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On the probability that two elements of a finite semigroup have the same right matrix

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Abstract. We study the probability that two elements which are selected at random with replacement from a finite semigroup S have the same right matrix.

Keywords: congruence; equivalence relation; probability; semigroup

Classification: 20M10, 60B99

1. Introduction and motivation

There are many papers in the mathematical literature which use probabilistic methods to study special algebraic structures. Only the papers [1], [3]–[6] and [8] are cited here, because we refer only to them. All of these papers deal with special cases of the following problem. For a given finite algebraic structure A and a given binary relation σ on A, find the probability $P_\sigma(A)$ that $(a,b)\in\sigma$ is satisfied for two elements a and b of A which are selected at random with replacement. We note that random elements are chosen independently with the uniform distribution on A. Thus every couple $(a,b)\in A\times A$ has the same probability $1/|A|^2$ of being chosen and so

$$P_\sigma(A)=\frac{|\{(a,b)\in A\times A\colon (a,b)\in\sigma\}|}{|A|^2}.$$

In [1], [6] and [8] the probability $P_\sigma(A)$ is examined in the cases, when A is a finite noncommutative semigroup, a noncommutative group and a noncommutative ring, respectively. In all three cases σ is defined by $(x,y)\in\sigma$ for $x,y\in A$ if and only if $xy=yx$. In [4], the probability $P_\sigma(A)$ is investigated in that case when A is a finite simple group and σ is defined by $(x,y)\in\sigma$ for $x,y\in A$ if and only if $w(x,y)=e$, where w is a given nontrivial element of the free group F_2 and e is the identity element of the group A. In both [3] and [5], the probability $P_\sigma(A)$ is examined in that case when A is the symmetric group S_n of degree n. In [3], σ is defined by $(x,y)\in\sigma$ for $x,y\in A$ if and only if x and y generate the group S_n.

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In [5], σ is defined by $(x,y)\in\sigma$ for $x,y\in A$ if and only if x and y generate the alternating group A_n or the group S_n.

In the theory of semigroups the right regular matrix representation plays a very important role. The above investigations motivate us to examine the probability $P_{\theta_S}(S)$, where S is a finite semigroup and θ_S is the kernel of the right regular matrix representation of S. In other words, we examine the following problem. Two elements a and b are selected at random with replacement from a finite semigroup S. What is the probability that a and b have the same right matrix? We show that $P_{\theta_S}(S)\geq 1/|S/\theta_S|$ for every finite semigroup S, where S/θ_S denotes the factor semigroup of S modulo θ_S. In the paper we also deal with the following question. How does the structure of a finite semigroup S depend on the probability $P_{\theta_S}(S)$. This question is very general to answer. For example, if G is a group of order 3 and S is a semigroup defined by the Cayley table then $P_{\theta_G}(G)=$

	e	a	b	x	x^2	x^3
e	e	e	e	x	x^2	x^3
a	e	e	e	x	x^2	x^3
b	e	e	e	x	x^2	x^3
x	x	x	x	x^2	x^3	e
x^2	x^2	x^2	x^2	x^3	e	x
x^3	x^3	x^3	x^3	e	x	x^2

TABLE 1.

$P_{\theta_S}(S)=1/3$, but the structures of G and S are very different. In this paper we deal with a special case of the above question. Our main goal is to describe the structure of a finite semigroup S if $P_{\theta_S}(S)=1/|S/\theta_S|$. In this paper we give solutions in two cases. In the first case S is an arbitrary finite semigroup with $|S/\theta_S|=1$; in the second case S is a finite commutative semigroup with $|S/\theta_S|=2$.

2. Preliminaries

By a semigroup we mean a nonempty set together with an associative multiplication. Let S be a semigroup and G^0 be a semigroup arising from a one-element group $G=\{1\}$ by the adjunction of a zero element 0. By an *$S\times S$ matrix over G^0*, we mean a mapping of $S\times S$ into G^0. Let A be an $S\times S$ matrix over G^0. For an element $s\in S$, the set $\{A(s,x)\colon x\in S\}$ is called a *row* (the *s-row*) of A. An

$S \times S$ matrix A over G^0 is called *strictly row-monomial* if each row of A contains exactly one nonzero element of G^0.

For an element a of a semigroup S, let $R^{(a)}$ denote the strictly row-monomial S-matrix over G^0 defined by

$$R^{(a)}((x,y)) = \begin{cases} 1, & \text{if } \ xa = y, \\ 0, & \text{otherwise.} \end{cases}$$

This matrix is called the *right matrix over G^0 defined by a*. (The right matrices were investigated, for example, in [11] and [14], in the case when 1 is the identity element and 0 is the zero element of a field.) It is known that

$$a \mapsto R^{(a)}$$

is a homomorphism of a semigroup S into the multiplicative semigroup of all strictly row-monomial S-matrices over G^0, see [2, Exercise 4 (b) for Section 3.5]. This homomorphism is called the *right regular matrix representation* of the semigroup S.

A semigroup S is said to be *left reductive*, see [10], if the following condition is satisfied for arbitrary elements a and b of S: $sa = sb$ for all $s \in S$ implies $a = b$. The right regular matrix representation of a semigroup S is *faithful* (*injective*) if and only if S is left reductive.

Let θ_S denote the kernel of the right regular matrix representation of a semigroup S. It is obvious that

$$\theta_S = \{(a,b) \in S \times S\colon\ \forall x \in S \ \ xa = xb\}.$$

In this paper, we investigate the probability

$$P_{\theta_S}(S) = \frac{|\{(a,b) \in S \times S\colon (a,b) \in \theta_S\}|}{|S|^2}.$$

In our investigation, the following construction (which is a special case of the construction defined in part (a) of [12, Theorem 2]) plays an important role.

Construction 2.1 ([13, Construction 1])**.** Let T be a left cancellative semigroup (that is, a semigroup with the property that $xa = xb$ implies $a = b$ for every $x, a, b \in T$). For each $t \in T$, associate a nonempty set S_t such that $S_x \cap S_y = \emptyset$ for all $x, y \in T$ with $x \neq y$.

For an arbitrary couple $(t,x) \in T \times T$, let $(\cdot)\varphi_{t,tx}$ be a mapping of S_t into S_{tx} acting on the right. For all $t, x, y \in T$, and $a \in S_t$, assume

$$(a)(\varphi_{t,tx} \circ \varphi_{tx,txy}) = (a)\varphi_{t,txy}.$$

On the set $S = \bigcup_{t\in T} S_t$ define an operation "$\star$" as follows: for arbitrary $a \in S_t$ and $b \in S_x$ let

$$a \star b = (a)\varphi_{t,tx}.$$

As every left cancellative semigroup is left reductive, [12, Theorem 2] implies that $(S;\star)$ is a semigroup such that the sets S_t, $t \in T$, are the θ_S-classes of S.

3. Results

Let A be a nonempty set and σ be a binary relation on A. Let $P_\sigma(A)$ denote the probability that $(a,b) \in \sigma$, where a and b are selected at random with replacement from the set A.

Theorem 3.1. *Let p be an arbitrary rational number with $0 \leq p \leq 1$. Then the following assertions are equivalent:*

(i) *There is a finite semigroup S such that $P_{\theta_S}(S) = p$.*

(ii) *There is a nonempty finite set A and an equivalence relation σ on A such that $P_\sigma(A) = p$.*

Proof: It is sufficient to show that (ii) implies (i). Assume (ii). We use Construction 2.1. Let T be a commutative group of order $|A/\sigma|$. Such group always exists. Let S_t, $t \in T$, denote the σ-classes of A. For every $t \in T$ fix an element s_t in S_t. For every $t, x \in T$ let $(\cdot)\varphi_{t,tx}$ be the mapping of S_t into S_{tx} which maps the elements of S_t to s_{tx}. It is easy to see that the family $\{\varphi_{t,tx} \colon t, x \in T\}$ of mappings satisfies the following condition: for every $t, x, y \in T$ and every $a \in S_t$

$$(a)(\varphi_{t,tx} \circ \varphi_{tx,txy}) = (a)\varphi_{t,txy}.$$

Thus $S = \bigcup_{t\in T} S_t$ forms a semigroup under the operation "$\star$" defined by the following way: for every $t, x \in T$ and $a \in S_t$, $b \in S_x$

$$a \star b = (a)\varphi_{t,tx} = s_{tx}.$$

As T is left reductive, [12, Theorem 2] implies that the θ_S-classes of the semigroup S are the sets S_t $(t \in T)$. Consequently $P_{\theta_S}(S) = P_\sigma(A) = p$. □

We note that the semigroup S defined in the proof of Theorem 3.1 is commutative, because

$$a \star b = (a)\varphi_{t,tx} = s_{tx} = s_{xt} = (b)\varphi_{x,xt} = b \star a$$

is satisfied for every $t, x \in T$ and $a \in S_t$, $b \in S_x$.

Let A be a nonempty finite set and σ be an equivalence relation on A. If m denotes the cardinality of the factor set A/σ (which is called the index of σ) and

$t_1, \ldots, t_m$ are the cardinalities of the σ-classes of A, then

$$P_\sigma(A) = \frac{t_1^2 + \cdots + t_m^2}{(t_1 + \cdots + t_m)^2}.$$

By the well known connection between the root mean square and the arithmetic mean, we have

$$\sqrt{\frac{t_1^2 + \cdots + t_m^2}{m}} \geq \frac{t_1 + \cdots + t_m}{m},$$

that is,

$$\frac{t_1^2 + \cdots + t_m^2}{m} \geq \frac{(t_1 + \cdots + t_m)^2}{m^2}$$

from which we get

$$P_\sigma(A) = \frac{t_1^2 + \cdots + t_m^2}{(t_1 + \cdots + t_m)^2} \geq \frac{1}{m} = \frac{1}{|A/\sigma|}.$$

The equation $P_\sigma(A) = 1/|A/\sigma|$ holds if and only if

$$t_1 = \cdots = t_m.$$

In particular, $P_{\theta_S}(S) \geq 1/|S/\theta_S|$ for every finite semigroup S. In addition, $P_{\theta_S}(S) = 1/|S/\theta_S|$ if and only if each θ_S-class contains the same number of elements.

In the next part of the paper, we deal with a special case of the following problem: How does the structure of a finite semigroup S depend on the probability $P_{\theta_S}(S)$? Our question is the following: What can we say about the structure of a finite semigroup S, if the index of θ_S is m and $P_{\theta_S}(S) = 1/m$?

We deal with this question for $m = 1, 2$. In the case of $m = 1$, we give a solution to the question in the class of all semigroups. For $m = 2$, we answer the question in the class of all commutative semigroups.

3.1 Case $m = 1$**.** For a finite semigroup S, the assumption $P_{\theta_S}(S) = 1$ is satisfied if and only if the index of θ_S is 1, that is, θ_S is the universal relation on S. The next theorem characterizes not necessarily finite semigroups S in which θ_S is the universal relation on S. We shall use the following notions.

A homomorphism φ of a semigroup S onto an ideal $I \subseteq S$ is called a *retract homomorphism* [9, Definition 1.44] if φ leaves the elements of I fixed. An ideal I of a semigroup S is called a *retract ideal* if there is a retract homomorphism of S onto I. In this case, we say that S is a *retract* (*ideal*) *extension of* I *by the Rees factor semigroup* S/I.

A semigroup satisfying the identity $ab = a$ is called a *left zero semigroup*, see [7]. A semigroup with a zero element 0 is called a *zero semigroup* if it satisfies the identity $ab = 0$.

Theorem 3.2. *For a semigroup S, θ_S is the universal relation on S if and only if S is a retract extension of a left zero semigroup by a zero semigroup.*

PROOF: Let S be a semigroup in which θ_S is the universal relation on S. Then $xa = xb$ holds for every $x, a, b \in S$. Let $a \in S$ be an arbitrary element. Then

$$a^2 = aa = aa^2 = a^3,$$

and so

$$(a^2)^2 = aa^3 = aa^2 = a^3 = a^2,$$

that is, a^2 is an idempotent element. Let $E(S)$ denote the set of all idempotent elements of S. As $ab = a^2 \in E(S)$ for every $a, b \in S$, the set $E(S)$ is an ideal of S, and the Rees factor semigroup $Q = S/E(S)$ is a zero semigroup. For arbitrary $e, f \in E(S)$,

$$ef = ee = e.$$

Hence $E(S)$ is a left zero semigroup. For every $a \in S$, we have $aS \subseteq E(S)$ and $|aS| = 1$. For every $a \in S$, let $(a)\varphi$ denote the element of aS. By the above,

$$(a)\varphi \in E(S)$$

for every $a \in S$. Moreover, $(e)\varphi = e$ for every idempotent element e of S. Let $x^* \in S$ be an arbitrary fixed element. Then, for every $a, b \in S$ we have

$$(ab)\varphi = abx^* = ax^*bx^* = (a)\varphi(b)\varphi.$$

Hence φ is a homomorphism of S onto $E(S)$. As φ leaves the elements of $E(S)$ fixed, it is a retract homomorphism of S onto $E(S)$. Thus S is a retract extension of the left zero semigroup $E(S)$ by the zero semigroup $Q = S/E(S)$.

Conversely, let S be a semigroup and I be an ideal of S such that I is a left zero semigroup, the Rees factor semigroup S/I is a zero semigroup, and there is a retract homomorphism φ of S onto I. Then, for arbitrary $x, a, b \in S$, we have $xa, xb \in I$ and so

$$xa = (xa)\varphi = (x)\varphi(a)\varphi = (x)\varphi = (x)\varphi(b)\varphi = (xb)\varphi = xb.$$

Hence θ_S is the universal relation on S. Thus the theorem is proved. □

In the next part of this subsection, we show how to construct semigroups S in which θ_S is the universal relation.

Construction 3.3. Let S be a nonempty set and L be a nonempty subset of S. Let $(\cdot)\varphi$ be an arbitrary mapping of S onto L which leaves the elements of L fixed. Define an operation "$\star$" on S as follows: for arbitrary $a, b \in S$, let

$a \star b = (a)\varphi$. For every $a, b, c \in S$,

$$a \star (b \star c) = a \star (b)\varphi = (a)\varphi = ((a)\varphi)\varphi = (a \star b)\varphi = (a \star b) \star c,$$

that is, S is a semigroup with the operation "$\star$". This semigroup is denoted by $(S, L, \varphi, \star)$.

Theorem 3.4. *In the semigroup $S = (S, L, \varphi, \star)$, the equation $\theta_S = \omega_S$ is satisfied. Conversely, every semigroup S in which $\theta_S = \omega_S$ is satisfied is isomorphic to a semigroup defined in Construction* 3.3.

Proof: For every $a, b \in (S, L, \varphi, \star)$, we have $a \star b \in L$. Thus L is an ideal of S and the Rees factor semigroup S/L is a zero semigroup. For every $a, b \in L$, we have

$$a \star b = (a)\varphi = a.$$

Thus L is a left zero semigroup. As

$$(a \star b)\varphi = ((a)\varphi)\varphi = (a)\varphi = ((a)\varphi)((b)\varphi),$$

φ is a retract homomorphism of S onto L. Thus $S = (S, L, \varphi, \star)$ is a retract extension of the left zero semigroup L by the zero semigroup S/L. Consequently $\theta_S = \omega_S$ by Theorem 3.2.

Conversely, assume that S is a semigroup in which $\theta_S = \omega_S$. By Theorem 3.2, there is an ideal L of S such that L is a left zero semigroup, the Rees factor semigroup S/L is a zero semigroup, and there is a retract homomorphism φ of S onto L. Consider the semigroup $(S, L, \varphi, \star)$ defined as in Construction 3.3. As

$$ab = (ab)\varphi = (a)\varphi(b)\varphi = (a)\varphi = a \star b$$

for every $a, b \in S$, the semigroups S and $(S, L, \varphi, \star)$ are isomorphic. □

3.2 Case $m = 2$. In our investigation, the following three examples play an important role.

Example 3.5. Let A and B be zero semigroups such that $A \cap B = \emptyset$. Let e and f denote the zero elements of A and B, respectively. Let $S = A \cup B$. We define an operation on S as follows:

$$xy = \begin{cases} e, & \text{if } x, y \in A; \\ f, & \text{otherwise.} \end{cases}$$

It is easy to see that S is a commutative semigroup whose θ_S-classes are A and B. The factor semigroup S/θ_S is a two-element semilattice, see [15], (that is, a two-element commutative semigroup in which every element is an idempotent element).

Example 3.6. Let A be a zero semigroup with a zero element e. Let B be a nonempty set with $A \cap B = \emptyset$. Let $S = A \cup B$. We define an operation on S as follows: fix an element b^* in B, and let

$$xy = \begin{cases} e, & \text{if } x, y \in A \text{ or } x, y \in B; \\ b^*, & \text{otherwise.} \end{cases}$$

It is a matter of checking to see that S is a commutative semigroup whose θ_S-classes are A and B. The factor semigroup S/θ_S is a two-element group.

Example 3.7. Let A be a zero semigroup with a zero element e. Let B be a nonempty set with $A \cap B = \emptyset$. Let $S = A \cup B$. We define an operation on S as follows: fix an element $a^* \in A$ with $e \neq a^*$, and let

$$xy = \begin{cases} a^*, & \text{if } x, y \in B; \\ e, & \text{otherwise.} \end{cases}$$

It is a matter of checking to see that S is a commutative semigroup whose θ_S-classes are A and B. The factor semigroup S/θ_S is a two-element zero semigroup.

Theorem 3.8. *On an arbitrary semigroup S, the following conditions are equivalent:*

(1) *S is a commutative semigroup such that the index of θ_S is* 2.
(2) *S is isomorphic to one of the semigroups defined in Example* 3.5, *Example* 3.6, *and Example* 3.7.

Proof: As the semigroups S defined in Example 3.5, Example 3.6 and Example 3.7 are commutative such that the index of θ_S is 2, it is sufficient to show that (1) implies (2). Let S be a commutative semigroup such that the index of θ_S is 2. Then the factor semigroup S/θ_S is a two-element commutative semigroup. Thus S/θ_S is either a two-element semilattice or a two-element group or a two-element zero semigroup. Let A and B denote the θ_S-classes of S.

First we consider the case when S/θ_S is a two-element semilattice. Then A and B are subsemigroups of S such that one of A and B, say B, is an ideal of S. It is easy to see that $\theta_A = \omega_A$ and $\theta_B = \omega_B$. Then, by Theorem 3.2, A and B are zero semigroups. Let e and f denote the zero element of A and B, respectively. For every $x, y \in A$, we have $xy = e$. For every $x, y \in B$, we have $xy = f$. For every $x \in A$ and $y \in B$, we have

$$yx = xy = xf = xff = f,$$

because $(y, f) \in \theta_S$, $xf \in B$ and f is the zero element of B.

Summarizing our results, we have

$$xy = \begin{cases} e, & \text{if } x, y \in A; \\ f, & \text{otherwise.} \end{cases}$$

Thus S is isomorphic to a semigroup defined in Example 3.5.

In the next part of the proof, we consider the case when S/θ_S is a two-element group. Let A be the θ_S-class corresponding to the identity element of S/θ_S. Then A is a subsemigroup of S. Since $\theta_A = \omega_A$, Theorem 3.2 implies that A is a zero semigroup. Let e denote the zero element of A. As B is a θ_S-class, we have that $eB \subseteq B$ is a singleton. Let b^* denote the element of eB. Then, for every $x \in A$ and $y \in B$, we have

$$xy = yx = ye = ey = b^*,$$

because $(x, e) \in \theta_S$. For every $x, y \in B$, we get $ex \in B$ and $x^2 \in A$, hence

$$xy = xex = x^2 e = e.$$

Summarizing our results, we have

$$xy = \begin{cases} e, & \text{if } x, y \in A \text{ or } x, y \in B; \\ b^*, & \text{otherwise.} \end{cases}$$

Thus S is isomorphic to a semigroup defined in Example 3.6.

Consider the case when S/θ_S is a two-element zero semigroup. Let A be the θ_S-class corresponding to the zero element of S/θ_S. Then, by Theorem 3.2, A is a zero semigroup. Let e denote the zero element of A. Then $xy = e$ for every $x, y \in A$. For an arbitrary $y \in B$,

$$ey = eey = e,$$

because $ey \in A$ and e is the zero element of A. Thus, for every $x \in A$ and $y \in B$, we have

$$xy = yx = ye = ey = e,$$

because $(x, e) \in \theta_S$. Let $x_0, y_0 \in B$ be fixed arbitrary elements, and let

$$a^* = x_0 y_0 = y_0 x_0.$$

Then, for every $x, y \in B$, we have

$$xy = xy_0 = y_0 x = y_0 x_0 = a^*,$$

because B is a θ_S-class.

Summarizing our results, we have

$$xy = \begin{cases} a^*, & \text{if } x, y \in B; \\ e, & \text{otherwise.} \end{cases}$$

If a^* was the zero element of A, then we would have

$$xa = xb = e$$

for every $a \in A$, $b \in B$ and $x \in S$, which would imply that $\theta_S = \omega_S$. It would be a contradiction. Hence $a^* \neq e$. Thus S is isomorphic to a semigroup defined in Example 3.7.

In a finite semigroup S, both of the conditions that the index of θ_S is m and $P_{\theta_S}(S) = 1/m$ are satisfied if and only if each θ_S-class contains the same number of elements. Thus the following result is a consequence of Theorem 3.8. □

Theorem 3.9. *On an arbitrary semigroup S, the following conditions are equivalent:*

(1) *S is a finite commutative semigroup such that the index of θ_S is* 2 *and $P_{\theta_S}(S) = 1/2$.*

(2) *S is isomorphic to one of the semigroups defined in Example* 3.5*, Example* 3.6 *and Example* 3.7 *in which $|A| = |B| < \infty$.*

Remark 3.10. Commutative semigroups defined in Example 3.5, Example 3.6, and Example 3.7 can also be obtained by using the construction defined in part (a) of [12, Theorem 2]. It seems to be that this construction would also be a useful tool to investigate our problem in other cases.

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