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FUZZY SETS (IN)EQUATIONS WITH A COMPLETE CODOMAIN LATTICE

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The paper applies some properties of the monotonous operators on the complete lattices to problems of the existence and the construction of the solutions to some fuzzy relational equations, inequations, and their systems, taking a complete lattice for the codomain lattice. The existing solutions are extremal – the least or the greatest, thus we prove some extremal problems related to fuzzy sets (in)equations. Also, some properties of upper-continuous lattices are proved and applied to systems of fuzzy sets (in)equations, in a special case of a meet-continuous complete codomain lattice.

Keywords: fuzzy relations, fuzzy set equations, fuzzy set inequations, monotonous operator, upper continuous lattice

Classification: 03B52, 03E72, 06B23

1. INTRODUCTION

Fuzzy sets, fuzzy relations, fuzzy correspondences, and their compositions have, among other applications, often been used to describe certain iterative processes. In that context, the problem of solving different fuzzy set and fuzzy relational equations and inequations and their systems are connected to the problem of the process control.

Fuzzy sets and fuzzy relational equations and inequations are being considered in different cases, depending on the type of the codomain lattice. Early researches treated the problem in the framework of fuzzy sets as they were initially defined, with $[0, 1]$ codomain (see [4, 5, 6, 16, 17]), while more recent researches take residuated lattices as codomain lattices (see [1, 7, 8, 12, 13, 14, 15]) meet-continuous complete lattices (see [19]) or even more general complete lattices (see [9, 10, 11]).

Some fuzzy (in)equations may be expressed using operators on the set of fuzzy subsets of a given crisp set, which has a lattice structure derived from the initial codomain lattice. If the codomain lattice is complete, so is the lattice of fuzzy subsets. In some typical fuzzy sets (in)equations we deal with monotonous operators. This allows us to apply the Tarski fixed point theorem to prove the existence of solutions to some fuzzy sets (in)equations, and also to solve some extremal problems. Since some of them are straightforward (e. g. the least, i. e. zero fuzzy set is fixed under the composition with

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any fuzzy relation; also, the zero fuzzy relation is fixed under the composition with itself, etc.), we state some equations we couldn't deal with, without using the Tarski theorem.

Also, a constructive approach to proving the Tarski theorem may be used to construct some existing (extremal) solutions. This process is transfinite in general, but in the case of upper-continuous (or dually, down-continuous) operators it terminates after at most countably many steps. Some of these cases are already described in [19], here we include new ones, applying the methods used for single (in)equations to systems of (in)equations. Also, a method used in [19] to prove the existence of a maximal solution to single (in)equations is here generalized and applied to some systems of (in)equations.

We are interested not only in the results we can get but also in the limits of this most general approach to fuzzy sets, with a complete lattice as the codomain lattice. Therefore we give some negative results.

2. PRELIMINARIES

In the lattice theory, a lattice L is said to be complete if for any subset A of L there exists a supremum, as well as an infimum in L, related to the order in L. (see [3]). Clearly, both the supremum and the infimum are uniquely defined.

An element l of a complete lattice L is said to be compact if for any subset S of L such that $l \leq \bigvee S$ we have that $l \leq \bigvee S_0$ for some finite subset S_0 of S.

A complete lattice L is said to be algebraic if for all $l \in L$ we have that l is the supremum of the set of all the compact elements less than or equal to l.

Let L be a complete lattice. An element $a \in L$ is said to be **meet-continuous** ($\wedge$-continuous) if

$$a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \wedge x_i)$$

for every chain $\{x_i \mid i \in I\} \subseteq L$. ([21]).

If every element of a lattice is meet-continuous, we shall say that the lattice is meet-continuous.

Lemma 2.1. (see [18]) Every algebraic lattice is meet-continuous.

Definition 2.2. An operator on a lattice is any function mapping a lattice to itself. The identical operator on a lattice, mapping every element of the lattice to itself, is denoted by id.

Definition 2.3. An operator ϕ on a lattice L is monotonous if for all $x, y \in L$, $x \leq y$ implies $\phi(x) \leq \phi(y)$.

Definition 2.4. An operator ϕ on a lattice L is upper-continuous if for any chain S in L we have that $\phi(\vee S) = \vee_{l \in S} \phi(l)$.

An upper-continuous operator is monotonous, but a monotonous operator may not be upper-continuous.

If ϕ is an operator and r is an ordinal number, we define ϕ^r by induction:

$\phi^0 = id$

$\phi^{r+1} = \phi \circ \phi^r$

$\phi^s = \bigvee\{\phi^r \mid r < s\}$, if s is a limit ordinal.

Dually, we define ϕ_r by induction:

$\phi_0 = id$

$\phi_{r+1} = \phi \circ \phi_r$

$\phi_s = \bigwedge\{\phi_r \mid r < s\}$, if s is a limit ordinal.

If there exists l_0 such that $\phi^r(l) = l_0$ for all ordinals r greater than or equal to some ordinal r_0, we write $\overrightarrow{\lim}(\phi)l = l_0$. Analogously, if there exists l_0 such that $\phi_r(l) = l_0$ for all ordinals r greater than or equal to some ordinal r_0, we write $\underrightarrow{\lim}(\phi)l = l_0$.

Let L be a fixed lattice. If A is a set, a fuzzy set on A with membership values in L is any function μ from A to L. We shall also say that μ is a fuzzy subset of A, or an L-valued fuzzy set on A. We also define an L-valued fuzzy relation R on A as a fuzzy subset on A^2. The lattice L is called the codomain lattice for the fuzzy set μ, as well as for the fuzzy relation R.

The support of a fuzzy set μ is usually denoted by Suppμ and defined as Supp$\mu = \{a \in A \mid \mu(a) > 0\}$. If Supp$\mu = \{a\}$, we shall say that μ is a fuzzy point; if $\mu(a) = l$, we shall write $\mu = a_l$.

The set of all fuzzy sets on A with membership values in L we denote by $\mathcal{F}(A)$. Now, $\mathcal{F}(A^2)$ is the set of all L-valued fuzzy relations on A.

In $\mathcal{F}(A)$, the fuzzy set inclusion is defined in a natural way, component-wise:

$\mu \subseteq \nu \Leftrightarrow (\forall a \in A)\mu(a) \leq \nu(a)$.

It's straightforward that $\subseteq$ is an order, and $\mathcal{F}(A)$ a lattice related to it. The join operation in that lattice is called the fuzzy set union - in analogy to crisp sets; also the meet operation in that lattice is called the fuzzy set intersection. If $\mu \subseteq \nu$, we say that μ is a fuzzy subset of ν; we shall also - and usually - write $\mu \leq \nu$.

If L is complete, so is $\mathcal{F}(A)$, so the fuzzy set union and the fuzzy set intersection exist for an arbitrary family of fuzzy sets $(\mu_i)_{i \in I}$ and the following holds:

$\bigcup\{\mu_i \mid i \in I\}(a) = \bigvee\{\mu_i(a) \mid i \in A\}$

$\bigcap\{\mu_i \mid i \in I\}(a) = \bigwedge\{\mu_i(a) \mid i \in A\}$

Taking an L-valued fuzzy set μ and a fuzzy relation R on A we define the composition of μ with R, denoted by $\mu \circ R$:

$(\mu \circ R)(a) = \bigvee_{b \in A}(\mu(b) \wedge R(b,a))$

Analogously, we can introduce the composition of R with μ :

$(R \circ \mu)(a) = \bigvee_{b \in A}(R(a,b) \wedge \mu(b))$.

We shall say that μ is closed under the composition with R if $\mu \circ R \leq \mu$. Since $\bigvee_{b \in A}(\mu(b) \wedge R(b,a)) \leq \mu(a) \Leftrightarrow (\forall b)(\mu(b) \wedge R(b,a) \leq \mu(a))$, we have that

$\mu \circ R \leq \mu \Leftrightarrow (\forall a, b \in A)\mu(b) \wedge R(b,a) \leq \mu(a)$

This generalizes the concept of the closedness of a (crisp) set under a (crisp) relation or, more especially, under a function: if μ and R are crisp (i. e. characteristic functions of a crisp subset of A and a crisp relation on A respectively), then μ is closed under the composition with R if it is closed in the usual sense.

If $\mu(a) \wedge R(a,b) \leq \mu(b)$, we have that $a \in S$ and aRb together imply $\mu(b) = 1$, i.e. $b \in S$, thus S is closed under the composition with R.

A fuzzy set μ for which $\mu \circ R = \mu$ is also closed under the composition with R. For such a special fuzzy set closed under the composition with R, we shall say that it is exactly closed under the composition with R.

Given a fuzzy set ν, the minimum of all fuzzy sets containing ν, closed under the composition with R will be called the closure of ν under composition with R. Indeed, such a minimum always exists and it is equal to the infimum of the family of all fuzzy sets closed under the composition with R containing ν (and the family is non-empty since $\mu = 1$ belongs to it).

Analogously, taking two L-valued fuzzy relations P and R, we define the composition of P with R, denoted by $P \circ R$:

$$(P \circ R)(a, b) = \bigvee_{c \in A}(P(a, c) \wedge R(c, b)).$$

The composition need not be associative on the set of all L-valued fuzzy relations on A. However, if L is a completely distributive lattice, the composition is associative, which is easily proved directly by definitions.

We shall say that P is closed under the composition with R if $P \circ R \leq P$. Given a fuzzy relation Q, the minimum of all fuzzy relations containing Q, closed under the composition with R will be called the closure of Q under composition with R. Such a closure exists and equals the intersection of all fuzzy relations P for which the inequality $P(x, y) \wedge R(y, z) \leq P(x, z)$ is true for all $x, y, z \in A$ (since $P(x, y) \wedge R(y, z) \leq P(x, z)$ is equivalent to $\bigvee_{y \in A}(P(x, y) \wedge R(y, z)) \leq P(x, z)$).

A fuzzy relation R is transitive if $R(x, y) \wedge R(y, z) \leq R(x, z)$ for all $x, y, z \in A$.

Given a fuzzy relation Q, the minimum of all transitive fuzzy relations containing Q will be called the transitive closure of Q.

A fuzzy relation R is reflexive if $R(a, a) = 1$ for all $a \in A$. Clearly, this generalizes the notion of reflexivity in the crisp case.

In certain cases, solutions of fuzzy set equations are fixed points of a monotonous operator on the lattice of all fuzzy sets which is – in the case of a complete codomain lattice – also complete; in those cases, we may apply the following theorem by Tarski.

Theorem 2.5. (Tarski [20]) Let ϕ be a monotonous operator on a complete lattice L. Its set of fixed points form a non-empty complete lattice under inclusion inherited from L, with $\bigvee\{l \in L \mid \phi(l) \geq l\}$ and $\bigwedge\{l \in L \mid \phi(l) \leq l\}$ being its greatest and the least element respectively.

To find solutions to some other types of fuzzy relational equations, we would need another theorem, adapted from Tarski [20].

Theorem 2.6. Let ϕ and ψ be monotonous operators on a complete lattice L, commuting under composition. The set of their joint fixed points form a non-empty complete lattice under inclusion inherited from L, with $\bigvee\{l \in L \mid \phi(l) \geq l$ and $\psi(l) \geq l\}$ and $\bigwedge\{l \in L \mid \phi(l) \leq l$ and $\psi(l) \leq l\}$ being its greatest and the least elements respectively.

When it comes to constructing solutions to some fuzzy set equations, we need a constructive approach to the Tarski theorem. In Corollary 3.3 of [2], Cousot and Cousot state the following:

Theorem 2.7. (Cousot and Cousot [2]) If ϕ is a monotonous operator on a complete lattice L and $l \in L$ is such that $\phi(l) \geq l$, then there exists $\overrightarrow{\lim}(\phi)l$, and it is equal to the least fixed point of ϕ containing l.

In a special case of the upper-semi-continuous operator, we shall reach $\overrightarrow{\lim}(\phi)l$ in at most countably many steps.

Theorem 2.8. (Cousot and Cousot [2]) Let ϕ be an upper-semi-continuous operator on the complete lattice L. If $\phi(l) \geq l$, then $\bigvee\{\phi^n(l) \mid n \in \omega\}$ is equal to $\overrightarrow{\lim}(\phi)l$ and to the least of the fixed points of ϕ greater than or equal to l.

To prove some more properties of the upper-semi-continuous operator, we shall use the following Set theory lemma, known as Zorn's lemma:

Lemma 2.9. (Zorn, Kuratowski) If a partially ordered set P has the property that every chain in P has an upper bound in P, then P contains at least one maximal element.

More generally than in Theorem 2.7, starting with an arbitrary lattice element $l \in L$ and a monotonous operator ϕ, we have:

Theorem 2.10. (Cousot and Cousot [2]) If ϕ is a monotonous operator on a complete lattice L, $l \in L$, $\xi = \phi \vee l$ and $\psi = \phi \vee id$, there exist $\overrightarrow{\lim}(\psi)l$ and $\overrightarrow{\lim}(\xi)l$, and they are both equal to the least element of L greater than or equal to l and mapped by ϕ to an element less than or equal to it.

3. RESULTS

3.1. Relational inequations and equations where a fuzzy set is unknown

We start from one of the typical fuzzy sets inequations and the related equation:

$$\mu \circ P \leq \mu$$
$$\mu \circ P = \mu$$

Here P is a known fuzzy relation, and μ is an unknown fuzzy set. Solutions to the given inequation are fuzzy sets that are closed under the composition with the given fuzzy relation P.

The above inequation has at least two trivial solutions: $\mu = 1$ and $\mu = 0$, the latter of which is also a solution of the corresponding equation – the least one. Does there exist the greatest solution to that equation? Applying Theorem 2.5 to the operator $\Phi : \mathcal{F}(A) \to \mathcal{F}(A)$ defined by $\Phi(\mu) = \mu \circ P$, we get the following result:

Theorem 3.1. There exists the greatest of all solutions to the equation $\mu \circ P = \mu$, as well as the greatest of all solutions to the inequation $\mu \circ P \geq \mu$, and the two of them coincide.

Thus, solving the equation $\mu \circ P = \mu$ is also connected to solving the following inequation:

$$\mu \circ P \geq \mu.$$

For an arbitrary fuzzy set α, taking the operator $\Phi : \mathcal{F}(A) \to \mathcal{F}(A)$ to be $\Phi(\mu) = \mu \circ P \wedge \alpha$ – which is clearly monotonous, we get a more general result:

Theorem 3.2. Let P be an arbitrary fuzzy relation and α an arbitrary fuzzy set. There exists the greatest of all solutions to the inequation $\mu \circ P \geq \mu$ contained in α. It is the greatest solution to the equation $\mu \circ P \wedge \alpha = \mu$.

Proof. A fuzzy set is a solution to the inequation $\mu \circ P \wedge \alpha \geq \mu$ if and only if it is a solution to the inequation $\mu \circ P \geq \mu$ which is less than or equal to α. Thus, the existing (by Theorem 2.5) greatest solution to the equation $\mu \circ P \wedge \alpha = \mu$ is (by the same theorem) the greatest solution to the inequation $\mu \circ P \wedge \alpha \geq \mu$ and thus the greatest of all solutions to the inequation $\mu \circ P \geq \mu$ contained in α. □

We can also consider the inequation and equation: $P \circ \mu \leq \mu$ and $P \circ \mu = \mu$ and, by Theorem 2.5, obtain the analogous results.

Similarly, we obtain solutions to this inequation which are contained in α and several other results.

We may generalize the results above for a system of inequations of the form $\mu \circ P \geq \mu$, by proving the following two theorems.

Theorem 3.3. If $\{P_i \mid i \in I\}$ is a family of relations from $\mathcal{F}(A^2)$, there exists the greatest solution to the system of inequations $(\mu \circ P_i \geq \mu)_{i \in I}$ contained in a given fuzzy set $\alpha \in \mathcal{F}(A)$. In particular, there exists the greatest solution to the system of inequations $(\mu \circ P_i \geq \mu)_{i \in I}$.

Proof. We define an operator ϕ on $\mathcal{F}(A)$ by $\phi(\mu) = \bigwedge\{\mu \circ P_i \mid i \in I\} \wedge \alpha$. We have that $\phi(\mu) \geq \mu \Leftrightarrow [(\mu \leq \alpha) \wedge (\forall i \in I)\mu \circ P_i \geq \mu]$. Since there exists, by Theorem 2.5, the greatest $\mu \in \mathcal{F}(A)$ such that $\phi(\mu) \geq \mu$, that same set is the greatest solution to the system of inequations $(\mu \circ P_i \geq \mu)_{i \in I}$ contained in α. For $\alpha = 1$ we get, as a special case, the second assertion of the theorem. □

When it comes to the construction of the greatest solution to the inequation $\mu \circ P \geq \mu$, or the system of inequations of the form $(\mu \circ P_i \geq \mu)_{i \in I}$, which is contained in α, we use the constructive approach of Cousot and Cousot. From the dual of Theorem 2.10 and the proof of Theorem 3.2, we get the following.

Theorem 3.4. Let $\{P_i \mid i \in I\}$ be a family of relations in $\mathcal{F}(A^2)$, α an arbitrary fuzzy set, ϕ and ψ operators on $\mathcal{F}(A)$ defined by $\psi(\mu) = \bigwedge\{\mu \circ P_i \mid i \in I\} \wedge \alpha$, $\xi(\mu) = \bigwedge\{\mu \circ P_i \mid i \in I\} \wedge \mu$, we have that there exist $\underrightarrow{\lim}(\psi)\alpha$ and $\underrightarrow{\lim}(\xi)\alpha$ and the two of them are equal, and equal to the greatest solution to the system of inequations $(\mu \circ P_i \geq \mu)_{i \in I}$ which is contained in α and to the greatest solution to the equation $\bigwedge\{\mu \circ P_i \mid i \in I\} \wedge \alpha = \mu$.

Since there exists the greatest solution to the inequation $\mu \circ P \geq \mu$ which is contained in a given (arbitrary) fuzzy set α, it is interesting to know if there exists the greatest of all solutions to the inequation $\mu \circ P \leq \mu$ which is contained in a given (arbitrary) fuzzy set α. That would be the greatest fuzzy set closed under the composition with the given relation contained in a given fuzzy set. But the following example proves that this may not hold: if we take the two-element set $A = \{a, b\}$ to be the initial set and the lattice given below to be the codomain lattice, there does not exist the greatest fuzzy set contained in a_1 closed under the composition with the fuzzy relation P given below:

P	a	b
a	0	l
b	l	0

$a_{l'} \circ P = 0 \leq a_{l'}$
$a_{l''} \circ P = 0 \leq a_{l''}$
So, $a_{l'} \leq a_1, a_{l''} \leq a_1$, $a_{l'} \circ P \leq a_{l'}, a_{l''} \circ P \leq a_{l''}$.

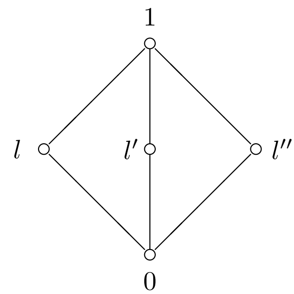

Fig. 1.

If there exists the greatest fuzzy set closed under the composition with P and less than or equal to a_1, it contains $a_{l'}$ and $a_{l''}$; therefore it contains $a_{l'} \vee a_{l''} = a_{l' \vee l''} = a_1$, and thus it equals a_1. But $a_1 \circ P = b_l$ and $b_l \not\leq a_1$, so we get a contradiction.

Thus, there may not exist the greatest fuzzy set closed under the composition with the given fuzzy relation, contained in a given (arbitrary) fuzzy set. In a special case of meet-continuous complete codomain lattice - thus also in the case of an algebraic codomain lattice - it is proved ([19]) that there exists a maximal solution to such an inequation $\mu \circ P \leq \mu$ which is contained in a given (arbitrary) fuzzy set α. However, as it is well known, a maximal solution need not be the greatest, so it is clear that in the finite case there are maximal solutions as well (since there is at least one solution, $\mu = 0$).

That assertion can be generalized to get a general result about the upper-semi-continuous operators on a complete lattice.

Proposition 3.5. Let ϕ be an upper-semi-continuous operator on a complete lattice L, such that $\phi(a) \leq a \leq b$, $\phi(c) = c \leq h$, $\phi(d) = e \leq f$ and $d \leq g$ for some $a, b, c, d, e, f, g, h \in L$. There exists a maximal element in each of the following sets:

a) $A = \{l \in L \mid \phi(l) \leq l \leq b\}$
b) $B = \{l \in L \mid \phi(l) = e \text{ and } l \leq g\}$
c) $C = \{l \in L \mid \phi(l) \leq f \text{ and } l \leq g\}$
d) $D = \{l \in L \mid \phi(l) = l \leq h\}$.

Proof. a) A is non-empty, since $\phi(a) \leq a \leq b$. Let $(l_i)_{i \in I}$ be a chain of elements from A. Since b is an upper bound $(l_i)_{i \in I}$, we have that $\bigvee(l_i)_{i \in I} \leq b$. Also $\phi(\bigvee l_i)_{i \in I} = \bigvee(\phi(l_i))_{i \in I} \leq (\bigvee l_i)_{i \in I}$, thus $\bigvee(l_i)_{i \in I} \in A$, and $(l_i)_{i \in I}$ has an upper bound in A. By Lemma 2.9, A has a maximal element. b) B is non-empty, since $\phi(d) = e$ and $h \leq d \leq g$. Let $(l_i)_{i \in I}$ be a chain of elements from B. As in (a) we have $\bigvee(l_i)_{i \in I} \leq g$. Also

$\phi(\bigvee l_i)_{i\in I} = \bigvee(\phi(l_i))_{i\in I} = e$, thus $\bigvee(l_i)_{i\in I} \in A$, and $(l_i)_{i\in I}$ has an upper bound in B. By Lemma 2.9, B has a maximal element. c) Analogously to (b). d) Analogously to (a). □

For a given fuzzy set α closed under the composition with a given fuzzy relation P, we have that the operator on $\mathcal{F}(A)$, given by $\phi(\mu) = \mu \circ P$, is also an operator on $\downarrow \alpha$ – principal ideal of $\mathcal{F}(A)$ generated by α – since $\alpha \circ P \leq \alpha$ and – due to monotony – $\phi(\downarrow \alpha) \subseteq \downarrow \alpha$. Applying Theorem 2.5, we get

Theorem 3.6. If $\alpha \circ P \leq \alpha$ for a fuzzy set α and a fuzzy relation P, then there exists the greatest solution to the equation $\mu \circ P = \mu$ contained in α.

Of course, there also exists the least fuzzy set contained in α closed under the composition with P, but it is a zero fuzzy set. In other terms, the previous theorem proves that there exists the greatest exactly closed fuzzy set under composition with a given fuzzy relation, contained in a given fuzzy set closed under the same fuzzy relation.

The dual of Theorem 2.7 gives a way to construct the existing fuzzy set:

Theorem 3.7. If $\alpha \circ P \leq \alpha$, then $\underrightarrow{\lim}(\phi)\alpha$, where $\phi(\mu) = \mu \circ P$ is the greatest fuzzy set μ contained in α such that $\mu \circ P = \mu$. In particular, $\underrightarrow{\lim}(\phi)1$ is the greatest fuzzy set μ such that $\mu \circ P = \mu$.

In the previous two theorems, the given fuzzy set α is closed under composition with P. This condition is essential and may not be skipped. In the above example with $A = \{a, b\}$ and the 5-element lattice in Figure 1, we take for P the following relation

P	a	b
a	1	l
b	l	1

$a_{l'} \circ P(a) = l' \wedge P(a, a) = l'$
$a_{l'} \circ P(b) = l' \wedge P(a, b) = 0.$

Thus, $a_{l'} \circ P = a_{l'}$; analogously, $a_{l''} \circ P = a_{l''}$. So, $a_{l'}$ and $a_{l''}$ are exactly closed under composition with P. If there exists the greatest exactly closed fuzzy set contained in a_1, the same would have to contain $a_{l'}$ and $a_{l''}$ and therefore also $a_{l'} \vee a_{l''} = a_1$. So, the greatest exactly closed fuzzy set contained in a_1 would have to be a_1; but $a_1 \circ P(a) = a_1(a) \wedge P(a, a) = 1$ and $a_1 \circ P(b) = a_1(a) \wedge P(a, b) = l$, thus $a_1 \circ P = a_1 \vee b_l \neq a_1$, which gives a contradiction.

If for a fuzzy set α we have that $\alpha \circ P \geq \alpha$, then taking $\phi(\mu) = \mu \circ P$ we have that $\phi(\uparrow \alpha) \subseteq \uparrow \alpha$, where $\uparrow \alpha$ is the principal filter in $\mathcal{F}(A)$ generated by α. Thus, ϕ may be seen as an operator on $\uparrow \alpha$, and we may prove the following.

Theorem 3.8. If $\alpha \circ P \geq \alpha$, there exist the least and the greatest fuzzy sets containing α, exactly closed under composition with P; the former is also the closure of α under composition with P, and the latter is the greatest of all fuzzy sets exactly closed under composition with P.

P r o o f. Applying Theorem 2.5, we get that there exist the least and the greatest of all fixed points of the operator $\phi(\mu) = \mu \circ P$ on $\uparrow \alpha$, and – by Theorem 2.5 – the least fixed point is also the least fuzzy set in $\uparrow \alpha$ for which $\phi(\mu) \leq \mu$, i. e. $\mu \circ P \leq \mu$, i. e. the closure of α under composition with P. The existing least and the greatest fixed points are the least and the greatest exactly closed fuzzy sets (under composition with P) containing α; the first one is also the closure of α under composition with P; we shall denote the greatest fuzzy set containing α exactly closed under composition with P by α'. But since ϕ is also an operator on $\mathcal{F}(A)$, there also exists the greatest of all fuzzy sets exactly closed under composition with P; we shall denote it by α''. It contains α' since it is the maximum of a greater (or equal) set. Thus $\alpha'' \supseteq \alpha' \supseteq \alpha$, so $\alpha'' \supseteq \alpha$ and $\alpha'' \in \{\mu \mid \mu \supseteq \alpha \text{ and } \mu \circ P = \mu\}$, thus $\alpha'' \subseteq max\{\mu \mid \mu \supseteq \alpha \text{ and } \mu \circ P = \mu\} = \alpha'$ and, thus, $\alpha' = \alpha''$. □

If P is reflexive, $\mu \circ P(x) = \bigvee_{y \in A} \mu(y) \wedge P(y, x) \geq \mu(x) \wedge P(x, x) = \mu(x)$, i.e. $\mu \circ P \geq \mu$, and applying the previous theorem we get the following corollary.

Corollary 3.9. For any fuzzy set $\alpha \in \mathcal{F}(A)$, there exists the least fuzzy set containing α, exactly closed under composition with a given reflexive fuzzy relation, which equals the closure of α under composition with the same fuzzy relation.

Applying Theorem 2.7, we may construct the existing least fuzzy set (exactly) closed under composition with P.

Theorem 3.10. If $\alpha \circ P \geq \alpha$ and ϕ is given by $\phi(\mu) = \mu \circ P$, then $\overrightarrow{\lim}(\phi)\alpha$ is the least fuzzy set containing α which is exactly closed under composition with P, as well as the closure of α under composition with P.

Now, take the operator $\Phi : \mathcal{F}(A) \to \mathcal{F}(A)$ defined by $\Phi(\mu) = \mu \circ P \vee \alpha$ for a given fuzzy set α and a given fuzzy relation P. Applying Theorem 2.5, arguments used to prove Theorem 3.6 and arguments dual to those used in the proof of Theorem 3.2, we get the following theorem.

Theorem 3.11. Let $\alpha \in \mathcal{F}(A)$ be an arbitrary fuzzy set and $P \in \mathcal{F}(A^2)$ be an arbitrary fuzzy relation on A. There exists the least of all solutions to the equation $\mu \circ P \vee \alpha = \mu$ which equals the closure of α under composition with P. If μ' is a fuzzy set containing α, closed under the composition with P, there exists the greatest subset of μ' which is a solution to the equation $\mu \circ P \vee \alpha = \mu$. In particular, there exists the greatest solution to the equation $\mu \circ P \vee \alpha = \mu$.

If $\alpha > 0$, the existing solution to the equation $\mu \circ P \vee \alpha = \mu$ is different from 0. If $1 \circ P \vee \alpha < 1$ or if there exists a fuzzy set containing α, different from 1 and closed under composition with P, there exists a solution different from 1.

Thus, using the Tarski theorem, we get another proof of the existence of the closure of an arbitrary fuzzy set α under the composition with a given fuzzy relation P. By Theorem 2.10 and the dual of Theorem 2.7, we get an algorithm providing the existing closure, as well as an algorithm giving the greatest solution to the equation $\mu \circ P \vee \alpha = \mu$.

Theorem 3.12. If P is an arbitrary fuzzy relation, α an arbitrary fuzzy set and ϕ and ϕ' operators on $\mathcal{F}(A)$ given by $\phi(\mu) = \mu \circ P \vee \alpha$, $\phi'(\mu) = \mu \circ P \vee \mu$, then there exist $\overrightarrow{\lim}(\phi)\alpha$ and $\overrightarrow{\lim}(\phi')\alpha$, and they equal the closure of α under composition with P. There also exists $\underrightarrow{\lim}(\phi)1$ and it equals the greatest solution to the equation $\mu \circ P \vee \alpha = \mu$. If μ' is a fuzzy set containing α, closed under the composition with P, there also exists $\underrightarrow{\lim}(\phi)\mu'$ and it equals the greatest solution to the equation $\mu \circ P \vee \alpha = \mu$ contained in μ'.

In [19] it is proved that in the case of a meet-continuous complete codomain lattice, the above transfinite process of getting the closure of α under composition with P will terminate after at most countably many steps.

We have already mentioned that there exists the closure of any fuzzy set under composition with any given fuzzy relation, and we don't have to use the Tarski theorem to prove that. But such an approach enables us to generalize the above results and to prove the following theorems.

Theorem 3.13. There exists the least fuzzy set containing a given one, which is closed under composition with all the relations from a given set of fuzzy relations from $\mathcal{F}(A^2)$.

Proof. Let α be a given fuzzy set and $\{P_i \mid i \in I\}$ a given set of fuzzy relations from $\mathcal{F}(A^2)$. We define an operator ϕ by $\phi(\mu) = \alpha \vee \bigvee_{i \in I}(\mu \circ P_i)$. We have that $\phi(\mu) \leq \mu \Leftrightarrow \alpha \leq \mu$ and $\mu \circ P_i \leq \mu$, i. e. $\phi(\mu) \leq \mu$ iff μ is a fuzzy set containing α which is closed under composition with any P_i. Since there exists the least fuzzy set μ for which $\phi(\mu) \leq \mu$, there also exists the least fuzzy set, closed under composition with any fuzzy relation from a given subset of $\mathcal{F}(A^2)$. □

Theorem 3.14. If $\{P_i \mid i \in I\}$ is a given set of fuzzy relations and operators ϕ and ψ are defined by $\phi(\mu) = \alpha \vee \bigvee_{i \in I}(\mu \circ P_i)$ and $\psi(\mu) = \mu \vee \bigvee_{i \in I}(\mu \circ P_i)$, then there exist $\overrightarrow{\lim}(\phi)\alpha$ and $\overrightarrow{\lim}(\psi)\alpha$, and they equal the least fuzzy set containing α, closed under composition with all P_i.

Since there exists the closure of any fuzzy set under composition with any relation, i. e. the least fuzzy set containing a given one which under the composition with a given relation becomes less than or stays equal, another question which we are interested in is: does there exist the least fuzzy set containing a given one which under composition with a given fuzzy relation becomes greater or stays equal? This does not hold even in the most special crisp case, let alone in the general case of a complete codomain. The following example illustrates that:

P	a	b	c
a	0	1	0
b	1	0	1
c	0	1	0

$$\begin{aligned} \{a,b\} \circ P &= \{a,b,c\} \supset \{a,b\} \\ \{c,b\} \circ P &= \{a,b,c\} \supset \{c,b\} \\ \{b\} \circ P &= \{a,c\} \not\supseteq \{b\}. \end{aligned}$$

There does not exist the least fuzzy set containing $\{b\}$ which under composition with P becomes greater, since such a set must be contained in $\{a,b\}$ and $\{b,c\}$ and thus also in $\{b\}$, i. e. it must be equal to $\{b\}$ which is impossible.

3.2. Relational inequations and equations where a fuzzy relation is unknown

In the following, we present some analogous results to those already proved in the case of unknown fuzzy sets and we present some results for which there are no analogous results for fuzzy sets. For those that are completely analogous to the case of fuzzy sets, they are just formulated without proof.

Theorem 3.15. Let P, R be arbitrary fuzzy relations over a given crisp set A. In the set of fuzzy relations $\mathcal{F}(A^2)$ there exist:

(i) the greatest of all solutions to the equation $Q \circ P \wedge R = Q$, and the greatest of all solutions to the inequation $Q \circ P \geq Q$ which are contained in R, and the two of them coincide,

(ii) the least solution to the equation $Q \circ P \vee R = Q$, which equals the closure of R under the composition with P,

(iii) the greatest solution to the equation $Q \circ P = Q$ contained in R, provided that R is closed under the composition with P (i.e. $R \circ P \leq R$),

(iv) the least solution to the equation $Q \circ P = Q$ containing R (which equals the closure of R under the composition with P), provided that $R \circ P \geq R$.

Theorem 3.16. Let $\phi_0(Q) = Q \circ P$, $\phi(Q) = Q \circ P \wedge R$, $\phi_1(Q) = Q \circ P \wedge Q$, $\psi(Q) = Q \circ P \vee R$, $\psi_1(Q) = Q \circ P \vee Q$, where P and R are given (arbitrary) fuzzy relations. There exist $\underrightarrow{\lim}(\phi)R$ and $\underrightarrow{\lim}(\phi_1)R$, and the two of them coincide and equal the greatest fuzzy relation S contained in R such that $S \circ P \geq S$ and the greatest solution to the equation $S \circ P \wedge R = S$. If $R \circ P \leq R$, then there exists $\underrightarrow{\lim}(\phi_0)R$, and it equals the greatest fuzzy relation S contained in R for which $S \circ P = S$. There, likewise, exist $\overrightarrow{\lim}(\psi)R$ and $\overrightarrow{\lim}(\psi_1)R$, and the two of them coincide and equal the closure of R under the composition with P. If $R \circ P \geq R$, there exists $\overrightarrow{\lim}(\phi_0)R$ and it equals the least fuzzy set S containing R for which $S \circ P = S$.

Consider the fuzzy relational operators ϕ, ψ, given by $\phi(R) = R \circ Q$ and $\psi(R) = Q \circ R$, where $Q \in \mathcal{F}(A^2)$ is a given fuzzy relation. Both of them are monotonous. Moreover, we have that $\phi(\psi(R)) = Q \circ (R \circ Q)$ and $\psi(\phi(R)) = (Q \circ R) \circ Q$. In case L is a completely distributive lattice, we have that $\phi(\psi(R)) = \psi(\phi(R))$.

In the following theorem, we consider the relational equation $R \circ Q = Q \circ R$, where $Q \in \mathcal{F}(A^2)$ is a given fuzzy relation. Obviously, $R = 0$ is the least solution to this equation.

Applying Theorem 2.6, we get:

Theorem 3.17. Let $Q \in \mathcal{F}(A^2)$ be an arbitrary fuzzy relation over a crisp set A with the codomain being a completely distributive lattice. There exists the greatest solution to the system of relational equations $R \circ Q = R; Q \circ R = R$.

Consider the fuzzy relational operators ϕ, ψ, given by $\phi(R) = R \circ R \wedge Q$ and $\psi(R) = R \circ R \vee Q$, where $Q \in \mathcal{F}(A^2)$ is a given fuzzy relation. Both of them are monotonous and thus, applying Theorem 2.5, we get

Theorem 3.18. Let $Q \in \mathcal{F}(A^2)$ be an arbitrary fuzzy relation over a crisp set A. There exists the least solution to the inequation $R \circ R \leq R$ containing Q which is at the same time the least solution to the equation: $R \circ R \vee Q = R$. There exists the greatest solution to the inequation $R \circ R \geq R$ contained in Q which is at the same time the greatest solution to the equation $R \circ R \wedge Q = R$.

We have that $R \circ R \leq R$ iff for all $x, y \in A$ we have $(R \circ R)(x, y) \leq R(x, y)$ iff for all $x, y \in A$ we have $\bigvee_{z \in A} R(x, z) \wedge R(z, y) \leq R(x, y)$ iff for all $x, y, z \in A$ we have $R(x, z) \wedge R(z, y) \leq R(x, y)$ iff R is transitive fuzzy relation. Moreover, a relation R for which $R \circ R = R$ is also transitive, a special case of a transitive relation; such a relation we shall call **exactly transitive**. Now, $R \circ R \vee Q \leq R$ if and only if $R \circ R \leq R$ and $Q \leq R$, iff R is transitive relation containing Q. Therefore, the least solution to the inequation $R \circ R \vee Q \leq R$ is the least transitive relation containing Q, i.e. the transitive closure of Q. Thus, the previous theorem proves, among other things, the existence of the transitive closure of an arbitrary relation, in the case of a complete codomain lattice. We may, however, prove this without using the Tarski theorem, as it is proven in [19]. But using the mentioned constructive versions of the Tarski theorem, i.e. Theorem 2.10 and 2.7, we also get an algorithm for constructing the existing transitive closure.

Theorem 3.19. Let Q be an arbitrary fuzzy relation, ϕ, ψ and ψ_1 operators on $\mathcal{F}(A^2)$ defined respectively by $\phi(R) = R \circ R$, $\psi(R) = R \circ R \vee Q$ and $\psi_1(R) = R \circ R \vee R$, then there exist $\overrightarrow{\lim}(\psi)Q$ and $\overrightarrow{\lim}(\psi_1)Q$, and they are both equal to the transitive closure of Q. If $Q \circ Q \geq Q$, then its transitive closure is an exactly transitive relation which also equals $\overrightarrow{\lim}(\phi)Q$.

Corollary 3.20. If Q is a reflexive fuzzy relation, there exists $\overrightarrow{\lim}(\phi)Q$, where ϕ is defined by $\phi(R) = R \circ R$, and it equals the transitive closure of Q, which is also the least exactly transitive relation containing Q.

Proof. If Q is reflexive, then $Q \circ Q(x, y) \geq Q(x, x) \wedge Q(x, y) = Q(x, y)$, since $Q(x, x) = 1$. Thus $Q \circ Q \geq Q$ and the assertion follows from the previous theorem. □

In [19] it is proved that in the case of a meet-continuous codomain lattice, the above transfinite process of getting transitive closure will terminate after at most countably many sets.

If Q is transitive, we may consider the operator ϕ given by $\phi(X) = X \circ X$ as an operator on $\downarrow Q$ – principal ideal in $\mathcal{F}(A^2)$, since $\phi(\downarrow Q) \subseteq \downarrow Q$. Applying Theorem 2.5 to the operator ϕ on $\downarrow Q$ and to the operator $\phi'(X) = X \circ X \wedge Q$ on $\mathcal{F}(A^2)$ we get:

Theorem 3.21. There exists the greatest exactly transitive relation contained in a given transitive relation R. It coincides with the greatest solution to the inequation $X \circ X \geq X$

contained in R, and also with the greatest solution of the inequation $X \circ X \wedge R \geq X$ and with the greatest solution to the equation $X \circ X \wedge R = X$

Using the dual of Theorem 2.7 and the previous theorem, we get the following theorem.

Theorem 3.22. If R is a transitive relation, there exists $\underrightarrow{\lim}(\phi)R$ – where $\phi(X) = X \circ X$ – and it equals the greatest exactly transitive relation contained in R, the greatest solution of the inequation $X \circ X \wedge R \geq X$ and the greatest solution to the equation $X \circ X \wedge R = X$.

To obtain some more results, namely to solve some other extremal problems, we shall need the following lemma.

Lemma 3.23. If $(Q_i)_{i \in I}$ is a chain of fuzzy relations over a meet-continuous complete codomain lattice, then $\bigvee_{i \in I}(Q_i \circ Q_i) = (\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i)$.

P r o o f. Since $Q_i \leq \bigvee_{i \in I} Q_i$, we have that $Q_i \circ Q_i \leq (\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i)$ for every $i \in I$, thus $\bigvee_{i \in I}(Q_i \circ Q_i) \leq (\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i)$

$(\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i)(x, z) = \bigvee_{y \in A}\{(\bigvee\{Q_i \mid i \in I\})(x, y) \wedge (\bigvee\{Q_j \mid j \in I\})(y, z)\} = \bigvee_{y \in A}\{\bigvee_{j \in I}((\bigvee\{Q_i \mid i \in I\})(x, y) \wedge Q_j(y, z))\}$, since $\bigvee\{Q_i \mid i \in I\})(x, y)$ is meet-continuous for any $x, y \in A$.

Further, $\bigvee_{y \in A}\{\bigvee_{j \in I}((\bigvee\{Q_i \mid i \in I\})(x, y) \wedge Q_j(y, z))\}$
$= \bigvee_{y \in A}\{\bigvee_{j \in I}(\bigvee\{Q_i(x, y) \mid i \in I\} \wedge Q_j(y, z))\}$
$= \bigvee_{y \in A}\{\bigvee_{j \in I}(\bigvee\{Q_i(x, y) \mid i \in I\} \wedge Q_j(y, z))\}$

$\bigvee_{y \in A}\{\bigvee_{j \in I}(\bigvee_{i \in I}(Q_i(x, y) \wedge Q_j(y, z)))\}$

$= \bigvee_{y \in A}\{\bigvee_{(i,j) \in I \times I}(Q_i(x, y) \wedge Q_j(y, z))\}$

$\leq \bigvee_{y \in A}\{\bigvee_{(i,j) \in I \times I}(Q_{s(i,j)}(x, y) \wedge Q_{s(i,j)}(y, z))\}$, where $s(i, j) \in \{i, j\}$ such that $Q_{s(i,j)} = \max\{Q_i, Q_j\}$.

$\bigvee_{y \in A}\{\bigvee_{(i,j) \in I \times I}(Q_{s(i,j)}(x, y) \wedge Q_{s(i,j)}(y, z))\}$

$= \bigvee_{(i,j) \in I \times I}\{\bigvee_{y \in A}(Q_{s(i,j)}(x, y) \wedge Q_{s(i,j)}(y, z))\} \leq \bigvee_{(i,j) \in I \times I}(Q_{s(i,j)} \circ Q_{s(i,j)})(x, z)$.

Connecting the beginning with the end, we get

$(\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i)(x, z) \leq \bigvee_{(i,j) \in I \times I}(Q_{s(i,j)} \circ Q_{s(i,j)})(x, z)$.

This holds for all $(x, z) \in A^2$, thus $(\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i) \leq \bigvee_{(i,j) \in I \times I}(Q_{s(i,j)} \circ Q_{s(i,j)}) \leq \bigvee_{k \in I}(Q_k \circ Q_k)$; thus $(\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i) \leq \bigvee_{i \in I}(Q_i \circ Q_i)$. Since we have already proven the reverse inequality, we get the assertion of the lemma. □

Using Proposition 3.5 and applying it on the operator $\phi(X) = X \circ X$ we solve another extremal problem.

Theorem 3.24. If $P \in \mathcal{F}(A^2)$, and the codomain lattice of $\mathcal{F}(A^2)$ is complete and meet-continuous, then there exists a maximal solution to the inequation $X \circ X \leq P$ contained in a given fuzzy relation R, and a maximal solution to the equation $X \circ X = P$ contained in R.

P r o o f. By Lemma 2.9, it's enough to prove that in the set of solutions of the above inequation, as well as in the set of solutions of the corresponding equation – i. e. in $\Sigma = \{S \in \mathcal{F}(A^2) \mid S \circ S \leq P \text{ and } S \subseteq R\}$ and $\Omega = \{S \in \mathcal{F}(A^2) \mid S \circ S = P \text{ and } S \subseteq R\}$ – any chain has an upper bound.

Let $(Q_i)_{i \in I}$ be a chain in Σ. Since R is an upper bound for that chain, we have that $(\bigvee_{i \in I} Q_i) \leq R$. By Lemma 3.23 we have that $(\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i) = \bigvee_{i \in I}(Q_i \circ Q_i) \leq P$, thus $\bigvee_{i \in I} Q_i \in \Sigma$, which proves that $(Q_i)_{i \in I}$ has an upper bound in Σ.

In a similar way, if $(Q_i)_{i \in I}$ is a chain in Ω, we have that $(\bigvee_{i \in I} Q_i) \leq R$ and $(\bigvee_{i \in I} Q_i) \circ (\bigvee_{i \in I} Q_i) = \bigvee_{i \in I}(Q_i \circ Q_i) = P$, thus $\bigvee_{i \in I} Q_i \in \Omega$, i. e. $(Q_i)_{i \in I}$ has an upper bound in Ω. □

3.3. Systems of (in)equations

Take, for example, the following system of inequations:

$$\begin{matrix} \mu \circ Q \leq \nu \\ \nu \circ P \leq \mu \end{matrix} \qquad (1)$$

and the corresponding system of equations:

$$\begin{matrix} \mu \circ Q = \nu \\ \nu \circ P = \mu \end{matrix} \qquad (2)$$

where P and Q are given fuzzy relations and μ, ν are unknown fuzzy sets. Any solution to these systems may be viewed as a solution to the inequation $\Phi(m) \leq m$, or the corresponding equation $\Phi(m) = m$, where Φ is an operator on $\mathcal{F}(A) \times \mathcal{F}(A)$ given by:

$$\Phi(\mu, \nu) = (\nu \circ P, \mu \circ Q).$$

Notice that $\mathcal{F}(A) \times \mathcal{F}(A)$ is a complete lattice if $\mathcal{F}(A)$ is complete, and the latter is complete if its codomain lattice is complete.

The above system of inequations has two obvious solutions: $(0, 0)$ and $(1, 1)$, the first of which is the least solution and the latter is the greatest solution. Another extremal problem – which is not straightforward – is whether there exists the least solution to that system containing a given par of fuzzy sets (μ', ν'). Any solution to that system containing (μ', ν') would also be a solution to the system

$$\begin{matrix} \mu \circ Q \vee \nu' \leq \nu \\ \nu \circ P \vee \mu' \leq \mu \end{matrix} \qquad (3)$$

and the other way round. The least solution to the first system containing (μ', ν') exists if and only if there exists the least solution to system (3).

Now, applying Theorem 2.5, we get

Theorem 3.25. There exists the least solution to system (1) of inequations containing a given pair of fuzzy sets (μ', ν'), which equals the least solution to system (3), as well as to the corresponding system of equations:

$$\begin{matrix} \mu \circ Q \vee \nu' = \nu \\ \nu \circ P \vee \mu' = \mu \end{matrix} \qquad (4)\ .$$

There also exists the greatest solutions to that system and, in particular, to system (2).

To construct the existing solutions we use Theorem 2.10 and the dual of Theorem 2.7. Thus we get the following.

Theorem 3.26. If $\mu', \nu' \in \mathcal{F}(A)$ are given fuzzy sets, $\Phi(\mu,\nu) = (\nu \circ P, \mu \circ Q)$, $\phi(\mu,\nu) = (\nu \circ P \vee \mu', \mu \circ Q \vee \nu')$ and $\phi'(\mu,\nu) = (\nu \circ P \vee \mu, \mu \circ P \vee \nu)$ are operators on $\mathcal{F}(A)^2$, we have that $\overrightarrow{\lim}(\phi)(\mu',\nu')$ and $\overrightarrow{\lim}(\phi')(\mu',\nu')$ equal the least solution to system (1) containing (μ',ν'), as well as the least solution to system (3) of inequations and the corresponding system of equations (4). Also, $\underrightarrow{\lim}(\phi)(1,1)$ equals the greatest solution to system (3). In particular, $\underrightarrow{\lim}(\Phi)(1,1)$ equals the greatest solution to system (2).

Using Theorem 2.8 and the approach of [19], we may prove that the construction of the least solution of system (1) containing a given pair of fuzzy sets will – in case of a meet-continuous complete codomain lattice, terminate after at most countably many sets. We need the following two lemmas.

Lemma 3.27. If the codomain lattice of $\mathcal{F}(A)$ is complete and meet-continuous, $P \in \mathcal{F}(A^2)$ a given fuzzy relations and $\{\mu_i \mid i \in I\}$ any chain of fuzzy sets from $\mathcal{F}(A)$, then $(\bigvee_{i\in I} \mu_i) \circ P = \bigvee_{i\in I}(\mu_i \circ P)$.

Proof. We have that

$$((\bigvee_{i\in I} \mu_i) \circ P)(x) = \bigvee_{y\in A} ((\bigvee_{i\in I} \mu_i)(y) \wedge P(y,x))$$
$$= \bigvee_{y\in A} ((\bigvee_{i\in I} \mu_i(y)) \wedge P(y,x)) = \bigvee_{y\in A} (\bigvee_{i\in I} (\mu_i(y) \wedge P(y,x)))$$

(since $P(y,x)$ is meet-continuous).

Further,

$$\bigvee_{y\in A} (\bigvee_{i\in I} (\mu_i(y) \wedge P(y,x))) = \bigvee_{i\in I} (\bigvee_{y\in A} (\mu_i(y) \wedge P(y,x))) = \bigvee_{i\in I} (\mu_i \circ P)(x),$$

thus $((\bigvee_{i\in I} \mu_i) \circ P)(x) = \bigvee_{i\in I}(\mu_i \circ P)(x)$ for any $x \in A$, and the assertion of the lemma follows. □

Lemma 3.28. If the codomain lattice of $\mathcal{F}(A)$ is complete and meet-continuous, $\mu', \nu' \in \mathcal{F}(A)$ are fuzzy sets, $P, Q \in \mathcal{F}(A^2)$ are given fuzzy relations, then the operator ϕ on $\mathcal{F}(A)^2$ given by $\phi(\mu,\nu) = (\nu \circ P \vee \mu', \mu \circ Q \vee \nu')$ is upper-semi-continuous.

Proof. Let $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ be two chains of fuzzy sets. We prove $\phi(\bigvee_{i\in I}(\mu_i,\nu_i)) = \phi(\bigvee_{i\in I} \mu_i, \bigvee_{i\in I} \nu_i) = ((\bigvee_{i\in I} \nu_i) \circ P \vee \mu', (\bigvee_{i\in I} \mu_i) \circ Q \vee \nu')$.

Now, by Lemma 3.27, we have $(\bigvee_{i\in I} \nu_i \circ P \vee \mu', \bigvee_{i\in I} \mu_i \circ Q \vee \nu') = (\bigvee_{i\in I}(\nu_i \circ P) \vee \mu', \bigvee_{i\in I}(\mu_i \circ Q) \vee \nu') = (\bigvee_{i\in I}(\nu_i \circ P \vee \mu'), \bigvee_{i\in I}(\mu_i \circ Q \vee \nu')) = \bigvee_{i\in I}(\nu_i \circ P \vee \mu', \mu_i \circ Q \vee \nu') = \bigvee_{i\in I} \phi(\mu_i,\nu_i)$, thus $\phi(\bigvee_{i\in I}(\mu_i,\nu_i)) = \bigvee_{i\in I} \phi(\mu_i,\nu_i)$ and the assertion of the lemma follows. □

Since the assumption of Theorem 2.8 is proven for the operator ϕ on $\mathcal{F}(A)^2$ defined by $\phi(\mu,\nu) = (\nu \circ P \vee \mu, \mu \circ Q \vee \nu)$, we have the following theorem.

Theorem 3.29. Let (μ_n, ν_n) be defined by induction on $n \in \omega$:
$(\mu_0, \nu_0) = (\mu', \nu')$
$(\mu_{k+1}, \nu_{k+1}) = (\mu' \vee \nu_k \circ P, \nu' \vee \mu_k \circ Q)$
If the codomain lattice is complete and meet-continuous, $(\bigvee_{n\in\omega} \mu_n, \bigvee_{n\in\omega} \nu_n)$ is the least solution to system (1) containing (μ', ν'), or the least solution to system (3) of inequations, and the corresponding system of equations (4).

Since $(0, 0)$ is a solution to systems (1) and (2), applying *a*) and *d*) of Proposition 3.5, we get the following theorem.

Theorem 3.30. For $(\alpha, \beta) \in \mathcal{F}(A)^2$, there exist, in case of a meet-continuos complete codomain lattice, a maximal solution to system (1) contained in (α, β) and a maximal solution to system (2) contained in (α, β).

Applying *b*) and *c*) of Proposition 3.5 we get more results.

Theorem 3.31. If $P, Q \in \mathcal{F}(A^2)$, $\gamma, \delta \in \mathcal{F}(A)$, then for any $(\alpha, \beta) \in \mathcal{F}(A)^2$, there exists, in case of a meet-continuous complete codomain lattice, a maximal solution to the system:
$\mu \circ P \vee \nu \circ Q \leq \gamma$
$\mu \circ R \vee \nu \circ S \leq \delta$
contained in (α, β).

Theorem 3.32. If $P, Q \in \mathcal{F}(A^2)$, $\mu_0, \nu_0, \alpha, \beta, \gamma, \delta \in \mathcal{F}(A)$, $\mu_0 \leq \alpha$ and $\nu_0 \leq \beta$, then there exists - in case of a meet-continuous complete codomain lattice - a maximal solution to the system:
$\mu \circ P \vee \nu \circ Q \vee \gamma = \mu_0 \circ P \vee \nu_0 \circ Q \vee \gamma$
$\mu \circ R \vee \nu \circ S \vee \delta = \mu_0 \circ R \vee \nu_0 \circ S \vee \delta$
contained in (α, β).

There also exist maximal solutions to some similar systems that might not be quadratic. For instance, using Lemma 3.27 and Lemma 2.9, we may prove the following theorem.

Theorem 3.33. Let $\{P_j \mid j \in J\}$ and $\{Q_k \mid k \in K\}$ be families of fuzzy relations from $\mathcal{F}(A^2)$ and $\{\alpha_j \mid j \in J\}$ a family of fuzzy sets from $\mathcal{F}(A)$. If the codomain lattice is complete and meet-continuous, then there exists a maximal solution to the system of inequations $(\mu \circ P_j \leq \alpha_j)_{j\in J}$ closed under the composition with all Q_k. In particular, there exist a maximal fuzzy set closed under composition with all Q_k and a maximal solution to the system of inequations $(\mu \circ P_j \leq \alpha_j)_{j\in J}$.

Now, consider the system of equations:

$$\begin{aligned} P \circ \nu \wedge \alpha &= \mu \\ Q \circ \mu \wedge \beta &= \nu \end{aligned} \qquad (5)$$

where P and Q are given fuzzy relations, α, β are given fuzzy sets, and μ, ν are the unknown fuzzy sets in the system. Any solution to the system is an invariant of the operator ψ on $\mathcal{F}(A)^2$, defined by $\psi(\mu, \nu) = (P \circ \nu \wedge \alpha, Q \circ \mu \wedge \beta)$. Applying Theorem 2.5, we get the following result:

Theorem 3.34. If the codomain lattice of $\mathcal{F}(A)$ is complete, $\alpha, \beta \in \mathcal{F}(A)$, $P, Q \in \mathcal{F}(A^2)$, then there exists the greatest of all solutions to the system:

$$\begin{aligned} P \circ \nu &\geq \mu \\ Q \circ \mu &\geq \nu \end{aligned} \qquad (6)$$

contained in (α, β). The same is the greatest solution to the system of inequations:

$$\begin{aligned} P \circ \nu \wedge \alpha &\geq \mu \\ Q \circ \mu \wedge \beta &\geq \nu \end{aligned} \qquad (7)$$

as well as to the corresponding system of equations (5).

In particular, there exists the greatest of all solutions to system (6), which is also the greatest of all solutions to the corresponding system of equations:

$$\begin{aligned} P \circ \nu &= \mu \\ Q \circ \mu &= \nu. \end{aligned} \qquad (8)$$

Applying the duals of Theorem 2.10 and Theorem 2.7 we may construct the existing solutions.

Theorem 3.35. If $\alpha, \beta \in \mathcal{F}(A)$ are given fuzzy sets, $\Psi(\mu, \nu) = (\nu \circ P, \mu \circ Q)$, $\psi(\mu, \nu) = (\nu \circ P \wedge \alpha, \mu \circ Q \wedge \beta)$ and $\psi'(\mu, \nu) = (\nu \circ P \wedge \mu, \mu \circ P \wedge \nu)$ are operators on $\mathcal{F}(A)^2$, we have that $\underrightarrow{\lim}(\psi)(\alpha, \beta)$ and $\underrightarrow{\lim}(\psi')(\alpha, \beta)$ equal the greatest solution to system (6) contained in (α, β) and the greatest solution to system (7) of inequations and the corresponding system of equations (5). In particular, $\underrightarrow{\lim}(\Psi)(1, 1)$ is the greatest solution to systems (6) and (8).

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