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SUM OF HIGHER DIVISOR FUNCTION WITH PRIME SUMMANDS

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Abstract. Let $l \geqslant 2$ be an integer. Recently, Hu and Lü offered the asymptotic formula for the sum of the higher divisor function

$$\sum_{1\leqslant n_1,n_2,\ldots,n_l\leqslant x^{1/2}} \tau_k(n_1^2+n_2^2+\ldots+n_l^2),$$

where $\tau_k(n)$ represents the kth divisor function. We give the Goldbach-type analogy of their result. That is to say, we investigate the asymptotic behavior of the sum

$$\sum_{1\leqslant p_1,p_2,\ldots,p_l\leqslant x} \tau_k(p_1+p_2+\ldots+p_l),$$

where $p_1, p_2, \ldots, p_l$ are prime variables.

Keywords: higher divisor function; circle method; prime

MSC 2020: 11N37, 11A41, 11P55

1. Introduction

The mean values of arithmetic functions are well studied in number theory. About 180 years ago, Dirichlet proved that

$$\sum_{n\leqslant x} \tau(n) = x\log x + (2c_0-1)x + O(\sqrt{x}),$$

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where c_0 is the Euler constant and $\tau(n)$ represents the number of positive divisors of n. The classical divisor function can be generalised to higher divisor functions. For an integer $k \geqslant 2$, put

$$\tau_k(n) = \sum_{\substack{m_1 m_2 \ldots m_k = n \\ 1 \leqslant m_1, m_2, \ldots, m_k \leqslant n}} 1$$

to be the kth divisor function. When $k = 2$, it represents the classical divisor function. Inspired by Dirichlet's theorem, Ingham in [11], Bellman in [1] and Hooley in [8], [9] investigated the sum of divisor function of quadratic polynomials $\sum_{n \leqslant \sqrt{x}} \tau(n^2 + a)$. Suppose that a is not a perfect square number, Hooley in [9] proved that

$$\sum_{n \leqslant \sqrt{x}} \tau(n^2 + a) = A_1(a)\sqrt{x} \log x + A_2(a)\sqrt{x} + O(x^{4/9} \log^3 x),$$

where $A_1(a)$ and $A_2(a)$ are two constants. Later, Gafurov in [5], [6] and Calderón and de Velasco in [2] studied the sums of divisor functions of two and three variables, respectively. Calderón and de Velasco in [2] showed that

$$\sum_{m,n,t \leqslant \sqrt{x}} \tau(m^2 + n^2 + t^2) = \frac{4\zeta(3)}{5\zeta(4)} x^{3/2} \log x + O(x^{3/2}).$$

Guo and Zhai in [7] provided the second term of Calderón and de Velasco's result. And the error term obtained by Guo and Zhai was further sharpened by Zhao, see [16]. Following this line, Sun and Zhang in [15] began to consider the higher divisor function. They studied the asymptotic behavior of $\sum_{m,n,t \leqslant \sqrt{x}} \tau_3(m^2 + n^2 + t^2)$. The method adapted by Sun and Zhang (see [15]) is also valid for the sum

$$\sum_{n_1, n_2, \ldots, n_l \leqslant \sqrt{x}} \tau_3(n_1^2 + n_2^2 + \ldots + n_l^2).$$

Recently, the above results have been generalized by Hu and Lü (see [10]), who showed that the leading term of

$$\sum_{n_1, \ldots, n_l \leqslant \sqrt{x}} \tau_k(n_1^2 + \ldots + n_l^2)$$

is of the magnitude $x^{l/2} \log^{k-1} x$ for $k \geqslant 4$ and $l \geqslant 3$. The magnitude obtained by them is surely consistent with the already known case for $k = 2, 3$.

In a former paper, the formula of Hu and Lü (see [10]) was further extended to the sum

$$\sum_{1\leqslant n_1,n_2,\ldots,n_l\leqslant x^{1/r}} \tau_k(n_1^r+n_2^r+\ldots+n_l^r)$$

for $r\geqslant 2$ by the authors of [17]. In this subsequent article, the authors turn to study the sums of higher divisor function with prime summands. Now we fix some notations. For a positive integer a and $0\leqslant j\leqslant k-1$, let

$$A_j(q)=\sum_{b=1}^{q} e\Big(-\frac{ab}{q}\Big)c_{j+1}(b,q), \tag{1.1}$$

where the coefficients $c_j(b,q)$ are of the form

$$\sum_{b_1b_2\equiv b \pmod q} f(b_1)$$

for some function f and we know that the number of terms in $c_j(b,q)$ depends only on k. The accurate definition of $A_j(q)$ is a bit cumbersome and we do not need it in the proof. So we omit its definition and recommend the reader to [3], (2.13) for the details. In [4], the bound

$$A_j(q)\ll_k \frac{1}{q} \tag{1.2}$$

of $A_j(q)$ was given. For $0\leqslant i\leqslant j\leqslant k-1$ and $l\geqslant 2$, put

$$\mathfrak{S}_{k,l,j}=\frac{1}{j!}\sum_{q=1}^{\infty}\frac{\mu(q)^l}{\varphi(q)^l}\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} A_j(q) \tag{1.3}$$

and

$$\mathfrak{L}_{k,l,i}=\int_{-\infty}^{\infty}\Big(\int_0^1 e(\beta t)\,\mathrm{d}t\Big)^l\Big(\int_0^l e(-\beta u)\log^i u\,\mathrm{d}u\Big)\,\mathrm{d}\beta, \tag{1.4}$$

where $\mu(n)$ is the Möbius function and $\varphi(n)$ is Euler's totient function. By equation (1.2), we know that $\mathfrak{S}_{k,l,j}\ll 1$. And it is clear that $\mathfrak{L}_{k,l,i}\ll 1$.

Based on some ideas developed by Hu and Lü (see [10]) and some standard techniques displayed in [13], we established the following asymptotic formulas. For the simplicity of calculations, we first give a weighted sum of higher divisor function with prime variables. Then, we use this weighted sum to calculate the sum mentioned in the abstract.

Theorem 1.1. *Let $l \geqslant 2$ and $k \geqslant 2$ be integers. Let*

$$R(x) = \sum_{1\leqslant p_1,p_2,\ldots,p_l\leqslant x} \tau_k(p_1+p_2+\ldots+p_l)\log p_1 \log p_2 \ldots \log p_l,$$

where $p_1, p_2, \ldots, p_l$ are primes. Then for any positive number A, we have

$$R(x) = \sum_{j=0}^{k-1} \mathfrak{S}_{k,l,j} \sum_{i=0}^{j} \binom{j}{i} \mathfrak{L}_{k,l,i} \cdot x^l (\log x)^{j-i} + O(x^l(\log x)^{-A}).$$

Corollary 1.1. *Let $l \geqslant 2$ and $k \geqslant 2$ be integers. Let*

$$r(x) = \sum_{1\leqslant p_1,p_2,\ldots,p_l\leqslant x} \tau_k(p_1+p_2+\ldots+p_l),$$

where $p_1, p_2, \ldots, p_l$ are primes. Then we have

$$r(x) = \mathfrak{S}_{k,l,k-1}\mathfrak{L}_{k,l,0} \cdot x^l (\log x)^{k-1-l}\Big(1+O\Big(\frac{\log\log x}{\log x}\Big)\Big).$$

2. Notations and auxiliary lemmas

We first fix some notations which are used in the proof. Throughout our paper, ε is an arbitrary small positive number which may vary at different instances. For large x, let

$$Q = \log^B x, \quad I = \Big[\frac{1}{Q}, 1+\frac{1}{Q}\Big),$$

where $B > 0$ is a parameter to be decided later. The major arcs $\mathfrak{M}$ are consisted of the subintervals

$$\mathfrak{M}(q,a) = \Big\{\alpha\colon\ \alpha = \frac{a}{q} + \beta,\ |\beta| \leqslant \frac{Q}{x}\Big\}$$

and the minor arcs $\mathfrak{m}$ are the complementary set of $\mathfrak{M}$ with respect to I, i.e.,

$$\mathfrak{M} = \bigcup_{q\leqslant Q} \bigcup_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}} \mathfrak{M}(q,a), \quad \mathfrak{m} = I \setminus \mathfrak{M}.$$

It is sure that the subintervals $\mathfrak{M}(q,a)$ in the major arcs are pairwise disjoint for large x. For $l \geqslant 2$, put

$$f(\alpha,x) = \sum_{1\leqslant p\leqslant x} e(p\alpha)\log p, \quad g(\alpha,x) = \sum_{1\leqslant n\leqslant lx} \tau_k(n)e(-n\alpha).$$

We now introduce several lemmas.

Lemma 2.1 ([13])**.** *Let B and C be positive numbers with $C > 2B$, then for $\alpha = a/q + \beta \in \mathfrak{M}(q,a)$, we have*

$$f(\alpha,x) = \frac{\mu(q)}{\varphi(q)} u(\beta) + O\Big(\frac{Q^2 x}{\log^C x}\Big) \quad \text{and} \quad f^l(\alpha,x) = \frac{\mu(q)^l}{\varphi(q)^l} u(\beta)^l + O\Big(\frac{Q^2 x^l}{\log^C x}\Big),$$

where $u(\beta) = \int_0^x e(t\beta)\,\mathrm{d}t$ satisfies

$$u(\beta) \ll \min\{x, |\beta|^{-1}\}.$$

Lemma 2.2 (Vinogradov, [13])**.** *Let a and q be integers with $1 \leqslant q \leqslant x$ and $(a,q) = 1$. Suppose that $|\alpha - aq^{-1}| \leqslant q^{-2}$, then*

$$f(\alpha,x) \ll \Big(\frac{x}{q^{1/2}} + x^{4/5} + x^{1/2}q^{1/2}\Big) \log^4 x.$$

Lemma 2.3. *Let $(a,q) = 1$ with $q \leqslant Q$ and $|\beta| \leqslant Q/x$. Then*

$$g\Big(\frac{a}{q} + \beta, x\Big) = \sum_{j=0}^{k-1} A_j(q) I_j(\beta) + O(x^{(k-1)/(k+2)+\varepsilon}),$$

where

$$I_j(\beta) = \int_1^{lx} e(-\beta u) \frac{\log^j u}{j!}\,\mathrm{d}u.$$

The proof of the lemma follows from the proof of Hu and Lü's, see [10], Lemma 3.1. Integrating by parts, we can get

$$I_j(\beta) \ll_l \log^k x \cdot \min\{x, |\beta|^{-1}\}.$$

Lemma 2.4 ([14])**.** *For $g(\alpha,x)$, we have*

$$\int_0^1 |g(\alpha,x)|^2\,\mathrm{d}\alpha = \sum_{n \leqslant lx} |\tau_k(n)|^2 \ll x(\log x)^{k^2-1},$$

where the implied constant depends only on l.

3. Proof of Theorem 1.1

P r o o f of Theorem 1.1. The proof of the theorem starts with

$$R(x)=\int_{1/Q}^{1+1/Q} f^l(\alpha,x)g(\alpha,x)\,\mathrm{d}\alpha=\int_{\mathfrak{M}} f^l(\alpha,x)g(\alpha,x)\,\mathrm{d}\alpha+\int_{\mathfrak{m}} f^l(\alpha,x)g(\alpha,x)\,\mathrm{d}\alpha.$$

For $\alpha\in I$, by Dirichlet's theorem, there are positive numbers $1\leqslant a\leqslant q\leqslant x/Q$ with $(a,q)=1$ such that

$$\Big|\alpha-\frac{a}{q}\Big|\leqslant\frac{Q}{qx}\leqslant\frac{Q}{x}.$$

By the definition of $\mathfrak{M}$ and $\mathfrak{m}$, we have $Q<q\leqslant x/Q$ for $\alpha\in\mathfrak{m}$. So, by Lemma 2.2, we have

$$f(\alpha,x)\ll\Big(\frac{x}{\log^{B/2}x}+x^{4/5}+x^{1/2}(x\log^{-B}x)^{1/2}\Big)\log^4 x\ll x\log^{4-B/2}x$$

for $\alpha\in\mathfrak{m}$. Also, from Paserval's identity and Chebyshev's estimate, we find that

$$\int_0^1|f(\alpha,x)|^2\,\mathrm{d}\alpha=\sum_{p\leqslant x}\log^2 p\ll\log x\sum_{p\leqslant x}\log p\ll x\log x.$$

Using the above estimates together with Cauchy's inequality and Lemma 2.4, we get

(3.1)
$$\begin{aligned}\int_{\mathfrak{m}} f^l(\alpha,x)g(\alpha,x)\,\mathrm{d}\alpha&\ll\sup_{\alpha\in\mathfrak{m}}|f(\alpha,x)|^{l-1}\Big(\int_0^1|f(\alpha,x)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_0^1|g(\alpha,x)|^2\,\mathrm{d}\alpha\Big)^{1/2}\\&\ll x^{l-1}(\log x)^{(l-1)(4-B/2)}\cdot x^{1/2}\log^{1/2}x\cdot x^{1/2}(\log x)^{(k^2-1)/2}\\&=x^l(\log x)^{(l-1)(4-B/2)+k^2/2}.\end{aligned}$$

We now turn to the treatment of the integral of the major arcs. Employing Lemmas 2.1 and 2.3, we obtain that

$$\begin{aligned}&\int_{\mathfrak{M}} f^l(\alpha,x)g(\alpha,x)\,\mathrm{d}\alpha\\&\quad=\sum_{q\leqslant Q}\frac{\mu(q)^l}{\varphi(q)^l}\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\sum_{j=0}^{k-1}A_j(q)\int_{|\beta|\leqslant Q/x}u(\beta)^l I_j(\beta)\,\mathrm{d}\beta+\mathcal{R}'+\mathcal{R}''+\mathcal{R}''',\end{aligned}$$

where the first error term is

$$\begin{aligned}\mathcal{R}' &\ll x^{(k-1)/(k+2)+\varepsilon} \sum_{q\leqslant Q} \frac{1}{\varphi(q)^l} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \int_{|\beta|\leqslant Q/x} u(\beta)^l \,\mathrm{d}\beta \\ &\ll x^{(k-1)/(k+2)+\varepsilon} \sum_{q\leqslant Q} \frac{1}{\varphi(q)^l} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} x^l \cdot \frac{Q}{x} \\ &= x^{l-3/(k+2)+\varepsilon} \sum_{q\leqslant Q} \frac{1}{\varphi(q)^{l-1}} \ll x^{l-3/(k+2)+\varepsilon}\end{aligned}$$

and the other two error terms are

$$\begin{aligned}\mathcal{R}'' &\ll \frac{Q^2 x^l}{\log^C x} \sum_{q\leqslant Q} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \sum_{j=0}^{k-1} A_j(q) \int_{|\beta|\leqslant Q/x} I_j(\beta)\,\mathrm{d}\beta \\ &\ll \frac{Q^2 x^l}{\log^C x} \sum_{q\leqslant Q} \frac{\varphi(q)}{q} \cdot \frac{Q}{x} \cdot \log^k x \cdot x \ll x^l (\log x)^{4B+k-C}\end{aligned}$$

and

$$\mathcal{R}''' \ll x^{(k-1)/(k+2)+\varepsilon} \cdot \frac{Q^2 x^l}{\log^C x} \sum_{q\leqslant Q} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \int_{|\beta|\leqslant Q/x} 1 \,\mathrm{d}\beta \ll x^{l-3/(k+2)+\varepsilon}.$$

Thus, we have

$$\begin{aligned}(3.2)\quad &\int_{\mathfrak{M}} f^l(\alpha,x) g(\alpha,x)\,\mathrm{d}\alpha \\ &= \sum_{q\leqslant Q} \frac{\mu(q)^l}{\varphi(q)^l} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \sum_{j=0}^{k-1} A_j(q) \int_{|\beta|\leqslant Q/x} u(\beta)^l I_j(\beta)\,\mathrm{d}\beta + O(x^l (\log x)^{4B+k-C}).\end{aligned}$$

Note that

$$\int_{|\beta|>Q/x} u(\beta)^l I_j(\beta)\,\mathrm{d}\beta \ll (\log x)^k \int_{Q/x}^{\infty} |\beta|^{-(l+1)}\,\mathrm{d}\beta \ll x^l (\log x)^{k-lB},$$

so we can extend the integral from $(-Q/x, Q/x)$ to $(-\infty,\infty)$ as

$$(3.3)\qquad \int_{|\beta|\leqslant Q/x} u(\beta)^l I_j(\beta)\,\mathrm{d}\beta = \int_{-\infty}^{\infty} u(\beta)^l I_j(\beta)\,\mathrm{d}\beta + O(x^l (\log x)^{k-lB}).$$

The integral from $-\infty$ to ∞ above can be treated as

$$\begin{aligned}\int_{-\infty}^{\infty} u(\beta)^l I_j(\beta)\,\mathrm{d}\beta &= \int_{-\infty}^{\infty}\Big(\int_0^x e(t\beta)\,\mathrm{d}t\Big)^l\Big(\int_1^{lx} e(-\beta u)\frac{\log^j u}{j!}\,\mathrm{d}u\Big)\,\mathrm{d}\beta\\ &\overset{y=t/x}{\underset{v=u/x}{=\!=}} \frac{x^{l+1}}{j!}\int_{-\infty}^{\infty}\Big(\int_0^1 e(xy\beta)\,\mathrm{d}y\Big)^l\Big(\int_{1/x}^{l} e(-\beta vx)\log^j(vx)\,\mathrm{d}v\Big)\,\mathrm{d}\beta\\ &\overset{\beta x\to\beta}{=\!=} \frac{x^l}{j!}\int_{-\infty}^{\infty}\Big(\int_0^1 e(\beta y)\,\mathrm{d}y\Big)^l\Big(\int_{1/x}^{l} e(-\beta v)\log^j(vx)\,\mathrm{d}v\Big)\,\mathrm{d}\beta.\end{aligned}$$

Since

$$\begin{aligned}&\frac{x^l}{j!}\int_{-\infty}^{\infty}\Big(\int_0^1 e(\beta y)\,\mathrm{d}y\Big)^l\Big(\int_0^{1/x} e(-\beta v)\log^j(vx)\,\mathrm{d}v\Big)\,\mathrm{d}\beta\\ &\qquad\ll x^l\log^j x\int_{-\infty}^{\infty}\frac{1}{|\beta|^l}\Big(\int_0^{1/x}|\log v|^j\,\mathrm{d}v\Big)\,\mathrm{d}\beta \ll x^l\log^k x\cdot\frac{\log^k x}{x} = x^{l-1}\log^{2k} x,\end{aligned}$$

we find that

$$\begin{aligned}(3.4)\quad &\int_{-\infty}^{\infty} u(\beta)^l I_j(\beta)\,\mathrm{d}\beta\\ &\quad= \frac{x^l}{j!}\int_{-\infty}^{\infty}\Big(\int_0^1 e(\beta y)\,\mathrm{d}y\Big)^l\Big(\int_0^{l} e(-\beta v)\log^j(vx)\,\mathrm{d}v\Big)\,\mathrm{d}\beta + O(x^{l-1}\log^{2k} x)\\ &\quad= \frac{x^l}{j!}\sum_{i=0}^{j}\binom{j}{i}(\log x)^{j-i}\mathfrak{L}_{k,l,i} + O(x^{l-1}\log^{2k} x).\end{aligned}$$

By equations (3.2), (3.3) and (3.4), we conclude that

$$\begin{aligned}\int_{\mathfrak{M}} f^l(\alpha,x)g(\alpha,x)\,\mathrm{d}\alpha = &\sum_{q\leqslant Q}\frac{\mu(q)^l}{\varphi(q)^l}\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\sum_{j=0}^{k-1}\frac{A_j(q)}{j!}\sum_{i=0}^{j}\binom{j}{i}\mathfrak{L}_{k,l,i}\cdot x^l(\log x)^{j-i}\\ &+ O(x^l(\log x)^{4B+k-C} + x^l(\log x)^{k-lB}).\end{aligned}$$

It is easy to see that

$$\sum_{q>Q}\frac{\mu(q)^l}{\varphi(q)^l}\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\sum_{j=0}^{k-1}\frac{A_j(q)}{j!}\sum_{i=0}^{j}\binom{j}{i}\mathfrak{L}_{k,l,i} \ll \sum_{q>Q}\frac{1}{\varphi(q)^{l-1}q} \ll \frac{1}{Q^{1-\varepsilon}} \ll (\log x)^{-B/2}.$$

Therefore, we obtain that

$$(3.5)\qquad \int_{\mathfrak{M}} f^l(\alpha,x)g(\alpha,x)\,\mathrm{d}\alpha = \sum_{j=0}^{k-1}\mathfrak{S}_{k,l,j}\sum_{i=0}^{j}\binom{j}{i}\mathfrak{L}_{k,l,i}\cdot x^l(\log x)^{j-i} + \mathcal{R},$$

where

$$\mathcal{R} \ll x^l(\log x)^{4B+k-C} + x^l(\log x)^{k-B/2} + x^l(\log x)^{k-lB}$$
$$\ll x^l(\log x)^{4B+k-C} + x^l(\log x)^{k-B/2}.$$

Combing equations (3.1) and (3.5), we have

$$R(x) = \sum_{j=0}^{k-1} \mathfrak{S}_{k,l,j} \sum_{i=0}^{j} \binom{j}{i} \mathfrak{L}_{k,l,i} \cdot x^l(\log x)^{j-i} + \widetilde{\mathcal{R}},$$

where

$$\widetilde{\mathcal{R}} \ll x^l(\log x)^{4B+k-C} + x^l(\log x)^{k-B/2} + x^l(\log x)^{(l-1)(4-B/2)+k^2/2}.$$

Our theorem immediately follows from taking

$$B = \max\Big\{\frac{2A+k^2}{l-1} + 8, 2A+2k\Big\} \quad \text{and} \quad C = 4B+k+A.$$

□

Using more elaborate arguments invented by Montgomery and Vaughan (see [12]), it is certain that one can improve the error term in the theorem to $x^{l-\delta}$ for some positive number δ.

4. Proof of Corollary 1.1

Proof of Corollary 1.1. For $0 < \gamma < 1/2$, it is clear that

$$\begin{aligned} R(x) &\geqslant \sum_{\substack{1\leqslant p_1,p_2,\ldots,p_l\leqslant x\\ p_1,p_2,\ldots,p_l>x^{1-\gamma}}} \tau_k(p_1+p_2+\ldots+p_l)\log p_1 \log p_2 \ldots \log p_l \\ &\geqslant (1-\gamma)^l(\log x)^l \sum_{\substack{1\leqslant p_1,p_2,\ldots,p_l\leqslant x\\ p_1,p_2,\ldots,p_l>x^{1-\gamma}}} \tau_k(p_1+p_2+\ldots+p_l) \\ &= (1-\gamma)^l(\log x)^l \Bigg(r(x) - \sum_{\substack{1\leqslant p_1,p_2,\ldots,p_l\leqslant x\\ \exists\, p_i\leqslant x^{1-\gamma}}} \tau_k(p_1+p_2+\ldots+p_l)\Bigg). \end{aligned}$$

We now deal with the restricted summation

$$\sum_{\substack{1\leqslant p_1,p_2,\ldots,p_l\leqslant x\\ \exists\, p_i\leqslant x^{1-\gamma}}} \tau_k(p_1+p_2+\ldots+p_l).$$

Note that

$$
\begin{aligned}
\sum_{\substack{1\leqslant p_1,p_2,\ldots,p_l\leqslant x\\ \exists\, p_i\leqslant x^{1-\gamma}}} & \tau_k(p_1+p_2+\ldots+p_l)\\
&\leqslant l \sum_{\substack{1\leqslant p_1,p_2,\ldots,p_l\leqslant x\\ p_l\leqslant x^{1-\gamma}}} \tau_k(p_1+p_2+\ldots+p_l)\\
&= l \sum_{p_l\leqslant x^{1-\gamma}} \sum_{p_2\leqslant x}\cdots \sum_{p_{l-1}\leqslant x}\sum_{p_1\leqslant x} \tau_k(p_1+p_2+\ldots+p_l)\\
&= l \sum_{p_l\leqslant x^{1-\gamma}} \sum_{p_2\leqslant x}\cdots \sum_{p_{l-1}\leqslant x}\sum_{p_1\leqslant x}\sum_{d_1\ldots d_k=p_1+\ldots+p_l} 1\\
&\leqslant l \sum_{p_l\leqslant x^{1-\gamma}} \sum_{p_2\leqslant x}\cdots \sum_{p_{l-1}\leqslant x}\sum_{d_1\leqslant lx}\cdots\sum_{d_{k-1}\leqslant lx}\sum_{d_k\leqslant lx/(d_1\ldots d_{k-1})} 1\\
&\ll l\frac{x^{l-1-\gamma}}{(\log x)^{l-1}}\cdot lx\cdot\Big(\sum_{d\leqslant lx}\frac{1}{d}\Big)^{k-1}\\
&\ll l^2(\log l)^{k-1}x^{l-\gamma}(\log x)^{k-l},
\end{aligned}
$$

so we have

$$R(x)\gg(1-\gamma)^l(\log x)^l(r(x)-l^2(\log l)^{k-1}x^{l-\gamma}(\log x)^{k-l}).$$

Thus, we deduce that

$$
\begin{aligned}
(\log x)^l r(x)-R(x) &\ll [(1-\gamma)^{-l}-1]R(x)+l^2(\log l)^{k-1}x^{l-\gamma}(\log x)^k\\
&\leqslant l2^{l+1}\gamma R(x)+l^2(\log l)^{k-1}x^{l-\gamma}(\log x)^k \ll_{k,l}\gamma R(x)+x^{l-\gamma}(\log x)^k,
\end{aligned}
$$

where the inequality $(1-\gamma)^{-l}-1\leqslant l2^{l+1}\gamma$ follows from Lagrange's mean value theorem. On the other hand, it is easy to see that

$$R(x)\leqslant(\log x)^l\sum_{1\leqslant p_1,p_2,\ldots,p_l\leqslant x}\tau_k(p_1+p_2+\ldots+p_l)=(\log x)^l r(x).$$

Therefore, we have shown that

$$0\leqslant(\log x)^l r(x)-R(x)\ll\gamma R(x)+x^{l-\gamma}(\log x)^k,$$

which means that

$$r(x)=\frac{R(x)}{(\log x)^l}\Big(1+O\Big(\gamma+\frac{x^{l-\gamma}}{R(x)}(\log x)^k\Big)\Big).$$

By Theorem 1.1,

$$\frac{x^{l-\gamma}}{R(x)}(\log x)^k\ll x^{-\gamma}\log x,$$

hence, the corollary follows from taking $\gamma=\log\log x/\log x$. □

References

[1] *R. Bellman*: Ramanujan sums and the average value of arithmetic functions. Duke Math. J. *17* (1950), 159–168. zbl MR doi

[2] *C. Calderón, M. J. de Velasco*: On divisors of a quadratic form. Bol. Soc. Bras. Mat., Nova Sér. *31* (2000), 81–91. zbl MR doi

[3] *C. E. Chace*: The divisor problem for arithmetic progressions with small modulus. Acta Arith. *61* (1992), 35–50. zbl MR doi

[4] *C. E. Chace*: Writing integers as sums of products. Trans. Am. Math. Soc. *345* (1994), 367–379. zbl MR doi

[5] *N. Gafurov*: On the sum of the number of divisors of a quadratic form. Dokl. Akad. Nauk Tadzh. SSR *28* (1985), 371–375. (In Russian.) zbl MR

[6] *N. Gafurov*: On the number of divisors of a quadratic form. Proc. Steklov Inst. Math. *200* (1993), 137–148. zbl MR

[7] *R. Guo, W. Zhai*: Some problems about the ternary quadratic form $m_1^2+m_2^2+m_3^2$. Acta Arith. *156* (2012), 101–121. zbl MR doi

[8] *C. Hooley*: On the representation of a number as the sum of a square and a product. Math. Z. *69* (1958), 211–227. zbl MR doi

[9] *C. Hooley*: On the number of divisors of a quadratic polynomials. Acta Math. *110* (1963), 97–114. zbl MR doi

[10] *G. Hu, G. Lü*: Sums of higher divisor functions. J. Number Theory *220* (2021), 61–74. zbl MR doi

[11] *A. E. Ingham*: Some asymptotic formulae in the theory of numbers. J. Lond. Math. Soc. *2* (1927), 202–208. zbl MR doi

[12] *H. L. Montgomery, R. C. Vaughan*: The exceptional set in Goldbach's problem. Acta Arith. *27* (1975), 353–370. zbl MR doi

[13] *M. B. Nathanson*: Additive Number Theory: The Classical Bases. Graduate Texts in Mathematics 164. Springer, New York, 1996. zbl MR doi

[14] *P. Shiu*: A Brun-Titchmarsh theorem for multiplicative functions. J. Reine Angew. Math. *313* (1980), 161–170. zbl MR doi

[15] *Q. Sun, D. Zhang*: Sums of the triple divisor function over values of a ternary quadratic form. J. Number Theory *168* (2016), 215–246. zbl MR doi

[16] *L. Zhao*: The sum of divisors of a quadratic form. Acta Arith. *163* (2014), 161–177. zbl MR doi

[17] *G.-L. Zhou, Y. Ding*: Sums of the higher divisor function of diagonal homogeneous forms. Ramanujan J. *59* (2022), 933–945. zbl MR doi

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