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A SMOOTHING LEVENBERG-MARQUARDT METHOD FOR THE COMPLEMENTARITY PROBLEM OVER SYMMETRIC CONE

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Abstract. In this paper, we propose a smoothing Levenberg-Marquardt method for the symmetric cone complementarity problem. Based on a smoothing function, we turn this problem into a system of nonlinear equations and then solve the equations by the method proposed. Under the condition of Lipschitz continuity of the Jacobian matrix and local error bound, the new method is proved to be globally convergent and locally superlinearly/quadratically convergent. Numerical experiments are also employed to show that the method is stable and efficient.

Keywords: complementarity problem; symmetric cone; Levenberg-Marquardt method; Euclidean Jordan algebra; local error bound

MSC 2020: 65K05, 90C33

1. Introduction

We consider the symmetric cone complementarity problem (SCCP): find $x \in \mathcal{V}$ such that

$$x \in \mathcal{K}, \ F(x) \in \mathcal{K}, \ \langle x, F(x) \rangle = 0, \tag{1.1}$$

where $\mathcal{V}$ is an n-dimensional vector space with inner product $\langle \cdot, \cdot \rangle$, $\mathcal{K} \subset \mathcal{V}$ is a symmetric cone and $F\colon \mathcal{V} \to \mathcal{V}$ is a continuously differentiable transformation. If there exists a bilinear mapping $(x, y) \to x \circ y\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ such that for any $x, y, z \in \mathcal{V}$,

$$x \circ y = y \circ x; \quad \langle x \circ y, z \rangle = \langle x, y \circ z \rangle \quad \text{and} \quad x \circ (x^2 \circ y) = x^2 \circ (x \circ y),$$

where $x^2 = x \circ x$, then $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra.

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The SCCP, as an important optimization problem, includes the nonlinear complementarity problem (NCP) on R^n [10], second-order cone complementarity problem (SOCCP) [4] and semi-definite complementarity problem (SDCP) [21] as special cases and is closely associated with uncertain optimization, combination optimization, pattern recognition and equilibrium theory. In addition, it arises from various applications in economics, management science, transportation and communication, see [1], [3], [8], [9], [11], [12]. Many scholars have studied the SCCP and proposed many algorithms, such as interior point methods [13], [15], [17] and smooth Newton smoothing methods [14], [20].

The basic idea of the smoothing Newton method is to transform the SCCP into a set of equivalent equations by using a smoothing function. Consider the following smoothing CHKS function

$$\varphi(\varepsilon, x, y) = x + y - \sqrt{(x-y)^2 + 2\varepsilon e}, \quad \varepsilon > 0,\ x \in \mathcal{V},\ y \in \mathcal{V}, \tag{1.2}$$

and set

$$H(z) = H(\varepsilon, x, y) = \begin{pmatrix} \varepsilon \\ y - F(x) \\ \varphi(\varepsilon, x, y) \end{pmatrix}. \tag{1.3}$$

Then $H(\varepsilon, x, y) = 0$ if and only if $\varepsilon = 0$ and (x, y) is the solution to the SCCP (1.1).

For solving the system of equations $G(x) = 0$, the iteration of the Newton method is

$$x^{k+1} = x^k + \alpha_k d_k,$$

where d_k satisfies $G'(x^k)d_k = G(x^k)$ and $G'(x^k)$ is the Jacobian matrix of $G(x)$ at x^k. The Newton method possesses quadratic convergence property if the Jacobian matrix is Lipschitz continuous and nonsingular at the solution. However, the Newton method may not be well-defined when $G'(x)$ is singular or nearly singular. To overcome this difficulty, the Levenberg-Marquardt (LM) method computes d_k by

$$[G'(x^k)^\top G'(x^k) + \mu_k I]d_k = -G'(x^k)^\top G(x^k), \tag{1.4}$$

where μ_k is a positive parameter. When the Jacobian matrix $G'(x^k)$ is singular or nearly singular, $G'(x^k)^\top G'(x^k) + \mu_k I$ can be made positive definite by appropriately selecting the parameter μ_k, so that (1.4) has a unique solution, which is one of the advantages of LM methods.

Facchinei and Kanzow [6] proposed an inexact LM method for the large-scale nonlinear complementarity problem and proved that the LM method possesses global and local superlinear/quadratic convergence based on the assumptions of strict complementarity and uniform nonsingularity. Yamashita and Fukushima [18] introduced a new update rule for μ_k and proposed an LM method for solving a system of nonlinear equations. This work is of great significance. The LM method is shown to be locally quadratically convergent under a local error bound assumption which is weaker than the nonsingularity condition. The norm $\|G(x)\|$ is said to provide a local error bound on a neighborhood N of $x^* \in X^*$ if there exists a positive constant c such that

$$\operatorname{dist}(x, X^*) \leqslant c\|G(x)\| \quad \forall\, x \in N, \tag{1.5}$$

where X^* is the solution set of $G(x) = 0$. Note, that if $G'(x)$ is nonsingular at x^*, then x^* is an isolated solution. Thus, there exists a constant $\beta > 0$ such that

$$\|G(x)\| = \|G(x) - G(x^*)\| \geqslant \beta\|x - x^*\| = \beta \cdot \operatorname{dist}(x, X^*).$$

So, $\|G(x)\|$ provides a local error bound by letting $c = 1/\beta$, see [18]. The converse is not necessarily true. For example, let $G(x)\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$G(x) = (\mathrm{e}^{x_1+x_2} - 1, (x_1 + x_2)^2)^\top.$$

The solution set is $X^* = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}$ and we have

$$\operatorname{dist}(x, X^*) = \frac{\sqrt{2}}{2}|x_1 + x_2|.$$

Then, (1.5) holds for any $c \in (1, \infty)$ when N is chosen as

$$N = \Big\{x \in \mathbb{R}^2 \mid |x_1 + x_2| \leqslant \frac{\sqrt{2}}{2}\Big\}.$$

However, $G'(x)$ is singular at any $x^* \in X^*$. Therefore, the condition that $\|G(x)\|$ provides a local error bound in a neighborhood of x^* is weaker than the condition that $G'(x)$ is nonsingular.

Zhang [19] utilized a new update rule $\mu_k = \|G(x^k)\|^\delta$, $\delta \in (0, 2]$, introduced by Dan [5] and proposed a smoothing LM method for the NCP. Because of the non-negativity of the smoothing parameter, they solved a constraint minimization problem instead of a linear system of equations to obtain the search direction. The local convergence of the method is also analyzed based on the local error bound condition.

Motivated by the research above, we propose a smoothing LM method for the SCCP. In order to guarantee the non-negativity of the smoothing parameter ε, a constraint condition is added when the line search is carried out. The proposed method is shown to be globally convergent and locally superlinearly/quadratically convergent under the condition of Lipschitz continuity of the Jacobian matrix and local error bound.

This paper is organized as follows. Some basic definitions and results are illustrated briefly in Section 2. Section 3 gives an LM algorithm for the SCCP and discusses its feasibility and global convergence. The local convergence of the proposed LM algorithm is analyzed is Section 4. Some numerical experimental results which show the effectiveness of the proposed method are reported in Section 5. Some conclusions are made in the last section.

2. Preliminaries

In this section, we review some basic concepts and results on Euclidean Jordan algebra. For a deeper discussion, the reader can be referred to [7].

Assume that $\mathcal{V}$ has a unit element, that is, there exists an element $e \in \mathcal{V}$ such that $x \circ e = x$ for any $x \in \mathcal{V}$. We define the degree of x as the minimal positive integer $m(x)$ such that $\{e, x, x^2, \ldots, x^{m(x)}\}$ is linearly dependent and the rank of $\mathcal{V}$ is defined by $\operatorname{rank}(\mathcal{V}) = \max\{m(x)\colon\ x \in \mathcal{V}\}$. An element $x \in \mathcal{V}$ is an idempotent if $x^2 = x$ and it is called a primitive idempotent if it is nonzero and cannot be written as a sum of two nonzero idempotents. A finite set of primitive idempotents $\{e_1, e_2, \ldots, e_r\}$ in $\mathcal{V}$ is a Jordan frame if $e_i \circ e_j = 0$ for all $i \neq j$ and $\sum\limits_{i=1}^{r} e_i = e$.

The following theorem is the famous spectral decomposition theorem of [7].

Theorem 2.1. *Suppose that $\mathcal{V}$ is a Euclidean Jordan algebra with rank r, then for any $x \in \mathcal{V}$, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$ such that*

$$x = \lambda_1(x)e_1 + \lambda_2(x)e_2 + \ldots + \lambda_r(x)e_r.$$

The numbers $\lambda_i(x)$ $(i = 1, 2, \ldots, r)$ are the eigenvalues of x which are uniquely determined by x. We write $x \in \mathcal{K}$ $(x \in \operatorname{int}\mathcal{K})$ for $x \geqslant 0$ $(x > 0)$. The inequalities $x \geqslant y$ and $x > y$ mean that $x - y \geqslant 0$ and $x - y > 0$, respectively. Define the trace and determinant of x by $\operatorname{tr}(x) = \sum\limits_{i=1}^{r} \lambda_i(x)$ and $\det(x) = \prod\limits_{i=1}^{r} \lambda_i(x)$, respectively. The inner product $\langle\cdot,\cdot\rangle$ is $\langle x, y\rangle = \operatorname{tr}(x \circ y)$ and the norm induced by $\langle\cdot,\cdot\rangle$ is defined by

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\operatorname{tr}(x^2)} = \sqrt{\sum_{i=1}^{r} \lambda_i^2(x)}.$$

Let $f(\cdot)\colon\ R \to R$ be a real-valued function. Then we can define a vector-valued function $F(\cdot)\colon\ \mathcal{V} \to \mathcal{V}$ associated with the Euclidean Jordan algebra by

$$F(x) = f(\lambda_1(x))e_1 + f(\lambda_2(x))e_2 + \ldots + f(\lambda_r(x))e_r,$$

where $x \in \mathcal{V}$ has the spectral decomposition $x = \sum\limits_{i=1}^{r} \lambda_i(x) e_i$. When $f(\cdot)$ is taken as $t_+ = \max\{0, t\}$ and $t_- = \min\{0, t\}$ for $t \in R$, we have

$$x_+ = \sum_{i=1}^{r} \lambda_i(x)_+ e_i \quad \text{and} \quad x_- = \sum_{i=1}^{r} \lambda_i(x)_- e_i.$$

Moreover, if $x \geqslant 0$, then $\lambda_i(x) \geqslant 0$, $i = 1, 2, \ldots, r$. When $x \geqslant 0$, we can define the square root of x by $\sqrt{x} = \sum\limits_{i=1}^{r} \sqrt{\lambda_i(x)} e_i$.

3. The LM algorithm and its global convergence

In this section, we propose a smoothing LM method for the SCCP (1.1) and discuss its global convergence. Let $H(z)$ be defined by (1.3) and

$$\Psi(z) = \frac{1}{2} \|H(z)\|^2.$$

Upon Theorem 3.1 of [16], we have the following lemma which shows the semi-smoothness of $H(z)$.

Lemma 3.1. *Let $\varphi(\varepsilon, x, y)$ and $H(z)$ be defined by* (1.2) *and* (1.3), *respectively. Then $H(z)$ is semi-smooth on any $z \in R \times \mathcal{V} \times \mathcal{V}$.*

According to Lemma 3.1, $H(z)$ is Lipschitz continuous on $\mathbb{R}_+ \times \mathcal{V} \times \mathcal{V}$, i.e., there exists $L_1 > 0$ such that

$$\|H(z) - H(w)\| \leqslant L_1 \|z - w\| \quad \forall\, z, w \in \mathbb{R}_+ \times \mathcal{V} \times \mathcal{V}. \tag{3.1}$$

Now we describe the LM method for the SCCP.

Algorithm 1 (A smoothing LM method)**.**

Initial step. Choose $\varepsilon > 0$, $\delta \in (0, 2]$, $\varrho \in (0, 1)$, and $\sigma \in (0, 1)$. Let $z^0 = (\varepsilon_0, x^0, y^0) \in \mathbb{R}_+ \times \mathcal{V} \times \mathcal{V}$ be an arbitrary point and $\mu_0 = \|H(z^0)\|^\delta$. Set $k = 0$ and go to Step 1.

Step 1. If $\|H(z^k)\| \leqslant \varepsilon$, stop. Otherwise, go to Step 2.

Step 2. Solve the following system of equations to obtain $\Delta z^k = (\Delta\varepsilon_k, \Delta x^k, \Delta y^k)$:

$$[H'(z^k)^\top H'(z^k) + \mu_k I] \Delta z^k = -H'(z^k)^\top H(z^k). \tag{3.2}$$

Step 3. Find the smallest positive integer m_k with $\alpha_k = \varrho^{m_k}$ such that

$$\Psi(z^k + \alpha_k \Delta z^k) \leqslant \Psi(z^k) - \sigma \alpha_k \mu_k \|\Delta z^k\|^2 \tag{3.3}$$

and

$$|\alpha_k \Delta \varepsilon_k| < \varepsilon_k. \tag{3.4}$$

Step 4. Let $z^{k+1} = z^k + \alpha_k \Delta z^k$, $\mu_{k+1} = \|H(z^{k+1})\|^\delta$. Set $k = k+1$ and go to Step 1.

Remark.

(1) It is easy to see that $H'(z^k)^\top H'(z^k) + \mu_k I$ is symmetric positive definite, thus the system of equations (3.2) has a unique solution Δz^k. Thus Step 2 is feasible.
(2) By (3.2) and (3.3), $\varepsilon_{k+1} = \varepsilon_k + \alpha_k \Delta \varepsilon_k$ which does not guarantee the non-negativity of $\{\varepsilon_k\}$. So (3.4) is essential.
(3) Upon (3.3), we obtain that $\{\Psi(z^k)\}$ is decreasing monotonically, which implies that $\{\|H(z^k)\|\}$ is bounded, i.e., there exists $C > 0$ such that

$$\|H(z^k)\| \leqslant C \quad \forall\, k \geqslant 0. \tag{3.5}$$

Theorem 3.2. *Algorithm* 1 *is well-defined.*

Proof. From the above discussion, the system of linear equations (3.2) in Step 2 is solvable. So it suffices to show that Step 3 is feasible. From the fact that Δz^k is the optimal solution to the unconstrained minimization problem

$$\min \theta(\Delta z) = \frac{1}{2}\|H'(z^k)\Delta z + H(z^k)\|^2 + \frac{1}{2}\mu_k \|\Delta z\|^2, \tag{3.6}$$

it follows that

$$\theta'(\Delta z^k)^\top (\Delta z - \Delta z^k) \geqslant 0 \quad \forall\, \Delta z \in R \times \mathcal{V} \times \mathcal{V}.$$

Let $\Delta z = (0,0,0) \in R \times \mathcal{V} \times \mathcal{V}$. We get that

$$\{H'(z^k)^\top [H'(z^k)\Delta z^k + H(z^k)] + \mu_k \Delta z^k\}^\top \Delta z^k \leqslant 0.$$

If $\Delta z^k \neq 0$, then

$$H(z^k)^\top H'(z^k)\Delta z^k \leqslant -\|H'(z^k)\Delta z^k\|^2 - \mu_k \|\Delta z^k\|^2.$$

Hence,

$$\begin{aligned}\Psi(z^k+\alpha\Delta z^k)-\Psi(z^k) &= \alpha\cdot\Psi'(z^k)\Delta z^k+o(\alpha)\\ &= \alpha\cdot H(z^k)^\top H'(z^k)\Delta z^k+o(\alpha)\\ &\leqslant -\alpha\cdot(\|H'(z^k)\Delta z^k\|^2+\mu_k\|\Delta z^k\|^2)+o(\alpha)\\ &\leqslant -\alpha\mu_k\|\Delta z^k\|^2+o(\alpha),\end{aligned}$$

which implies that there exists a constant $\overline{\alpha}\in(0,1)$ such that

$$\Psi(z^k+\alpha\Delta z^k)\leqslant\Psi(z^k)-\sigma\alpha\mu_k\|\Delta z^k\|^2$$

holds for any $\alpha\in(0,\overline{\alpha}]$ and $\sigma\in(0,1)$. Hence, Step 3 can be carried out. The proof is completed. □

Now we show the global convergence. For this purpose, we need the following assumption.

Assumption 3.3. The function $H'(z)$ is Lipschitz continuous on $\mathbb{R}_+\times\mathcal{V}\times\mathcal{V}$, i.e.,

$$\|H'(z)-H'(w)\|\leqslant L_2\|z-w\|\quad\forall\,z,w\in\mathbb{R}_+\times\mathcal{V}\times\mathcal{V}. \tag{3.7}$$

Theorem 3.4. *Suppose that Assumption* 3.3 *holds, then Algorithm* 1 *terminates in a finite number of iterations or the sequence* $\{z^k=(\varepsilon_k,x^k,y^k)\}$ *generated by Algorithm* 1 *satisfies* $\liminf\limits_{k\to\infty}\|H'(z^k)^\top H(z^k)\|=0$.

Proof. We prove the theorem by contradiction. Assume that there exists a positive integer $\overline{k}>0$ such that

$$\|H'(z^k)^\top H(z^k)\|\geqslant\tau_1>0\quad\forall\,k\geqslant\overline{k}. \tag{3.8}$$

Thus, for any $k\geqslant\overline{k}$, there exist $\tau_2>0$ and $\tau_3>0$ such that

$$\|H'(z^k)\|\geqslant\tau_2>0\quad\text{and}\quad\|H(z^k)\|\geqslant\tau_3>0. \tag{3.9}$$

By the line search (3.3),

$$\sum_{k=0}^{\infty}\sigma\alpha_k\mu_k\|\Delta z^k\|^2<\infty,\quad\text{i.e.,}\quad\lim_{k\to\infty}\sigma\alpha_k\mu_k\|\Delta z^k\|^2=0.$$

Combining this with $\mu_k=\|H(z^k)\|^\delta$ and (3.9) yields

$$\lim_{k\to\infty}\alpha_k\|\Delta z^k\|^2=0.$$

Consider the following two cases:

(1) $\lim\limits_{k\to\infty} \|\Delta z^k\| = 0$. According to (3.2), we get

$$\liminf_{k\to\infty} \|H'(z^k)^\top H(z^k)\| = \liminf_{k\to\infty} \|[H'(z^k)^\top H'(z^k) + \mu_k I]\Delta z^k\| = 0,$$

which is a contradiction.

(2) $\lim\limits_{k\to\infty} \alpha_k = 0$. Set $\alpha_{k'} = \alpha_k/\varrho$. It follows from the line search (3.3) that

$$\Psi(z^k + \alpha_{k'}\Delta z^k) > \Psi(z^k) - \sigma\alpha_{k'}\mu_k\|\Delta z^k\|^2.$$

Owing to Assumption 3.3, we have

$$\|H(z) - H(w) - H'(w)(z-w)\| \leqslant L_2\|z-w\|^2. \tag{3.10}$$

Then,

$$\begin{aligned}
&\sigma\alpha_{k'}\mu_k\|\Delta z^k\|^2 \\
&\quad > -[\Psi(z^k + \alpha_{k'}\Delta z^k) - \Psi(z^k)] \\
&\quad = -\frac{1}{2}[\|H(z^k + \alpha_{k'}\Delta z^k)\|^2 - \|H(z^k)\|^2] \\
&\quad = -\frac{1}{2}\{\|H(z^k + \alpha_{k'}\Delta z^k) - H(z^k)\|^2 + 2H(z^k)^\top[H(z^k + \alpha_{k'}\Delta z^k) - H(z^k)]\} \\
&\quad = -\frac{1}{2}\{\|H(z^k + \alpha_{k'}\Delta z^k) - H(z^k)\|^2 \\
&\qquad + 2H(z^k)^\top[H(z^k + \alpha_{k'}\Delta z^k) - H(z^k) - \alpha_{k'}H'(z^k)\Delta z^k + \alpha_{k'}H'(z^k)\Delta z^k]\} \\
&\quad \geqslant -\frac{1}{2}\{\|H(z^k + \alpha_{k'}\Delta z^k) - H(z^k)\|^2 + 2L_2\alpha_{k'}^2\|H(z^k)\|\cdot\|\Delta z^k\|^2 \\
&\qquad + 2\alpha_{k'}[H'(z^k)^\top H(z^k)]^\top\Delta z^k\}\} \\
&\quad = -\frac{1}{2}\{\|H(z^k + \alpha_{k'}\Delta z^k) - H(z^k)\|^2 + 2L_2\alpha_{k'}^2\|H(z^k)\|\cdot\|\Delta z^k\|^2 \\
&\qquad - 2\alpha_{k'}(\Delta z^k)^\top[H'(z^k)^\top H'(z^k) + \mu_k I]\Delta z^k\} \\
&\quad \geqslant -\frac{1}{2}[L_1^2\alpha_{k'}^2\|\Delta z^k\|^2 + 2L_2C\alpha_{k'}^2\|\Delta z^k\|^2 - 2\alpha_{k'}\mu_k\|\Delta z^k\|^2],
\end{aligned}$$

where the second inequality comes from (3.10) and the last inequality comes from (3.1) and (3.5).

Notice that $\mu_k = \|H(z^k)\|^\delta \geqslant \tau_3^\delta$. By a simple calculation, we get

$$\alpha_{k'} > \frac{2\tau_3^\delta(1-\sigma)}{L_1^2 + 2L_2C}.$$

Taking limits on both sides of the above inequality yields

$$0 = \lim_{k\to\infty} \alpha_{k'} > 0,$$

which is a contradiction. The proof is completed. □

4. Local convergence

In this section, we discuss the local convergence of Algorithm 1. Without loss of generality, assume that $\{z^k\}$ is an infinite sequence generated by Algorithm 1 which converges to $z^* \in \Omega$, where Ω is the solution set of $H(z) = 0$. We first make an assumption.

Assumption 4.1. $\|H(z)\|$ provides a local error bound on some neighborhood of z^*, i.e., there exist two constants $b_1 > 0$ and $r > 0$ such that

$$\operatorname{dist}(z, \Omega) \leqslant r \cdot \|H(z)\| \quad \forall\, z \in N(z^*, b_1). \tag{4.1}$$

In view of Assumption 3.3 and the Lipschitz continuity of $H(z)$, the functions $H(z)$ and $H'(z)$ are both Lipschitz continuous at z^*, i.e., there exists a constant $b_2 > 0$ such that

$$\|H(z) - H(w)\| \leqslant L_1 \|z - w\| \quad \forall\, z, w \in N(z^*, b_2) \tag{4.2}$$

and

$$\|H'(z) - H'(w)\| \leqslant L_2 \|z - w\| \quad \forall\, z, w \in N(z^*, b_2). \tag{4.3}$$

Upon the Lipschitz continuity of $H'(z)$, we have

$$\|H(z) - H(w) - H'(w)(z - w)\| \leqslant L_2 \|z - w\|^2 \quad \forall\, z, w \in N(z^*, b_2). \tag{4.4}$$

Lemma 4.2. *Suppose that Assumptions* 3.3 *and* 4.1 *hold,* $\{z^k\}$ *is generated by Algorithm* 1 *and* $z^k \in N(z^*, b)$, *where* $b = \min\{b_1, b_2/2\}$. *Then there exist two constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\|\Delta z^k\| \leqslant c_1 \cdot \operatorname{dist}(z^k, \Omega),$$
$$\|H(z^k) + H'(z^k)\Delta z^k\| \leqslant c_2 \cdot \operatorname{dist}(z^k, \Omega)^{1+\delta/2}.$$

Proof. Let $\overline{z}^k = (0, \bar{x}^k, \overline{y}^k) \in \Omega$ be such that $\|\overline{z}^k - z^k\| = \operatorname{dist}(z^k, \Omega)$. Then

$$\|\overline{z}^k - z^*\| \leqslant \|\overline{z}^k - z^k\| + \|z^k - z^*\| \leqslant \|z^* - z^k\| + \|z^k - z^*\| \leqslant 2b \leqslant b_2.$$

In view of Assumption 4.1 and the Lipschitz continuity of $H(z)$,

$$\mu_k = \|H(z^k)\|^\delta = \|H(z^k) - H(\overline{z}^k)\|^\delta \leqslant L_1^\delta \|\overline{z}^k - z^k\|^\delta \tag{4.5}$$

and

$$\mu_k = \|H(z^k)\|^\delta \geqslant \left(\frac{1}{r}\right)^\delta \cdot \operatorname{dist}(z^k, \Omega)^\delta = \left(\frac{1}{r}\right)^\delta \cdot \|\overline{z}^k - z^k\|^\delta. \tag{4.6}$$

It is easy to verify that $\overline{z}^k - z^k = (-\varepsilon^k, \bar{x}^k - x^k, \overline{y}^k - y^k)$ is a feasible solution of (3.6). Combining (4.4) and (4.6) yields

$$\begin{aligned}
\|\Delta z^k\|^2 &\leqslant \frac{2}{\mu_k}\theta(\overline{z}^k - z^k) \\
&= \frac{2}{\mu_k}\Big[\frac{1}{2}\|H'(z^k)(\overline{z}^k - z^k) + H(z^k)\|^2 + \frac{\mu_k}{2}\|\overline{z}^k - z^k\|^2\Big] \\
&= \frac{1}{\mu_k}\|H'(z^k)(\overline{z}^k - z^k) + H(z^k) - H(\overline{z}^k)\|^2 + \|\overline{z}^k - z^k\|^2 \\
&\leqslant \frac{1}{\mu_k}\cdot L_2^2\|\overline{z}^k - z^k\|^4 + \|\overline{z}^k - z^k\|^2 \leqslant (r^\delta b^{2-\delta} L_2^2 + 1)\|\overline{z}^k - z^k\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|H(z^k) + H'(z^k)\Delta z^k\|^2 &\leqslant 2\theta(\overline{z}^k - z^k) \\
&= \|H'(z^k)(\overline{z}^k - z^k) + H(z^k)\|^2 + \mu_k\|\overline{z}^k - z^k\|^2 \\
&= \|H'(z^k)(\overline{z}^k - z^k) + H(z^k) - H(\overline{z}^k)\|^2 + \mu_k\|\overline{z}^k - z^k\|^2 \\
&\leqslant L_2^2\|\overline{z}^k - z^k\|^4 + \mu_k\|\overline{z}^k - z^k\|^2 \\
&\leqslant (b^{2-\delta}L_2^2 + L_1^\delta)\|\overline{z}^k - z^k\|^{2+\delta}.
\end{aligned}$$

Let $c_1 = \sqrt{r^\delta b^{2-\delta} L_2^2 + 1}$ and $c_2 = \sqrt{b^{2-\delta}L_2^2 + L_1^\delta}$. We have

$$\|\Delta z^k\| \leqslant c_1 \cdot \operatorname{dist}(z^k, \Omega) \quad \text{and} \quad \|H(z^k) + H'(z^k)\Delta z^k\| \leqslant c_2 \cdot \operatorname{dist}(z^k, \Omega)^{1+\delta/2}.$$

The proof is completed. □

Lemma 4.3. *Suppose that Assumptions* 3.3 *and* 4.1 *hold. For all sufficient large k, if $z^k \in N(z^*, b)$, where b is defined as in Lemma* 4.2*, then $z^k + \Delta z^k \in N(z^*, b)$.*

P r o o f. It follows from Lemma 4.2 that

$$\begin{aligned}
\|z^k + \Delta z^k - z^*\| &\leqslant \|z^k - z^*\| + \|\Delta z^k\| \\
&\leqslant \|z^k - z^*\| + c_1 \cdot \operatorname{dist}(z^k, \Omega) \\
&= (1 + c_1)\|z^k - z^*\|.
\end{aligned}$$

By the fact that $\{z^k\}$ converges to z^*, we get for any sufficiently large k, that if $\|z^k + \Delta z^k - z^*\| \leqslant b$, then $z^k + \Delta z^k \in N(z^*, b)$. The proof is completed. □

Lemma 4.4. *Suppose that Assumptions* 3.3 *and* 4.1 *hold. If $z^k, z^k + \Delta z^k \in N(z^*, b)$, where b is defined as in Lemma* 4.2*, then*

$$\operatorname{dist}(z^k + \Delta z^k, \Omega) \leqslant c_3 \operatorname{dist}(z^k, \Omega)^{1+\delta/2}.$$

P r o o f. It follows from Assumption 4.1 and Lemma 4.2 that

$$\begin{aligned}\operatorname{dist}(z^k+\Delta z^k,\Omega) &\leqslant r\|H(z^k+\Delta z^k)\|\\ &\leqslant r\|H(z^k+\Delta z^k)-H(z^k)-H'(z^k)\Delta z^k\|\\ &\quad + r\|H(z^k)+H'(z^k)\Delta z^k\|\\ &\leqslant rL_2\|\Delta z^k\|^2 + rc_2\cdot\operatorname{dist}(z^k,\Omega)^{1+\delta/2}\\ &\leqslant rL_2c_1^2\cdot\operatorname{dist}(z^k,\Omega)^2 + rc_2\cdot\operatorname{dist}(z^k,\Omega)^{1+\delta/2}\\ &\leqslant (rL_2c_1^2b^{1-\delta/2}+rc_2)\cdot\operatorname{dist}(z^k,\Omega)^{1+\delta/2}.\end{aligned}$$

Let $c_3 = rL_2c_1^2b^{1-\delta/2}+rc_2$. The proof is completed. □

Theorem 4.5. *Suppose that Assumptions* 3.3 *and* 4.1 *hold and* $\{z^k\}$ *is an infinite sequence generated by Algorithm* 1 *converging to* $z^*\in\Omega$. *Then* $\{z^k\}$ *converges to* z^* *superlinearly for* $\delta\in(0,2)$ *and quadratically for* $\delta=2$.

P r o o f. By Lemma 4.4, it suffices to show that $z^{k+1}=z^k+\Delta z^k$ for any sufficient k, i.e.,

$$\Psi(z^k+\Delta z^k)\leqslant\Psi(z^k)-\sigma\mu_k\|\Delta z^k\|^2 \tag{4.7}$$

holds for sufficiently large k.

Since $\{z^k\}$ converges to z^*, it follows from Lemma 4.3 that there exists $\hat{k}>0$ such that $z^k, z^k+\Delta z^k\in N(z^*,b)$ for all $k\geqslant\hat{k}$.

By (4.4) and Theorem 4.2, we have

$$\begin{aligned}&\Psi(z^k+\Delta z^k)\\ &\quad=\frac{1}{2}\|H(z^k+\Delta z^k)\|^2\\ &\quad=\frac{1}{2}\|H(z^k+\Delta z^k)-H(z^k)-H'(z^k)\Delta z^k+H(z^k)+H'(z^k)\Delta z^k\|^2\\ &\quad\leqslant\frac{1}{2}[\|H(z^k+\Delta z^k)-H(z^k)-H'(z^k)\Delta z^k\|^2+\|H(z^k)+H'(z^k)\Delta z^k\|^2]\\ &\qquad+\|H(z^k+\Delta z^k)-H(z^k)-H'(z^k)\Delta z^k\|\cdot\|H(z^k)+H'(z^k)\Delta z^k\|\\ &\quad\leqslant\frac{1}{2}[L_2^2\|\Delta z^k\|^4+\|H(z^k)+H'(z^k)\Delta z^k\|^2]+L_2\|\Delta z^k\|^2\cdot\|H(z^k)+H'(z^k)\Delta z^k\|\\ &\quad\leqslant\frac{1}{2}[L_2^2c_1^4\cdot\operatorname{dist}(z^k,\Omega)^4+c_2^2\cdot\operatorname{dist}(z^k,\Omega)^{2+\delta}]+L_2c_1^2c_2\cdot\operatorname{dist}(z^k,\Omega)^{2+\delta/2}\\ &\quad\leqslant\frac{1}{2}\eta_k\cdot\operatorname{dist}(z^k,\Omega)^2\leqslant\eta_kr^2\Psi(z^k),\end{aligned}$$

which shows that (4.7) holds for any sufficiently large k, where $\{\eta_k\}$ is a sequence which converges to 0. The proof is completed. □

5. Numerical experiments

In this section, we give some numerical results of Algorithm 1 for solving several SOCCPs. All experiments have been done on a PC of 2.40 GHz CPU and 6.00 GB memory. The computer codes are written in Matlab R2017a. We use $\|H(z^k)\| \leqslant 10^{-6}$ as the stopping rule. Throughout our experiments, the parameters used in Algorithm 1 are chosen as

$$\varrho = 0.85, \quad \sigma = 0.01, \quad \text{and} \quad \varepsilon_0 = 0.8.$$

Notice that the choice of δ controls the rate of convergence. We set

$$\delta = \begin{cases} \dfrac{1}{\Psi(z^k)}, & \|H(z^k)\| \geqslant 1, \\ 2, & \|H(z^k)\| < 1, \end{cases} \tag{5.1}$$

which is a variation of the adaptive LM parameter in [2].

The test results are listed in Tables 1, 2, and 3, where IT denotes the number of iterations, CPU is the CPU time in seconds, GAP and ERO represent the value of $\|H(z^k)\|$ and $|\langle x^k, F(x^k)\rangle|$ at the final iteration, respectively. The starting points in the first five examples are chosen as $(0, \dots, 0)^\top$ of suitable dimensions.

Example 5.1. $\mathcal{V} = \mathbb{R}^5$, $\mathcal{K} = \mathcal{K}^5$ is the corresponding second-order cone and $F\colon \mathbb{R}^5 \to \mathbb{R}^5$ is given by $F(x) = Mx + q$ with

$$M = \begin{pmatrix} 15 & -5 & -1 & 4 & -5 \\ 0 & 5 & 0 & 0 & 1 \\ -1 & -3 & 8 & 2 & -3 \\ 2 & -4 & 2 & 9 & -4 \\ 0 & -5 & 0 & 0 & 10 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

The problem has the unique solution

$$x^* \approx (0.449185, -0.0030997, 0.0096024, 0.0031883, 0.048033)^\top.$$

Example 5.2. $\mathcal{V} = \mathbb{R}^3$, $\mathcal{K} = \mathcal{K}^3$ is the corresponding second-order cone and $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $F(x) = Mx + q$ with

$$M = \begin{pmatrix} 21 & -9 & 18 \\ -9 & 4 & -7 \\ 18 & -7 & 19 \end{pmatrix}, \quad q = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}.$$

The problem has the unique solution

$$x^* \approx (0.183606, -0.154346, -0.099440)^\top.$$

Example 5.3. $\mathcal{V} = \mathbb{R}^4$, $\mathcal{K} = \mathcal{K}^4$ is the corresponding second-order cone and $F(x)\colon\ \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} e^{x_1} + x_1^2 \\ e^{x_2} + x_2^2 \\ e^{x_3} + x_3^2 \\ e^{x_4} + x_4^2 \end{pmatrix}.$$

This problem has the unique solution

$$x^* = (0.327830, -0.189273, -0.189273, -0.189273)^\top.$$

Example 5.4. $\mathcal{V} = \mathbb{R}^3$, $\mathcal{K} = \mathcal{K}^3$ is the corresponding second-order cone and $F(x)\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$F(x) = \begin{pmatrix} 0.07x_1^3 - 4 \\ 0.04x_2^2 - 3.93 \\ 0.03x_3^3 - 5.72 \end{pmatrix}.$$

This problem has the unique solution $x^* = (5, 3, 4)^\top$.

Results of the above four experiments are listed in Table 1.

Q	IT	GAP	CPU	ERO
Ex. 5.1	11	7.3092×10^{-12}	0.0249	4.3957×10^{-13}
Ex. 5.2	15	1.2004×10^{-8}	0.0275	2.6947×10^{-9}
Ex. 5.3	8	2.3200×10^{-12}	0.0179	1.1102×10^{-16}
Ex. 5.4	12	3.5511×10^{-10}	0.0239	1.4614×10^{-11}

Table 1. Numerical results.

Example 5.5. $\mathcal{V} = \mathbb{R}^n$, $\mathcal{K} = \mathcal{K}^n$ is the corresponding second-order cone and $F\colon\ \mathbb{R}^n \to \mathbb{R}^n$ is given by $F(x) = Mx + q$ with

$$M = \begin{pmatrix} 1 & 2 & \dots & 2 & 2 \\ 0 & 1 & \dots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 2 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \end{pmatrix}.$$

This problem has the solution $x^* = (\frac{1}{2}, 0, \dots, 0, \frac{1}{2})^\top$. We have also coded the inexact smoothing Newton method in [20] for comparison purpose. Results of the numerical comparison are shown in Table 2, where LM denotes Algorithm 1 of this paper and Smoothing denotes the inexact smoothing Newton method of [20].

n		LM			Smoothing	
	Iter	ERO	CPU	Iter	ERO	CPU
100	6	3.2628×10^{-9}	0.0299	5	1.2008×10^{-7}	0.5423
200	6	3.6426×10^{-9}	0.1020	5	1.2005×10^{-7}	0.8275
500	6	3.7024×10^{-9}	0.8956	5	1.2240×10^{-7}	2.5024
1000	6	3.6887×10^{-9}	5.4909	6	1.2145×10^{-9}	12.9209
1500	6	3.6792×10^{-9}	20.9476	8	3.6163×10^{-10}	39.4867
2000	6	3.6732×10^{-9}	56.8582	9	6.6828×10^{-9}	78.7763
3000	6	3.6663×10^{-9}	230.1145	13	1.7917×10^{-9}	281.6014

Table 2. Comparison of the results of LM and Smoothing for Example 5.5.

n	IT	GAP	CPU	ERO
100	11	4.9668×10^{-7}	0.0557	2.2493×10^{-6}
	12	3.9537×10^{-14}	0.0556	7.9936×10^{-14}
	12	2.7209×10^{-7}	0.0453	1.3457×10^{-6}
	12	3.0030×10^{-12}	0.0481	1.0572×10^{-11}
200	13	7.3041×10^{-12}	0.2276	4.1506×10^{-11}
	13	3.6666×10^{-13}	0.2463	1.8296×10^{-12}
	13	1.5178×10^{-14}	0.2324	5.3290×10^{-14}
	13	5.0066×10^{-14}	0.2219	2.3803×10^{-13}
500	14	2.9683×10^{-9}	2.0452	3.0025×10^{-8}
	14	4.1513×10^{-10}	2.0547	4.1058×10^{-9}
	14	1.9481×10^{-10}	2.0302	1.8921×10^{-9}
	14	4.0587×10^{-11}	2.0321	3.8831×10^{-10}
1000	15	9.7527×10^{-11}	12.2889	1.3694×10^{-9}
	15	5.1356×10^{-12}	12.1903	6.9121×10^{-11}
	15	8.7585×10^{-11}	12.2506	1.2150×10^{-9}
	15	9.9627×10^{-11}	12.3782	1.3891×10^{-9}
1500	15	1.4521×10^{-7}	36.1350	2.0164×10^{-6}
	15	9.2302×10^{-8}	36.2356	1.3361×10^{-6}
	15	2.2984×10^{-7}	39.7649	2.9866×10^{-6}
	15	5.3295×10^{-7}	39.0144	6.2286×10^{-6}
2000	16	3.9668×10^{-12}	95.9855	7.6795×10^{-11}
	16	8.6998×10^{-12}	95.4721	1.7274×10^{-10}
	16	1.0061×10^{-11}	92.8790	1.9667×10^{-10}
	16	3.3956×10^{-12}	94.2072	6.5995×10^{-11}

Table 3. Numerical results with different dimensions for Example 5.6.

Example 5.6. $\mathcal{V} = \mathbb{R}^n$, $\mathcal{K} = \mathcal{K}^n$ is the corresponding second-order cone and $F(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by $F(x) = Mx + q$ where $q = (-1, -1, \ldots, -1)^\top$ and M is generated by the following procedure: let $M = V\Sigma V^\top$, where V is a Householder matrix and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a diagonal matrix whose diagonal elements are generated by taking

$$\sigma_i = \cos\frac{i\pi}{n+1} + 1 + \frac{\cos\frac{\pi}{n+1} + 1 - \operatorname{cond}(M)\Bigl(\cos\frac{\pi}{n+1} + 1\Bigr)}{\operatorname{cond}(M) - 1}, \quad i = 1, 2, \ldots, n,$$

By this means, M possesses a prescribed condition number. We set $\operatorname{cond}(M) = 100$. The matrix V can be obtained by letting

$$V = I - 2\frac{vv^\top}{\|v\|^2},$$

where I is the unit matrix and v is uniformly distributed in $(-1, 1)$.

All the components of the starting point x^0 are randomly selected from $(0, 1)$. The results are listed in Table 3.

From the tables above, we see that the LM method is feasible and effective. As the dimension of the problem increases, the LM method is more stable than the smoothing Newton method.

6. Conclusions

In this paper, we have presented a smoothing Levenberg-Marquardt method for solving the symmetric cone complementarity problem. Under the condition of Lipschitz continuity of the Jacobian matrix and the local error bound condition which is weaker than the nonsingularity at the Jacobian and the condition of Lipschitz continuity of the Jacobian matrix, we show that the proposed method possesses global convergence and locally superlinear/quadratic convergence. Results of numerical experiments show that the method is efficient.

References

[1] *F. Alizadeh, D. Goldfarb*: Second-order cone programming. Math. Program. *95* (2003), 3–51. zbl MR doi

[2] *K. Amini, F. Rostami*: A modified two steps Levenberg-Marquardt method for nonlinear equations. J. Comput. Appl. Math. *288* (2015), 341–350. zbl MR doi

[3] *X. Chen, H. Qi, P. Tseng*: Analysis of nonsmooth symmetric-matrix-valued functions with applications to semidefinite complementarity problems. SIAM J. Optim. *13* (2003), 960–985. zbl MR doi

[4] *J.-S. Chen, P. Tseng*: An unconstrained smooth minimization reformulation of the second-order cone complementarity problem. Math. Program. *104* (2005), 293–327. zbl MR doi

[5] *H. Dan, N. Yamashita, M. Fukushima*: Convergence properties of the inexact Levenberg-Marquardt method under local error bound conditions. Optim. Methods Softw. *17* (2002), 605–626. zbl MR doi

[6] *F. Facchinei, C. Kanzow*: A nonsmooth inexact Newton method for the solution of large-scale nonlinear complementarity problems. Math. Program. *76* (1997), 493–512. zbl MR doi

[7] *J. Faraut, A. Korányi*: Analysis on Symmetric Cones. Oxford Mathematical Monographs. Oxford University Press, Oxford, 1994. zbl MR

[8] *M. Fukushima, Z.-Q. Luo, P. Tseng*: Smoothing functions for second-order-cone complementarity problems. SIAM J. Optim. *12* (2002), 436–460. zbl MR doi

[9] *D. Goldfarb, W. Yin*: Second-order cone programming methods for total variation-based image restoration. SIAM J. Sci. Comput. *27* (2005), 622–645. zbl MR doi

[10] *P. T. Harker, J.-S. Pang*: Finite-dimensional variational inequalities and nonlinear complementarity problems: A survey of theory, algorithms and applications. Math. Program., Ser. B *48* (1990), 161–220. zbl MR doi

[11] *S. Hayashi, N. Yamashita, M. Fukushima*: Robust Nash equilibria and second-order cone complementarity problems. J. Nonlinear Convex Anal. *6* (2005), 283–296. zbl MR

[12] *Y. Kanno, J. A. C. Martins, A. Pinto Da Costa*: Three-dimensional quasi-static frictional contact by using second-order cone linear complementarity problem. Int. J. Numer. Methods Eng. *65* (2006), 62–83. zbl MR doi

[13] *B. Kheirfam, N. Mahdavi-Amiri*: A new interior-point algorithm based on modified Nesterov-Todd direction for symmetric cone linear complementarity problem. Optim. Lett. *8* (2014), 1017–1029. zbl MR doi

[14] *N. Lu, Z.-H. Huang*: A smoothing Newton algorithm for a class of non-monotonic symmetric cone linear complementarity problems. J. Optim. Theory Appl. *161* (2014), 446–464. zbl MR doi

[15] *M. Sayadi Shahraki, H. Mansouri, M. Zangiabadi, N. Mahdavi-Amiri*: A wide neighborhood primal-dual predictor-corrector interior-point method for symmetric cone optimization. Numer. Algorithms *78* (2018), 535–552. zbl MR doi

[16] *D. Sun, J. Sun*: Löwner's operator and spectral functions in Euclidean Jordan algebras. Math. Oper. Res. *33* (2008), 421–445. zbl MR doi

[17] *G. Q. Wang, Y. Q. Bai*: A class of polynomial interior point algorithms for the Cartesian P-matrix linear complementarity problem over symmetric cones. J. Optim. Theory Appl. *152* (2012), 739–772. zbl MR doi

[18] *N. Yamashita, M. Fukushima*: On the rate of convergence of the Levenberg-Marquardt method. Topics in Numerical Analysis. Computing Supplementa 15. Springer, Wien, 2001, pp. 239–249. zbl MR doi

[19] *J.-L. Zhang, X. Zhang*: A smoothing Levenberg-Marquardt method for NCP. Appl. Math. Comput. *178* (2006), 212–228. zbl MR doi

[20] *J. Zhang, K. Zhang*: An inexact smoothing method for the monotone complementarity problem over symmetric cones. Optim. Methods Softw. *27* (2012), 445–459. zbl MR doi

[21] *L. Zhang*: Solvability of semidefinite complementarity problems. Appl. Math. Comput. *196* (2008), 86–93. zbl MR doi

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