gamma))\\
&\ll \log N\sum_{b\leqslant B}\Big((a^{k\gamma}b^{r\gamma})^{\kappa_1/(1+\kappa_1)}\Big(\frac{N^c}{a^kb^r}\Big)^{(\lambda_1+\gamma\kappa_1)/(1+\kappa_1)}\\
&\qquad +(a^{k\gamma}b^{r\gamma})^{-1}\Big(\frac{N^c}{a^kb^r}\Big)^{1-\gamma}\Big)\\
&\ll a^{-k\lambda_1/(1+\kappa_1)}N^{(c\lambda_1+\kappa_1)/(1+\kappa_1)}\log N\sum_{b\leqslant B}b^{-r\lambda_1/(1+\kappa_1)}+a^{-k}N^{c-1}\log N\\
&\ll a^{-k\lambda_1/(1+\kappa_1)}B^{1-r\lambda_1/(1+\kappa_1)}N^{(c\lambda_1+\kappa_1)/(1+\kappa_1)}\log N+a^{-k}N^{c-1}\log N.
\end{aligned}
$$

In view of $k\lambda_1/(1+\kappa_1)>1$ and (3.5), the first term in (3.4) is bounded by

$$
\begin{aligned}
(3.6)\quad & \sum_{a\leqslant N^{c/k}}\sum_{b\leqslant B}\mu(b)\sum_{m\leqslant N^c/a^kb^r}(\psi(-(a^kb^rm+1)^\gamma)-\psi(-a^{k\gamma}b^{r\gamma}m^\gamma))\\
&\ll B^{1-r\lambda_1/(1+\kappa_1)}N^{(c\lambda_1+\kappa_1)/(1+\kappa_1)}\log N+N^{c-1}\log N\\
&\ll N^{(c/r)+\delta(\kappa_1,\lambda_1,\kappa_2,\lambda_2)}\log N,
\end{aligned}
$$

where

$$
\delta(\kappa_1,\lambda_1,\kappa_2,\lambda_2)=\frac{(\lambda_2\kappa_1-\lambda_1\kappa_2)+r^{-1}(\kappa_2-\kappa_1)}{\lambda_2(1+\kappa_1)-\lambda_1(1+\kappa_2)}.
$$

Now we bound the second sum in (3.4). In view of Lemma 2.3 with the exponent pair $(\kappa,\lambda)=(\kappa_2,\lambda_2)$, we have

$$
\begin{aligned}
(3.7)\quad & \sum_{B<b\leqslant N^{c/r}/a^{k/r}}\mu(b)\sum_{m\leqslant N^c/a^kb^r}(\psi(-(a^kb^rm+1)^\gamma)-\psi(-a^{k\gamma}b^{r\gamma}m^\gamma))\\
&\ll \log N\sum_{B<b\leqslant N^{c/r}/a^{k/r}}\Big((a^{k\gamma}b^{r\gamma})^{\kappa_2/(1+\kappa_2)}\Big(\frac{N^c}{a^kb^r}\Big)^{(\lambda_2+\gamma\kappa_2)/(1+\kappa_2)}\\
&\qquad +(a^{k\gamma}b^{r\gamma})^{-1}\Big(\frac{N^c}{a^kb^r}\Big)^{1-\gamma}\Big)\\
&\ll a^{-k\lambda_2/(1+\kappa_2)}N^{(c\lambda_2+\kappa_2)/(1+\kappa_2)}\\
&\qquad\times\log N\sum_{B<b\leqslant N^{c/r}/a^{k/r}}b^{-r\lambda_2/(1+\kappa_2)}+a^{-k}N^{c-1}\log N\\
&\ll a^{-k\lambda_2/(1+\kappa_2)}B^{1-r\lambda_2/(1+\kappa_2)}N^{(c\lambda_2+\kappa_2)/(1+\kappa_2)}\log N\\
&\qquad +a^{-k}N^{c-1}\log N.
\end{aligned}
$$

In view of $k\lambda_2/(1+\kappa_2) > 1$ and (3.7), the second term in (3.4) is bounded by

$$
\begin{aligned}
(3.8) \quad & \sum_{a\leqslant N^{c/k}} \sum_{B<b\leqslant N^{c/r}/a^{k/r}} \mu(b) \sum_{m\leqslant N^c/a^k b^r} (\psi(-(a^k b^r m+1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\
&\ll B^{1-r\lambda_2/(1+\kappa_2)} N^{(c\lambda_2+\kappa_2)/(1+\kappa_2)} \log N + N^{c-1}\log N \\
&\ll N^{(c/r)+\delta(\kappa_1,\lambda_1,\kappa_2,\lambda_2)} \log N.
\end{aligned}
$$

From $r\lambda_1/(1+\kappa_1) < 1 < r\lambda_2/(1+\kappa_2)$, we have

$$0 < \frac{r\lambda_2}{1+\kappa_2} - \frac{r\lambda_1}{1+\kappa_1} = \frac{r(\lambda_2(1+\kappa_1) - \lambda_1(1+\kappa_2))}{(1+\kappa_1)(1+\kappa_2)}.$$

Then

$$(3.9) \qquad \lambda_2(1+\kappa_1) - \lambda_1(1+\kappa_2) > 0.$$

From $\kappa_1 > r\lambda_1 - 1$ and $\kappa_2 < r\lambda_2 - 1$, we have

$$
\begin{aligned}
(3.10) \qquad (\lambda_2\kappa_1 - \lambda_1\kappa_2) + r^{-1}(\kappa_2 - \kappa_1) &= \frac{1}{r}(r\lambda_2\kappa_1 - r\lambda_1\kappa_2 + \kappa_2 - \kappa_1) \\
&= \kappa_1(r\lambda_2 - 1) - \kappa_2(r\lambda_1 - 1) \\
&> \kappa_1\kappa_2 - \kappa_2\kappa_1 = 0.
\end{aligned}
$$

From (3.9) and (3.10), we have

$$(3.11) \qquad \delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2) > 0.$$

In view of (3.6) and (3.8) we have

(3.12)

$$\sum_{a^k b^r m\leqslant N^c} \mu(b)(\psi(-(a^k b^r m+1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) = O(N^{(c/r)+\delta(\kappa_1,\lambda_1,\kappa_2,\lambda_2)} \log N).$$

In view of (3.11), the bound (3.12) dominates the error term in (3.3). Thus, the conclusion follows from (3.1), (3.3) and (3.12).

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