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MAJORITY CHOOSABILITY OF 1-PLANAR DIGRAPH

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Abstract. A majority coloring of a digraph D with k colors is an assignment $\pi\colon V(D) \to \{1, 2, \dots, k\}$ such that for every $v \in V(D)$ we have $\pi(w) = \pi(v)$ for at most half of all out-neighbors $w \in N^+(v)$. A digraph D is majority k-choosable if for any assignment of lists of colors of size k to the vertices, there is a majority coloring of D from these lists. We prove that if $U(D)$ is a 1-planar graph without a 4-cycle, then D is majority 3-choosable. And we also prove that every NIC-planar digraph is majority 3-choosable.

Keywords: majority choosable; OD-3-choosable; 1-planar digraph

MSC 2020: 05C15

1. Introduction

Let $D = (V(D), A(D))$ be a digraph, and diendgraphs considered in this paper are finite and simple (loopless, have no parallel edges, or anti-parallel pairs of arcs). A digraph D is called 1-*planar* (see [5]) if it can be drawn in the plane so that each arc is crossed by at most one other edge. A digraph is NIC-planar (near-independent-crossing-planar) if it admits a drawing in the plane with at most one crossing per edge and such that two pairs of crossing arcs share at most one common end-vertex. If there exists a crossing c in a digraph D drawn on the plane, this crossing induces a function $\theta\colon\ c \to \{v_1, v_2, v_3, v_4\}$, where $\{v_1, v_2, v_3, v_4\}$ is a set of four vertices that are the ones incident with the two crossed arcs generating the crossing c, see [4]. For any two distinct crossings c_1 and c_2 in D, if $|\theta(c_1) \cap \theta(c_2)| \leqslant 2$, D is a 1-planar digraph; if $|\theta(c_1) \cap \theta(c_2)| \leqslant 1$, D is a NIC-planar digraph. The underlying graph of a digraph D, denoted by $U(D)$, is the simple undirected graph with vertex set $V(D)$ in which two vertices $x \neq y$ are adjacent if and only if $(x, y) \in A(D)$ or $(y, x) \in A(D)$.

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We use $F(D)$, $\Delta(D)$ and $\delta(D)$ to denote the set of faces, the maximum degree and the minimum degree of D, respectively. For every vertex $v \in V(D)$, let $N(v)$ $(N^+(v), N^-(v))$ denote the neighborhood (out-neighborhood, in-neighborhood) of v in D. The symbol $d(v)$ $(d^+(v), d^-(v))$ stands for the size of $N(v)$ $(N^+(v), N^-(v))$, and is called *degree* (outdegree, indegree) of v in D. A majority coloring of a digraph D with k colors is an assignment $\pi\colon V(D) \to \{1, 2, \ldots, k\}$ such that for every $v \in V(D)$ we have $\pi(w) = \pi(v)$ for at most half of all out-neighbors $w \in N^+(v)$. This concept was introduced by Kreutzer et al. (see [8]) in connection to neural networks, who showed that every digraph is majority 4-colorable. Kreutzer et al. proved this result on the basis that every acyclic digraph is majority 2-colorable. And they proposed the following conjecture.

Conjecture 1.1 ([8]). *Every digraph is majority* 3*-colorable.*

And they also gave the concept of majority k-choosability of digraphs if for any assignment of lists of size at least k to the vertices, we can choose colors from the respective lists such that the arising coloring is a majority coloring. Anholcer, Bosek and Grytczuk in [2] proved that every digraph is majority 4-choosable. The following conjecture naturally arises:

Conjecture 1.2 ([2]). *Every digraph is majority* 3*-choosable.*

Anastos et al. in [1] studied sparse digraphs in particular, and showed that if the chromatic number of underlying graph $U(D)$ is less than 6, then the digraph D is majority 3-colorable. And they introduced a new notion, a digraph D is OD-3-choosable if for any assignment of color lists $L(x)$, $x \in V(D)$ of size 3 to the vertices, there exists a choice function ϕ (i.e., $\phi(x) \in L(x)$ for all $x \in V(D)$) such that no odd directed cycle in (D, ϕ) is monochromatic. And they also gave some results about majority 3-choosability.

Theorem 1.1 ([1]). *Let D be a digraph. If D is* OD-3*-choosable, then D is majority* 3*-choosable.*

Theorem 1.2 ([1]). *Let D be a digraph whose underlying graph $U(D)$ is* 6*-choosable. Then D is majority* 3*-choosable.*

The majority coloring conjecture has been proved for a few special cases, see [6], [7]. Borodin in [3] gave the following result.

Theorem 1.3 ([3]). *If a graph is* 1*-planar, then it is vertex* 6*-colorable.*

Wang et al. in [9] put forward the following theorem.

Theorem 1.4 ([9]). *If a graph is* 1*-planar, then it is vertex* 7*-choosable.*

And they also gave the following conjecture.

Conjecture 1.3 ([9]). *If a graph is* 1*-planar, then it is vertex* 6*-choosable.*

So, if that is true, 1-planar digraph would also follow from the result by Anastos et al., see [1].

Therefore, 1-planar digraph is majority 3-colorable. But the majority choosability of 1-planar digraph still showed no better results.

In this paper, we studied the majority choosability of 1-planar digraphs and proved the following results.

Theorem 1.5. *Let D be a digraph. If $U(D)$ is a* 1*-planar graph without* 4*-cycle, then D is* OD-3*-choosable.*

Theorem 1.6. *Let D be a digraph. If $U(D)$ is a* NIC*-planar graph, then D is* OD-3*-choosable.*

By Theorem 1.1, it is easy to see that if every digraph D is OD-3-choosable, then D is majority 3-choosable. Thus, we have the following corollaries.

Corollary 1.4. *Let D be a digraph. If $U(D)$ is a* 1*-planar graph without* 4*-cycle, then D is majority* 3*-choosable.*

Corollary 1.5. *Let D be a digraph. If $U(D)$ is a* NIC*-planar graph, then D is majority* 3*-choosable.*

2. Preliminaries

If the underlying graph $U(D)$ is OD-3-choosable, it is obvious that the digraph D is also OD-3-choosable. Orientations of edges are not important at this point, so we shall consider only undirected graphs throughout the whole section. The main tool we will use in our proof is the discharging method. In the proof, we will use the following terminology. A k-, k^+- and k^--vertex (or face) is a vertex (or face) of degree k, at least k and at most k, respectively.

Assume that G is a plane representation (see [10]) of a 1-planar graph such that each edge has at most one crossing. So, for each pair of crossing edges

$x_1y_1, x_2y_2 \in E(G)$, the four end-vertices x_1, y_1, x_2, y_2 are pairwise distinct. Let $X(G)$ denote the set of crossings in G. The associated plane graph, denoted G^*, of G is a plane graph with

$$V(G^*) = V(G) \cup X(G), \quad E(G^*) = E_0(G) \cup E_1(G),$$

where $E_0(G)$ stands for the set of non-crossed edges in G and

$$E_1(G) = \{xz, zy\colon\ xy \in E(G) \setminus E_0(G) \text{ and z is a crossing on } xy\}.$$

3. Proof of Theorem 1.5

We are now ready to prove Theorem 1.5.

Proof. Our proof is proceeded by reduction and absurdum. Assume that $G = U(D)$ is a counterexample to Theorem 1.5 such that $|V(G)|$ is as small as possible. Obviously, G is connected. We have the following claims.

Claim 3.1. *We have* $\delta(G) \geqslant 6$.

Proof. Suppose on the contrary that G contains a vertex v such that $d(v) \leqslant 5$, $N(v) = \{v_1, v_2, \ldots, v_k\}$ $(1 \leqslant k \leqslant 5)$. Let $G' = G - v$, by the minimality of G. There is an OD-3-choosable ϕ of G'. Let $L(v)$ be the color list of vertex v. Since $|L(v)| = 3$, there must be a color s in $L(v)$ that appears in $L(N(v))$ at most once. Assign color s to vertex v. Therefore, we can extend coloring ϕ to graph G, which is a contradiction. □

Fact 3.1. In graph G^*, no 4-vertex is adjacent to any 4-vertex.

Claim 3.2. *In graph G^*, each* 4*-vertex is incident with at most three triangles, see Figure* 1.

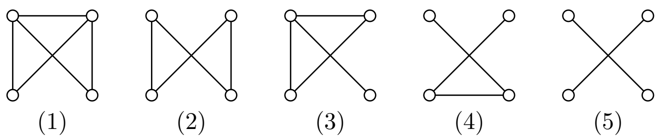
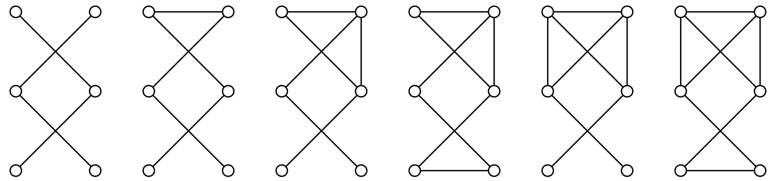


Figure 1. Without 4-face in G.

Claim 3.3. *In graph G^*, each k-face is incident with at most* $\lfloor \frac{k}{2} \rfloor$ 4*-vertices,* $k \geqslant 4$.

Proof. Since each 4-vertex in k-face is adjacent to two 6^+-vertices, and each 6^+-vertex in k-face is adjacent to at most two 4-vertices, then each k-face is incident with at most $\lfloor \frac{k}{2} \rfloor$ 4-vertices, $k \geqslant 4$. □

Since G is without 4-face, G^* contains a 4-face only in the following cases, see Figure 2.



(1) (2) (3) (4) (5) (6)

Figure 2. 4-face in G^*.

A 4-vertex v with at least one incident 5^+-face is called a good 4-vertex. Otherwise it is called a bad 4-vertex.

In order to complete the proof, we used the discharging method. By Euler's formula $|V(G^*)| - |E(G^*)| + |F(G^*)| = 2$, we have

$$\sum_{x\in V(G^*)\cup F(G^*)} w(x) \sum_{v\in V(G^*)} \Big(\frac{1}{2}d(v) - 3\Big) + \sum_{f\in F(G^*)} (d(f) - 3) = -6.$$

First, we give an initial charge function $w(v) = \frac{1}{2}d(v) - 3$ for every $v \in V(G^*)$ and $w(f) = d(f) - 3$ for every $f \in F(G^*)$. Next, we design discharging rules and redistribute weights accordingly. Let $w'(x)$ be the new charge after the discharging, $x \in V(G^*) \cup F(G^*)$. We will show that $w'(x) \geqslant 0$ for all $x \in V(G^*) \cup F(G^*)$. This leads to the following obvious contradiction:

$$-6 = \sum_{x\in V(G^*)\cup F(G^*)} w(x) = \sum_{x\in V(G^*)\cup F(G^*)} w'(x) \geqslant 0.$$

We are going to apply the following discharging rules:

(R1) Each 5^+-face in G^* sends charge of 1 to each incident 4-vertex.

(R2) If a 4-face is incident with exactly one 4-vertex in G^* sends charge of 1 to incident 4-vertex. If a 4-face is incident with two 4-vertex in G^* sends charge of 1 to incident bad 4-vertex.

We will show that $w'(x) \geqslant 0$ for all $x \in V(G^*) \cup F(G^*)$.

(1) If $k \geqslant 5$, by Claim 3.3, each k-face f is incident with at most $\lfloor \frac{k}{2} \rfloor$ 4-vertices. By rule (R1), a 5^+-face sends charge of 1 to each incident 4-vertex. So $w'(f) \geqslant k - 3 - 1 \times \lfloor \frac{k}{2} \rfloor$, since $k \geqslant 5$, $w'(f) \geqslant 0$.

(2) If there is a 4-face in G^*, except Figure 3 (5) (6), every 4-vertex is incident to a 5^+-face, by (1), $w'(f) \geqslant 0$. If Figure 3 (5) (6) exists, one of the 4-vertices, say v, is a bad 4-vertex, and the other 4-vertices, say u, is a good 4-vertex. Then by rule (R2), the 4-face f_1 sends charge 1 to bad 4-vertex v, and the 5^+-face f_2 sends charge 1 to good 4-vertex u. So for each 4-face f_1, $w'(f_1) \geqslant 4 - 3 - 1 = 0$.

(3) For each 4-vertex v, by (1) (2) and rule (R1) (R2), so $w'(v) \geqslant \frac{1}{2} \times 4 - 3 + 1 = 0$.

From the above discussion, we have for all $x \in V(G^*) \cup F(G^*)$,

$$\sum_{x \in V(G^*) \cup F(G^*)} w'(x) \geqslant 0.$$

It is a contradiction, which completes the proof of Theorem 1.5. □

4. Proof of Theorem 1.6

For any two distinct 4-vertices c_1, c_2 in G^*, let $|\theta(c_1) \cap \theta(c_2)|$ denote the number of two 4-vertices c_1, c_2 which share common end-vertices, and in Theorem 1.6, $|\theta(c_1) \cap \theta(c_2)| \leqslant 1$.

We are now ready to prove Theorem 1.6.

Proof. Our proof is proceeded by reduction and absurdum. Assume that $G = U(D)$ is a counterexample to Theorem 1.6 such that $|V(G)|$ is as small as possible. Obviously, G is connected. We have the following claims.

Claim 4.1. *We have $\delta(G) \geqslant 6$.*

Fact 4.1. In graph G^*, no 4-vertex is adjacent to any 4-vertex.

Claim 4.2. *In graph G^*, each k-vertex v is adjacent to at most $\lfloor \frac{k}{3} \rfloor$ 4-vertices, $k \geqslant 6$.*

Proof. Let v be a k-vertex in G, $k \geqslant 6$. Since G is a NIC-planar graph, two pairs of crossing edges share one common end-vertex at most, then v is the only common end-vertex. Therefore, no crossing adjacent to v shares the common end-vertex. We also conclude that three edges confirm a crossing, then v has at most $\lfloor \frac{k}{3} \rfloor$ adjacent 4-vertices, $k \geqslant 6$. □

Claim 4.3. *In graph G^*, each 4-face is incident with at most one 4-vertex, and each k-face is incident with at most $\lfloor \frac{k}{2} \rfloor$ 4-vertices, $k \geqslant 5$.*

Proof. Let $v_1v_2v_3v_4$ be a 4-face. If v_1 and v_3 are 4-vertices, then v_2 and v_3 are common end-vertices, which is a contradiction. Since each 4-vertex in the k-face is adjacent to two 6^+-vertices, and 6^+-vertex in the k-face is adjacent to at most two 4-vertices, then each k-face is incident with at most $\lfloor \frac{k}{2} \rfloor$ 4-vertices, $k \geqslant 5$. □

Claim 4.4. *In graph G^*, let c_i be every 4-vertex expect for 4-vertex v. When $|\theta(v) \cap \theta(c_i)| = 0$ and 4-vertex v is incident with four triangles, then v is adjacent to at least two 7^+-vertices.*

P r o o f. The vertices degree of G^* are the same as G since G^* is the associated plane graph of G. So we consider G instead.

Suppose from the contrary that G contains one 7-vertex v_1 which is adjacent to seven 6-vertices $\{v_2, v_3, \ldots, v_8\}$. Let $v_1v_3v_2v_4$ be a pair of crossing edges, and the crossing vertex is adjacent to one 7^+-vertex, see Figure 3 (1). Let $G' = G - v_1$, by the minimality of G. There is an OD-3-choosable φ of G'. Let $L(v_i)$ be the color list of vertex v_i. If $L(v_1) \setminus \{\varphi(v_2), \varphi(v_3), \ldots, \varphi(v_8)\} \neq \emptyset$, then let v_1 be colored by a color in $L(v_1)$ but out of $\{\varphi(v_2), \varphi(v_3), \ldots, \varphi(v_8)\}$. Obviously, G has OD-3 choosable. If $L(v_1) \setminus \{\varphi(v_2), \varphi(v_3), \ldots, \varphi(v_8)\} = \emptyset$, consider that $L(v_1)$ is of the same color as $\{\varphi(v_2), \varphi(v_3), \ldots, \varphi(v_8)\}$, and G does not have OD-3-choosable only if two adjacent vertices are of the same color and each color appears at least twice.

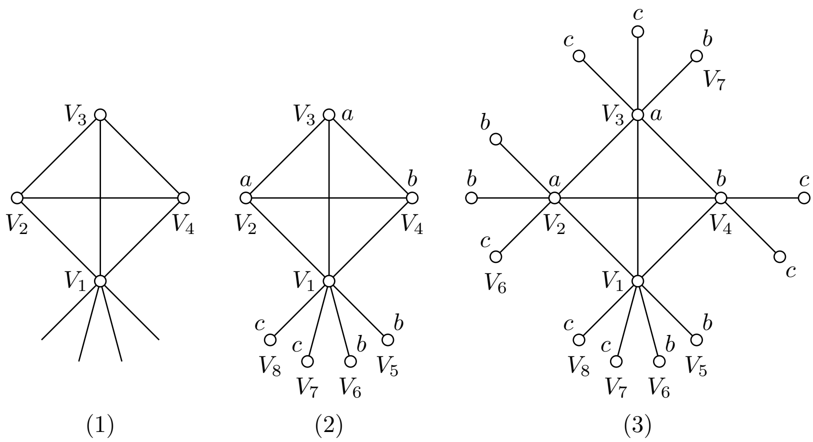


Figure 3. (1) Counterexample 1, (2) counterexample 2, (3) counterexample 3.

Let $L(v_1) = \{a, b, c\}$. In G, there must exist three monochromatic odd cycles over v_1, and be colored in different colors from $L(v_1)$. Assume $\varphi(v_2) = a$, $\varphi(v_3) = a$, $\varphi(v_4) = b$, $\varphi(v_5) = b$, $\varphi(v_6) = c$, $\varphi(v_7) = c$, $\varphi(v_8) = c$. Then G has the structure in Figure 3 (2).

Since $d(v_i) = 6$, $i = 2, 3, \ldots, 8$, the following structure exists.

By Figure 3 (3), it can be seen that at most two monochromatic odd cycles pass through vertex v_2 and v_3, so change the color of v_2 to c or change the color of v_3 to b, then v_1 can be colored as a. Therefore, G has an OD-3-choosable, a contradiction. So the crossing vertex is adjacent to at least two 7^+-vertices. In G^*, 4-vertex v is adjacent to at least two 7^+-vertices. $\square$

Claim 4.5. *In graph G^*, if $|\theta(c_1) \cap \theta(c_2)| = 1$ and 4-vertices c_1, c_2 are incident with four triangles, then 4-vertices c_1, c_2 are adjacent to four 7^+-vertices.*

P r o o f. The vertices degree of G^* are the same as G since G^* is the associated plane graph of G. So we consider G instead. Suppose to the contrary that G is a NIC-planar graph and $|\theta(c_1) \cap \theta(c_2)| = 1$. Let $d(v) = 6$, $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $d(v_i) = 7, i = 1, 2, \ldots, 6$. The crossing vertices c_1, c_2 are adjacent to three 7^+-vertices. Let $G' = G - v$, by the minimality of G. There is an OD-3-choosable φ of G'. Let $L(v)$ be the color list of vertex v. If $L(v) \setminus \{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_6)\} \neq \emptyset$, then let v be colored by a color in $L(v)$ but out of $\{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_6)\}$. Obviously, G has OD-3 choosable.

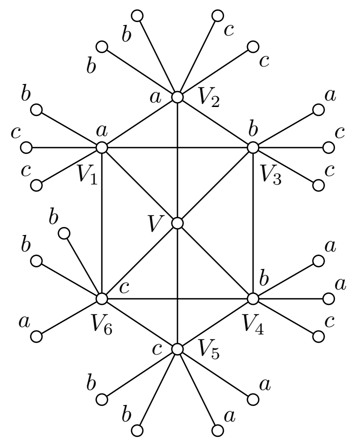


Figure 4. Counterexample 4.

If $L(v) \setminus \{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_6)\} = \emptyset$, consider that $L(v)$ is of the same color as $\{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_6)\}$, and G does not have OD-3-choosable only if two adjacent vertices are of the same color, and each color appears twice.

Let $L(v) = \{a, b, c\}$. In G there must exist three monochromatic odd cycles over v, and they must be colored in different colors in $L(v)$. Assume $\varphi(v_1) = a$, $\varphi(v_2) = a$, $\varphi(v_3) = b$, $\varphi(v_4) = b$, $\varphi(v_5) = c$, $\varphi(v_6) = c$. Since $d(v_i) = 7$, $i = 1, 2, \ldots, 6$, the following structure exists.

By Figure 4, it shows that at most two monochromatic odd cycles pass through the vertices v_1 and v_6, so change the color of v_1 to b, or change the color of v_6 to a, then v can be colored as b or a. Therefore, G has an OD-3-choosable, a contradiction. So the crossing vertex c_1, c_2 is adjacent to four 7^+-vertices. In G^*, 4-vertices c_1, c_2 is adjacent to four 7^+-vertices. □

Claim 4.6. *In graph G^*, if $|\theta(c_1) \cap \theta(c_2)| = 1$ and one of the 4-vertices, say c_1, is incident with four triangles, another 4-vertex, say c_2, is incident with three triangles, then 4-vertex c_1 is adjacent to four 7^+-vertices.*

P r o o f. By Claim 4.5, we change Figure 4 to Figure 5. It has similar structure. It does not change the degree of the neighbourhood of 4-vertex.

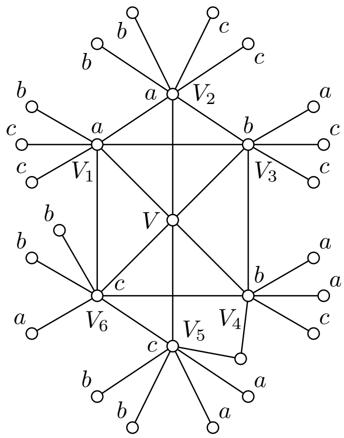


Figure 5. Counterexample 5.

□

In order to complete the proof, we used the discharging method. By Euler's formula $|V(G^*)| - |E(G^*)| + |F(G^*)| = 2$, we have

$$\sum_{x \in V(G^*) \cup F(G^*)} w(x) = \sum_{v \in V(G^*)} \Bigl(\frac{1}{2}d(v) - 3\Bigr) + \sum_{f \in F(G^*)} (d(f) - 3) = -6.$$

First, we give an initial charge function $w(v) = \frac{1}{2}d(v) - 3$ for every $v \in V(G^*)$, and $w(f) = d(f) - 3$ for every $f \in F(G^*)$. Next, we design discharging rules and redistribute weights accordingly. Let $w'(x)$ be the new charge after the discharging, $x \in V(G^*) \cup F(G^*)$. We will show that $w'(x) \geqslant 0$ for all $x \in V(G^*) \cup F(G^*)$. This leads to the following obvious contradiction:

$$-6 = \sum_{x \in V(G^*) \cup F(G^*)} w(x) = \sum_{x \in V(G^*) \cup F(G^*)} w'(x) \geqslant 0.$$

We are going to apply the following discharging rules:

(R1) A 4^+-face in G^* sends charge of 1 to each incident 4-vertex.

(R2) Let c_i be every 4-vertex expect for 4-vertex v, if $|\theta(v) \cap \theta(c_i)| = 0$, and 4-vertex v incident to four triangles, then each adjacent 7^+-vertex sends charge $\frac{1}{2}$ to 4-vertex v.

(R3) If $|\theta(c_1) \cap \theta(c_2)| = 1$, one of the 4-vertices, say c_1, is incident with four triangles, another 4-vertex, say c_2, is incident with at least three triangles, then each adjacent 7^+-vertex sends charge $\frac{1}{4}$ to 4-vertex v.

(R4) If $|\theta(c_1)\cap\theta(c_2)| = 1$, one of the 4-vertices, say c_1, is incident with four triangles, another 4-vertex, say c_2, is incident with at most two triangles, then c_2 sends charge 1 to c_1.

We will show that $w'(x) \geqslant 0$ for all $x \in V(G^*) \cup F(G^*)$.

(1) If $k \geqslant 5$, by Claim 4.3, each k-face f has at most $\lfloor \frac{k}{2} \rfloor$ incident 4-vertices. By rule (R1), a 5^+-face sends charge of 1 to each incident 4-vertex. So $w'(f) \geqslant k - 3 - 1 \times \lfloor \frac{k}{2} \rfloor$. Since $k \geqslant 5$, $w'(f) \geqslant 0$.

(2) If $k = 4$, by Claim 4.4, each 4-face f has at most one incident 4-vertex. By rule (R1), a 4-face sends charge of 1 to each incident 4-vertex. So $w'(f) \geqslant 4 - 3 - 1 = 0$.

(3) Let c_i be every 4-vertex expect for 4-vertex v. If $|\theta(v)\cap\theta(c_i)| = 0$, for 4-vertex v incident to four triangles, by rule (R2) and Claims 4.3, 4.4, each adjacent 7^+-vertex u sends charge $\frac{1}{2}$ to 4-vertex v. So $w'(u) \geqslant \frac{1}{2} \times 7 - 3 - 1 \times \frac{1}{2} = 0$.

(4) If $|\theta(c_1) \cap \theta(c_2)| = 1$.

(i) One of the 4-vertices, say c_1, is incident to four triangles and another 4-vertex, say c_2, is incident with at most two triangles, by rule (R1), (R4), then the charge of $c_2 \geqslant \frac{1}{2} \times 4 - 3 + 2 = 1$, so we need c_2 to send charge 1 to c_1, so $w'(c_1) = \frac{1}{2} \times 4 - 3 + 1 = 0$ and $w'(c_2) = 0$.

(ii) One of the 4-vertices, say c_1, is incident with four triangles, another 4-vertex, say c_2, is incident with at least three triangles, by Claims 4.5, 4.6 and rule (R3), for each 7^+-vertex u, $w'(u) \geqslant \frac{1}{2} \times k - 3 - \lfloor \frac{k}{3} \rfloor \times \frac{1}{4}$, since $k \geqslant 7$, $w'(u) \geqslant 0$, for each 4-vertex v, $w'(v) \geqslant \frac{1}{2} \times 4 - 3 + \frac{1}{4} \times 4 = 0$.

(5) For each 4-vertex v, by rule (R1), (R2), (R3), (R4), $w'(v) \geqslant 0$.

From the above discussion, we have

$$\sum_{x \in V(G^*)\cup F(G^*)} w'(x) \geqslant 0 \quad \text{for all } x \in V(G^*) \cup F(G^*).$$

It is a contradiction, which completes the proof of Theorem 1.6. $\square$

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