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A GEOMETRIC CONSTRUCTION FOR SPECTRALLY ARBITRARY SIGN PATTERN MATRICES AND THE $2n$-CONJECTURE

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Abstract. We develop a geometric method for studying the spectral arbitrariness of a given sign pattern matrix. The method also provides a computational way of computing matrix realizations for a given characteristic polynomial. We also provide a partial answer to $2n$-conjecture. We determine that the $2n$-conjecture holds for the class of spectrally arbitrary patterns that have a column or row with at least $n-1$ nonzero entries.

Keywords: spectrally arbitrary sign pattern; $2n$-conjecture

MSC 2020: 15B35

1. Introduction

A *sign pattern matrix* of order n is an $n \times n$ matrix whose entries belong to the sign set $\{+,-,0\}$. A *qualitative class* of a sign pattern matrix S is denoted by $Q(S)$ and is defined as

$$Q(S) := \{A = (a_{ij}) \in M_n(\mathcal{R})\colon\ \operatorname{sign} a_{ij} = s_{ij} \text{ for all } i, j\}.$$

If there exists a matrix in the qualitative class of a given sign pattern matrix S such that it satisfies a given property P of a matrix then property P is said to be an *allowable* property. If every matrix in the qualitative class of S satisfies a property P then sign pattern S *requires* a property P.

A *permutation pattern* is a square sign pattern matrix with entries 0 and $+$, where the entry $+$ occurs precisely once in each row and in each column. A square sign pattern S is said to be a *signature sign pattern* if diagonal entries in S are either $+$ or $-$ and off-diagonal entries are 0. Two $n \times n$ sign patterns S_1 and S_2 are said to be *permutationally similar* if there exists a permutation matrix P such

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that $S_1 = P^\top S_2 P$. Two $n \times n$ sign patterns S_1 and S_2 are said to be *signature similar* if there exists a signature pattern matrix S such that $S_1 = SS_2S$. Two sign patterns are said to be *equivalent* if one of them can be obtained from other by using any combination of transposition, permutation similarity and signature similarity.

A study of sign pattern matrices was initiated by Samuelson. In 1947, Samuelson encountered matrices, where rather than entries, only sign of the entries is known. Such matrices are called the *sign pattern matrices.* In the year 1965, two economist Quirk and Rupert wrote a paper entitled "Qualitative economics and the stability of equilibrium" which demands further understanding of sign pattern matrices, see [9]. Since then, several researchers have come up with several ideas related to sign pattern matrices. A significantly studied concept for sign pattern matrices is spectrally arbitrariness, as we can see in [1], [3], [4], [5], [8]. Most of the studies also dealt with inertially arbitrary sign patterns, potential stability and sign stability of sign patterns.

In this paper, we are interested in studying spectral arbitrariness of sign patterns form a geometric point of view. In Section 2 we have developed a geometric method for studying spectral properties of a sign pattern matrix.

2. Spectrally arbitrary sign pattern matrices

Definition 2.1. A sign pattern matrix S of order $n \times n$ is said to be *spectrally arbitrary* if every monic polynomial of degree n is a characteristic polynomial of some matrix A in the qualitative class of S.

We assign a one-to-one correspondence between vectors in $\mathbb{R}^n$ and the coefficients of the characteristic polynomials for an $n \times n$ matrix in $M_n(\mathbb{R})$. For a vector $v = (a_1, a_2, \ldots, a_{n-1}, a_n)$ there is a characteristic polynomial $x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n$ for some square matrix A of order n.

Definition 2.2. Let S be a sign pattern matrix of order n. For a given vector $v = (a_1, a_2, \ldots, a_{n-1}, a_n)$ in $\mathbb{R}^n$ if there exists a matrix $A \in Q(S)$ whose characteristic polynomial is $x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n$, then A is called a *realization* of vector v.

Here we discuss about two open questions, Question 2.3 as given in [4] and Question 2.4 which has not directly appeared in literature before.

Question 2.3. How can we identify if a sign pattern S is spectrally arbitrary?

Question 2.4. If a sign pattern S is spectrally arbitrary, then, for any given monic polynomial with appropriate order, how do we find a matrix in the qualitative class of S whose characteristic polynomial is the given monic polynomial?

Example 2.5. In $\mathbb{R}^2$, $e_1 = (1,0)$, $e_2 = (0,1)$, $-e_1 = (-1,0)$ and $-e_2 = (0,-1)$ are the special vectors. From [4], 2×2 sign pattern $S = \begin{pmatrix} - & + \\ - & + \end{pmatrix}$ is spectrally arbitrary. The characteristic polynomials corresponding to e_1, e_2, $-e_1$, $-e_2$ are, respectively, $x^2 - 1x$, x^2+1, x^2+1x, x^2-1. As S is spectrally arbitrary, there exist matrices

$$A = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} -2 & 2 \\ -3/2 & 2 \end{pmatrix}$$

in $Q(S)$ whose characteristic polynomials are, respectively,

$$x^2 - 1x \quad x^2+1 \quad x^2+1x \quad x^2-1.$$

Now we seek to find a matrix in $Q(S)$ whose characteristic polynomial is x^2-5x+6 which corresponds to a vector $(5,6)$ in the first quadrant. Consider the curve $(5t, 6t^2)$ for $0 \leqslant t \leqslant 1$, which passes through the origin and $(5,6)$. Also consider the line segment $(1-s, s)$ for $0 \leqslant s \leqslant 1$ joining $(1,0)$ and $(0,1)$. These two curves intersect each other exactly at one point, as shown in Figure 1. Solving $(5t, 6t^2)$ with $x+y=1$ simultaneously we get $t = 1/6$.

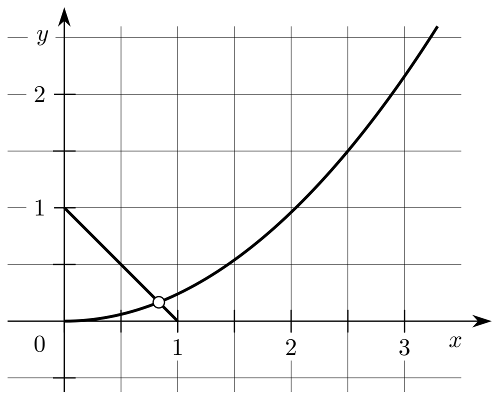

Figure 1. Intersection of the curve $(5t, 6t^2)$ for $0 \leqslant t \leqslant 1$ and the line $x+y=1$.

Equating the points $(5t, 6t^2)$ and $(1-s, s)$ for $t = 1/6$, we get $s = 1/6$. The matrix $E(s) = (1-s)A + sB = \begin{pmatrix} -1 & 2 \\ -1 & 2-s \end{pmatrix} \in Q(S)$ gives a realization for any vector $(1-s, s)$ with $0 \leqslant s \leqslant 1$, where $(1,0)$ corresponds to a matrix A and $(0,1)$ corresponds to a matrix B.

Setting $s = 1/6$, we get $E(1/6) = \begin{pmatrix} -1 & 2 \\ -1 & 11/6 \end{pmatrix} \in Q(S)$. Note that $E(1/6)$ is a matrix realization for a vector $(5/6, 1/6)$. The vector $(5/6, 1/6)$ is obtained by substituting $t = 1/6$ in $(5t, 6t^2)$.

Now we need a matrix realization for the vector $(5,6)$. Observe that vector $(5,6)$ corresponds to $t = 1$ in $(5t, 6t^2)$. Thus, the required matrix realization for the vector $(5,6)$ is $(1/t) \times E(1/6) = 6 \times E(1/6) = \begin{pmatrix} -6 & 12 \\ -6 & 11 \end{pmatrix}$ whose characteristic polynomial is $x^2 - 6 \times 5x/6 + 36 \times 1/6 = x^2 - 5x + 6$.

Thus, for any nonzero vector in the first quadrant, we can always find a matrix realization in the qualitative class of S realizing the vector. Now, for vectors in the second quadrant, we can make use of vectors $(0,1)$, $(-1,0)$ with matrices B and C, i.e., line segment $(-s, 1-s)$ for $0 \leqslant s \leqslant 1$ with $(1/t)[(1-s)B + sC]$. For vectors in the third quadrant we use line segment $(s-1, -s)$ for $0 \leqslant s \leqslant 1$ with $(1/t)[(1-s)C + sD]$ and for the fourth quadrant we can use line segment $(s, s-1)$ for $0 \leqslant s \leqslant 1$ with $(1/t)[(1-s)D + sA]$. So every nonzero vector in $\mathbb{R}^2$ can be realized by a matrix in $Q(S)$. However, this process can be extended to $\mathbb{R}^3$ and in general for $\mathbb{R}^n$.

Example 2.6. In $\mathbb{R}^3$, $\pm e_1 = (\pm 1, 0, 0)$, $\pm e_2 = (0, \pm 1, 0)$, $\pm e_3 = (0, 0, \pm 1)$ are the special vectors. From [4], a sign pattern $S = \begin{pmatrix} + & + & - \\ + & 0 & - \\ + & 0 & - \end{pmatrix}$ is spectrally arbitrary. It is easy to verify that for $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$, a matrix realization is $A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & -6 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix}$, respectively. Now, the affine linear combination of vectors e_1, e_2, e_3 will be given by $\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1)$ for $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geqslant 0$. The same can also be parametrized by $(u, v, 1-u-v)$ for $0 \leqslant u, v \leqslant 1$ and Cartesian equation of the corresponding plane is $x + y + z = 1$. If we take a linear combination $uA + vB + (1-u-v)C$, we get the matrix $A(u,v) = \begin{pmatrix} 2 & 1 & 2u - v - 5 \\ 1 & 0 & -v - 1 \\ 1 & 0 & u - 2 \end{pmatrix} \in Q(S)$ as $0 \leqslant u, v \leqslant 1$. Thus, $A(u,v)$ is a realization for a vector $(u, v, 1-u-v)$ for $0 \leqslant u, v \leqslant 1$.

Now consider the curve $(t, t^2, 2t^3)$ for $t \geqslant 0$, which passes through the origin and the point $(1,1,2)$. However, this curve only intersects the plane $x+y+z = 1$ once. Solving the curve $(t, t^2, 2t^3)$ with $x + y + z = 1$ for t, we get $t = 0.5$. Hence, the point on the plane is $(0.5, 0.25, 0.25)$ and its matrix realization is $A(0.5, 0.25) = \begin{pmatrix} 2 & 1 & -4.25 \\ 1 & 0 & -1.25 \\ 1 & 0 & -1.5 \end{pmatrix}$. Observe that the point $(t, t^2, 2t^3)$ corresponds to $(0.5, 0.25, 0.25)$ for $t = 0.5$ and corresponds to $(1,1,2)$ for $t = 1$. Hence, the required matrix realization for $(1,1,2)$ is $(1/t)A(u,v) = (1/0.5)A(0.5, 0.25) = \begin{pmatrix} 4 & 2 & -8.5 \\ 2 & 0 & -2.5 \\ 2 & 0 & -3 \end{pmatrix}$.

We have given matrix realizations for the octants surrounded by the unit vectors $\pm e_1$, $\pm e_2$ and $\pm e_3$ for the sign pattern matrix S discussed in Example 2.2 as displayed in the Appendix.

Construction 2.7. A method for finding matrix realizations for a given characteristic polynomial of a spectrally arbitrary sign pattern.

(1) Find matrix realizations for vectors $\pm e_1, \pm e_2, \ldots, \pm e_n$ with the property that matrices surrounding one hyperoctant can be allowed to differ only in one row (or column).

(2) Consider a hyperoctant H surrounded by arbitrary vectors $v_1, v_2, \ldots, v_n$, where $\{v_1, v_2, \ldots, v_n\} \subseteq \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ and also consider the convex linear combination of these vectors $x_1 v_1 + x_2 v_2 + \ldots + (1 - x_1 - \ldots - x_{n-1}) v_n$, where $0 \leqslant x_i \leqslant 1$ for all $1 \leqslant i \leqslant n-1$.

(3) To identify a matrix realization for a nonzero vector $v = (a_1, a_2, \ldots, a_n)$ lying in the hyperoctant surrounded by $v_1, v_2, \ldots, v_n$: Consider the curve $(ta_1, t^2 a_2, \ldots, t^n a_n)$ for $t \geqslant 0$ which passes through the origin and the vector v. Find the point of intersection t_0 of the curve and the hyperplane spanned by vectors $v_1, v_2, \ldots, v_n$. It should be noted that the curve and the plane intersect only in one point.

(4) Equating $(ta_1, t^2 a_2, \ldots, t^n a_n) = x_1 v_1 + x_2 v_2 + \ldots + (1 - x_1 - \ldots - x_{n-1}) v_n$ for particular $t = t_0$ to obtain $x_1, x_2, \ldots, x_{n-1}$.

(5) Using an affine linear combination of matrices corresponding to the standard unit vectors $v_1, v_2, \ldots, v_n$ surrounding the hyperoctant H, obtain the matrix realization A for $x_1 v_1 + x_2 v_2 + \ldots + (1 - x_1 - \ldots - x_{n-1}) v_n$.

(6) Finally, compute $(1/t_0)A$, a required matrix realization for the vector $v = (a_1, a_2, \ldots, a_n)$.

In view of a geometric construction for spectrally arbitrary sign pattern matrices as described above, we prove our main Theorem 2.10 of this paper. Throughout, the characteristic polynomial of a square matrix A is denoted by $ch(A)$.

Lemma 2.8. *Let A and B be two $n \times n$ matrices such that they differ only in a row (or column). Then*

$$ch(sA + (1-s)B) = sch(A) + (1-s)ch(B) \quad \text{for } 0 \leqslant s \leqslant 1.$$

P r o o f. Without loss of generality, assume that A and B differ only in the ith row. Let ith row of A and B be $(a_{i1}, a_{i2}, \ldots, a_{in})$ and $(b_{i1}, b_{i2}, \ldots, b_{in})$, respectively. Now the characteristic polynomial of $sA + (1-s)B$ is $\det(s(xI - A) + (1-s)(xI - B))$. By hypothesis, A and B vary in the ith row, hence, $(xI - A)$ and $(xI - B)$ will also vary only in the ith row. Thus, we get

$$\begin{aligned}
&s(xI-A)+(1-s)(xI-B)\\
&= s\begin{pmatrix} x-a_{11} & -a_{12} & \dots & -a_{1n}\\ \vdots & \vdots & \vdots & \vdots\\ -a_{i1} & -a_{i2} & \dots & -a_{in}\\ \vdots & \vdots & \vdots & \vdots\\ -a_{n1} & -a_{n2} & \dots & x-a_{nn}\end{pmatrix} + (1-s)\begin{pmatrix} x-a_{11} & -a_{12} & \dots & -a_{1n}\\ \vdots & \vdots & \vdots & \vdots\\ -b_{i1} & -b_{i2} & \dots & -b_{in}\\ \vdots & \vdots & \vdots & \vdots\\ -a_{n1} & -a_{n2} & \dots & x-a_{nn}\end{pmatrix}\\
&= \begin{pmatrix} x-a_{11} & -a_{12} & \dots & -a_{1n}\\ \vdots & \vdots & \vdots & \vdots\\ -sa_{i1}-(1-s)b_{i1} & -sa_{i2}-(1-s)b_{i2} & \dots & -sa_{in}-(1-s)b_{in}\\ \vdots & \vdots & \vdots & \vdots\\ -a_{n1} & -a_{n2} & \dots & x-a_{nn}\end{pmatrix}.
\end{aligned}$$

By splitting the determinant along the ith row, we get

$$ch(sA+(1-s)B)=sch(A)+(1-s)ch(B).$$

□

Remark 2.9. We can extend the above lemma naturally to an affine linear combination of n matrices.

As our construction does not take care of the characteristic polynomial x^n, we prove the following for potentially nilpotent sign patterns of order n.

Theorem 2.10. *Let S be a potentially nilpotent sign pattern matrix of order n and let $\pm e_1, \pm e_2, \dots, \pm e_n$ be the unit vectors along the axis. Suppose there exist at least $2n$ matrices which are a realization of these $2n$ vectors corresponding to a sign pattern S. If n matrices corresponding to n vectors surrounding each hyperoctant differ only in one fixed row (or column), then the sign pattern S is spectrally arbitrary. Moreover, any particular non-nilpotent matrix realization can be constructed as an affine combination of the matrices corresponding to a hyperoctant (i.e., the unit vectors).*

Proof. Suppose $p(x)=x^n-a_1x^{n-1}+\dots+(-1)^{n-1}a_{n-1}x+(-1)^na_n$ is a monic polynomial of degree n. Our aim is to find a matrix in $Q(S)$ whose characteristic polynomial is the monic polynomial $p(x)$. By Definition 2.2, the polynomial $p(x)$ corresponds to a vector $v=(a_1,a_2,\dots,a_n)$ in $\mathbb{R}^n$. Assume that this vector belongs to the hyperoctant surrounded by the unit vectors $v_1,v_2,\dots,v_n$, where $\{v_1,v_2,\dots,v_n\}\subseteq\{\pm e_1,\pm e_2,\dots,\pm e_n\}$. Consider the curve $(ta_1,t^2a_2,\dots,t^na_n)$ for $t\geqslant 0$ and the hyperplane say H spanned by the vectors $v_1,v_2,\dots,v_n$. The curve and the hyperplane H intersects only at one point; let it be $t = t_0$ (it should be noted that the value of t is always positive even though signs of v_i's are arbitrary). By equating the point $(t_0a_1, t_0^2a_2, \ldots, t_0^na_n)$ with $x_1v_1 + x_2v_2 + \ldots + (1 - x_1 - x_2 - \ldots - x_n)v_n$, where $0 \leqslant x_i \leqslant 1$, compute the coefficients $x_1, x_2, \ldots, x_{n-1}$. Now by hypothesis we have n matrices, say $A_1, A_2, \ldots, A_n$ in $Q(S)$, realizations for the unit vectors $v_1, v_2, \ldots, v_n$, respectively. Also, all these matrices vary either in a row or column. Consider affine linear combinations of all these n matrices with the above computed coefficients $x_1, x_2, \ldots, x_{n-1}$, i.e., $E(x_1, x_2, \ldots, x_{n-1}) = x_1A_1 + x_2A_2 + \ldots + (1-x_1-x_2-\ldots-x_n)A_n$. As all the coefficients are nonnegative, $E(x_1, x_2, \ldots, x_{n-1})$ belongs to $Q(S)$. In view of Lemma 2.8, it is clear that characteristic polynomial of $E(x_1, x_2, \ldots, x_{n-1})$ is $x_1ch(A_1) + x_2ch(A_2) + \ldots + (1 - x_1 - x_2 - \ldots - x_{n-1})ch(A_n)$. The vector corresponding to this characteristic polynomial is $x_1v_1 + x_2v_2 + \ldots + (1 - x_1 - x_2 - \ldots - x_n)v_n = (t_0a_1, t_0^2a_2, \ldots, t_0^na_n)$. Hence, the characteristic polynomial of $(1/t_0)E(x_1, x_2, \ldots, x_{n-1})$ corresponds to the vector $v = (a_1, a_2, \ldots, a_n)$. Observe that $(1/t_0)E(x_1, x_2, \ldots, x_{n-1})$ belongs to the qualitative class of S as $t_0 > 0$. Hence, for any vector $v = (a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n \setminus \{(0, 0, \ldots, 0)\}$, we can find a matrix in a qualitative class of a sign pattern matrix S whose realization is v. This proves that S is a spectrally arbitrary sign pattern matrix and any particular non-nilpotent matrix realization can be constructed as an affine combination. $\square$

Finally, we provide an application of Theorem 2.10.

Example 2.11. Consider an $n \times n$ Hessenberg sign pattern for $n \geqslant 3$ with first column positive, the superdiagonal and (n, n) entry negative and zeros elsewhere.

$$V_n = \begin{pmatrix} + & - & 0 & 0 & \ldots & 0 & 0 \\ + & 0 & - & 0 & \ldots & 0 & 0 \\ + & 0 & 0 & - & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ + & 0 & 0 & 0 & \ldots & 0 & - \\ + & 0 & 0 & 0 & \ldots & 0 & - \end{pmatrix}.$$

In the above sign pattern matrix V_n, if we replace the first column by $a_1, a_2, \ldots, a_n$ positive real numbers, $-$ sign by -1 and fixing its (n, n) entry by -2, then it will look like

$$V_n = \begin{pmatrix} a_1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ a_2 & 0 & -1 & 0 & \ldots & 0 & 0 \\ a_3 & 0 & 0 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 0 & 0 & 0 & \ldots & 0 & -1 \\ a_n & 0 & 0 & 0 & \ldots & 0 & -2 \end{pmatrix}.$$

It should be noted that from [2], the trace of V_n is $\sum\limits_1 = a_1 - 2$, the sum of all 2×2 principal minors is $\sum\limits_2 = a_2 - 2a_1$, the sum of all 3×3 principal minors is $\sum\limits_3 = a_3 - 2a_2$ and determinant of V_n is $\sum\limits_n = a_n - 2a_{n-1}$. By choosing $a_1 = 2, a_2 = 4, \ldots, a_n = 2^n$, we get a realization for a vector $(0, 0, \ldots, 0)$, which implies that the given sign pattern is a potentially nilpotent sign pattern. However, a realization of a vector $e_1 = (1, 0, 0, \ldots, 0)$ can be obtained by choosing $a_1 = 3, a_2 = 6, a_3 = 12, \ldots, a_n = 3\times 2^{n-1}$. For $e_2 = (0, 1, 0, 0, \ldots, 0)$, choose $a_1 = 2, a_2 = 5, a_3 = 10, \ldots, a_n = 5\times 2^{n-2}$ and so on. For $-e_n = (-1, 0, \ldots, 0)$, choose $a_1 = 1, a_2 = 2, \ldots, a_n = 2^{n-1}$. The remaining realizations can be worked out in a similar manner. Thus, for all $\pm e_i$'s , there exist matrices in a qualitative class of V_n. Observe that all these matrices differ only in the first column. Therefore, by Theorem 2.10, V_n is a spectrally arbitrary sign pattern and we can find realizations for every vector in $\mathbb{R}^n$.

Lemma 2.12. *The matrix*

$$A = \begin{pmatrix} 1-\alpha_1 & -\alpha_1 & \ldots & -\alpha_1 & -\alpha_1 \\ -\alpha_2 & 1-\alpha_2 & \ldots & -\alpha_2 & -\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_{n-1} & -\alpha_{n-1} & \ldots & 1-\alpha_{n-1} & -\alpha_{n-1} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

has rank at least $n-1$ for all $\alpha_i \in \mathbb{R}$.

Proof. The given matrix A can also be written as follows:

$$A = I - \begin{pmatrix} \alpha_1 & \alpha_1 & \ldots & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \ldots & \alpha_2 & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & \alpha_{n-1} & \ldots & \alpha_{n-1} & \alpha_{n-1} \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} = I - B.$$

Since the rank of B is at most 1, that the kernel of A has dimension at most 1. Thus, the rank of A is at least $n-1$. □

Lemma 2.13. *If S is a SAP of order n, that satisfies the conditions in Theorem* 2.10, *then there exists a column (or row) in S such that n entries in that column (or row) are nonzero.*

Proof. Suppose that S is a SAP as per Theorem 2.10. Then realizations for the unit vectors $\pm e_1, \pm e_2, \ldots, \pm e_n$ exist in the qualitative class of S. Also, any n realizations surrounding a hyperoctant of these unit vectors vary either in a row or in a column. Let $A_1, A_2, \ldots, A_n$ be the realizations for unit vectors $e_1, e_2, \ldots, e_n$. Assume that the matrices $A_1, A_2, \ldots, A_n$ vary in the jth column. Now we will prove that the sign pattern matrix S has n nonzero entries in its jth column.

Firstly, assume that a sign pattern S has m nonzero entries in its jth column. Consider a realization A of a sign pattern S obtained as follows. Replace the kth place entry in the jth column by a_k if that entry is positive and by $-a_k$ if that entry is negative. Replace other column entries in a sign pattern S by corresponding entries in A_i for any i. As $A_1, A_2, \ldots, A_n$ vary only in the jth column, the remaining columns are identical in the matrices $A_1, A_2, \ldots, A_n$. Then the characteristic polynomial of a matrix A will look like

$$ch(A) = x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots + (-1)^{n-1} p_{n-1} x + (-1)^n p_n,$$

where $p_1, p_2, \ldots, p_n$ are linear polynomials in the variables $a_1, a_2, \ldots, a_m$. Also, p_1 is the trace of matrix A, p_2 is the sum of all 2×2 principal minors and so on with p_n is the determinant of matrix A. Note that the polynomials $p_1, p_2, \ldots, p_{n-1}$ may contain some constant terms. However, p_n will not have a constant term as p_n is the determinant of matrix A. We can observe this fact by expanding the determinant of A along the jth column which contains all nonzero entries as variable. Now, since A_1 is a realization for the vector e_1, the characteristic polynomial of A_1 is $x^n - x^{n-1}$. Hence, the nonzero entries in the jth column of A_1, say $c_{i_1,1}, c_{i_2,1}, \ldots, c_{i_m,1}$ (as only m of them are nonzero) are the solutions to the system of equations $p_1(a_1, a_2, \ldots, a_m) = 1$, $p_2(a_1, a_2, \ldots, a_m) = 0, \ldots, p_n(a_1, a_2, \ldots, a_m) = 0$. Similarly A_2 is a realization for the vector e_2, and the characteristic polynomial of A_2 is $x^n + x^{n-2}$. Hence, the nonzero entries in the jth column of A_2, say $c_{i_1,2}$, $c_{i_2,2}, \ldots, c_{i_m,2}$ are the solutions to the system of equations $p_1(a_1, a_2, \ldots, a_m) = 0$, $p_2(a_1, a_2, \ldots, a_m) = 1, \ldots$, $p_n(a_1, a_2, \ldots, a_m) = 0$. Therefore, in general the nonzero entries from the jth column of A_i, i.e., $c_{i_1,i}$, $c_{i_2,i}, \ldots,$ $c_{i_m,i}$ are the solutions to the system of equations $p_1(a_1, a_2, \ldots, a_m) = 0, \ldots,$ $p_{i-1}(a_1, a_2, \ldots, a_m) = 0$, $p_i(a_1, a_2, \ldots, a_m) = 1$, $p_{i+1}(a_1, a_2, \ldots, a_m) = 0, \ldots,$ $p_n(a_1, a_2, \ldots, a_m) = 0$ for all $1 \leqslant i \leqslant n$. All these systems of equations can be represented by

$$\begin{aligned} (p_1(c_{i_1,1}, \ldots, c_{i_m,1}), p_2(c_{i_1,1}, \ldots, c_{i_m,1}), \ldots, p_n(c_{i_1,1}, \ldots, c_{i_m,1})) &= e_1, \\ (p_1(c_{i_1,2}, \ldots, c_{i_m,2}), p_2(c_{i_1,2}, \ldots, c_{i_m,2}), \ldots, p_n(c_{i_1,2}, \ldots, c_{i_m,2})) &= e_2, \\ &\vdots \\ (p_1(c_{i_1,n}, \ldots, c_{i_m,n}), p_2(c_{i_1,n}, \ldots, c_{i_m,n}), \ldots, p_n(c_{i_1,n}, \ldots, c_{i_m,n})) &= e_n. \end{aligned} \tag{2.1}$$

The systems in equation (2.1) can be written in matrix form as follows:

$$\begin{pmatrix} B & u \end{pmatrix}_{n\times(m+1)} \begin{pmatrix} C \\ J_{1,n} \end{pmatrix} = I_{n\times n}, \tag{2.2}$$

where entries in the ith row of a matrix B are the coefficients of $a_1, a_2, \ldots, a_m$ in the polynomial p_i for $1 \leqslant i \leqslant n$ and the vector u is $(\alpha_1, \alpha_2 \ldots, \alpha_{n-1}, 0)^\top$. Note that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the constant terms in the polynomials $p_1, p_2, \ldots, p_{n-1}$, respectively. Observe that the $(n, m+1)$th entry in matrix $(\, B \quad u\,)$ is zero as polynomial p_n has zero constant term. Also, matrix C is an $m \times n$ matrix, with the kth column being $(c_{i_1,k}, c_{i_2,k}, \ldots, c_{i_m,k})^\top$ for $1 \leqslant k \leqslant n$ and $J_{1,n}$ is an $1 \times n$ matrix of entries as 1. For transferring those constant terms to the right hand side, we have to subtract the column of constant terms from each column of the identity matrix and remove the last row of $J_{1,n}$ from $\begin{pmatrix} C \\ J_{1,n} \end{pmatrix}$. Then the equation (2.2) gets converted to the following form:

$$BC = \begin{pmatrix} 1-\alpha_1 & -\alpha_1 & \ldots & -\alpha_1 & -\alpha_1 \\ -\alpha_2 & 1-\alpha_2 & \ldots & -\alpha_2 & -\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_{n-1} & -\alpha_{n-1} & \ldots & 1-\alpha_{n-1} & -\alpha_{n-1} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} = D.$$

By Lemma 2.12, the rank of a matrix D is at least $n-1$. Thus, the rank of B is also at least $n-1$. Hence, we have $m \geqslant n-1$. Now if $m = n-1$, then we consider the map $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by $\varphi(x) = Bx + u$, where the matrix B and the vector u are as constructed above. By definition of φ, it is clear that it is an affine map. However, we know that the dimension of the image space under an affine map is always less than or equal to the dimension of a domain. Thus, the image space of the map φ is at most of dimension $n-1$. So $e_1, e_2, \ldots, e_n$ will not lie in the image space of φ. When we say that the realizations for the vectors $e_1, e_2, \ldots, e_n$ lie in the qualitative class of sign pattern S satisfying the hypothesis of Theorem 2.10, in particular, the inverse image of the vectors $e_1, e_2, \ldots, e_n$ under the map φ exists in $(0,\infty)^{n-1} \subset \mathbb{R}^{n-1}$, which is not possible. Hence, $m = n$ and the jth column of sign pattern S must contain n nonzero entries. Thus, if the spectrally arbitrary sign pattern S satisfies the hypothesis of Theorem 2.10, then it should contain a row or a column having exactly n nonzero entries in it. □

Remark 2.14. Lemma 2.13 says that if S is a spectrally arbitrary sign pattern matrix as per Theorem 2.10, then there exists a row (or column) in S which contains exactly n nonzero entries. This says that it is not possible to apply the Theorem 2.10 to the sign pattern matrices which have no rows (or columns) with n nonzero entries. For example the sign pattern matrix T_n which is the tridiagonal sign pattern with the superdiagonal and (n, n) entry positive and subdiagonal and $(1, 1)$ entry negative and zeros elsewhere. We note that from [7], T_n is spectrally arbitrary, but it has no rows (or columns) with n nonzero entries for $n \geqslant 3$.

Remark 2.15. Importance of the Lemma 2.13 lies in the fact that it is proof gives a partial indication for the construction of realizations for the unit vectors $\pm e_1, \pm e_2, \ldots, \pm e_n$ with the property that any n realizations surrounding a hyperoctant vary either in a row (or a column). Choose a row (or a column) in a sign pattern S which has n nonzero entries. Working out the process stated in Lemma 2.13, construct the matrices B, C and D and arrive at the equation $BC = D$. If we assume that determinant of the matrix B is nonzero and the solution matrix C has all its entries positive, then $C = B^{-1}D$ is the solution of $BC = D$. Consequently each column of C gives one realization for the matrix A. Thus, all columns of matrix C will give realizations for $e_1, e_2, \ldots, e_n$ corresponding to one of the hyperoctants. The following example gives an illustration for the construction of realizations.

Example 2.16. A sign pattern matrix $S = \begin{pmatrix} + & + & - \\ + & 0 & - \\ + & 0 & - \end{pmatrix}$, see Example 2.6. The realizations P, T and R for the vectors e_1, e_2, e_3 were

$$P = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 2 & 1 & -6 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix},$$

respectively. Let us see how to construct them. Consider a realization $A = \begin{pmatrix} 2 & 1 & -a \\ 1 & 0 & -b \\ 1 & 0 & -c \end{pmatrix} \in Q(S)$, where a, b, c are positive real numbers. The characteristic polynomial of A is

$$ch(A) = x^3 - (2 - c)x^2 + (a - 2c - 1)x - (c - b).$$

Here we have $p_1(a, b, c) = 2 - c$, $p_2(a, b, c) = a - 2c - 1$ and $p_3(a, b, c) = c - b$. Now we seek realization of vector $e_1 = (1, 0, 0)$ that means we want a, b, c such that $p_1(a^1, b^1, c^1) = 2 - c^1 = 1$, $p_2(a^1, b^1, c^1) = a^1 - 2c^1 - 1 = 0$, $p_3(a^1, b^1, c^1) = c^1 - b^1 = 0$. It means

$$-c^1 = -1, \quad a^1 - 2c^1 = 1, \quad c^1 - b^1 = 0 \tag{2.3}$$

which is a system of linear equations. Similarly, we seek a realization of a vector $e_2 = (0, 1, 0)$, i.e., $p_1(a^2, b^2, c^2) = 2 - c^2 = 0$, $p_2(a^2, b^2, c^2) = a^2 - 2c^2 - 1 = 1$, $p_3(a^2, b^2, c^2) = c^2 - b^2 = 0$. This gives the following system of linear equations:

$$-c^2 = -2, \quad a^2 - 2c^2 = 2, \quad c^2 - b^2 = 0. \tag{2.4}$$

Similarly, one can get the following system of linear equations while seeking a realization of a vector $e_3 = (0, 0, 1)$:

$$-c^3 = -2, \quad a^3 - 2c^3 = 1, \quad c^3 - b^3 = 1. \tag{2.5}$$

By considering all these three systems (2.3), (2.4) and (2.5) together, we get

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The above system resembles to $BC = D$ in Lemma 2.13. By solving the above system we get

$$\begin{pmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 5 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$$

The first column of the resulting matrix gives the values of $a^1 = 3$, $b^1 = 1$, $c^1 = 1$, which gives a realization for vector e_1 as $P = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$. Similarly, other realizations exist as $T = \begin{pmatrix} 2 & 1 & -6 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, $R = \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix}$.

Note that the realization matrix A considered in the above Example 2.16 is constructed such that the Jacobian matrix B is invertible and final solution matrix C has each of its entries as positive real numbers.

3. On $2n$-conjecture

A matrix A is said to be *reducible* if there exists a permutation matrix P such that

$$P^\top AP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where A_1 and A_3 are square matrices of order at least one and 0 is the zero matrix. If A is not reducible, then we say A is *irreducible* matrix.

The following conjecture was verified for $n = 2, 3, 4$ and 5, see for example [4]. Recently it was also settled for values of $n = 6, 7$, see for example [6]. Here we provide a partial answer to the $2n$ conjecture for $n > 3$.

Conjecture 3.1 ([2]). *For $n \geqslant 2$, an $n \times n$ irreducible sign pattern that is spectrally arbitrary has at least $2n$ nonzero entries.*

Theorem 3.2. *Let S be an irreducible spectrally arbitrary sign pattern of order $n > 3$. If the sign pattern S has one column (or row) with at least $n-1$ nonzero entries, then it has at least $2n$ nonzero entries.*

Proof. Let S be an irreducible sign pattern of order $n > 3$. From [2], a spectrally arbitrary sign pattern has at least $2n-1$ nonzero entries. Assume that S has exactly $2n-1$ nonzero entries. By hypothesis, S is spectrally arbitrary and S has one column (or row) having at least $n-1$ nonzero entries. This means that S has one column (or row) with exactly $n-1$ nonzero entries or S has one column (or row) with n nonzero entries.

Case A: Assume that S has one column (or row) with exactly $n-1$ nonzero entries. By using permutation similarity, we can exchange this column (or row) with the last column (or the first row). Without loss of generality, assume that S has the last column with $n-1$ nonzero entries. Out of the remaining n entries, $n-1$ entries must lie with one in each first $n-1$ columns; otherwise the determinant of S will be zero. Thus, exactly one column can have two nonzero entries in it for the first $n-1$ columns. Also, one nonzero entry must lie in a row whose nth column entry is zero to avoid a zero row in S. If we have more than one row containing only zeroes except their nth column entries, then by expanding the determinant of S along the nth column, the determinant will be zero, hence not a SAP. Thus, at most one zero row will be allowed except the last column entry in that row. We conclude that S has no zero rows except the last column entry or S has one zero row and one row with exactly two nonzero entries except the last column entry in those rows.

Now, as per the position of the zero in the nth column, we have the following two cases.

Subcase A1: Assume the (n,n)th entry in S is zero. In this case there exist at least two nonzero entries along the diagonal such that the trace is not fixed. Let the (i,i)th and the (j,j)th entries for $i \neq j$ be nonzero. If the ith column has only one nonzero entry, then by permutation similarity, swap the 1st and the ith row and column with each other so that S will look like the following:

$$\left(\begin{array}{c|cccc} \times & \# & \dots & \# & \times \\ \hline 0 & \# & \dots & \# & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \# & \dots & \# & \times \\ 0 & \# & \dots & \# & 0 \end{array}\right).$$

We observe S is an reducible sign pattern contradicting the hypothesis.

Subcase A2: Assume the (i, n)th entry is zero in S for $i \neq n$. By using permutation similarity, swap the ith and 1st row and column. Thus, the $(1, n)$ entry is zero. If there are two nonzero entries along the diagonal apart from the (n, n) entry, then by Subcase A1, S will be permutationally similar to the reducible pattern. Suppose there is only one nonzero diagonal entry at the (k, k)th place and all other entries in the kth column are zero. By permutational similarity, swap the 1st and the kth row and column with each other. Then the matrix becomes reducible as in Subcase A1. Assume that the kth column contains one more nonzero entry at the (l, k)th place, then by using permutation similarity, swapping firstly the first and the kth row and column with each other and secondly the second and the lth row and column with each other, it can be brought to the following form:

$$\begin{pmatrix} \times & \# & \dots & \# & \times \\ \times & \# & \dots & \# & \times \\ 0 & \# & \dots & \# & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \# & \dots & \# & \times \end{pmatrix}, \tag{3.1}$$

where $\#$ is either zero or nonzero and the (k, n)th entry is zero in the last column. All diagonal entries at the places of $\#$ are zero. Now if $n - 2$ nonzero entries are to be placed at $\#$ so two rows of $\#$ will be zero. If these two rows are placed such that the first column entry in those two rows is zero then the determinant of S will be zero. If the second column nonzero entry is in the first (or second) row then it is a reducible pattern, so the second column entry must lie from the 3rd row onwards. By using permutation similarity that nonzero entry can be brought to the place $(3, 2)$. By arguing similarly for columns $3, 4, \dots, n - 1$, S can be made permutationally similar to the sign pattern in which the $(1, 1)$th, $(i + 1, i)$th entry to be nonzero for $1 \leqslant i \leqslant n-1$, and the last column contains $n-1$ nonzero entries and one zero. If that zero lies at the position of $(1, n)$th or $(2, n)$th then the determinant will be nonzero. Suppose zero lies in the (i, n)th place for $3 \leqslant i \leqslant n-1$, then the coefficient of x^{i-1} in the characteristic polynomial of any matrix in its qualitative class will be a product of nonzero entries which can not be made zero. Hence, S will not be a SAP.

Case B: Assume that S has one column (or row) with n nonzero entries. By using permutation similarity we can exchange this column (or row) with the last column (or the first row). Without loss of generality, assume that S has the last column with n nonzero entries. Remaining $n - 1$ entries must lie one each in first $n - 1$ columns otherwise determinant of S will be zero. Amongst these first $n-1$ columns at least one nonzero entry must lie on diagonal so that trace is not fixed. Suppose (k, k)th entry is nonzero for some $k \neq n$. But then by exchanging first and kth rows and columns with each other, S can be made permutationally similar to the following sign pattern

$$\left(\begin{array}{ccccc} \times & \# & \dots & \# & \times \\ \hline 0 & \# & \dots & \# & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \# & \dots & \# & \times \\ 0 & \# & \dots & \# & \times \end{array}\right).$$

Which is reducible contradicting the hypothesis. Thus, S has at least $2n$ nonzero entries. Hence, the theorem. □

Appendix: Matrix realizations

Sr.No.	Octant surrounding vectors	Matrix realizations in sign pattern $S = \begin{pmatrix} + & + & - \\ + & 0 & - \\ + & 0 & - \end{pmatrix}$
1	e_1, e_2, e_3	$e_1 \to \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $e_2 \to \begin{pmatrix} 2 & 1 & -6 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, $e_3 \to \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix}$
2	$-e_1, -e_2, -e_3$	$-e_1 \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 4 & 0 & -2 \end{pmatrix}$, $-e_2 \to \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & -1 \\ 6 & 0 & -2 \end{pmatrix}$, $-e_3 \to \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & -1 \\ 7 & 0 & -2 \end{pmatrix}$
3	$-e_1, e_2, e_3$	$-e_1 \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 4 & 0 & -2 \end{pmatrix}$, $e_2 \to \begin{pmatrix} 2 & 1 & -1 \\ 5 & 0 & -1 \\ 10 & 0 & -2 \end{pmatrix}$, $e_3 \to \begin{pmatrix} 2 & 1 & -1 \\ 5 & 0 & -1 \\ 9 & 0 & -2 \end{pmatrix}$
4	$-e_1, -e_2, e_3$	$-e_1 \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 4 & 0 & -2 \end{pmatrix}$, $-e_2 \to \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & -1 \\ 6 & 0 & -2 \end{pmatrix}$, $e_3 \to \begin{pmatrix} 2 & 1 & -1 \\ 5 & 0 & -1 \\ 9 & 0 & -2 \end{pmatrix}$
5	$-e_1, e_2, -e_3$	$-e_1 \to \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 4 & 0 & -2 \end{pmatrix}$, $e_2 \to \begin{pmatrix} 2 & 1 & -1 \\ 5 & 0 & -1 \\ 10 & 0 & -2 \end{pmatrix}$, $-e_3 \to \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & -1 \\ 7 & 0 & -2 \end{pmatrix}$
6	$e_1, -e_2, -e_3$	$e_1 \to \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $-e_2 \to \begin{pmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, $-e_3 \to \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -3 \\ 1 & 0 & -2 \end{pmatrix}$
7	$e_1, -e_2, e_3$	$e_1 \to \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $-e_2 \to \begin{pmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, $e_3 \to \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix}$
8	$e_1, e_2, -e_3$	$e_1 \to \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $e_2 \to \begin{pmatrix} 2 & 1 & -6 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, $-e_3 \to \begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -3 \\ 1 & 0 & -2 \end{pmatrix}$

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