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G-SUPPLEMENTED PROPERTY IN THE LATTICES

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Abstract. Let L be a lattice with the greatest element 1. Following the concept of generalized small subfilter, we define g-supplemented filters and investigate the basic properties and possible structures of these filters.

Keywords: filter; g-small; g-supplemented; lattice

MSC 2020: 06C05; 06C15

1. Introduction

In this paper, we extend several concepts from module theory to lattice theory. With a careful generalization, we can cover some basic corresponding results in the former setting. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [1], [2], [4], [5], [6]).

Since Kasch and Mares (see [7]) defined the notions of perfect and semiperfect for modules, the notion of a supplemented module has been used extensively by many authors. In a series of papers, Zöschinger has obtained detailed information about supplemented and related modules, see [13]. Supplemented modules are also discussed in [9]. Recently, the study of the supplemented property in the rings, modules, and lattices has become quite popular (see for example [3], [4], [8], [10], and [11]). Wisbauer calls a module M supplemented if, for every submodule N of M, there is a submodule K of M such that $M = N + K$ and $N \cap K$ is a small submodule of K. In [11], the basic properties of supplemented modules are given. A submodule N of an R-module M is called generalized small in M (denoted by $N \ll_g M$), if $N + K = M$ with K essential in M implies $K = M$ (see [12]). Let N, K

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be submodules of M. Module K is called a generalized supplement of N in M if $M = N + K$ and $N \cap K \ll_g K$. A module M is called generalized supplemented if every submodule of M has a generalized supplement in M (see [8], [10]).

Let L be a distributive lattice with 1. In the present paper, we are interested in investigating (amply) generalized supplemented filters to use other notions of generalized supplemented, and find out which exist in the literature as laid forth in [8]. We shortly summarize the content of the paper. If A is a subset of a lattice L, then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that contains A. Among many results in this paper, in Section 2, we introduce the class of all essential subfilters to generalize small subfilters and the class of all small subfilters to generalize essential subfilters, respectively (see [12]). It is defined (Definition 2.2) that a subfilter U of a filter F of L is said to be g-small in F, written $U \ll_g F$, if $T(U \cup V) = F$ with $V \trianglelefteq F$ implies $V = F$ (U is said to be g-essential in F, written $U \trianglelefteq_g F$, if $U \cap V = \{1\}$ with $V \ll F$ implies $V = F$). In Theorem 2.3, we show that for a subfilter U of a filter F of L the following assertions are equivalent:

(1) $U \ll_g F$;

(2) If $F = T(U \cup V)$, then there is a semisimple subfilter V' of F such that $F = V \oplus V'$.

Some basic properties of g-small subfilters and g-essential subfilters are given in Lemma 2.5, Theorem 2.7, Theorem 2.8, Lemma 2.10 and Theorem 2.11. Moreover, the generalized maximal subfilter and the generalized radical of a filter F (denoted by $\mathrm{Rad}_g(F)$) are defined, and the relationship between the generalized radical and the radical of F is investigated. Using these, we observe in Theorem 2.14 that if F is a filter of L, then $\mathrm{Rad}_g(F) = T(\cup_{V \ll_g F} V)$. We also prove in Theorem 2.18, that if F is a finitely generated filter of L and F has a proper essential subfilter, then every proper essential subfilter of F is contained in a generalized maximal subfilter. In Section 3, we use the concepts of g-small subfilters (see [12]) to introduce a generalized supplemented filter or briefly a g-supplemented filter (Definition 3.1). Some basic properties of g-supplement subfilters are given in Proposition 3.3, Theorem 3.5 and Corollary 3.9. We show in Theorem 3.4 that if V is a subfilter of a filter F of L such that V is a g-supplement of an essential subfilter of F, then $\mathrm{Rad}_g(V) = V \cap \mathrm{Rad}_g(F)$. We also prove in Theorem 3.13 that if $F = T(F_1 \cup F_2)$ with F_1 and F_2 being g-supplemented filters, then F is also g-supplemented. Moreover, it is shown that if F is a g-supplemented filter of L, then there exist a semisimple subfilter K and a subfilter V with $\mathrm{Rad}_g(V) \trianglelefteq V$ such that $F = K \oplus V$ (Theorem 3.17). Finally, the definition of amply generalized supplemented filters is given with some properties of these filters. Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. There are many different definitions of a quotient lattice appearing in the literature. In Section 4, quotient filters are defined and some possible properties of these filters are investigated. It is proved that every quotient filter of a g-supplemented filter is g-supplemented (Theorem 4.7). We also prove in Theorem 4.8 that if F is a g-supplemented filter of L, then $F/\operatorname{Rad}_g(F)$ is a semisimple filter.

Let us briefly review some definitions and tools that are used later (see [1], [2]). By a lattice we mean a poset $(L, \leqslant)$ in which every couple of elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and an l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when every of its subsets X has an l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains the least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A nonempty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leqslant b$ implies $b \in F$ and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter P of L is said to be maximal if it holds that if E is a filter in L with $P \subsetneq E$, then $E = L$. If F is a filter of a lattice L, then the radical of F, denoted by $\operatorname{Rad}(F)$, is the intersection of all maximal subfilters of F.

Let L be a lattice. If H is a subset of L, then the filter generated by H, denoted by $T(H)$, is the intersection of all filters that contains H. A filter F is called finitely generated if there is a finite subset H of F such that $F = T(H)$. A subfilter G of a filter F of L is called *small* in F, written $G \ll F$, if, for every subfilter H of F, the equality $T(G \cup H) = F$ implies $H = F$. A subfilter G of F is called *essential* in F (written $G \trianglelefteq F$) if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of F. Let G be a subfilter of a filter F of L. A subfilter $H \subseteq F$ is called a *supplement* of G in F if H is a minimal element in the set of subfilters $U \subseteq F$ with $T(G \cup U) = F$. A filter F of L is called *supplemented* if every subfilter of F has a supplement in F. A subfilter G of a filter F of L has *ample supplements* in F if, for every subfilter H of F with $F = T(H \cup G)$, there is a supplement H' of G with $H' \subseteq H$. If every subfilter of a filter F has ample supplements in F, then we call F *amply supplemented.* A filter F of a lattice L is called *hollow* if $F \neq \{1\}$ and every proper subfilter G of F is small in F. Filter F is called *local* if it has exactly one maximal subfilter that contains all proper subfilters.

Proposition 1.1. *Let L be a lattice.*

(1) *A nonempty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$, see* [6], [5].

(2) *If F is a filter of L, then $\operatorname{Rad}(F) = T\Big(\bigcup\limits_{G \ll F} G\Big)$, see* [4].

2. Generalized small subfilters

Throughout this paper, we assume, unless otherwise stated, that L is a distributive lattice with 1. In this section, generalizations of small subfilters and essential subfilters, g-small subfilters and g-essential subfilters are introduced, and some their properties are investigated. We need the following lemma proved in [4], Proposition 2.1.

Lemma 2.1.

(1) *Let A be an arbitrary nonempty subset of L. Then $T(A) = \{x \in L\colon a_1 \wedge a_2 \wedge \ldots \wedge a_n \leqslant x$ for some $a_i \in A\}$ $(1 \leqslant i \leqslant n)$. Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$.*

(2) *Let A, B and C be subfilters of a filter F of L. Then $T(T(A \cup B) \cup C) \subseteq T(A \cup T(B \cup C))$. In particular, if $F = T(T(A \cup B) \cup C)$, then $F = T(T(C \cup B) \cup A) = T(T(A \cup C) \cup B)$.*

(3) *(Modular law) If F_1, F_2, F_3 are filters of L with $F_2 \subseteq F_1$, then $F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3))$.*

Let U be a subfilter of a filter F of L. If subfilter V of F is maximal with respect to $U \cap V = \{1\}$, then we say that V is a complement of U. Using the maximal principle we readily see that if U is a subfilter of F, then the set of those subfilters of F whose intersection with U is $\{1\}$ contains the maximal element V. Thus every subfilter U of F has a complement.

Definition 2.2. Let U be a subfilter of a filter F of L.

(1) U is said to be *generalized small in* F (or, briefly, g-small in F), written $U \ll_g F$, if $T(U \cup V) = F$ with $V \trianglelefteq F$ implies $V = F$.

(2) U is said to be *generalized essential in* F (or, briefly, g-essential in F), written $U \trianglelefteq_g F$, if $U \cap V = \{1\}$ with $V \ll F$ implies $V = F$.

It is clear that if F is a filter of L, then $\{1\} \ll_g F$.

A lattice L is called *semisimple*, if for every proper filter F of L, there exists a filter G of L such that $L = T(F \cup G)$ and $F \cap G = \{1\}$. In this case, we say that F is a *direct summand* of L and we write $L = F \oplus G$. A filter F of L is called a *semisimple* filter, if every subfilter of F is a direct summand. A *simple filter* is a filter that has no filters besides the $\{1\}$ and itself (see [4]).

We are now in a position to prove necessary and sufficient conditions on a subfilter U of a filter F of L such that $U \ll_g F$. Compare the next theorem with Proposition 2.3 in [12].

Theorem 2.3. *Let U be a subfilter of a filter F of L. Then the following statements are equivalent:*

(1) *$U \ll_g F$;*

(2) *If $F = T(U \cup V)$, then there is a semisimple subfilter V' of F such that $F = V \oplus V'$.*

P r o o f. (1) $\Rightarrow$ (2): Let V' be a complement of V in F. We first show that $T(V \cup V') \trianglelefteq F$. If $\{1\} \neq K \subseteq F$ and $T(V \cup V') \cap K = \{1\}$, then we prove that $V \cap T(V' \cup K) = \{1\}$. Let $x \in V \cap T(V' \cup K)$. Then $x \in V$ and $x = (a \wedge b) \vee x = (x \vee a) \wedge (x \vee b)$ for some $a \in V'$ and $b \in K$. As $a \vee x \in V \cap V' = \{1\}$, we get $x = b \vee x \in K$. Thus $x \in K \cap T(V \cup V') = \{1\}$, contrary to the maximality of V'. Thus $T(V \cup V') \trianglelefteq F$. Since $F = T(F \cup V') = T(T(U \cup V) \cup V') = T(U \cup T(V \cup V'))$ and $U \ll_g F$, it follows that $T(V \cup V') = F$. To see that V' is semisimple, let H be a subfilter of V'. Then $F = T(T(U \cup V) \cup H) = T(T(V \cup H) \cup U)$. Arguing as above with $T(V \cup H)$ replacing V, there exists a subfilter K of F such that $F = T(T(V \cup H) \cup K) = T(H \cup T(V \cup K))$ and $T(V \cup H) \cap K = \{1\}$. By the modular law, $V' = V' \cap T(H \cup T(V \cup K)) = T(H \cup (V' \cap T(V \cup K)))$. Now it is enough to show that $H \cap (V' \cap T(V \cup K)) = H \cap T(V \cup K) = \{1\}$. Let $x \in H \cap T(V \cup K)$. Then there are elements $k \in K$ and $v \in V$ such that $x = (k \wedge v) \vee x = (x \vee k) \wedge (x \vee v)$. Since H and V are filters, $x \vee v \in H \cap V \subseteq V \cap V' = \{1\}$ which implies that $x = x \vee k \in K \cap H \subseteq K \cap T(V \cup H) = \{1\}$.

(2) $\Rightarrow$ (1): Let $K \trianglelefteq F$ and $F = T(U \cup K)$. Then there is a subfilter K' of F such that $F = T(K \cup K')$ and $K \cap K' = \{1\}$. Then $K \trianglelefteq F$ gives $K = F$; hence $U \ll_g F$. $\square$

A filter F is called *indecomposable* if it holds that if $F \neq \{1\}$ and $F = T(G \cup H)$ with $H \cap H = \{1\}$, then either $G = \{1\}$ or $H = \{1\}$, see [4].

Corollary 2.4. *Let F be an indecomposable filter of L. A proper subfilter U of F is small if and only if it is g-small.*

P r o o f. Clearly, every small subfilter of F is g-small. Conversely, assume that $U \ll_g F$ and $F = T(U \cup V)$ for some subfilter V of F. By Theorem 2.3, there exists a subfilter V' of F such that $F = V \oplus V'$. But F is indecomposable and $V \neq \{1\}$, so $V = F$. Thus $U \ll F$. $\square$

Compare the next lemma with Lemma 1 in [8].

Lemma 2.5. *Let F be a filter of L. Then the following assertions are true:*

(1) *If $U \ll_g F$ and $U' \subseteq U$, then $U' \ll_g F$.*

(2) *If U and U' are subfilters of F with $U \ll_g U'$, then U is a generalized small subfilter in subfilters of F that contains the subfilter of U'. In particular, $U \ll_g F$.*

(3) *U_1, U_2 are generalized small subfilters of F if and only if $T(U_1 \cup U_2)$ is generalized small in F.*

(4) *If U_1, U_2, V_1 and V_2 are subfilters of F with $U_1 \ll_g U_2$ and $V_1 \ll_g V_2$, then $T(U_1 \cup V_1) \ll_g T(U_2 \cup V_2)$.*

P r o o f. (1) Let $T(U' \cup V) = F$ for an essential subfilter V of F. Then $F = T(U' \cup V) \subseteq T(U \cup V) \subseteq F$ gives $T(U \cup V) = F$; so $V = F$. Thus $U' \ll_g F$.

(2) Assume that V is a subfilter of F with $U' \subseteq V$ and let $T(U \cup K) = V$ for an essential subfilter K of V. Since $U \subseteq U'$,

$$U' = U' \cap V = U' \cap (T(U \cup K)) = T(U \cup (U' \cap K))$$

by the modular law. Now $U \ll_g U'$ and $K \cap U' \trianglelefteq U'$ gives $U' = U' \cap K$; so $U \subseteq U' \subseteq K$. Hence $V = T(U \cup K) = T(K) = K$. Thus $U \ll_g V$. The particular statement is clear.

(3) Let $U_1 \ll_g F$ and $U_2 \ll_g F$. Let G be an essential subfilter of F such that $T(T(U_1 \cup U_2) \cup G) = F$. By Lemma 2.1, $F = T(T(U_1 \cup U_2) \cup G) = T(U_1 \cup T(U_2 \cup G))$. As $G \subseteq T(U_2 \cup G)$ and $G \trianglelefteq F$, we have $T(G \cup U_2) \trianglelefteq F$. Now $U_1 \ll_g F$ gives $F = T(U_2 \cup G)$; hence $G = F$ since $U_2 \ll_g F$. Thus $T(U_1 \cup U_2) \ll_g F$. Conversely, since for each i $(i = 1, 2)$, $U_i \subseteq T(U_1 \cup U_2)$, $U_i \ll_g F$ by (1).

(4) By (2), $U_1 \subseteq U_2 \subseteq T(U_2 \cup V_2)$ gives $U_1 \ll_g T(U_2 \cup V_2)$. Similarly, $V_1 \ll_g T(U_2 \cup V_2)$. Thus $T(U_1 \cup V_1) \ll_g T(U_2 \cup V_2)$ by (3). $\square$

At this stage it is useful to give an elementary remark about essential subfilters of a filter which we will use without further comment.

R e m a r k 2.6 ([4]). Let G be a subfilter of a filter F of L. Then $G \trianglelefteq F$ if and only if for every $1 \neq x \in F$ there exists an element $a \in L$ such that $1 \neq a \vee x \in G$. To see that, assume $G \trianglelefteq F$ and $1 \neq x \in F$. Then $T(\{x\}) \cap G \neq \{1\}$; so there is an element $1 \neq y \in G$ with $y = y \vee x \in G$. Conversely, if the condition holds and $1 \neq x \in H \subseteq F$, there is an element $a \in L$ such that $1 \neq a \vee x \in G \cap H$.

Theorem 2.7. *Let U, V be subfilters of a filter F of L such that V is a direct summand of F with $U \subseteq V$. Then $U \ll_g F$ if and only if $U \ll_g V$.*

P r o o f. If $U \ll_g V$, then $U \ll_g F$ by Lemma 2.5 (2). Conversely, assume that $U \ll_g F$. By assumption, there is a subfilter V' of F such that $F = T(V \cup V')$ and $V \cap V' = \{1\}$. To see that $U \ll_g V$, assume $V = T(U \cup K)$ for some $K \trianglelefteq V$. Then $F = T(V' \cup T(U \cup K)) = T(U \cup T(V' \cup K))$ by Lemma 2.1. We claim that $T(V' \cup K) \trianglelefteq F$. Let $1 \neq x \in F$. Then $x = (a \wedge b) \vee x = (a \vee x) \wedge (b \vee x)$ for some $a \in V$ and $b \in V'$. If $a \vee x = 1$, then $b \neq 1$ and $1 \neq b \vee x = x \in V' \subseteq T(V' \cup K)$. So we can assume that $a \vee x \neq 1$. Then $a \vee x \in V$ gives that there is an element $1 \neq c \in L$ such that $1 \neq a \vee x \vee c \in K$ which implies that $c \vee x = (c \vee x \vee a) \wedge (c \vee x \vee b) \neq 1$. Now $c \vee x = c \vee x((c \vee a \vee x) \wedge (c \vee b \vee x))$ gives $c \vee x \in T(V' \cup K)$; hence $T(V' \cup K) \trianglelefteq F$ by Remark 2.6. Since $U \ll_g F$, we get $T(V' \cup K) = F$. Let $z \in V \subseteq F$. There are elements $v' \in V$ and $k \in K$ such that $z = (v' \wedge k) \vee z = (z \vee v') \wedge (z \vee k)$. As $z \vee v' \in V \cap V' = \{1\}$, we have $z = z \vee k \in K$. Thus $K = V$ and so $U \ll_g V$. $\square$

Compare the next theorem with Proposition 2.5 (3) in [12].

Theorem 2.8. *Assume that U_1, V_1, U_2 and V_2 are subfilters of a filter F of L and let $U_1 \subseteq U_2$, $V_1 \subseteq V_2$ and $F = U_2 \oplus V_2$. Then $U_1 \oplus V_1 \ll_g U_2 \oplus V_2$ if and only if $U_1 \ll_g U_2$ and $V_1 \ll_g V_2$.*

Proof. If $U_1 \ll_g U_2$ and $V_1 \ll_g V_2$, then $T(U_1 \cup V_1) \ll_g T(U_2 \cup V_2)$ by Lemma 2.5 (4). To see the other implication, $U_1 \subseteq T(U_1 \cup V_1) \ll_g F = T(U_2 \cup V_2)$ gives $U_1 \ll_g F$ by Lemma 2.5 (1). Since U_2 is a direct summand of F and $U_1 \subseteq U_2$, we get $U_1 \ll_g U_2$ by Theorem 2.7. Similarly, $V_1 \ll_g V_2$. $\square$

Corollary 2.9. *Assume that U_1, V_1, U_2 and V_2 are subfilters of a filter F of L and let $U_1 \subseteq U_2$, $V_1 \subseteq V_2$ and $F = U_2 \oplus V_2$. Then $U_1 \oplus V_1 \ll U_2 \oplus V_2$ if and only if $U_1 \ll U_2$ and $V_1 \ll V_2$.*

Proof. If $U_1 \ll U_2$ and $V_1 \ll V_2$, then $T(U_1 \cup V_1) \ll T(U_2 \cup V_2)$ by [4], Lemma 2.5 (4). To see the other implication, $U_1 \subseteq T(U_1 \cup V_1) \ll F = T(U_2 \cup V_2)$ gives $U_1 \ll F$ by [4], Lemma 2.5 (1). Since U_2 is a direct summand of F (so it is a supplement in F) and $U_1 \subseteq U_2$, we get $U_1 \ll U_2$ by [4], Proposition 3.6. Similarly, $V_1 \ll V_2$. $\square$

Lemma 2.10.

(1) *If $U \neq \{1\}$ is a subfilter of a filter F of L, then $U \trianglelefteq_g F$ if and only if for every $1 \neq x \in F$; if $T(\{x\}) \ll F$, then there exists $a \in L$ such that $1 \neq a \vee x \in U$.*

(2) *Let U, V, K be subfilters of a filter F of L with $K \subseteq U$.*
  - (a) *If $K \trianglelefteq_g F$, then $K \trianglelefteq_g U$ and $U \trianglelefteq_g F$.*
  - (b) *$U \cap V \trianglelefteq_g F$ if and only if $U \trianglelefteq_g F$ and $V \trianglelefteq_g F$.*

Proof. (1) Let $U \trianglelefteq_g F$. For every $1 \neq x \in F$, if $T(\{x\}) \ll F$, then $T(\{x\}) \neq \{1\}$ and $T(\{x\}) \cap U \neq \{1\}$. Therefore, there is an element $a \in L$ such that $1 \neq a \vee x \in U$. Conversely, assume that H is a small subfilter of F and $1 \neq x \in H$. By Lemma 2.5 (1), $T(\{x\}) \ll F$; so there exists $c \in L$ such that $1 \neq c \vee x \in U \cap H$. Thus $U \trianglelefteq_g F$.

(2a) If $K \cap K' = \{1\}$ with $K' \ll U$, then [4], Lemma 2.5 (1) gives $K' \ll F$; hence $K' = U$. Thus $K \trianglelefteq_g U$. Moreover, if $U \cap G = \{1\}$ with $G \ll F$, then $K \cap G = \{1\}$ gives $G = F$, and so $U \trianglelefteq_g F$.

(2b) Assume that $U \cap V \trianglelefteq_g F$ and let $U \cap V' = \{1\}$ for some small subfilter V' of F. Then $U \cap V \cap V' = \{1\}$ gives $V' = F$. So $U \trianglelefteq_g F$. Similarly, $V \trianglelefteq_g F$. Conversely, assume that $U \cap V \cap K = \{1\}$ for some $K \ll F$. Then $V \cap K = F$ since $U \trianglelefteq_g F$; hence $K = F$. Thus $U \cap V \trianglelefteq_g F$. □

Compare the next theorem with Proposition 2.7 in [12].

Theorem 2.11. *Assume that U_1, V_1, U_2 and V_2 are subfilters of F and let $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ and $F = V_1 \oplus V_2$. Then $U_1 \oplus U_2 \trianglelefteq_g V_1 \oplus V_2$ if and only if $U_1 \trianglelefteq_g V_1$ and $U_2 \trianglelefteq_g V_2$.*

P r o o f. Suppose, say, that U_1 is not g-essential in V_1; so $U_1 \cap K = \{1\}$ for some small subfilter $K \neq \{1\}$ of V_1. Let $x \in T(U_1 \cup U_2) \cap K$. Then $x \in K$ and $x = (u_1 \wedge u_2) \vee x = (x \vee u_1) \wedge (x \vee u_2)$ for some $u_1 \in U_1$ and $u_2 \in U_2$. Since K and U_1 are filters, $x \vee u_1 \in K \cap U_1 = \{1\}$; hence $x \in U_2$. Therefore $x \in V_1 \cap V_2 = \{1\}$. Thus $T(U_1 \cup U_2) \cap K = \{1\}$ which is impossible. Thus $U_1 \trianglelefteq_g V_1$ and $U_2 \trianglelefteq_g V_2$.

Conversely, assume that $1 \neq x = (v_1 \wedge v_2) \vee x = (v_1 \vee x) \wedge (v_2 \vee x) \in T(V_1 \cup V_2)$ for some $v_i \in V_i$ such that $T(\{x\}) \ll T(V_1 \cup V_2)$. We can easily show that $T(\{v_1 \vee x\}) \cap T(\{v_2 \vee x\}) = \{1\}$ and $T(T(\{v_1 \vee x\}) \cup T(\{v_2 \vee x\})) \subseteq T(\{x\}) \ll T(V_1 \cup V_2)$, which implies that $T(T(\{v_1 \vee x\}) \cup T(\{v_2 \vee x\})) \ll T(V_1 \cup V_2)$; hence $T(\{v_1 \vee x\}) \ll V_1$ and $T(\{v_2 \vee x\}) \ll V_2$ by Corollary 2.9. Then by Lemma 2.10 (1), there is some $a_1 \in L$ such that $1 \neq a_1 \vee (v_1 \vee x) \in U_1$. If $a_1 \vee (v_2 \vee x) \in U_2$, then $1 \neq a_1 \vee x = a_1 \vee ((v_1 \vee x) \wedge (v_2 \vee x)) = (a_1 \vee v_1 \vee x) \wedge (a_1 v_2 \vee x) \in T(U_1 \cup U_2)$. If $a_1 \vee (v_2 \cup x) \notin U_2$, then again by Lemma 2.10 (1), there is $a_2 \in L$ with $1 \neq a_2 \vee a_1 \vee (v_2 \vee x) \in U_2$ and we have $1 \neq a_1 \vee a_2 \vee x \in T(U_1 \cup U_2)$. Thus $T(U_1 \cup U_2) \trianglelefteq T(V_1 \cup V_2)$. □

Corollary 2.12 ([4], Lemma 2.15 (2))**.** *Assume that U_1, V_1, U_2 and V_2 are subfilters of F and let $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ and $F = V_1 \oplus V_2$. Then $U_1 \oplus U_2 \trianglelefteq V_1 \oplus V_2$ if and only if $U_1 \trianglelefteq V_1$ and $U_2 \trianglelefteq V_2$.*

Definition 2.13. Let K be a subfilter of a filter F of L. If K is both maximal and essential in F, then K is called a *generalized maximal* subfilter of F. The intersection of all generalized maximal subfilters of F is called the *generalized radical* of F denoted by $\mathrm{Rad}_g(F)$. If F does not have generalized maximal subfilters, then we write $\mathrm{Rad}_g(F) = F$.

Compare the next theorem with Theorem 2.10 in [12].

Theorem 2.14. *Let F be a filter of L such that it has at least one generalized maximal subfilter. Then the following statements hold:*
(1) $x \in \mathrm{Rad}_g(F)$ *if and only if* $T(\{x\}) \ll_g F$.
(2) $\mathrm{Rad}_g(F) = T\Bigl(\bigcup\limits_{V \ll_g F} V\Bigr)$.

Proof. (1) Suppose that $T(\{x\})$ is not generalized small in F and set

$$\Omega = \{U\colon\ x \notin U,\ U \trianglelefteq F,\ \text{and}\ T(U \cup T(\{x\})) = F\}.$$

As $T(\{x\})$ is not generalized small in F, we conclude that $\Omega \neq \emptyset$. Clearly, every chain has an upper bound by inclusion in Ω; hence Ω contains a maximal element K by Zorn's lemma. Let U be a subfilter of F such that $K \subsetneq U \subseteq F$. Then $x \in U$ by maximality of K and so $F = T(G \cup T(\{x\})) \subseteq U$; hence $F = U$. Thus K is a generalized maximal subfilter of F with $x \notin K$. Since $\mathrm{Rad}_g(F) \subseteq K$, we get $x \notin \mathrm{Rad}_g(F)$, which is impossible. Therefore $T(\{x\}) \ll_g F$. The other implication is clear.

(2) Let $V \ll_g F$. If K is a generalized maximal subfilter of F and $V \nsubseteq K$, then $T(V \cup K) = F$; but since $V \ll_g F$, we have $K = F$, which is a contradiction. Therefore, V is contained in every generalized maximal subfilter of F and hence $T\Bigl(\bigcup\limits_{V \ll F} V\Bigr) \subseteq \mathrm{Rad}_g(F)$. For the reverse inclusion, assume that $x \in \mathrm{Rad}_g(F)$. Then $x \in T(\{x\}) \subseteq T\Bigl(\bigcup\limits_{V \ll_g F} V\Bigr)$ by (1), and so we have equality. □

Corollary 2.15. *Let F be a filter of L. Then the following statements hold:*
(1) *If F does not have generalized maximal subfilters, then* $\mathrm{Rad}_g(F) = T\Bigl(\bigcup\limits_{V \ll_g F} V\Bigr)$.
(2) $\mathrm{Rad}(F) \subseteq \mathrm{Rad}_g(F)$.

Remark 2.16. Let F be a simple filter of L. Then $\mathrm{Rad}_g(F) = F$ and $\mathrm{Rad}(F) = \{1\}$; hence $\mathrm{Rad}_g(F) \neq \mathrm{Rad}(F)$.

Proposition 2.17. *Let F be a filter of L. Then the following statements hold:*
(1) $\mathrm{Rad}_g(F) = F$ *if and only if all finitely generated subfilters are g-small subfilters of F.*
(2) *Let* $\mathrm{Rad}(F) \neq F$. *If every proper essential subfilter F is contained in a generalized maximal subfilter, then* $\mathrm{Rad}(F) \ll_g F$

Proof. (1) Assume that $\mathrm{Rad}_g(F) = F$ and let $H = T(A)$, where $A = \{a_1, a_2, \ldots, a_n\} \subseteq H$. By assumption, $T(\{a_i\}) \ll_g F$ $(1 \leqslant i \leqslant n)$, and so by Lemma 2.5 (3), $S = T(T(\{a_1\}) \cup \ldots \cup T(\{a_n\})) \ll_g F$. Now by Lemma 2.5 (1), $H \subseteq S$ gives $H \ll_g F$. Conversely, assume that $x \in F$. Then by assumption, $T(\{x\}) \ll_g F$; hence $x \in T(\{x\}) \subseteq \mathrm{Rad}_g(F)$ by Theorem 2.14.

(2) Let G be an essential subfilter of F such that $F = T(\mathrm{Rad}(F) \cup G)$. If $F \neq G$, then there is a generalized maximal subfilter H of F such that $G \subseteq H$; hence $F \subseteq T(\mathrm{Rad}(F) \cup H) = T(H) = H$ which is impossible. Thus $G = F$, and so $\mathrm{Rad}(F) \ll_g F$. $\square$

Compare the next theorem with Theorem 5 in [8].

Theorem 2.18. *If F is a finitely generated filter of L and F has a proper essential subfilter, then every proper essential subfilter of F is contained in a generalized maximal subfilter.*

P r o o f. Assume that H is a proper essential subfilter of F and let $F = T(A)$, where $A = \{a_1, a_2, \ldots, a_n\} \subseteq F$. Since $H \neq F$, it cannot contain all of the generators $a_1, \ldots, a_n$. By reordering the generators, if necessary, it is possible to find $a_1, \ldots, a_k$ such that $F = T(H \cup T(\{a_1, \ldots, a_k\}))$ but $F \neq T(H \cup T(\{a_2, \ldots, a_k\}))$. Set $K = T(H \cup T(\{a_2, \ldots, a_k\}))$; so $a_1 \notin K$. At first we show that F has a subfilter G maximal with respect to $K \subseteq G$ and $a_1 \notin G$. Consider the set $\Omega = \{U\colon U$ is a subfilter of F, $K \subseteq U$ and $a_1 \notin U\}$. This set is not empty since $K \in \Omega$. Clearly, Ω is closed under taking unions of chains and so the result follows by Zorn's lemma. Let G be the maximal element of Ω. Let V be a subfilter of F such that $G \subsetneq V \subseteq F$. Then $a_1 \in V$ by the maximality of G and so $F = T(K \cup T(\{a_1\})) \subseteq V$; hence $F = V$. Thus $H \subseteq K$ is contained in a maximal subfilter G and $G \trianglelefteq F$ because $H \trianglelefteq F$. $\square$

Definition 2.19. A filter F of L is called a *generalized hollow filter* if every proper subfilter of F is generalized small in F.

It is clear that every hollow filter is a generalized hollow filter. Compare the next theorem with Theorem 4 in [8].

Theorem 2.20. *Let F be a filter of L such that $\mathrm{Rad}_g(F) \neq F$. The following conditions are equivalent:*

(1) *F is a generalized hollow filter;*
(2) *F is a local filter;*
(3) *F is a hollow filter.*

P r o o f. (1) $\Rightarrow$ (2): Let G be a proper subfilter of a generalized hollow filter F. Then $G \ll_g F$ gives $G \subseteq \mathrm{Rad}_g(F)$ by Theorem 2.14 (2). Since $\mathrm{Rad}_g(F) \neq F$, F is local, as needed.

(2) $\Rightarrow$ (3): Assume that F is a local filter with unique maximal subfilter of K and let U be a proper subfilter of F with $T(U \cup V) = F$ for some subfilter V of F. If $V \neq F$, then $F \subseteq T(K \cup U) = T(K) = K$, a contradiction. Thus $F = V$.

(3) $\Rightarrow$ (1): Clear. $\square$

3. Generalized supplemented filters

In this section, we define the concept of generalized supplemented filters of a lattice and we prove some basic properties concerning such filters. We begin with the key definition of this section.

Definition 3.1. Let U and V be subfilters of a filter F of L. If $F = T(U \cup V)$ and $F = T(U \cup K)$ with $K \trianglelefteq V$ implies that $V = K$, then V is called a *generalized supplement* (or briefly a *g-supplement*) of U in F. If every subfilter of F has a g-supplement in F, then F is called a *generalized supplemented* (or briefly a *g-supplemented*) *filter*.

The supplemented filters are g-supplemented. Compare the next lemma with Lemma 2 in [8].

Lemma 3.2. *Let U, V be subfilters of a filter F of L. V is a g-supplement of U in F if and only if $T(U \cup V) = F$ and $U \cap V \ll_g V$.*

Proof. Let V be a g-supplement of U in F (so $T(U \cup V) = F$). Let $Y \trianglelefteq V$ with $T(Y \cup (U \cap V)) = V$. Then by Lemma 2.1, we have

$$\begin{aligned} F &= T(U \cup V) = T(T((U \cap V) \cup Y) \cup U) \\ &= T(T(U \cup (U \cap V)) \cup Y) = T(T(U) \cup Y) = T(U \cup Y), \end{aligned}$$

which implies that $V = Y$ because V is a g-supplement of U in F and $Y \trianglelefteq V$. Thus $U \cap V \ll_g V$. Conversely, assume that $T(U \cup V) = F$ and $U \cap V \ll_g V$. For $X \trianglelefteq V$ with $T(X \cup U) = F$, we have $V = V \cap F = V \cap T(X \cup U) = T(X \cup (V \cap U))$ by the modular law. Now $U \cap V \ll_g V$ gives $X = V$. Hence V is a g-supplement of U in F. $\square$

Proposition 3.3. *Let U, V be subfilters of a filter F of L. Assume V to be a g-supplement of U. Then the following assertions are true:*
(1) *If $T(V \cup U') = F$ for some $U' \subseteq U$, then V is a g-supplement of U'.*
(2) *If $K \ll_g F$ and $V \trianglelefteq F$, then V is a g-supplement of $T(U \cup K)$.*
(3) *If K is a subfilter of V and $U \trianglelefteq F$, then $K \ll_g V$ if and only if $K \ll_g F$.*

Proof. (1) By Lemma 3.2, it is enough to show that $U' \cap V \ll_g V$. Assume that X is an essential subfilter of V such that $T(X \cup (U' \cap V)) = V$. Now $V = T(X \cup (U' \cap V)) \subseteq T(X \cup (U \cap V)) \subseteq V$ gives $V = T(X \cup (U \cap V))$; hence $X = V$ since $U \cap V \ll_g V$. Thus V is a g-supplement of U'.

(2) By Lemma 2.1, we have that $F = T(U \cup V) \subseteq T(T(U \cup K) \cup V) \subseteq F$; so $T(T(U \cup K) \cup V) = F$. Assume that Y is an essential subfilter of V (so $Y \trianglelefteq F$)

such that $T(T(U \cup K) \cup Y) = F$; we show that $Y = V$. By Lemma 2.1, $F = T(T(U \cup K) \cup Y) = T(T(U \cup Y) \cup K)$. Since $Y \trianglelefteq F$ and $Y \subseteq T(Y \cup U)$, we clearly see that $T(Y \cup U)$ is essential in F. Now $K \ll_g F$ gives $T(U \cup Y) = F$; hence $Y = V$ since V is a g-supplement of U. Thus V is a g-supplement of $T(U \cup K)$.

(3) If $K \ll_g V$, then $K \ll_g F$ by Lemma 2.5 (2). Assume that $K \ll_g F$ and let $G \trianglelefteq V$ with $V = T(G \cup K)$. $F = T(U \cup V)$ gives

$$F = T(U \cup T(G \cup K)) = T(K \cup T(G \cup U)).$$

As $U \trianglelefteq F$ and $U \subseteq T(G \cup U)$, we get $T(U \cup G) \trianglelefteq F$ which implies that $T(G \cup U) = F$. Since V is a g-supplement of U in F, $G = V$. Thus $K \ll_g V$. $\square$

Compare the next theorem with 41.1 (5) in [11].

Theorem 3.4. *Let V be a subfilter of a filter F of L such that V is a g-supplement of an essential subfilter of F. Then* $\mathrm{Rad}_g(V) = V \cap \mathrm{Rad}_g(F)$.

P r o o f. By Proposition 3.3 (3) and Theorem 2.14, it is clear that $\mathrm{Rad}_g(V) \subseteq V \cap \mathrm{Rad}_g(F)$. For the reverse inclusion, assume that $x \in \mathrm{Rad}_g(F) \cap V$. Since $x \in \mathrm{Rad}_g(F)$, by Theorem 2.14 (1) then $T(\{x\}) \ll_g F$, which implies that $T(\{x\}) \ll_g V$ by Proposition 3.3 (3); hence $x \in T(\{x\}) \subseteq \mathrm{Rad}_g(V)$, and so we have equality. $\square$

Compare the next theorem with 41.1 (3) in [11].

Theorem 3.5. *Let V be a subfilter of a filter F of L such that U is an essential maximal subfilter of F and V is a g-supplement of U in F. Then* $\mathrm{Rad}_g(V) = U \cap V$.

P r o o f. Since $T(U \cup V) = F$ and U is a maximal subfilter of F, then $V \nsubseteq U$; so $U \cap V \neq V$. Let K be a subfilter of V such that $U \cap V \subsetneq K \subseteq V$. Then there is an element $x \in K \subseteq V$ with $x \notin U$. Now $U \subsetneq T(T(\{x\}) \cup U) \subseteq F$ gives $F = T(T(\{x\}) \cup U)$. By the modular law, we conclude that

$$V = V \cap T(T(\{x\}) \cup U) = T(T(\{x\}) \cup (U \cap V)) \subseteq K;$$

so $V = K$. Thus $U \cap V$ is a maximal subfilter of V. Since U is essential in F, we clearly see that $U \cap V$ is essential in V. So $\mathrm{Rad}_g(V) \subseteq U \cap V$. As V is a g-supplement of U, $U \cap V \ll_g V$; hence $U \cap V \subseteq \mathrm{Rad}_g(V)$ by Theorem 2.14 (2). Hence $\mathrm{Rad}(V)_g = U \cap V$. $\square$

Proposition 3.6. *Let V be a g-supplement of U in a filter F of L. If H is a subfilter of V and $K \trianglelefteq V$, then H is a g-supplement of K in V if and only if H is a g-supplement of $T(K \cup U)$ in F.*

P r o o f. Let H be a g-supplement of K in V. Then $V = T(K \cup H)$ gives $F = T(U \cup T(K \cup H)) = T(H \cup T(U \cup K))$. Let $F = T(G \cup T(U \cup K)) = T(U \cup T(G \cup K))$ with $G \trianglelefteq H$. Since $K \trianglelefteq V$ and $K \subseteq T(G \cup K) \subseteq V$, then $T(G \cup K) \trianglelefteq V$. Now V is a g-supplement of U, which gives $V = T(G \cup K)$. Since $G \trianglelefteq H$ and H is a g-supplement of K in V, $G = H$.

Conversely, let H be a g-supplement of $T(K \cup U)$ in F; so $F = T(H \cup T(K \cup U)) = T(U \cup T(K \cup H))$. Since $K \trianglelefteq V$ and $K \subseteq T(H \cup K) \subseteq V$, thus $T(K \cup H) \trianglelefteq V$. Then by V being a g-supplement of U in F, $V = T(H \cup K)$. Let $V = T(K \cup G')$ with $G' \trianglelefteq H$. Then $F = T(U \cup V)$ gives

$$F = T(U \cup T(K \cup G')) = T(G' \cup T(K \cup U)).$$

Since $G' \trianglelefteq H$ and H is a g-supplement of $T(U \cup K)$ in F, $G' = H$. Thus H is a g-supplement of K in V. $\square$

Theorem 3.7. *Let U and V be mutual g-supplements in a filter F of L. If $G \trianglelefteq U$, $G' \trianglelefteq V$, H is a g-supplement of G in U and H' is a g-supplement of G' in V, then $T(H \cup H')$ is a g-supplement of $T(G \cup G')$ in F.*

P r o o f. Since $U = T(G \cup H)$ and $V = T(G' \cup H')$, Lemma 2.1 gives

$$\begin{aligned} F &= T(U \cup V) = T(T(G \cup H) \cup V) = T(G \cup T(H \cup V)) \\ &\subseteq T(G \cup T(G' \cup T(H \cup H'))) = T(T(G \cup G') \cup T(H \cup H')) \subseteq F; \end{aligned}$$

hence $F = T(T(G \cup G') \cup T(H \cup H'))$. Since V is a g-supplement of U in F, $G' \trianglelefteq V$ and H' is a g-supplement G' in V, then by Proposition 3.6, H' is a g-supplement of $T(U \cup G')$ in F; so $T(U \cup G') \cap H' \ll_g H'$. Similarly, $T(V \cup G) \cap H \ll_g H$. To simplify our notation let

$$T(G \cup G' \cup H) \cap H' = A, \quad T(G \cup G' \cup H') \cap H = B \quad \text{and} \quad T(G \cup G') \cap T(H \cup H') = C.$$

We first show that $C \subseteq T(A \cup B)$. Let $x \in C$. Then there are elements $g \in G$, $g' \in G'$, $h \in H$ and $h' \in H'$ such that $x = (g \wedge g') \vee x = (h \wedge h') \vee x$; so $x = (g \wedge g') \vee (h \wedge h') \vee x = ((g \vee h) \wedge (g \vee h') \wedge (g' \vee h) \wedge (g' \vee h')) \vee x \in T(A \cup B)$. Thus $C \subseteq T(A \cup B)$. Since $G \cup H \subseteq T(G \cup H)$, we have

$$A \subseteq T(G' \cup T(G \cup H)) \cap H' = T(G' \cup U) \cap H'.$$

Similarly, $B \subseteq T(G \cup V) \cap H$. So

$$C \subseteq T(A \cup B) \subseteq T((T(G' \cup U) \cap H') \cup (T(G \cup V) \cap H)) = D.$$

As $D \ll_g T(H \cup H')$ by Lemma 2.5 (4), $C \subseteq D$ gives $C \ll_g T(H \cup H')$ by Lemma 2.5 (1). $\square$

Corollary 3.8. *Let V be a supplement of U in a filter F of L. If H is a subfilter of V and $K \trianglelefteq V$, then H is a g-supplement of K in V if and only if H is a g-supplement of $T(K \cup U)$ in F.*

P r o o f. Since V is a supplement of U in F, V is a g-supplement of U in F. Now the assertion follows from Proposition 3.6. $\square$

Corollary 3.9. *Let $F = U \oplus V$. If H is a subfilter of V and $K \trianglelefteq V$, then H is a g-supplement of K in V if and only if H is a g-supplement of $T(K \cup U)$ in F.*

P r o o f. Since $F = T(U \cup V)$ and $U \cap V = \{1\} \ll V$, we get that V is a supplement U in F. Now the assertion follows from Corollary 3.8. $\square$

Corollary 3.10. *Let U and V be mutual supplements in a filter F of L. If $G \trianglelefteq U$, $G' \trianglelefteq V$, H is a g-supplement of G in U and H' is a g-supplement of G' in V, then $T(H \cup H')$ is a g-supplement of $T(G \cup G')$ in F.*

P r o o f. Since U and V are mutual supplements in F, we get that they are mutual g-supplements in F. Then the assertion follows from Theorem 3.7. $\square$

Corollary 3.11. *Let $F = U \oplus V$. If $G \trianglelefteq U$, $G' \trianglelefteq V$, H is a g-supplement of G in U and H' is a g-supplement of G' in V, then $T(H \cup H')$ is a g-supplement of $T(G \cup G')$ in F.*

P r o o f. Since $F = T(U \cup V)$, $U \cap V = \{1\} \ll U$ and $U \cap V = \{1\} \ll V$, we get U and V are mutual supplements in F. Now the assertion follows from Corollary 3.10. $\square$

Proposition 3.12. *Assume that F_1 and U are subfilters of a filter F of L and let F_1 be a g-supplemented filter. If $T(F_1 \cup U)$ has a g-supplement in F, then the same is true for U.*

P r o o f. Let X be a g-supplement of $T(F_1 \cup U)$ in F; so $T(X \cup T(F_1 \cup U)) = F$ and $X \cap T(F_1 \cup U) \ll_g X$. Since F_1 is g-supplemented, $B = T(X \cup U) \cap F_1 \subseteq T(X \cup U)$ has a g-supplement in F_1, say Y (so $T(Y \cup B) = F_1$). We now show that $T(X \cup Y)$ is a g-supplement of U in F. By Lemma 2.1, we have

$$\begin{aligned} F = T(X \cup T(F_1 \cup U)) &\subseteq T(F_1 \cup T(X \cup U)) = T(T(B \cup Y) \cup T(X \cup U)) \\ &\subseteq T(Y \cup T(B \cup T(X \cup U))) = T(Y \cup T(X \cup U)) \\ &\subseteq T(U \cup T(X \cup Y)) \subseteq F; \end{aligned}$$

hence $F = T(U \cup T(X \cup Y))$. It is enough to show that $T(X \cup Y) \cap U \ll_g T(X \cup Y)$. As Y is a g-supplement of $T(X \cup U) \cap F_1$ in F_1,

$$A = Y \cap T(X \cup U) = Y \cap (T(X \cup U) \cap F_1) \ll_g Y.$$

Since $T(U \cup Y) \subseteq T(F_1 \cup U)$ and $F = T(U \cup T(X \cup Y)) = T(X \cup T(U \cup Y))$, Lemma 2.5 (1) gives that X is also a g-supplement of $T(U \cup Y)$ in F which implies that $B = T(U \cup Y) \cap X \ll_g X$. We first show that $T(X \cup Y) \cap U \subseteq T(A \cup B)$. Let $x \in T(X \cup Y) \cap U$. Then there are elements $x' \in X$ and $y' \in Y$ such that $x = x \vee ((x \vee x') \wedge (x \vee y'))$, where $x \vee x' \in B$ and $x \vee y' \in A$; hence $x \in T(A \cup B)$. Now by Lemma 2.5 (4), $T(X \cup Y) \cap U \subseteq T(A \cup B) \ll_g T(X \cup Y)$; hence $T(X \cup Y) \cap U \ll_g T(X \cup Y)$ by Lemma 2.5 (1). $\square$

Compare the next theorem with Theorem 1 in [8].

Theorem 3.13. *Let $F = T(F_1 \cup F_2)$. If F_1 and F_2 are g-supplemented filters, then F is a g-supplemented filter.*

P r o o f. If U is any subfilter of F, then $T(F_2 \cup U \cup F_1) = F$. Let V be a g-supplement of $D = T(F_2 \cup U) \cap F_1 \subseteq T(F_2 \cup U)$ in F_1; so $T(V \cup D) = F_1$ and $D \cap V \ll_g V$. Moreover, $D, F_2 \cup U \subseteq T(F_2 \cup U)$ gives $T(D \cup F_2 \cup U) \subseteq T(F_2 \cup U)$. Now by Lemma 2.1, we have

$$\begin{aligned} F = T(F_2 \cup U \cup F_1) &= T(F_2 \cup U \cup T(V \cup D)) \\ &\subseteq T(V \cup T(F_2 \cup U \cup D)) \subseteq T(V \cup T(F_2 \cup U)) \subseteq F; \end{aligned}$$

hence $F = T(V \cup T(F_2 \cup U))$ which implies that V is a g-supplement of $T(F_2 \cup U)$ in F since $V \cap T(F_2 \cup U) = V \cap T(F_2 \cup U) \cap F_1 \ll_g V$. Now the assertion follows from Proposition 3.12. $\square$

Corollary 3.14. *If $F_1, \ldots, F_n$ are g-supplemented filters of L, then $T\Big(\bigcup\limits_{i=1}^{n} F_i\Big)$ is a g-supplemented filter.*

Proposition 3.15. *Let F be a g-supplemented filter of L. If V is a subfilter of F with $V \cap \mathrm{Rad}_g(F) = \{1\}$, then V is semisimple. In particular, if $\mathrm{Rad}_g(F) = \{1\}$, then F is semisimple.*

P r o o f. Let V' be any subfilter of V. By assumption, there is a subfilter K of F with $F = T(V' \cup K)$ and $V' \cap K \ll_g K$ (so $V' \cap K \subseteq \mathrm{Rad}_g(K)$). By the modular law, $V = V \cap T(V' \cup K) = T(V' \cup (V \cap K))$. As $(V \cap K) \cap V' = K \cap V' \subseteq V \cap \mathrm{Rad}_g(K) \subseteq V \cap \mathrm{Rad}_g(F) = \{1\}$, we get $(V \cap K) \cap V' = \{1\}$ and $V = T(V' \cup (V \cap K))$. Thus V is semisimple. Moreover, if $\mathrm{Rad}_g(F) = \{1\}$, then $F \cap \mathrm{Rad}_g(F) = \{1\}$; hence F is semisimple. $\square$

Proposition 3.16. *Let F be a filter of L. Then the following statements hold:*

(1) *If U, V are subfilters of F such that $F = U \oplus V$, then* $\mathrm{Rad}_g(F) = \mathrm{Rad}_g(U) \oplus \mathrm{Rad}_g(V)$.

(2) *If F is semisimple, then* $\mathrm{Rad}_g(F) = \{1\}$.

Proof. (1) By assumption, $\mathrm{Rad}_g(U) \cap \mathrm{Rad}_g(V) \subseteq U \cap V = \{1\}$ gives $\mathrm{Rad}_g(U) \cap \mathrm{Rad}_g(V) = \{1\}$. By Lemma 2.5 (2), $\mathrm{Rad}_g(U), \mathrm{Rad}_g(V) \subseteq \mathrm{Rad}_g(F)$, which implies that $T(\mathrm{Rad}_g(U) \cup \mathrm{Rad}_g(V)) \subseteq \mathrm{Rad}_g(F)$. For the reverse inclusion, assume that $x \in \mathrm{Rad}_g(F)$. By Theorem 2.14, $x = (x_1 \wedge x_2 \wedge \ldots \wedge x_k) \vee x$, where $x_1 \in F_1 \ll_g F, \ldots, x_k \in F_k \ll_g F$. By Lemma 2.5 (1), $T(\{x_1\}) \subseteq F_1 \ll_g F$ gives $T(\{x_1\}) \ll_g F$. Since $x_1 \in F = T(U \cup V)$, then $x_1 = (u \wedge v) \vee x_1 = (x_1 \vee u) \wedge (x_1 \vee v)$ for some $u \in U$ and $v \in V$. We can easily show that $T(\{x_1 \vee u\}) \cap T(\{x_1 \vee v\}) = \{1\}$ and

$$T(T(\{x_1 \vee u\}) \cup T(\{x_1 \vee v\})) \subseteq T(\{x_1\}) \ll_g T(V_1 \cup V_2),$$

which implies that $T(T(\{x_1 \vee u\}) \cup T(\{x_1 \vee v\})) \ll_g T(U \cup V)$ by Lemma 2.5 (1); hence $T(\{x_1 \vee u\}) \ll_g U$ and $T(\{x_1 \vee v\}) \ll_g V$ by Theorem 2.8. Therefore $x_1 \vee u \in \mathrm{Rad}_g(U)$ and $x_1 \vee v \in \mathrm{Rad}_g(V)$. Hence $x_1 = x_1 \vee (u \wedge v) \vee x_1 = ((x_1 \vee u) \wedge (x_1 \vee v)) \vee x_1 \in T(\mathrm{Rad}_g(U) \cup \mathrm{Rad}_g(V)) = A$. Similarly, $x_2, \ldots, x_k \in A$. Thus $x \in A$ and so we have equality.

(2) Since every proper subfilter of F is a direct summand, the only proper g-small subfilter of F can be $\{1\}$. Thus $\mathrm{Rad}_g(F) = \{1\}$. □

Theorem 3.17. *Let F be a g-supplemented filter of L. Then there exist a semisimple subfilter K and a subfilter V with* $\mathrm{Rad}_g(V) \trianglelefteq V$ *such that $F = K \oplus V$.*

Proof. Let K be a subfilter of F which is a complement of $\mathrm{Rad}_g(F)$. Then $K \cap \mathrm{Rad}_g(F) = \{1\}$ and $T(K \cup \mathrm{Rad}_g(F)) \trianglelefteq F$. Since F is g-supplemented, there is a subfilter V of F such that $F = T(V \cup K)$ and $V \cap K \ll_g V$ (so $V \cap K \subseteq \mathrm{Rad}_g(V)$). Since $V \cap K = K \cap (V \cap K) \subseteq K \cap \mathrm{Rad}_g(V) \subseteq K \cap \mathrm{Rad}_g(F) = \{1\}$; hence $F = T(K \cup V)$ with $V \cap K = \{1\}$. By Proposition 3.15, K is semisimple. By Proposition 3.16, $\mathrm{Rad}_g(F) = T(\mathrm{Rad}_g(V) \cup \mathrm{Rad}_g(K)) = T(\mathrm{Rad}_g(V) \cup \{1\}) = \mathrm{Rad}_g(V)$. Since $T(K \cup \mathrm{Rad}_g(V)) \trianglelefteq F = T(K \cup V)$, $\mathrm{Rad}_g(V) \trianglelefteq V$ by Corollary 2.12, as requried. □

Definition 3.18. Let U be a subfilter of a filter F of L. If for every subfilter V of F with $F = T(U \cup V)$ has a g-supplement H in F such that $H \subseteq V$, then we say that U has an *ample generalized supplement* (or briefly an *ample g-supplement*) in F. If every subfilter of F has ample g-supplement in F, then F is called an *amply generalized supplemented* (or briefly an *amply g-supplemented*) *filter.*

Compare the next theorem with Theorem 7 in [8].

Theorem 3.19. *Assume that U_1 and U_2 are subfilters of a filter F of L and let $F = T(U_1 \cup U_2)$. If U_1 and U_2 have ample g-supplements in F, then $U_1 \cap U_2$ has also ample g-supplements in F.*

Proof. Let H be a subfilter of F such that $F = T(H \cup (U_1 \cap U_2))$. Suppose now that $U_1 \cap U_2 = A$ and $U_1 \cap H = B$. Then by Lemma 2.1, $U_1 \cap U_2 \subseteq U_1$ gives

$$U_1 = U_1 \cap T(H \cup (U_1 \cap U_2)) = T((U_1 \cap U_2) \cup (U_1 \cap H)) = T(A \cup B),$$

which implies that $F = T(U_1 \cup U_2) = T(T(A \cup B) \cup U_2) = T(B \cup T(A \cup U_2)) = T(B \cup U_2) = T(U_2 \cup (U_1 \cap H))$. Similarly, $F = T(U_1 \cup (U_2 \cap H))$. Therefore there is a supplement H_2' of U_1 in F with $H_2' \subseteq U_2 \cap H$ and a supplement H_1' of U_2 in F with $H_1' \subseteq U_1 \cap H$ which implies that $T(H_1' \cup H_2') \subseteq T(H \cap (U_1 \cup U_2)) \subseteq H$. So $T(H_2' \cup U_1) = F$, $H_2' \cap U_1 \ll H_2'$, $T(H_1' \cup U_2) = F$ and $H_1' \cap U_2 \ll H_1'$. By Lemma 2.1, $U_1 = U_1 \cap T(H_1' \cup U_2) = T(H_1' \cup (U_1 \cap U_2))$; hence

$$F = T(U_1 \cup H_2') = T(H_2' \cup T(H_1' \cup (U_1 \cap U_2))) = T(T(H_1' \cup H_2') \cup (U_1 \cap U_2)).$$

By the modular law, $T(H_1' \cup H_2') \cap (U_1 \cap U_2) = T(H_1' \cup (H_2' \cap U_1)) \cap U_2 = T((H_2' \cap U_1) \cup (U_2 \cap H_1'))$. Now by Lemma 2.5 (4), $T(H_1' \cup H_2') \cap (U_1 \cap U_2) \ll_g T(H_1' \cup H_2')$. □

Theorem 3.20. *Let F be a filter of L. If every subfilter of F is a g-supplemented filter, then F is an amply g-supplemented filter.*

Proof. Let U and V be subfilters of F such that $F = T(U \cup V)$. By assumption, there exists a subfilter V' of V such that $V = T(V' \cup (V \cap U))$ and $(U \cap V) \cap V' = V' \cap U \ll_g V'$. Then $V = T(V' \cup (V \cap U)) \subseteq T(V' \cup U)$ gives $F = T(U \cup V) \subseteq T(U \cup T(V' \cup U)) = T(V' \cup U) \subseteq F$; hence $F = T(V' \cup U)$. □

Corollary 3.21. *The following statements are equivalent for a lattice L.*

(1) *Every filter is amply g-supplemented.*

(2) *Every filter is g-supplemented.*

Proof. (1) $\Rightarrow$ (2): Clearly, if a filter F is amply g-supplemented, then F is g-supplemented.

(2) $\Rightarrow$ (1): Follows from Theorem 3.20. □

4. Generalized supplemented quotient filters

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(L, \leqslant)$, we define a relation on L given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on L, and we denote the equivalence class of a by $a \wedge F$ and the collection of all equivalence classes by L/F. We set up a partial order $\leqslant_Q$ on L/F as follows: for every $a \wedge F, b \wedge F \in L/F$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. It is straightforward to check that $(L/F, \leqslant_Q)$ is a poset. The notation below (Lemma 4.1) will be kept in this section.

Lemma 4.1. *$(L/F, \leqslant_Q)$ is a lattice.*

Proof. Let $a \wedge F, b \wedge F \in L/F$ and set $X = \{a \wedge F, b \wedge F\}$. By definition of $\leqslant_Q$, $(a \vee b) \wedge F$ is an upper bound for the set X. If $c \wedge F$ is any upper bound of X, then we can easily show that $(a \vee b) \wedge F \leqslant_Q c \wedge F$. Thus $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$. $\square$

Remark 4.2. Let F be a filter of L.

(1) If $a \in F$, then $a \wedge F = F$. By the definition of $\leqslant_Q$, it is easy to see that $1 \wedge F = F$ is the greatest element of L/F.

(2) If $a \in F$, then $a \wedge F = b \wedge F$ (for every $b \in L$) if and only if $b \in F$. In particular, $c \wedge F = F$ if and only if $c \in F$. Moreover, if $a \in F$, then $a \wedge F = F = 1 \wedge F$.

(3) By the definition of $\leqslant_Q$, we can easily show that if L is distributive, then L/F is distributive.

Lemma 4.3. *Let G be a filter of L. Then the following statements hold:*

(1) *If $G \subseteq F$ is a filter of L, then $F/G = \{a \wedge G\colon a \in F\}$ is a filter of L/G.*

(2) *If K is a filter of L/G, then $K = F/G$ for some filter F of L.*

(3) *If F and H are filters of L such that $G \subseteq F$, $G \subseteq H$ and $F/G = H/G$, then $F = G$.*

(4) *If F, H and V are filters of L containing G, then $F/G \cap H/G = V/G$ if and only if $V = H \cap F$.*

(5) *If U, V are filters of L containing K, then $T(U \cup V)/K = T(U/K \cup V/K)$.*

(6) *Let H be a subfilter of F with $G \subseteq H$. If H is a maximal subfilter of F, then H/G is a maximal subfilter of F/G.*

Proof. (1) Since $1 \wedge G \in F/G$, then $F/G \neq \emptyset$. Let $a \wedge G, b \wedge G \in F/G$ (so $a, b \in F$) and $c \wedge G \in L/G$. Then $(a \wedge G) \wedge_Q (b \wedge G) = (a \wedge b) \wedge G \in F/G$ and $(a \wedge G) \vee_Q (c \wedge G) = (a \vee c) \wedge G \in F/G$ by Proposition 1.1. Thus F/G is a filter of L/G.

(2) Assume that $F = \{x \in L\colon x \wedge G \in K\}$ and let $g \in G$. Then by Remark 4.2, $g \wedge G = 1 \wedge G = G \in K$; so $G \subseteq F$. It is easy to see that F is a filter of L with $K = F/G$.

(3) If $x \in F$, then $x \wedge G = y \wedge G$ for some $y \in H$ which implies that $x \sim y$. Then $x \wedge c = y \wedge d$ for some $c, d \in G$. Since H is a filter and $x \wedge c \in H$, we get $x \in H$ by Proposition 1.1. So $F \subseteq H$. Similarly, $H \subseteq F$, and so we have equality.

(4) Let $x \in H \cap F$. Then $x \wedge G \in (F/G) \cap (H/G) = V/G$; so $x \wedge G = z \wedge G$ for some $z \in V$ which implies that $x \wedge a = z \wedge b$ for some $a, b \in G$. Now $x \wedge a \in V$ gives $x \in V$. Thus $H \cap F \subseteq V$. Similarly, $V \subseteq F \cap H$. The other implication is similar.

(5) Let $x \wedge K \in T(U/K \cup V/K)$. Then there are elements $u \in U$ and $v \in V$ such that $(u \wedge K) \wedge_Q (v \wedge K) \leqslant_Q x \wedge K$; so $u \wedge v \leqslant x$, which implies that $x = x \vee x \vee (u \wedge v) = x \vee ((x \vee u) \wedge (x \vee v))$. Then $(u \vee x) \wedge (v \vee x) \leqslant x$ gives $x \in T(U \cup V)$ and so $x \wedge K \in T(U \cup V)/K$. Thus $T(U/K \cup V/K) \subseteq T(U \cup V)/K$. The proof of the reverse inclusion is similar.

(6) If $H/G \subsetneq K/G \subseteq F/G$, then $H \subsetneq K \subseteq F$ gives $K = F$, as needed. $\square$

Lemma 4.4. *Let F be a filter of L. The following statements hold:*
(1) *Let K, H be subfilters of F with $K \subseteq H$. If $H/K \trianglelefteq F/K$, then $H \trianglelefteq F$.*
(2) *Let K, H be subfilters of F with $K \subseteq H$. If $H \ll F$, then $H/K \ll F/K$.*
(3) *Let K, H be subfilters of F with $K \subseteq H$. If $H \ll_g F$, then $H/K \ll_g F/K$.*
(4) *If K, H are subfilters of F with $H \ll F$, then $T(H \cup K)/K \ll F/K$.*

P r o o f. (1) is clear. To see (2), let $F/K = T(H/K \cup G/K)$ for some filter G/K of F/K; so $T(H \cup G)/K = F/K$ gives $T(H \cup G) = F$ by Lemma 4.3. Hence $G = F$ since $H \ll F$, as needed.

(3) Follows from (1) and (2).

(4) Assume that $A = T(H \cup K)$ and let $F/K = T(A/K \cup G/K) = T(A \cup G)/K$ for some subfilter G/K of F/K; so $F = T(T(H \cup K) \cup G) = T(H \cup T(K \cup G)) = T(H \cup G)$ by Lemma 4.3. Then $H \ll F$ gives $G = F$. $\square$

Compare the next proposition with 41.1 (7) in [11].

Proposition 4.5. *Let X, U be subfilters of a filter F of L with $X \subseteq U$. If V is a g-supplement of U in F, then $T(X \cup V)/X$ is a g-supplement of U/X in F/X.*

P r o o f. If $A = T(V \cup X)$, then

$$T(A \cup U) = T(U \cup T(V \cup X)) = T(V \cup T(U \cup X)) = T(U \cup V) = F$$

by Lemma 2.1. Now Lemma 4.3 gives $T(U/X \cup A/X) = T(U \cup A)/X = F/X$. For $X \subseteq U$, we have $U \cap T(X \cup V) = T(X \cup (U \cap V))$ by the modular law, and so $(U/X) \cap T(V \cup X)/X = T((U \cap V) \cup X)/X$ by Lemma 4.3. Since V is a g-supplement of U in F, we have $D = U \cap V \ll_g V$. By the above consideration, it is enough to show that $B = T(D \cup X)/X \ll_g A/X$. Let $T(B \cup K/X) = A/X$ for some $K/X \trianglelefteq A/X$ (so $K \trianglelefteq T(V \cup X) = A$). Then

$$A = T(V \cup X) = T(K \cup T(X \cup (U \cap V))) = T((U \cap V) \cup T(K \cup X)) = T(K \cup (U \cap V)).$$

Since $U \cap V \ll_g V \subseteq T(V \cup X)$, we get $U \cap V \ll_g T(V \cup X)$ by Lemma 2.5; hence $K = T(V \cup X)$, as required. $\square$

Theorem 4.6. *If F is a g-supplemented filter of L, then every quotient filter of F is g-supplemented.*

Proof. Clear from Proposition 4.5. $\square$

Theorem 4.7. *If F is an amply g-supplemented filter of L, then every quotient filter of F is amply g-supplemented.*

Proof. Let V/X be a subfilter of F/X such that $F/X = T(V/X \cup U/X)$ for some subfilter U/X of F/X. Then Lemma 4.3 gives $F = T(V \cup U)$. Since F is amply g-supplemented, there is a subfilter $H \subseteq U$ such that H is a g-supplement of V in F. Then by Proposition 4.5, $T(H \cup X)/X \subseteq U/X$ is a g-supplement of V/X in F/X. Thus F/X is amply g-supplemented. $\square$

Compare the next theorem with 41.2 (3) (ii) in [11].

Theorem 4.8. *If F is a g-supplemented filter of L, then $F/\operatorname{Rad}_g(F)$ is a semi-simple filter.*

Proof. Let G be any subfilter of F containing $\operatorname{Rad}_g(F)$. Then there is a supplement H of G in F; so $T(G \cup H) = F$ and $H \cap G \ll_g H$; so $G \cap H \ll_g F$ by Lemma 2.5. If K is a generalized maximal subfilter of F and $H \cap G \nsubseteq K$, then $T((H \cap G) \cup K) = F$; but since $H \cap G \ll_g F$, we have $K = F$, which is a contradiction. Therefore, $H \cap G$ is contained in every generalized maximal subfilter of F and hence $H \cap G \subseteq \operatorname{Rad}_g(F)$. Then $F = T(\operatorname{Rad}_g(F) \cup H \cup G) \subseteq T(G \cup T(\operatorname{Rad}_g(F) \cup H)) \subseteq F$ which implies that $F = T(G \cup T(\operatorname{Rad}_g(F) \cup H))$. Set $T(\operatorname{Rad}_g(F) \cup H) = A$. Thus by Lemma 4.3,

$$\frac{F}{\operatorname{Rad}_g(F)} = \frac{T(G \cup A)}{\operatorname{Rad}_g(F)} = T\Big(\frac{G}{\operatorname{Rad}_g(F)} \cup \frac{A}{\operatorname{Rad}_g(F)}\Big).$$

It suffices to show that $G/\operatorname{Rad}_g(F)\cap A/\operatorname{Rad}_g(F)=\{\bar{1}\}$, where $\bar{1}=1\wedge\operatorname{Rad}_g(F)=\operatorname{Rad}_g(F)$ is the greatest element of $L/\operatorname{Rad}_g(F)$. By the modular law and Lemma 4.3, we have

$$\begin{aligned}\frac{G}{\operatorname{Rad}_g(F)}\cap\frac{A}{\operatorname{Rad}_g(F)}&=\frac{G\cap A}{\operatorname{Rad}_g(F)}=\frac{T(\operatorname{Rad}_g(F)\cup(G\cap H))}{\operatorname{Rad}_g(F)}\\&=\frac{T(\operatorname{Rad}_g(F))}{\operatorname{Rad}_g(F)}=\frac{\operatorname{Rad}_g(F)}{\operatorname{Rad}_g(F)}=\{\bar{1}\}.\end{aligned}$$

□

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