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Wei Zhang
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ON k-FREE NUMBERS OVER BEATTY SEQUENCES

Wei Zhang, Henan

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Abstract. We consider k-free numbers over Beatty sequences. New results are given. In particular, for a fixed irrational number $\alpha > 1$ of finite type $\tau < \infty$ and any constant $\varepsilon > 0$, we can show that

$$\sum_{\substack{1 \leqslant n \leqslant x \\ [\alpha n + \beta] \in \mathcal{Q}_k}} 1 - \frac{x}{\zeta(k)} \ll x^{k/(2k-1)+\varepsilon} + x^{1-1/(\tau+1)+\varepsilon},$$

where $\mathcal{Q}_k$ is the set of positive k-free integers and the implied constant depends only on α, ε, k and β. This improves previous results. The main new ingredient of our idea is employing double exponential sums of the type

$$\sum_{1 \leqslant h \leqslant H} \sum_{\substack{1 \leqslant n \leqslant x \\ n \in \mathcal{Q}_k}} e(\vartheta h n).$$

Keywords: k-free number; exponential sum; Beatty sequence

MSC 2020: 11L07, 11B83

1. Introduction

In this paper, we are interested in k-free integers over Beatty sequences. The so-called *Beatty sequence* of integers is defined by $\mathcal{B}_{\alpha,\beta} := \{[\alpha n + \beta]\}_{n=1}^{\infty}$, where α and β are fixed real numbers and $[x]$ denotes the greatest integer not larger than x. The analytic properties of such sequences have been studied by many experts. For example, one can refer to [1], [2], [3] and the references therein. A number q is called a *k-free integer* if and only if $m^k | q \Rightarrow m = 1$. For a sufficiently large $x \geqslant 1$, it is well known that

$$\sum_{n \in \mathcal{Q}_k} n^{-s} = \frac{\zeta(s)}{\zeta(ks)}, \quad \operatorname{Re} s > 1,$$

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and

$$\sum_{\substack{n\leqslant x\\ n\in\mathcal{Q}_k}} 1=\frac{x}{\zeta(k)}+O(x^{1/k}), \tag{1.1}$$

where $\mathcal{Q}_k$ is the set of positive k-free integers. In this paper, we are interested in the sum

$$\sum_{\substack{1\leqslant n\leqslant x\\ [\alpha n+\beta]\in\mathcal{Q}_k}} 1.$$

In fact, this problem has been considered by many experts. For example, in 2008, Güloğlu and Nevans in [7] proved that

$$\sum_{\substack{1\leqslant n\leqslant x\\ [\alpha n+\beta]\in\mathcal{Q}_2}} 1=\frac{x}{\zeta(2)}+O\Big(\frac{x\log\log x}{\log x}\Big),$$

where $\alpha>1$ is the irrational number of finite type.

Now we recall some notion related to the type of α. The definition of an irrational number of constant type can be cited as follows. For an irrational number α, we define its type τ by the relation

$$\tau:=\sup\Big\{\theta\in\mathbb{R}\colon \liminf_{\substack{q\to\infty\\ q\in\mathbb{Z}^+}} q^{\theta}\|\alpha q\|=0\Big\}.$$

Let ψ be a nondecreasing positive function that is defined for integers. The irrational number α is said to be of type less than ψ if $q\|q\alpha\|\geqslant 1/\psi(q)$ holds for every positive integer q. If ψ is a constant function, then an irrational α is also said to be of constant type (finite type). The relation between these two definitions is that an individual number α is of type τ if and only if for every constant τ, there is a constant $c(\tau,\alpha)$ such that α is of type τ with $q\|q\alpha\|\geqslant c(\tau,\alpha)q^{-\tau-\varepsilon+1}$.

Recently, in [5] and [6], it is proved that

$$\sum_{\substack{1\leqslant n\leqslant x\\ [\alpha n+\beta]\in\mathcal{Q}_2}} 1=\frac{x}{\zeta(2)}+O(Ax^{5/6}(\log x)^5),$$

where $A=\max\{\tau(m),\,1\leqslant m\leqslant x^2\}$ and $\alpha>1$ is a fixed irrational algebraic number. More recently, Kim, Srichan and Mavecha in [9] improved the above result by showing that

$$\sum_{\substack{1\leqslant n\leqslant x\\ [\alpha n+\beta]\in\mathcal{Q}_k}} 1=\frac{x}{\zeta(k)}+O(x^{(k+1)/2k}(\log x)^3).$$

Recently, with some much more generalized arithmetic functions, in [1], [11], one may also get some other estimates for such type of sums. However, the estimates of [1], [11] cannot be applied to an individual α. In this paper, we give the following formula.

Theorem 1.1. *Let $\alpha > 1$ be a fixed irrational number of finite type $\tau < \infty$. Then for any constant $\varepsilon > 0$, we have*

$$\sum_{\substack{1\leqslant n\leqslant x\\ [\alpha n+\beta]\in\mathcal{Q}_k}} 1 - \alpha^{-1} \sum_{\substack{1\leqslant n\leqslant [\alpha x+\beta]\\ n\in\mathcal{Q}_k}} 1 \ll x^{k/(2k-1)+\varepsilon} + x^{1-1/(\tau+1)+\varepsilon},$$

where the implied constant depends only on α, ε, k and β.

Then by (1.1) and the above theorem, we can obtain the following.

Corollary 1.2. *Let $\alpha > 1$ be a fixed irrational number of finite type $\tau < \infty$. Then for any constant $\varepsilon > 0$, we have*

$$\sum_{\substack{1\leqslant n\leqslant x\\ [\alpha n+\beta]\in\mathcal{Q}_k}} 1 - \frac{x}{\zeta(k)} \ll x^{k/(2k-1)+\varepsilon} + x^{1-1/(\tau+1)+\varepsilon},$$

where the implied constant depends only on α, ε, k and β.

In fact, our result relies heavily on the following double sum.

Theorem 1.3. *Suppose for some positive integers $a, q, h, q \leqslant x$, $h \leqslant H \ll x$, $(a, q) = 1$ and*

$$\left|\vartheta - \frac{a}{q}\right| \leqslant \frac{1}{q^2}, \tag{1.2}$$

then for sufficiently large x and any $\varepsilon > 0$, we have

$$\sum_{1\leqslant h\leqslant H} \sum_{\substack{1\leqslant n\leqslant x\\ n\in\mathcal{Q}_k}} e(\vartheta hn) \ll \Big(Hx^{k/(2k-1)} + q + \frac{Hx}{q}\Big)x^{\varepsilon},$$

where the implied constant may depend on k and ε.

Remark 1.4. One can also compare this result with the results of Brüdern-Perelli, see [4] and Tolev, see [12]. By using the argument of [4], [12], one may get some better results for some special cases.

2. Proof of Theorem 1.1

We will start the proof by introducing some necessary lemmas.

Lemma 2.1. *Let $\alpha > 1$ be of finite type $\tau < \infty$ and let K be sufficiently large. For an integer $w \geqslant 1$, there exists $a, q \in \mathbb{N}$, $a/q \in \mathbb{Q}$ with $(a,q) = 1$ and q satisfying $K^{1/\tau-\varepsilon}w^{-1} < q \leqslant K$ such that*

$$\left|\alpha w - \frac{a}{q}\right| \leqslant \frac{1}{qK}.$$

Proof. By Dirichlet approximation theorem, there is a rational number a/q with $(a,q) = 1$ and $q \leqslant K$ such that $|\alpha w - a/q| < 1/qK$. Then we have $\|qw\alpha\| \leqslant 1/K$. Since α is of type $\tau < \infty$, for a sufficiently large K, we have $\|qw\alpha\| \geqslant (qw)^{-\tau-\varepsilon}$. Then we get $1/K \geqslant \|qw\alpha\| \geqslant (qw)^{-\tau-\varepsilon}$. This gives that $q \geqslant K^{1/\tau-\varepsilon}w^{-1}$. $\square$

In order to prove the theorem, we need definition of discrepancy. Suppose that we are given a sequence u_m, $m = 1, 2, \ldots, M$, of points of $\mathbb{R}/\mathbb{Z}$. Then the discrepancy $D(M)$ of the sequence is

$$D(M) = \sup_{\mathcal{I}\in[0,1)}\left|\frac{\mathcal{V}(\mathcal{I}, M)}{M} - |\mathcal{I}|\right|, \tag{2.1}$$

where the supremum is taken over all subintervals $\mathcal{I} = (c,d)$ of the interval $[0,1)$, $\mathcal{V}(\mathcal{I}, M)$ is the number of positive integers $m \leqslant M$ such that $a_m \in \mathcal{I}$, and $|\mathcal{I}| = d - c$ is the length of $|\mathcal{I}|$.

Without losing generality, let $D_{\alpha,\beta}(M)$ denote the discrepancy of the sequence $\{\alpha m + \beta\}$, $m = 1, 2, \ldots, M$, where $\{x\} = x - [x]$. The following lemma is from [2].

Lemma 2.2. *Let $\alpha > 1$. An integer m has the form $m = [\alpha n + \beta]$ for some integer n if and only if*

$$0 < \{\alpha^{-1}(m - \beta + 1)\} \leqslant \alpha^{-1}.$$

The value of n is determined uniquely by m.

Lemma 2.3 ([10], Chapter 2, Theorem 3.2)**.** *Let α be a fixed irrational number of type $\tau < \infty$. Then for all $\beta \in \mathbb{R}$, we have*

$$D_{\alpha,\beta}(M) \leqslant M^{-1/\tau+o(1)} \quad (M \to \infty),$$

where the function implied by $o(1)$ depends only on α.

Lemma 2.4 ([13], page 32). *For any $\Delta \in \mathbb{R}$ such that $0 < \Delta < \frac{1}{8}$ and $\Delta \leqslant \frac{1}{2}\min\{\gamma, 1-\gamma\}$, there exists a periodic function $\Psi_\Delta(x)$ of period 1 satisfying the following properties:*

- ▷ $0 \leqslant \Psi_\Delta(x) \leqslant 1$ *for all* $x \in \mathbb{R}$;
- ▷ $\Psi_\Delta(x) = \Psi(x)$ *if* $\Delta \leqslant x \leqslant \gamma - \Delta$ *or* $\gamma + \Delta \leqslant x \leqslant 1 - \Delta$;
- ▷ $\Psi_\Delta(x)$ *can be represented as a Fourier series*

$$\Psi_\Delta(x) = \gamma + \sum_{j=1}^{\infty} g_j e(jx) + h_j e(-jx),$$

where

$$\Psi(x) = \begin{cases} 1 & \text{if } 0 < x \leqslant \gamma, \\ 0 & \text{if } \gamma < x \leqslant 1, \end{cases}$$

and the coefficients g_j and h_j satisfy the upper bound

$$\max\{|g_j|, |h_j|\} \ll \min\{j^{-1}, j^{-2}\Delta^{-1}\} \quad (j \geqslant 1).$$

Suppose that $\alpha > 1$. Then we have that α and $\gamma = \alpha^{-1}$ are of the same type. This means that $\tau(\alpha) = \tau(\gamma)$, see [3], page 133. Let $\delta = \alpha^{-1}(1-\beta)$ and $M = [\alpha x + \beta]$. Then by Lemma 2.2, we have

$$\sum_{\substack{1 \leqslant n \leqslant x \\ [\alpha n + \beta] \in \mathcal{Q}_k}} 1 = \sum_{\substack{1 \leqslant m \leqslant M \\ 0 < \{\gamma m + \delta\} \leqslant \gamma \\ m \in \mathcal{Q}_k}} 1 + O(1) = \sum_{\substack{1 \leqslant m \leqslant M \\ m \in \mathcal{Q}_k}} \Psi(\gamma m + \delta) + O(1), \tag{2.2}$$

where $\Psi(x)$ is the periodic function with period one for which

$$\Psi(x) = \begin{cases} 1 & \text{if } 0 < x \leqslant \gamma, \\ 0 & \text{if } \gamma < x \leqslant 1. \end{cases}$$

By a classical result of Vinogradov (see Lemma 2.4), it is known that for any Δ such that $0 < \Delta < \frac{1}{8}$ and $\Delta \leqslant \frac{1}{2}\min\{\gamma, 1-\gamma\}$, there is a real-valued function $\Psi_\Delta(x)$ that satisfies the conditions of Lemma 2.4. Hence, by (2.2), we can obtain that

$$\sum_{\substack{1 \leqslant n \leqslant x \\ [\alpha n + \beta] \in \mathcal{Q}_k}} 1 = \sum_{\substack{1 \leqslant m \leqslant M \\ m \in \mathcal{Q}_k}} \Psi(\gamma m + \delta) + O(1) = \sum_{\substack{1 \leqslant m \leqslant M \\ m \in \mathcal{Q}_k}} \Psi_\Delta(\gamma m + \delta) + O(1 + V(I, M)M^{\varepsilon}), \tag{2.3}$$

where $V(I, M)$ denotes the number of positive integers $m \leqslant M$ such that

$$\{\gamma m + \delta\} \in I = [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).$$

Since $|I| \ll \Delta$, it follows from the definition (2.1) and Lemma 2.3 that

$$V(I,M) \ll \Delta x + x^{1-1/\tau+\varepsilon}, \tag{2.4}$$

where the implied constant depends only on α. By Fourier expansion for $\Psi_\Delta(\gamma m+\delta)$ (see Lemma 2.4) and changing the order of summation, we have

$$\begin{aligned}\sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} \Psi_\Delta(\gamma m+\delta) = \gamma \sum_{\substack{m\leqslant M\\ m\in\mathcal{Q}_k}} 1 + \sum_{k=1}^{\infty} g_k e(\delta k) \sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} e(\gamma k m)\\ + \sum_{k=1}^{\infty} h_k e(-\delta k) \sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} e(\gamma k m).\end{aligned} \tag{2.5}$$

By Theorem 1.3, Lemmas 2.1 and 2.4, we see that for $0 < k \ll x^{(4k-4)/(2k-1)+\varepsilon}$, we have

$$\sum_{1\leqslant k\leqslant x^{(4k-4)/(2k-1)+\varepsilon}} g_k e(\delta k) \sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} e(\gamma k m) \ll x^{k/(2k-1)+\varepsilon} + x^{1-1/(\tau+1)+\varepsilon}, \tag{2.6}$$

where we have also used the fact that α and α^{-1} are of the same type (finite type). Similarly, we have

$$\sum_{1\leqslant k\leqslant x^{(4k-4)/(2k-1)+\varepsilon}} h_k e(-\delta k) \sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} e(\gamma k m) \ll x^{k/(2k-1)+\varepsilon} + x^{1-1/(\tau+1)+\varepsilon}. \tag{2.7}$$

On the other hand, the trivial bound

$$\sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} e(\gamma k m) \ll x$$

implies that

$$\begin{aligned}\sum_{k\geqslant x^{(4k-4)/(2k-1)+\varepsilon}} g_k e(\delta k) \sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} e(\gamma k m) &\ll x^{1+\varepsilon} \sum_{k\geqslant x^{(4k-4)/(2k-1)+\varepsilon}} k^{-2}\Delta^{-1}\\ &\ll x^{k/(2k-1)+\varepsilon}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\sum_{k\geqslant x^{(4k-4)/(2k-1)+\varepsilon}} h_k e(-\delta k) \sum_{\substack{1\leqslant m\leqslant M\\ m\in\mathcal{Q}_k}} e(\gamma k m) &\ll x^{1+\varepsilon} \sum_{k\geqslant x^{(4k-4)/(2k-1)+\varepsilon}} k^{-2}\Delta^{-1}\\ &\ll x^{k/(2k-1)+\varepsilon}\end{aligned} \tag{2.9}$$

where $\Delta = x^{-(k-1)/(2k-1)+\varepsilon}$. Inserting the bounds (2.6)–(2.9) into (2.5), we have

$$\sum_{\substack{1\leqslant n\leqslant x\\ [\alpha n+\beta]\in\mathcal{Q}_k}} 1-\alpha^{-1}\sum_{\substack{1\leqslant n\leqslant[\alpha x+\beta]\\ n\in\mathcal{Q}_k}} 1 \ll x^{k/(2k-1)+\varepsilon}+x^{1-1/(\tau+1)+\varepsilon},$$

where the implied constant depends on α, k, β and ε. Substituting these bounds and (2.4) into (2.3) and choosing $\Delta = x^{-(k-1)/(2k-1)+\varepsilon}$, we complete the proof of Theorem 1.1. □

3. Proof of Theorem 1.3

By the Dirichlet hyperbolic method, we have

$$\begin{aligned}\sum_{1\leqslant h\leqslant H}\sum_{\substack{1\leqslant n\leqslant x\\ n\in\mathcal{Q}_k}} e(\alpha hn) &= \sum_{1\leqslant h\leqslant H}\sum_{1\leqslant m^k\leqslant y}\mu(m)\sum_{1\leqslant l\leqslant x/m^k} e(\alpha m^k lh)\\ &\quad+\sum_{1\leqslant h\leqslant H}\sum_{1\leqslant l\leqslant x/y}\sum_{1\leqslant m^k\leqslant x/l}\mu(m)e(\alpha m^k lh)\\ &\quad-\sum_{1\leqslant h\leqslant H}\sum_{1\leqslant m^k\leqslant y}\mu(m)\sum_{1\leqslant l\leqslant x/y} e(\alpha m^k lh),\end{aligned}$$

where y is a certain parameter to be chosen later. By the well known estimate

$$\sum_{1\leqslant n\leqslant x} e(n\alpha)\leqslant \min\Big(x,\frac{1}{2\|\alpha\|}\Big),$$

we have

$$\begin{aligned}\sum_{1\leqslant h\leqslant H}\sum_{1\leqslant m^k\leqslant y}\mu(m)\sum_{1\leqslant l\leqslant x/m^k} e(\alpha m^k lh) &\ll \sum_{1\leqslant h\leqslant H}\sum_{1\leqslant m^k\leqslant y}\min\Big(\frac{x}{m^k},\frac{1}{2\|\alpha hm^k\|}\Big)\\ &\ll \sum_{1\leqslant h\leqslant H}\sum_{1\leqslant m\leqslant y}\min\Big(\frac{x}{m},\frac{1}{2\|\alpha hm\|}\Big)\\ &\ll (Hy)^{\varepsilon}\sum_{1\leqslant n\leqslant Hy}\min\Big(\frac{Hx}{n},\frac{1}{2\|\alpha n\ |}\Big)\\ &\ll (Hy)^{\varepsilon}\Big(Hy+\frac{Hx}{q}+q\Big),\end{aligned}$$

where we have used the following lemma.

Lemma 3.1 ([8], Section 13.5)**.** *For*

$$\left|\theta - \frac{a}{q}\right| \leqslant q^{-2},$$

$a, q \in \mathbb{N}$ *and* $(a, q) = 1$, *we have*

$$\sum_{1 \leqslant n \leqslant M} \min\left\{\frac{x}{n}, \frac{1}{2\|n\theta\|}\right\} \ll (M + q + xq^{-1}) \log 2qx,$$

where $\|u\theta\|$ *denotes the distance of* u *from the nearest integer.*

For the second sum, we can use the exponential sum for Möbius function. We have

$$\sum_{1 \leqslant h \leqslant H} \sum_{1 \leqslant l \leqslant x/y} \sum_{1 \leqslant m^k \leqslant x/l} \mu(m) e(\alpha m^k l h) \ll Hxy^{1/k-1} (\log x)^{-A},$$

where A is any positive constant. For the third sum we need the following lemma.

Lemma 3.2 ([8], Section 13.5)**.** *For*

$$\left|\theta - \frac{a}{q}\right| \leqslant q^{-2},$$

$a, q \in \mathbb{N}$ *and* $(a, q) = 1$, *we have*

$$\sum_{1 \leqslant n \leqslant M} \min\left\{x, \frac{1}{2\|n\theta\|}\right\} \ll \left(M + x + \frac{Mx}{q} + q\right) \log 2qx,$$

where $\|u\theta\|$ *denotes the distance of* u *from the nearest integer.*

By Lemma 3.2, we have

$$\begin{aligned}
\sum_{1 \leqslant h \leqslant H} \sum_{1 \leqslant m^k \leqslant y} \mu(m) \sum_{1 \leqslant l \leqslant x/y} e(\alpha m^k l h) &\ll \sum_{1 \leqslant h \leqslant H} \sum_{1 \leqslant m^k \leqslant y} \min\left(\frac{x}{y}, \frac{1}{2\|\alpha h m^k\|}\right) \\
&\ll \sum_{1 \leqslant h \leqslant H} \sum_{1 \leqslant m \leqslant y} \min\left(\frac{x}{y}, \frac{1}{2\|\alpha h m\|}\right) \\
&\ll (Hy)^{\varepsilon} \sum_{1 \leqslant n \leqslant Hy} \min\left(\frac{x}{y}, \frac{1}{2\|\alpha n\|}\right) \\
&\ll (Hy)^{\varepsilon} \left(Hy + \frac{x}{y} + \frac{Hx}{q} + q\right).
\end{aligned}$$

Choosing $y = x^{k/(2k-1)}$ completes the proof of Theorem 1.3. □

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Author's address: Wei Zhang, School of Mathematics and Statistics, Henan University, Kaifeng 475004, Henan, P. R. China, e-mail: `zhangweimath@126.com`.