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FINITELY SILTING COMODULES IN QUASI-FINITE COMODULE CATEGORY

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Abstract. We introduce the notions of silting comodules and finitely silting comodules in quasi-finite category, and study some properties of them. We investigate the torsion pair and dualities which are related to finitely silting comodules, and give the equivalences among silting comodules, finitely silting comodules, tilting comodules and finitely tilting comodules.

Keywords: quasi-finite silting comodule; finitely silting comodule; finitely tilting comodule; torsion pair; duality

MSC 2020: 16T15, 18E40, 18G15

1. Introduction

Silting theory is vital for the study of representation theory. It has been developed gradually from many aspects. Silting complexes in the triangulated category were first introduced by Keller and Vossieck, see [9]. They are important tools to study t-structures in the bounded derived category of representations of Dynkin quivers. In [5], the authors introduced the notion of (partial) silting R-modules (where R is a unital associative ring) as a common generalization of tilting modules over an arbitrary ring (see [7]), as well as the notion of support τ-tilting modules over a finite dimensional algebra, see [1]. Moreover, they showed these modules generate torsion classes that provide left approximations, and every partial silting module admits an analogue of the Bongartz complement, all silting modules are quasitilting modules. In [4], Angeleri and Hrbek gave the classification of silting modules over commutative rings, and established a bijective correspondence with Gabriel filters of finite type.

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The coalgebra theory, dual to the representation of algebra, has attracted extensive attention of eminent scholars. Many scholars devoted themselves to investigating this aspect and came up with some open problems, see [8], [11]. For instance, Professor Simson in Copernicus University, pointed nine questions out, about the structure of coalgebra and the research for representation, see [13]. The fifth is in connection with constructing the tilting and cotilting theory in comodule category. In 1998, Wang in [16] defined the classical tilting comodules in comodule category. In 1999, Wang in [17] introduced the tilting comodules and partial tilting comodules over coalgebras, and showed that every tilting comodule can induce a torsion pair. In 2008, Simson in [14] gave a description of the f-cotilting comodules and cotilting procedures over coalgebras, and constructed a pair of cotilting functors of Brenner-Butler type over coalgebras. Yuan and Yao in [19] introduced the notions of silting comodules and partial silting comodules over coalgebras, and obtained the Bongartz theorem for the complement of partial silting comodules.

Inspired by the above researchers, in this paper, we investigate the quasi-finitely silting comodules induced by silting comodules. And we introduce the finitely silting comodules, also study many properties of it.

The paper is organized as follows. In Section 2, we present some necessary notations and conventions which are used throughout the paper. We also state some results which are used in next sections. In Section 3, we introduce the concept of quasi-finitely silting comodules. In Section 4, we discuss the finitely silting comodules, and study some nice properties of this new notion. In Section 5, we investigate the torsion pair and dualities which are in connection with finitely silting comodules and equivalences among silting comodules, finitely silting comodules, finitely tilting comodules, and tilting comodules.

2. Preliminaries

Throughout the paper, unless explicitly stated, let C, D be coalgebras over a commutative ring R, and under the mild hypothesis that the involved coalgebras are flat as R-modules. We always assume that C and D are right semiperfect when we use the derived functor $\operatorname{ext}^i_C(T,-)$ ($\operatorname{ext}^i_C(-,T)$) of cohom functor $h_C(T,-)$ ($h_C(-,T)$).

We give some notations at first.

We denote the category of all right C-comodules (or left D-comodules) by $\mathcal{M}^C$ (or ${}^D\mathcal{M}$). For a comodule T we denote by $\operatorname{Add}(T)$ (or by $\operatorname{add}(T)$) the class of all C-comodules which are isomorphic to direct summands of direct sums (or finite direct sums) of copies of T. Moreover, we consider the perpendicular class $T^{\perp}$ of a quasi-finite right C-comodule T, defined as $T^{\perp}=\{X\in\mathcal{M}^C\colon \operatorname{ext}^1_C(T,X)=0\}$. Let $\operatorname{Cogen}(T_C)=\{X\colon 0\to X\to T^{(I)},\ I \text{ is a set}\}$, see [16], Proposition 5.3. If X

is a comodule, we denote the reject of X in T by $\mathrm{Rej}_T(X)$, where $\mathrm{Rej}_T(X) = \bigcap_{f\in\mathrm{Com}(X,T)} \mathrm{Ker}\, f$. We denote the canonical natural transformation of a quasi-finite right C-comodule T, $\eta\colon 1_{\mathcal{M}^C} \to G\circ F$, by $\theta_X\colon X \to h_C(T,X) \,\square\, T$, where $G = - \,\square_D\, T_C$ and $F = h_C({}_DT_C, -)$. A quasi-finite comodule X is called *θ-torsionless* (or *θ-reflexive*) if θ_X is a monomorphism (or an isomorphism). We recall that Refl_θ is the class of all θ-reflexive comodules.

In order to define the notion of silting comodules, we have to give the class $\mathcal{B}_\gamma$ at first. Let $\gamma\colon Q_0 \to Q_1$ be a homomorphism. We have a class of comodules

$$\mathcal{B}_\gamma = \{X\colon \mathrm{Com}_C(X,\gamma) \text{ is an epimorphism}\}.$$

Definition 2.1 ([19]). We say that a right comodule T_C is silting (with respect to γ) if there exists an injective copresentation

$$0 \to T \xrightarrow{f} Q_0 \xrightarrow{\gamma} Q_1$$

of T such that $\mathrm{Cogen}(T_C) = \mathcal{B}_\gamma$.

Definition 2.2 ([2]). A right C-comodule X is called *quasi-finite* if the functor $-\otimes X\colon R\text{-Mod} \to \mathcal{M}^C$ has a left adjoint. A quasi-finite left D-comodule is defined analogously.

Notation 2.3. *For a quasi-finite right C-comodule X we denote the left adjoint of the functor $-\otimes X$ by $h_C(X,-)\colon \mathcal{M}^C \to R\text{-Mod}$. This functor is called the cohom functor.*

Lemma 2.4 ([18], Proposition 8.1.2)**.** *For a bicomodule ${}_DX_C$, the following statements are equivalent.*

(1) *X_C is quasi-finite.*

(2) *The functor $- \,\square_D\, X\colon \mathcal{M}^D \to \mathcal{M}^C$ has the left adjoint. In this case, the left adjoint is given by $h_C(X,-)\colon \mathcal{M}^C \to \mathcal{M}^D$.*

It follows that there is a very useful isomorphism for any Y_C and W_D:

$$\mathrm{Com}_D(h_C(X,Y),W) \cong \mathrm{Com}_C(Y, W\,\square_D\, X).$$

By [18], *Section* 8.1, *if R is a field, there is $(-)^* = \mathrm{Hom}_R(-,R) \cong \mathrm{Com}_C(-,C)$, and*

$$h_C(X,Y) = \varinjlim \mathrm{Com}_C(Y_i,X)^* \cong \varinjlim (X\,\square_C\, Y_i^*)^*.$$

If R is a commutative ring, we have that $h_C(X,Y)^* = \mathrm{Hom}_R(h_C(X,Y),R) \cong \mathrm{Com}_C(Y, R\otimes X) = \mathrm{Com}_C(Y,X)$ by [2], *Section* 3.4.

Definition 2.5 ([12]). Let X be a C-comodule. Then X is called *cocoherent* if X is finitely cogenerated and every finitely cogenerated quotient comodule of X is finitely copresented.

We say that the coalgebra C is right (or left) cocoherent if the C-comodule C_C (or ${}_CC$) is a cocoherent C-comodule.

Lemma 2.6. *Assume that C is cocoherent, $0 \to X \to Y \to Z \to 0$ is a short exact sequence with X, Y being finitely copresented. Then Z is finitely copresented.*

3. Quasi-finite silting comodules

Let T be a quasi-finite right C-comodule with co-endomorphism $D = e_C(T)$, and we have already known that T is a (D, C)-bicomodule by [2], Section 3.4. Let Q_0 and Q_1 be right C-comodules which are quasi-finite throughout this paper.

According to [19], we define the quasi-finite silting comodules similarly.

Definition 3.1. Let $\gamma\colon Q_0 \to Q_1$ be a homomorphism between right C-comodules Q_0 and Q_1. Then we have a class of comodules as:

$$\mathcal{C}_\gamma = \{X \text{ is a right } C\text{-comodule: } h_C(\gamma, X) \text{ is a monomorphism}\}.$$

Lemma 3.2. *Let $\gamma\colon Q_0 \to Q_1$ be a homomorphism and assume that $\gamma = \tau_\gamma \circ \pi_\gamma$ is the canonical factorization of γ. The following statements are equivalent for a right C-comodule X.*

(1) $X \in \mathcal{C}_\gamma$.

(2) *$h_C(\tau_\gamma, X)$ is an isomorphism and $h_C(\pi_\gamma, X)$ is a monomorphism.*

Proof. (1) $\Rightarrow$ (2) $h_C(\gamma, X) = h_C(\tau_\gamma \circ \pi_\gamma, X) = h_C(\pi_\gamma, X) \circ h_C(\tau_\gamma, X)$. Since $h_C(\gamma, X)$ is a monomorphism, we have that $h_C(\tau_\gamma, X)$ is a monomorphism. As $h_C(-, X)$ is a contravariant right exact functor, we obtain that $h_C(\tau_\gamma, X)$ is an epimorphism. Then $h_C(\tau_\gamma, X)$ is an isomorphism. We will prove that $h_C(\pi_\gamma, X)$ is a monomorphism next.

Consider the natural isomorphism in the following diagram:

$$\begin{tikzcd}
h_C(Q_0, X)^* \arrow[r, "h_C(\gamma, X)^*"] \arrow[d, "\cong"'] & h_C(Q_1, X)^* \arrow[d, "\cong"] \\
\mathrm{Com}_C(X, Q_0) \arrow[r, "\mathrm{Com}_C(X, \gamma)"'] & \mathrm{Com}_C(X, Q_1).
\end{tikzcd}$$

Since $h_C(\gamma, X)$ is a monomorphism, we have that $\mathrm{Com}_C(X, \gamma)$ is an epimorphism, i.e. for any $g\colon X \to Q_1$, there is an $f\colon X \to Q_0$ such that $\gamma f = g$. Then for any $h\colon X \to \operatorname{Im}\gamma$, $\tau_\gamma h \in \mathrm{Com}_C(X, Q_1)$, we have that there is an $f'\colon X \to Q_0$ such that $\tau_\gamma h = \gamma f'$. Therefore, $\mathrm{Com}_C(X, \pi_\gamma)$ is an epimorphism. We consider the natural isomorphism:

$$\begin{tikzcd} h_C(Q_0, X)^* \arrow[r, "h_C(\pi_\gamma, X)^*"] \arrow[d, "\cong"'] & h_C(\operatorname{Im}\gamma, X)^* \arrow[d, "\cong"] \\ \mathrm{Com}_C(X, Q_0) \arrow[r, "\mathrm{Com}_C(X,\pi_\gamma)"'] & \mathrm{Com}_C(X, \mathrm{Im}\gamma). \end{tikzcd}$$

So $h_C(\pi_\gamma, X)^*$ is an epimorphism. By the commutative diagram

$$\begin{tikzcd} h_C(\operatorname{Im}\gamma, X) \arrow[r, "h_C(\pi_\gamma, X)"] \arrow[d, "\varrho_1"'] & h_C(Q_0, X) \arrow[d, "\varrho_2"] \\ h_C(\operatorname{Im}\gamma, X)^{**} \arrow[r, "h_C(\pi_\gamma,X)^{**}"'] & h_C(Q_0, X)^{**} \end{tikzcd}$$

since ϱ_1 and ϱ_2 are injective, $h_C(\pi_\gamma, X)^{**}$ is a monomorphism, we have $h_C(\pi_\gamma, X)$ is a monomorphism.

(2) $\Rightarrow$ (1) It is obvious. $\square$

Corollary 3.3. *Let $\gamma\colon Q_0 \to Q_1$ be a homomorphism with $T = \mathrm{Ker}(\gamma)$ and assume that $\gamma = \tau_\gamma \circ \pi_\gamma$ is the canonical factorization. The following statements are equivalent for a quasi-finite C-comodule X which belongs to $T^\perp$.*

(1) *$X \in \mathcal{C}_\gamma$.*

(2) *$h_C(\tau_\gamma, X)$ is an isomorphism.*

Lemma 3.4. *Let $\gamma\colon Q_0 \to Q_1$ be a homomorphism, the following statements hold.*

(1) *The class $\mathcal{C}_\gamma$ is closed under direct sums.*

(2) *If Q_1 is injective, then the class $\mathcal{C}_\gamma$ is closed under subcomodules.*

(3) *If Q_0 is injective, then the class $\mathcal{C}_\gamma$ is closed under extensions.*

(4) *If Q_0 is injective and $T = \mathrm{Ker}(\gamma)$, then $\mathcal{C}_\gamma \subseteq T^\perp$.*

(5) *Assume that Q_1 is injective. If $T = \mathrm{Ker}(\gamma)$ and*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} X \to 0 \tag{3.1}$$

is an exact sequence in $\mathcal{M}^C$ such that A and B belong to $\mathcal{C}_\gamma$ and $X \in T^\perp$, then $X \in \mathcal{C}_\gamma$.

P r o o f. (1) This follows from the fact that the cohom functor respects direct sums.

(2) Let $Y \in \mathcal{C}_\gamma$ and $f\colon X \to Y$ be a monomorphism. Then $h_C(\gamma, Y)$ is a monomorphism by Lemma 3.2. By the natural isomorphism

$$\begin{tikzcd} h_C(Q_0, Y)^* \arrow[r, "h_C(\gamma,Y)^*"] \arrow[d, "\cong"'] & h_C(Q_1, Y)^* \arrow[d, "\cong"] \\ \mathrm{Com}_C(Y, Q_0) \arrow[r, "\mathrm{Com}_C(Y,\gamma)"'] & \mathrm{Com}_C(Y, Q_1) \end{tikzcd}$$

we learn that $\mathrm{Com}_C(Y, \gamma)$ is an epimorphism, i.e., $Y \in \mathcal{B}_\gamma$.

And by another natural isomorphism

$$\begin{tikzcd} h_C(Q_0, X)^* \arrow[r, "h_C(\gamma,X)^*"] \arrow[d, "\cong"'] & h_C(Q_1, X)^* \arrow[d, "\cong"] \\ \mathrm{Com}_C(X, Q_0) \arrow[r, "\mathrm{Com}_C(X,\gamma)"'] & \mathrm{Com}_C(X, Q_1) \end{tikzcd}$$

we need to show $\mathrm{Com}_C(X, \gamma)$ is an epimorphism.

Since $\mathcal{B}_\gamma$ is closed under subcomodules by [19], Lemma 2, we can get the $\mathrm{Com}_C(X, \gamma)$ is an epimorphism. By the commutative diagram

$$\begin{tikzcd} h_C(Q_0, X) \arrow[r, "h_C(\gamma,X)"] \arrow[d, "\varrho_1"'] & h_C(Q_1, X) \arrow[d, "\varrho_2"] \\ h_C(Q_0, X)^{**} \arrow[r, "h_C(\gamma,X)^{**}"'] & h_C(Q_1, X)^{**} \end{tikzcd}$$

since ϱ_1 and ϱ_2 are injective, $h_C(\gamma, X)^{**}$ is a monomorphism, we have $h_C(\gamma, X)$ is a monomorphism, i.e., $X \in \mathcal{C}_\gamma$.

(3) Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence with $X, Z \in \mathcal{C}_\gamma$, i.e., $h_C(\gamma, X)$ and $h_C(\gamma, Z)$ are monomorphisms. Then $\mathrm{Com}_C(X, \gamma)$ and $\mathrm{Com}_C(Z, \gamma)$ are epimorphisms. This means $X, Z \in \mathcal{B}_\gamma$. Since $\mathcal{B}_\gamma$ is closed under extensions, we have that $Y \in \mathcal{B}_\gamma$, i.e., $\mathrm{Com}_C(Y, \gamma)$ is an epimorphism. It follows that $h_C(\gamma, Y)$ is a monomorphism, i.e., $Y \in \mathcal{C}_\gamma$.

(4) By the assumption, $h_C(\gamma, X)$ and $h_C(\pi_\gamma, X)$ are monomorphisms. Applying the functor $h_C(-, X)$ on the exact sequence

$$0 \to T \xrightarrow{i} Q_0 \xrightarrow{\pi_\gamma} \mathrm{Im}\,\gamma \to 0,$$

we obtain an exact sequence as follows:

$$
\begin{aligned}
0 \longrightarrow \operatorname{ext}^1_C(T,X) \longrightarrow h_C(\operatorname{Im}\gamma, X) \xrightarrow{h_C(\pi_\gamma, X)} h_C(\operatorname{Im}\gamma, X) \\
\xrightarrow{h_C(i,X)} h_C(Q_0, X) \longrightarrow h_C(T,X) \longrightarrow 0.
\end{aligned}
$$

Then $\operatorname{ext}^1_C(T,X) = 0$, i.e., $X \in T^{\perp}$.

(5) Suppose that $\gamma = \tau_\gamma \circ \pi_\gamma$ is the canonical factorization. Applying the covariant functor $h_C(\operatorname{Im}\gamma, -)$ on the exact sequence (3.1), we can get the exact sequence

$$h_C(\operatorname{Im}\gamma, A) \to h_C(\operatorname{Im}\gamma, B) \to h_C(\operatorname{Im}\gamma, X) \to 0.$$

We also apply the covariant functor $h_C(Q_1, -)$ on (3.1), then we obtain the exact sequence

$$0 \to h_C(Q_1, A) \to h_C(Q_1, B) \to h_C(Q_1, X) \to 0.$$

Since A and B lie in $\mathcal{C}_\gamma$, it follows by Lemma 3.2 that both $h_C(\tau_\gamma, A)$ and $h_C(\tau_\gamma, B)$ are isomorphisms. So $h_C(\operatorname{Im}\gamma, f)$ is a monomorphism. Hence, we have the following commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & h_C(Q_1, A) \arrow[r] \arrow[d, "h_C(\tau_\gamma, A)"] & h_C(Q_1, B) \arrow[r] \arrow[d, "h_C(\tau_\gamma, B)"] & h_C(Q_1, X) \arrow[r] \arrow[d, "h_C(\tau_\gamma, X)"] & 0 \\
0 \arrow[r] & h_C(\operatorname{Im}\gamma, A) \arrow[r] & h_C(\operatorname{Im}\gamma, B)) \arrow[r] & h_C(\operatorname{Im}\gamma, X) \arrow[r] & 0.
\end{tikzcd}
$$

Since $A, B \in \mathcal{C}_\gamma$, we have that $h_C(\tau_\gamma, A), h_C(\tau_\gamma, B)$ are isomorphisms. Applying the Snake Lemma, we get that $h_C(\tau_\gamma, X)$ is an isomorphism. By Corollary 3.3, we can obtain the conclusion. □

Definition 3.5. We say that a quasi-finite right C-comodule T is

(1) a partial silting comodule (with respect to γ) if there exists a quasi-finite injective copresentation of T

$$0 \to T \to Q_0 \to Q_1$$

such that $T \in \mathcal{C}_\gamma$;

(2) a silting comodule (with respect to γ) if there exists a quasi-finite injective copresentation

$$0 \to T \to Q_0 \to Q_1$$

of T such that $\operatorname{Cogen}(T_C) = \mathcal{C}_\gamma$.

Remark 3.6. Note that the differences between silting comodules in [19] and quasi-finite silting comodules are as follows.

(1) Quasi-finite silting comodules are quasi-finite.

(2) Quasi-finite partial silting comodules are defined with respect to the class $\mathcal{C}_\gamma$. Since $\mathcal{C}_\gamma$ is closed under direct sums clearly, we omit this condition in the definition.

(3) Quasi-finite silting comodules are defined with respect to the class $\operatorname{Cogen}(T_C)$, which is defined in Preliminaries. It is different to the class $\operatorname{Cogen}(T_C)$, which is given in Definition 2.1 of silting comodules.

Proposition 3.7 ([16], Proposition 5.3)**.** *Let T_C be a quasi-finite right C-comodule with co-endomorphism $D = e_C(T)$. The comodule $X \in \operatorname{Cogen}(T_C)$ if and only if $\theta_X\colon X \to h_C(T,X) \mathbin{\square} T$ is a monomorphism.*

Lemma 3.8. *Let T_C be a quasi-finite right C-comodule. Then*

(1) $X/\operatorname{Ker}\theta_X \in \operatorname{Cogen}(T_C)$.

(2) $\operatorname{Ker}\theta_X = \operatorname{Rej}_T(X)$ *whenever T is faithfully coflat.*

Proof. For any $M = \operatorname{Rej}_T(X)$, it is clear that $f(M) = 0$, where $f \in \operatorname{Com}_C(X,T)$, i.e., $\operatorname{Com}_C(M,T) = 0$. By the fact that $h_C(T,M)^* \cong \operatorname{Com}_C(M,T)$ and $h_C(T,M) \hookrightarrow h_C(T,M)^{**}$, $h_C(T,M) \mathbin{\square} T = 0$, i.e., $\operatorname{Rej}_T(X) \subseteq \operatorname{Ker}\theta_X$. Since $X/\operatorname{Ker}\theta_X$ is a subcomodule of $X/\operatorname{Rej}_T(X)$ and by [3], Corollary 8.13 $X/\operatorname{Rej}_T(X)$ is cogenerated, we have $X/\operatorname{Ker}\theta_X \in \operatorname{Cogen}(T_C)$.

Moreover, let $M = \operatorname{Ker}\theta_X$, then $h_C(T,M) \mathbin{\square} T = 0$. Since T is faithfully coflat, we can obtain $h_C(T,M) = 0$. It follows that for any $f \in \operatorname{Com}_C(X,T)$, $f(M) = 0$, i.e., $M \subseteq \operatorname{Rej}_T(X)$. Therefore, $\operatorname{Ker}\theta_X = \operatorname{Rej}_T(X)$. $\square$

Proposition 3.9. *Let T be a quasi-finite right C-comodule and $T = \operatorname{Ker}(\gamma)$, where $\gamma\colon Q_0 \to Q_1$ is a homomorphism between injective right C-comodules. Then T is a silting comodule with respect to γ if and only if*

(1) $T^{(I)} \in \mathcal{C}_\gamma$ *for all sets I;*

(2) $\operatorname{Ker} h_C(T,-) \cap \mathcal{C}_\gamma = 0$.

Proof. Necessity. If T is a silting comodule with respect to γ, then we have the equality $\operatorname{Cogen}(T_C) = \mathcal{C}_\gamma$. It follows that $T^{(I)} \in \mathcal{C}_\gamma$ for all sets I. Now let $X \in \operatorname{Ker} h_C(T,-) \cap \mathcal{C}_\gamma$, i.e., $h_C(T,X) = 0$ and $X \in \mathcal{C}_\gamma$. Since $X \in \operatorname{Cogen}(T_C)$, there is a monomorphism $0 \to X \xrightarrow{f} T^{(I)}$. Now, we assume that $X \neq 0$, then $f \neq 0$, i.e., $\operatorname{Com}_C(X,T^{(I)}) \neq 0$. It is clear that $\operatorname{Com}_C(X,T)^{(I)} \neq 0$ and $\operatorname{Com}_C(X,T) \neq 0$. Therefore, $h_C(T,X) \neq 0$, which is a contradiction. It follows that $X = 0$.

Sufficiency. Suppose that statements (1) and (2) hold. Let $X \in \mathrm{Cogen}(T_C)$, then there is a monomorphism $0 \to X \xrightarrow{f} T^{(I)}$. By (1), we have that $T^{(I)} \in \mathcal{C}_\gamma$. And by Lemma 3.4 (2), we obtain that $X \in \mathcal{C}_\gamma$, i.e., $\mathrm{Cogen}(T_C) \subseteq \mathcal{C}_\gamma$.

In order to show the reverse inclusion $\mathcal{C}_\gamma \subseteq \mathrm{Cogen}(T)$, we consider $X \in \mathcal{C}_\gamma$. Then we have an exact sequence

$$0 \to \mathrm{Ker}\,\theta_X \xrightarrow{i} X \xrightarrow{p} X/\,\mathrm{Ker}\,\theta_X \to 0.$$

Since, in general, $X/\mathrm{Ker}\,\theta_X \in \mathrm{Cogen}(T)$, it follows from the above that $X/\mathrm{Ker}\,\theta_X \in \mathcal{C}_\gamma$. By Lemma 3.4 (4), we obtain that $X/\,\mathrm{Ker}\,\theta_X \in T^\perp$, i.e., $\mathrm{ext}^1_C(T, X/\,\mathrm{Ker}\,\theta_X) = 0$. Hence, we have the short exact sequence

$$0 \longrightarrow h_C(T, \mathrm{Ker}\,\theta_X) \xrightarrow{h_C(T,i)} h_C(T, X) \xrightarrow{h_C(T,p)} h_C(T, X/\,\mathrm{Ker}\,\theta_X) \longrightarrow 0.$$

Since $\mathrm{Com}_C(p, T)$ is an isomorphism, we obtain that $h_C(T, p)$ is also an isomorphism. Since $\mathrm{Im}\, h_C(T, i) = \mathrm{Ker}\, h_C(T, p) = 0$ and $h_C(T, i)$ is a monomorphism obviously, we learn that $h_C(T, \mathrm{Ker}\,\theta_X) = 0$. Therefore, $\mathrm{Ker}\,\theta_X \in \mathrm{Ker}\, h_C(T, -)$. On the other hand, applying Lemma 3.4 (2), it follows that $\mathrm{Ker}\,\theta_X \in \mathcal{C}_\gamma$. From statement (2), $\mathrm{Ker}\,\theta_X = 0$. And $X \in \mathrm{Cogen}(T)$ by Proposition 3.7. To sum up, $\mathcal{C}_\gamma = \mathrm{Cogen}(T)$, that is, T is a silting comodule. $\square$

In particular, if T is faithfully coflat, we have $\mathrm{Ker}\,\theta_X = \mathrm{Ker}\, h_C(T, -)$. Then $X \in \mathrm{Cogen}(T)$ clearly. The conclusion above is also right.

4. Quasi-finite finitely silting comodule

For a quasi-finite right C-comodule T_C, we consider the following classes $\mathcal{Y}_C = \mathrm{Cogen}(T_C) \cap \text{comod-}C$ and $\mathcal{X}_C = \mathrm{Ker}\, h_C(T, -) \cap \text{comod-}C$, where the comod-$C$ represents the category of finitely cogenerated right C-comodules. And let $X' = \mathrm{Ker}\,\theta_X$ from now on. Moreover, we denote $\mathrm{ext}^1_C(T, -)$ by $\Gamma_C(-)$ and, in the case the coalgebra C is understood, we simply write $\Gamma(-)$.

Lemma 4.1. *Let T be a quasi-finite C-comodule such that $T = \mathrm{Ker}(\gamma)$, where $\gamma\colon Q_0 \to Q_1$ is a C-homomorphism between injective right C-comodules. Suppose that all C-comodules in $\mathcal{Y}_C$ belong to the class $\mathcal{C}_\gamma$ and all C-comodules in $\mathcal{X}_C$ are finitely copresented. Let X be a finitely cogenerated C-comodule. Then the following statements hold:*

(1) *$X/X' \in \mathcal{Y}_C$ if and only if $h_C(T, X) \cong h_C(T, X/X')$ if and only if $X' \in \mathcal{X}_C$.*
(2) *If $\mathrm{inj.dim}\, T_C \leqslant 1$, then $\Gamma(X) \cong \Gamma(X')$.*
(3) *If R is QF, and there is an exact sequence $0 \to X \to C^n \to K \to 0$, then $\mathrm{ext}^{n+1}(T, K) \cong \mathrm{ext}^n(T, X)$.*

P r o o f. By the hypothesis, when $X/X' \in \mathcal{Y}_C$, we have that $X/X' \in \mathcal{C}_\gamma$. Then $\Gamma(X/X') = 0$, and we have the exact sequence

$$0 \to X' \xrightarrow{i} X \xrightarrow{p} X/X' \to 0.$$

Applying the functor $h_C(T, -)$, we can obtain that

$$0 = \Gamma(X/X') \longrightarrow h_C(T, X') \xrightarrow{h_C(T,i)} h_C(T, X) \xrightarrow{h_C(T,p)} h_C(T, X/X') \longrightarrow 0$$

and

$$\ldots \to \mathrm{ext}^2_C(T, X/X') \xrightarrow{\delta^1} \Gamma(X') \xrightarrow{\Gamma(i)} \Gamma(X) \to 0.$$

Since $h_C(T, p)$ is an isomorphism, we have that $h_C(T, X) \cong h_C(T, X/X')$ and $h_C(T, X') = 0$, i.e., $X' \in \operatorname{Ker} h_C(T, -)$. Therefore, $X' \in \mathcal{X}_C$.

Conversely, for $X' \in \mathcal{X}_C$, we have that $X' \in \operatorname{Ker} h_C(T, -)$, i.e., $h_C(T, X') = 0$. Therefore, $h_C(T, p)$ is an isomorphism, i.e., $h_C(T, X) \cong h_C(T, X/X')$. By the hypothesis, all C-comodules in $\mathcal{X}_C$ are finitely copresented, then X/X' is finitely cogenerated. And by $X/X' \in \mathrm{Cogen}(T_C)$, we learn that $X/X' \in \mathcal{Y}_C$.

In order to show (2), we show that $\mathrm{ext}^2_C(T, Y) = 0$ for any right C-comodule Y. We have already known that $h_C(T, Y)^* \cong \mathrm{Com}_C(Y, T)$ for any right C-comodule Y, and then $\mathrm{ext}^n_C(T, Y)^* \cong \mathrm{Ext}^n_C(Y, T)$. It follows that inj.dim $T_C \leqslant 1$ if and only if $\mathrm{ext}^2_C(T, Y) = 0$ for any right comodule Y, so is X/X', see [18], Proposition 12.2.2.

Since R is a QF-ring, we have that the coalgebra C over R is also projective. Applying the cohom functor on the exact sequence

$$0 \to X \to C^n \to K \to 0,$$

by the fact that $\mathrm{ext}^i_C(T, C) = 0$ for all $i \geqslant 1$, we can get the conclusion easily. □

Corollary 4.2. *Let T be a quasi-finite right C-comodule such that $T = \mathrm{Ker}(\gamma)$, where $\gamma\colon Q_0 \to Q_1$ is a homomorphism between injective comodules. Suppose that all C-comodules in $\mathcal{Y}_C$ belong to the class $\mathcal{C}_\gamma$. Let X be a conoetherian C-comodule and let $X' = \operatorname{Ker} \theta_X$. If X is finitely cogenerated, then the following statements hold.*

(1) $X' \in \mathcal{X}_C$.

(2) $X/X' \in \mathcal{Y}_C$.

(3) $h_C(T, X) \cong h_C(T, X/X')$.

P r o o f. Since $X/X' \in \mathrm{Cogen}(T_C)$ and X is conoetherian, we have $X/X' \in$ comod-C. Thus, $X/X' \in \mathcal{Y}_C$. By the hypothesis, we have $X/X' \in \mathcal{C}_\gamma$. And $\Gamma(X/X') = 0$ by Lemma 3.4 (4). Applying the covariant $h_C(T, -)$ functor on the canonical short exact sequence

$$0 \to X' \xrightarrow{i} X \xrightarrow{p} X/X' \to 0$$

we obtain the exact sequence

$$0=\Gamma(X/X') \longrightarrow h_C(T,X') \xrightarrow{h_C(T,i)} h_C(T,X) \xrightarrow{h_C(T,p)} h_C(T,X/X') \longrightarrow 0.$$

Since $h_C(T,p)$ is an isomorphism, i.e., $h_C(T,X)\cong h_C(T,X/X')$, we obtain that $h_C(T,X')=0$, i.e., $X'\in \operatorname{Ker} h_C(T,-)$. As X' is a subcomodule of X, we have that X' is finitely cogenerated, i.e., $X'\in$ comod-C. Therefore, $X'\in \mathcal{X}_C$. □

Lemma 4.3. *Let T be a quasi-finite C-comodule and $\gamma\colon Q_0\to Q_1$ be a homomorphism between injective comodules. Suppose that $T=\operatorname{Ker}(\gamma)$ and $\operatorname{Ker} h_C(T,-)\cap \mathcal{C}_\gamma=0$. If X is a right C-comodule such that $X\in\mathcal{C}_\gamma$ and $X/X'\in\mathcal{C}_\gamma$, then $X\in \operatorname{Cogen}(T_C)$.*

Proof. Since $X/X'\in\mathcal{C}_\gamma$, and by Lemma 3.4 (4), we obtain that $\Gamma(X/X')=0$. Applying the covariant $h_C(T,-)$ functor on the short exact sequence

$$0\to X'\xrightarrow{i} X\xrightarrow{p} X/X'\to 0$$

we obtain a short exact sequence

$$0 \longrightarrow h_C(T,X') \xrightarrow{h_C(T,i)} h_C(T,X) \xrightarrow{h_C(T,p)} h_C(T,X/X') \longrightarrow 0$$

and then, taking into account that $h_C(T,p)$ is an isomorphism, we have $h_C(T,X')$=0, i.e., $X'\in\operatorname{Ker} h_C(T,-)$. It follows that $X'\in\mathcal{C}_\gamma$, by Lemma 3.4 (2). From the hypothesis, we get that $X'=0$. And by Proposition 3.7, we learn that $X\in\operatorname{Cogen}(T_C)$. □

Proposition 4.4. *Let T_C be a quasi-finite C-comodule with $D=e_C(T)$ and let $\gamma\colon Q_0\to Q_1$ be a homomorphism between injective C-comodules. Assume that $T=\operatorname{Ker}(\gamma)$ and $T\in\mathcal{C}_\gamma$. Let $X\in\operatorname{Cogen}(T_C)$. Then $h_C(T,X)$ is a finitely cogenerated right D-comodule if and only if there is an exact sequence $0\to X\to T^n\to Z\to 0$ with $Z\in\mathcal{C}_\gamma$.*

Proof. Consider $h_C(T,X)$ is a finitely cogenerated right D-comodule. By [6], Proposition 4.2.2, there is a monomorphism $0\to X\xrightarrow{f} T^n$ such that $h_C(T,f)$ is a monomorphism. Hence, we have an exact sequence

(4.1)

$$\cdots \xrightarrow{\delta^1} \Gamma(X) \to \Gamma(T^n) \to \Gamma(Z) \xrightarrow{\delta^0} h_C(T,X) \xrightarrow{h_C(T,f)} h_C(T,T^n) \to h_C(T,Z) \to 0.$$

Since $T\in\mathcal{C}_\gamma$, it is immediate by Lemma 3.4 (1) that $T^n\in\mathcal{C}_\gamma$ and $X\in\mathcal{C}_\gamma$. Thus, $\Gamma(T^n)=0$. Taking into account that $h_C(T,f)$ is a monomorphism, we obtain that $\Gamma(Z)=0$, i.e., $Z\in T^\perp$. By Lemma 3.4 (5), we get that $Z\in\mathcal{C}_\gamma$.

Conversely, assume that there is an exact sequence

$$0\to X\xrightarrow{f} T^n\xrightarrow{g} Z\to 0$$

with $Z \in \mathcal{C}_\gamma$. Then there is a long exact sequence induced by $h_C(T,-)$ similar to (4.1). Since $Z \in \mathcal{C}_\gamma$, we have that $\Gamma(Z) = 0$. Therefore,

$$0 \longrightarrow h_C(T,X) \xrightarrow{h_C(T,f)} h_C(T,T^n) \cong D^n$$

is a monomorphism, i.e., $h_C(T,X)$ is a finitely cogenerated right D-comodule. $\square$

Now, we define the concept of finitely silting right C-comodules over a field R. Note that all finite cogenerated comodules are quasi-finitely by the example in [18], Section 8.1.

Definition 4.5. Let T be a right C-comodule with $T = \operatorname{Ker}(\gamma)$, where $\gamma\colon Q_0 \to Q_1$ is a homomorphism between injective C-comodules. We say that T is a finitely silting comodule with respect to γ if the following conditions are satisfied:
(1) $T \in \mathcal{C}_\gamma$;
(2) $\operatorname{Ker} h_C(T,-) \cap \mathcal{C}_\gamma = 0$;
(3) T is finitely cogenerated;
(4) $h_C(T,-)\colon \mathcal{M}^C \to \mathcal{M}^D$ carries finitely cogenerated comodules to finitely cogenerated comodules, where $D = e_C(T)$;
(5) if R is a commutative ring, we need T to be quasi-finite.

Remark 4.6. By Proposition 3.9 and the above definition, we observe that finitely silting comodules are silting comodules in quasi-finite comodule category. However, the converse is not true.

Proposition 4.7. *If T_C is a finitely silting comodule with respect to the injective copresentation $\gamma\colon Q_0 \to Q_1$, and T is faithfully coflat then $\mathcal{Y}_C = \mathcal{C}_\gamma \cap \operatorname{comod-}C$.*

Proof. Let $X \in \mathcal{Y}_C$. Then $X \in \operatorname{Cogen}(T_C)$ and $X \in \operatorname{comod-}C$. If T_C is a finitely silting comodule with respect to γ, we have that $T \in \mathcal{C}_\gamma$ and $X \in \mathcal{C}_\gamma$.

For the reverse inclusion, let $X \in \mathcal{C}_\gamma \cap \operatorname{comod-}C$. In general, we have $X/X' \in \operatorname{Cogen}(T_C)$. As T is faithfully coflat, it follows by Lemma 3.8 that $X' \in \operatorname{Ker} h_C(T,-)$ and moreover, $X' \in \mathcal{X}_C$. Then $h_C(T,X/X') \cong h_C(T,X)$ by Lemma 4.1. Since T_C is a finitely silting comodule with respect to γ, we have that $T \in \mathcal{C}_\gamma$ and $h_C(T,X) \in \operatorname{comod-}D$, which is the category of finitely cogenerated D-comodules. Consequently, $h_C(T,X/X') \in \operatorname{comod-}D$. By Proposition 4.4, there is an exact sequence

$$0 \to X/X' \xrightarrow{f} T^n \xrightarrow{g} Z \to 0$$

with $Z \in \mathcal{C}_\gamma$. From Lemma 3.4 (1), $X/X' \in \mathcal{C}_\gamma$, and hence, by Lemma 4.3, we have $X \in \operatorname{Cogen}(T_C)$. $\square$

Remark 4.8. Let comod-C be closed under quotients, i.e., all finitely cogenerated comodules are finitely copresented, or C is a conoetherian coalgebra, Proposition 4.7 is also true.

Proposition 4.9. *Let T_C be a finitely silting comodule with respect to the injective copresentation $\gamma\colon Q_0 \to Q_1$. If T is faithfully coflat, then $\mathcal{Y}_C \subseteq \mathrm{Refl}_\theta$.*

Proof. Let $X \in \mathcal{Y}_C$. Then $X \in \mathrm{Cogen}(T_C)$ and $X \in \text{comod-}C$. Since T is finitely silting with respect to γ, we have that $h_C(T, X)$ is a finitely cogenerated D-comodule, where $D = e_C(T)$. Indeed, by Proposition 4.4, there is an exact sequence

$$0 \to X \xrightarrow{f} T^n \xrightarrow{g} Z \to 0$$

with $Z \in \mathcal{C}_\gamma$. It follows by Lemma 3.4 (4) that $\Gamma(Z) = 0$. Hence, the induced sequence

$$0 \longrightarrow h_C(T,X) \xrightarrow{h_C(T,f)} h_C(T,T^n) \xrightarrow{h_C(T,g)} h_C(T,Z) \longrightarrow 0$$

is also exact. We have known that a cotensor functor is exact when one of the variables is coflat by [15]. Now, consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "\theta_X"] & T^n \arrow[r, "g"] \arrow[d, "\theta_{T^n}"] & Z \arrow[r] \arrow[d, "\theta_Z"] & 0 \arrow[d] \\
0 \arrow[r] & h_C(T,X) \,\square\, T \arrow[r, "h_C(T,f)\square T"] & h_C(T,T^n) \,\square\, T \arrow[r, "h_C(T,g)\square T"] & h_C(T,Z) \,\square\, T \arrow[r] & 0
\end{tikzcd}$$

Since T is a finitely silting comodule and $X \in \mathrm{Cogen}(T)$, also by [19], Theorem 1, we have $Z \in \mathrm{Cogen}(T)$, and θ_Z is a monomorphism. From the fact that θ_{T^n} is an isomorphism, it follows by applying the Snake Lemma to the above diagram that θ_X is an isomorphism. $\square$

Remark 4.10. If we suppose that all C-comodules in $\mathcal{Y}_C$ are finitely copresented, then Proposition 4.9 also holds.

Proposition 4.11. *Let T_C be a conoetherian and faithfully coflat C-comodule. If T_C is finitely silting with respect to the injective copresentation $\gamma\colon Q_0 \to Q_1$, then the class $\mathcal{Y}_C$ consists of the finitely copresented comodules belonging to the class $\mathcal{C}_\gamma$.*

Proof. Let $X \in \mathcal{Y}_C$. Then $X \in \mathrm{Cogen}(T_C)$ and $X \in \text{comod-}C$. Since T_C is finitely silting with respect to γ, we have that $T \in \mathcal{C}_\gamma$ and $h_C(T, X)$ is a finitely cogenerated right D-comodule, where $D = e_C(T)$. According to Proposition 4.4, it follows that there is an exact sequence $0 \to X \xrightarrow{f} T^n \xrightarrow{g} Z \to 0$ with $Z \in \mathcal{C}_\gamma$. From the facts that T^n is finitely cogenrated and conoetherian, we have that Z is finitely cogenerated. Thus, X is finitely copresented. Since $T \in \mathcal{C}_\gamma$, it follows that $X \in \mathcal{C}_\gamma$.

Let X be a finitely copresented C-comodule such that $X \in \mathcal{C}_\gamma$, then X is finitely cogenerated. Since T is faithfully coflat, we have that $X' \in \operatorname{Ker} h_C(T, -)$. As $X \in \mathcal{C}_\gamma$, we obtain $X' \in \mathcal{C}_\gamma$. Moreover, T_C is a quasi-finitely silting comodule, it follows that $X' = 0$, i.e., $X \in \operatorname{Cogen}(T_C)$. Therefore, $X \in \operatorname{Cogen}(T_C) \cap \text{comod-}C$, i.e., $X \in \mathcal{Y}_C$.

In particular, if finitely cogenerated comodules are closed under quotients, based on Proposition 3.7, we can also have the necessity. $\square$

5. Some relations

In this section we define the notion of finitely tilting comodules. And we discuss the relation between it and finitely silting right C-comodules.

Definition 5.1. Assume that T is a right C-comodule. Let $D = e_C(T_C)$. If T satisfies the following conditions, then we call T a finitely tilting comodule.

(1) $\operatorname{id}_C T \leqslant 1$.
(2) $\operatorname{ext}^1_C(T, T) = 0$.
(3) $\operatorname{Ker} h_C(T, -) \cap \operatorname{Ker} \operatorname{ext}^1_C(T, -) = 0$.
(4) T is finitely cogenerated.
(5) $h_C(T, -)$ respects finitely cogenerated.
(6) If R is a commutative ring, we need T is quasi-finite.

Proposition 5.2. *Let T be a right C-comodule and $\operatorname{id}_C T \leqslant 1$. Then T is a finitely silting comodule if and only if T is a finitely tilting comodule.*

Proof. Let

$$0 \to T \to Q_0 \xrightarrow{\zeta} Q_1 \to 0 \tag{5.1}$$

be an injective copresentation of T.

Necessity. Assume that T is a finitely silting comodule. It is enough to show that $\operatorname{ext}^1_C(T, T) = 0$ and $\operatorname{Ker} h_C(T, -) \cap \operatorname{Ker} \operatorname{ext}^1_C(T, -) = 0$. Firstly, since $T \in \mathcal{C}_\gamma$ and $\mathcal{C}_\gamma \subseteq {}^\perp T$, we have that $\operatorname{ext}^1_C(T, T) = 0$. Secondly, let $X \in \operatorname{Ker} h_C(T, -) \cap \operatorname{Ker} \operatorname{ext}^1_C(T, -)$. So, $\operatorname{ext}^1_C(T, X) = 0$. Applying the contravariant functor $h_C(-, X)$ on the exact sequence (5.1), we have that

$$0 \to \operatorname{ext}^1_C(T, X) \to h_C(Q_1, X) \to h_C(Q_0, X) \to h_C(T, X) \to 0.$$

Then $X \in \mathcal{C}_\gamma$. It follows that $X \in \operatorname{Ker} h_C(T, -) \cap \mathcal{C}_\gamma$. Since T is a finitely silting comodule, it is immediate that $X = 0$.

Sufficiency. Suppose that T is a finitely tilting comodule. Indeed, applying the contravariant functor $h_C(-,T)$ on (5.1), we get that

$$0 \to \mathrm{ext}^1_C(T,T) \to h_C(Q_1,T) \to h_C(Q_0,T) \to h_C(T,T) \to 0.$$

Since T is finitely tilting, it follows that $\mathrm{ext}^1_C(T,T) = 0$. Hence, $T \in \mathcal{C}_\gamma$. Let $X \in \mathrm{Ker}\, h_C(T,-) \cap \mathcal{C}_\gamma$. Applying the functor $h_C(-,X)$ on (5.1), we have the exact sequence

$$0 \longrightarrow \mathrm{ext}^1_C(T,X) \longrightarrow h_C(Q_1,X) \xrightarrow{h_C(\gamma,X)} h_C(Q_0,X) \longrightarrow h_C(T,X) \longrightarrow 0.$$

As $h_C(\gamma,X)$ is a monomorphism, we can obtain $\mathrm{ext}^1_C(T,X) = 0$. Then $X \in \mathrm{Ker}\, h_C(T,-) \cap \mathrm{Ker}\, \mathrm{ext}^1_C(T,-)$. Since T is a finitely tilting comodule, it follows that $X = 0$. $\square$

Proposition 5.3. *If T is finitely cogenerated and product-complete, then T is a finitely silting comodule if and only if T is a silting comodule.*

Proof. By Remark 4.6, we have that finitely silting comodules are silting.

Conversely, we assume that T is a silting comodule. We only need to prove that $h_C(T,-)$ carries finitely cogenerated comodules to finitely cogenerated comodules. Because T is product-complete, according to [10], Theorem 3.1, for any finitely cogenerated comodule X, there is a homomorphism $f\colon X \to T^{(I)}$ such that $h_C(T,f)$ is a monomorphism, where I is finite. Let $D = e_C(T)$, we have that $h_C(T,X)$ is finitely cogenerated. $\square$

Example 5.4 ([19], Example 1).

(1) It is well known that a right C-comodule T is partial tilting if and only if $\mathrm{Cogen}(T) \subseteq \ker \mathrm{Ext}^1_C(-,T)$ and the class $\ker \mathrm{Ext}^1_C(-,T)$ is a torsion-free class. Moreover, T is tilting if and only if $\mathrm{Cogen}(T) = \ker \mathrm{Ext}^1_C(-,T)$. Note that the class $\ker \mathrm{Ext}^1_C(-,T)$ is closed under subcomodules if and only if $\mathrm{id}_C(T) \leqslant 1$, where $\mathrm{id}_C(T)$ denotes the injective dimension of T. So every (partial) tilting comodule is of injective dimension at most 1.

Let T be a right C-comodule of injective dimension at most 1, and consider an exact sequence $0 \to T \xrightarrow{f} Q_0 \xrightarrow{\zeta} Q_1 \to 0$ with Q_0 and Q_1 injective. Then T is (partial) tilting if and only if T is (partial) silting with respect to ζ. The direct implications are obvious since $\mathcal{B}_\zeta = \ker \mathrm{Ext}^1_C(-,T)$. If we assume that T is partial silting with respect to ζ, we have $\mathrm{Cogen}(T) \subseteq \mathcal{B}_\zeta = \ker \mathrm{Ext}^1_C(-,T)$. Since $\mathcal{B}_\zeta$ is closed under direct products, it follows that T is a partial tilting comodule. Moreover, if T is silting with respect to ζ, we obtain $\mathrm{Cogen}(T) = \mathcal{B}_\zeta = \ker \mathrm{Ext}^1_C(-,T)$, hence T is tilting.

(2) Let C be a coalgebra. If $\zeta\colon\ 0 \to C$ is the trivial homomorphism, then $\mathcal{B}_\zeta$ contains only the trivial comodule, hence $0 = \mathrm{Ker}(\zeta)$ is a silting comodule.

Remark 5.5. Let us consider Example 5.4 in quasi-finite comodule categories.

(1) Suppose that all comodules are quasi-finite. A quasi-finite right C-comodule T is partial tilting if and only if $\mathrm{Cogen}(T) \subseteq T^\perp$, and $T^\perp$ is closed under subcomodules, direct sums and extensions. Moreover, T is tilting if and only if $\mathrm{Cogen}(T) = T^\perp$. Note that the class $T^\perp$ is closed under subcomodules if and only if $\mathrm{id}_C(T) \leqslant 1$. So every (partial) tilting comodule is of injective dimension at most 1.

Let T be a quasi-finite right C-comodule of injective dimension at most 1, and consider an exact sequence $0 \to T \xrightarrow{f} Q_0 \xrightarrow{\zeta} Q_1 \to 0$ with Q_0 and Q_1 injective. Then T is (partial) tilting if and only if T is (partial) silting with respect to ζ. In fact, the necessity can be obtained easily from $\mathcal{C}_\zeta = T^\perp$. On the other hand, if we assume that T is partial silting with respect to ζ, we have $\mathrm{Cogen}(T) \subseteq \mathcal{C}_\zeta = T^\perp$. Since $\mathcal{C}_\zeta$ is closed under direct sums, it follows that T is a partial tilting comodule. Moreover, if T is silting with respect to ζ, then we obtain $\mathrm{Cogen}(T) = \mathcal{C}_\zeta = T^\perp$, hence T is tilting.

(2) Let C be a semiperfect coalgebra and P be a projective generator of C. If $\gamma\colon\ P \to 0$ is the trivial homomorphism, then $\mathcal{C}_\gamma$ contains only the trivial comodule, hence $0 = \mathrm{Coker}(\gamma)$ is a quasi-finite silting comodule.

Before giving next example, we introduce the notion of endofinite comodules. A quasi-finite right C-comodule M is said to be endofinite if it has finite length as a comodule over its endomorphism coalgebra $e_C(M)$.

(3) Let K be a filed, and C be the upper 2×2 triangular matrix coalgebra. There are two simple comodules $S = (0, K)$ and $T = Q/S$, where $Q = (K, K)$. Let $\gamma\colon\ Q \to T \oplus T$ be a homomorphism such that $\mathrm{Ker}\,\gamma = S$, then $X \in \mathcal{C}_\gamma$ if and only if $\mathrm{ext}^1_C(S, X) = 0$ and $h_C(T, X) = 0$.

Note that every simple comodule is quasi-finite and quasi-finite comodules are closed under direct sums. Then S, T and $Q \oplus T$ are quasi-finite.

Since every indecomposable endofinite comodule is product-complete, and the direct sums of product-complete comodules are product-complete, we have that S and $Q \oplus T$ are product-complete.

Let $X \in \mathcal{C}_\gamma$. Since $Q \oplus T$ is an injective cogenerator of $\mathcal{M}^C$, there is a set I such that X can be embedded in $Q^{(I)} \oplus T^{(I)}$. By $h_C(T, X) = 0$, we have that X can be embedded in $Q^{(I)}$. Then we can consider X as a subcomodule of $Q^{(I)}$. Since the direct sums are exact in comodule category, we get that $S^{(I)}$ is a subcomodule of $Q^{(I)}$ such that $Q^{(I)}/S^{(I)} \cong T^{(I)}$. It follows that $X/(X \cap S^{(I)}) \cong (X + S^{(I)})/S^{(I)}$ can be embedded in $T^{(I)}$. By $h_C(T, X) = 0$, we can obtain $X \subseteq S^{(I)}$, i.e., $X \in \mathrm{Cogen}(S)$. Then $\mathcal{C}_\gamma \subseteq \mathrm{Cogen}(S)$.

Conversely, if $X \in \operatorname{Cogen}(S)$, it is easy to see that there is a set I such that $X \cong S^{(I)}$. Then $\operatorname{ext}^1_C(S, X) = 0$ and $h_C(T, X) = 0$. So $X \in \mathcal{C}_\gamma$.

In conclusion, $\mathcal{C}_\gamma = \operatorname{Cogen}(S)$ and S is finitely silting by Proposition 5.3.

Corollary 5.6. *Suppose that T is a product-complement and finitely cogenerated comodule. If $\operatorname{id}_C T \leqslant 1$, then the following statements are equivalent.*
(1) *T is a silting comodule.*
(2) *T is a finitely silting comodule.*
(3) *T is a finitely tilting comodule.*
(4) *T is a tilting comodule.*

Proof. By the above proposition, we have that (1)$\Leftrightarrow$(2). By Remark 5.5, (1)$\Leftrightarrow$(4) holds. According to Proposition 5.2, we can obtain (2)$\Leftrightarrow$(3). $\square$

Remark 5.7. By Remark 4.6, we have that finitely silting comodules are silting comodules in quasi-finite comodule category. The equivalence between them can be obtained by Proposition 5.3.

Proposition 5.8. *Let ${}_DT_C$ be a finitely silting comodule, also be faithfully coflat. Then the pair $(\mathcal{Y}_C, \mathcal{X}_C)$ is a torsion pair in* comod-C.

Proof. Firstly, we prove that for all $X \in$ comod-C and all $Y \in \mathcal{Y}_C$, $h_C(Y, X) = 0$ if and only if $X \in \mathcal{X}_C$. Let $X \in$ comod-C. If $h_C(Y, X) = 0$ for all $Y \in \mathcal{Y}_C$, then it follows from $T_C \in \mathcal{Y}_C$ that $h_C(T, X) = 0$, i.e., $X \in \operatorname{Ker} h_C(T, -)$. Thus, $X \in \mathcal{X}_C$.

Conversely, assume that $X \in \mathcal{X}_C$, then $X \in \operatorname{Ker} h_C(T, -)$, i.e., $h_C(T, X) = 0$. If $Y \in \mathcal{Y}_C$, then $Y \in \operatorname{Cogen}(T_C)$. Hence, there is a monomorphism $0 \to Y \xrightarrow{f} T^{(I)}$ for a set I. Applying the contravariant functor $h_C(-, X)$ on it, we obtain that

$$h_C(T^{(I)}, X) \xrightarrow{h_C(f,X)} h_C(Y, X) \longrightarrow 0$$

is an epimorphism. As $X \in \operatorname{Ker} h_C(T, -)$, we have that $h_C(Y, X) = 0$.

Secondly, we prove that for every $Y \in$ comod-C and for all $X \in \mathcal{X}_C$, $h_C(Y, X) = 0$ if and only if $Y \in \mathcal{Y}_C$.

Let $Y \in$ comod-C. Assume that $h_C(Y, X) = 0$ for all $X \in \mathcal{X}_C$. Let $Y' = \operatorname{Ker} \theta_Y$. As T is faithfully coflat, it follows that $Y' \subseteq \operatorname{Ker} h_C(T, -)$. Since $Y' \in$ comod-C, we get that $Y' \in \mathcal{X}_C$. Then $h_C(Y, Y') = 0$. And taking into account that Y' is a subcomodule of Y, so $Y' = 0$. It is enough to show $Y \in \operatorname{Cogen}(T_C)$. Moreover, $Y \in \mathcal{Y}_C$. Conversely, suppose that $Y \in \mathcal{Y}_C$, we obtain that $Y \in \operatorname{Cogen}(T_C)$.

Then there is a monomorphism $0 \to Y \xrightarrow{f} T^{(I)}$. Consider $X \in \mathcal{X}_C$, i.e., $X \in \operatorname{Ker} h_C(T,-) \cap \text{comod-}C$. Then $h_C(T,X) = 0$. Since the contravariant functor $h_C(-,X)$ is right exact, we have that $h_C(T^{(I)},X) \to h_C(Y,X) \to 0$ is an epimorphism. Thus, $h_C(Y,X) = 0$. $\square$

Proposition 5.9. *Let T_C be a finitely silting comodule. If T is injective, then*

$$h_C(T,-)\colon\ \mathcal{Y}_C \rightleftarrows \mathcal{Y}_D\colon\ -\,\square_D\, T_C$$

is a duality, where $\mathcal{Y}_D = \operatorname{cogen}(T^*) \cap \text{Comod-}D$.

Proof. Let $X \in \mathcal{Y}_C$. Then $X \in \operatorname{Cogen}(T_C) \cap \text{comod-}C$. Since T_C is a finitely silting comodule, we have $h_C(T,X) \in \text{comod-}D$. Since cohom functor $h_C(T,-)$ is closed under direct sums, applying $h_C(T,-)$ on $0 \to X \to C^{(m)}$ and by Lemma 2.4, we get that

$$0 \to h_C(T,X) \to h_C(T,C^{(m)}) = h_C(T,C)^{(m)} = (\operatorname{Com}_C(C,T)^*)^{(m)} = (T^*)^{(m)}.$$

It means that $h_C(T,X) \in \mathcal{Y}_D$.

In order to show the converse, let $X \in \mathcal{Y}_D$. Then we have two exact sequences, $0 \to X \to (T^*)^{(m)}$ and $0 \to X \to D^{(I)}$. Applying cotensor functor $-\,\square_D\, T_C$ on them, we obtain $0 \to X \,\square_D\, T_C \to (T^*)^{(m)} \,\square_D\, T_C$ and $0 \to X \,\square_D\, T_C \to D^{(I)} \,\square_D\, T_C$. As T is finitely silting, we have T and T^* are finitely cogenerated. Therefore, the first sequence is $0 \to X \,\square_D\, T_C \to D^{(n)} \,\square_D\, T_C \cong T_C^{(n)}$, and the second is $0 \to X \,\square_D\, T_C \to {T_C}^{(I)}$. So $X \,\square_D\, T_C \in \mathcal{Y}_C$. $\square$

Let C be a coalgebra, we denote by copres-C the subcategory of Comod-C consisting of finitely copresented C-comodules. Moreover, given a bicomodule ${}_DT_C$, we consider the classes $\mathcal{Y}'_C = \operatorname{Cogen}(T_C) \cap \text{copres-}C$ and $\mathcal{Y}'_D = \operatorname{cogen}((T^*)_D) \cap \text{copres-}D$.

Proposition 5.10. *Let T_C be a finitely silting comodule. If T is injective, then*

$$h_C(T,-)\colon\ \mathcal{Y}'_C \rightleftarrows \mathcal{Y}'_D\colon\ -\,\square_D\, T_C$$

is a duality. If T is also faithfully coflat and conoetherian, then $\mathcal{Y}_C = \mathcal{Y}'_C$.

Proof. The inclusion $\mathcal{Y}'_C \subseteq \mathcal{Y}_C$ is obvious. Let us prove the reverse inclusion. From Proposition 4.11, the class $\mathcal{Y}_C$ consists of finitely copresented comodules belonging to the class $\mathcal{C}_\gamma$. Hence, we have $\mathcal{Y}_C \subseteq \mathcal{Y}'_C$. Therefore, $\mathcal{Y}_C = \mathcal{Y}'_C$.

Similarly to Proposition 5.9, we can prove the duality is true. $\square$

Before giving the next proposition for a quasi-finite bicomodule ${}_DT_C$, we define T_C (${}_DT$) is cofaithful if $u_D\colon e_C(T) \to D$ ($u_C\colon e_D(T) \to C$) is surjective.

Proposition 5.11. *Let C be a cocoherent coalgebra. Suppose T_C is a finitely silting comodule which is also cofaithful and faithfully coflat. If T_C is finitely copresented, and let*

$$\mathrm{Sup}\{\mathrm{id}_C M\colon M \in \mathcal{Y}'_C\} = n,$$

where $n \in N$, then for any $X \in \mathcal{Y}'_C$, there is a sequence

$$0 \to X \to T_0 \to T_1 \to \ldots \to T_n \to 0,$$

where all $T_i \in \mathrm{add}(T_C)$.

Proof. Suppose $X \in \mathcal{Y}'_C$, by Proposition 4.4, there is a short exact sequence

$$0 \to X \to T^{(n)} \to Z_0 \to 0$$

with $Z_0 \in \mathcal{C}_\gamma$. According to Lemma 2.6 and Proposition 4.7, we have $Z_0 \in \mathcal{Y}'_C$. Repeating the above process, we can get the exact sequence

$$0 \to X \to T_0 \xrightarrow{f_0} T_1 \to \ldots \to T_{n-1} \xrightarrow{f_{n-1}} Z_{n-1} \to 0, \tag{5.2}$$

$$0 \to Z_{n-1} \to T_n \xrightarrow{f_n} Z_n \to 0 \tag{5.3}$$

with $T_i \in \mathrm{add}(T_C)$, $\mathrm{Ker}\, f_i \in \mathcal{Y}'_C$, $i = 1, 2, \ldots, n$.

We assert that the short exact sequence (5.3) is split. We only need to prove that $\mathrm{ext}^1_C(Z_{n-1}, Z_n) = 0$. We notice that Z_n is finitely copresented. Then there is an exact sequence

$$0 \to Z_n \xrightarrow{g_0} C_0 \to \mathrm{Coker}\, g_0 \to 0,$$

where C_0 is a finitely cogenerated free comodule. We know that $\mathrm{Coker}\, g_0$ is finitely copresented. Since T_C is cofaithful, we have $C \in \mathrm{Cogen}(T)$. Thus, $C_i \in \mathcal{Y}'_C$. And again, repeating the above process, we obtain an injective resolution of Z_n

$$0 \to Z_n \xrightarrow{g_0} C_0 \xrightarrow{g_1} \ldots \xrightarrow{g_i} C_i \to \ldots,$$

where C_i are finitely cogenerated free comodules, $\mathrm{Coker}\, g_i \in \mathcal{Y}'_C$, $i = 0, 1, 2, \ldots$ According to the Dimension-Shifting theorem, we can obtain that $\mathrm{ext}^{i+1}_C(Z_n, T) \cong \mathrm{ext}^1_C(\mathrm{Coker}\, g_{i-1}, T) = 0$, $i = 1, 2, 3, \ldots$ Applying $h_C(-, Z_n)$ on sequence (5.2), and by the Dimension-Shifting theorem, we get that

$$\begin{aligned} \mathrm{ext}^1_C(Z_{n-1}, Z_n) &\cong \mathrm{ext}^2_C(\mathrm{Ker}\, f_{n-1}, Z_n) \cong \mathrm{ext}^3_C(\mathrm{Ker}\, f_{n-2}, Z_n) \\ &\cong \ldots \cong \mathrm{ext}^{n+1}_C(X, Z_n) = 0. \end{aligned}$$

Indeed, in the above equation the last one is 0 by $\mathrm{id}_C X \leqslant n$. Therefore, $Z_{n-1} \in \mathrm{add}(T_C)$. $\square$

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