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A NEW INCLUSION INTERVAL FOR THE REAL EIGENVALUES OF REAL MATRICES

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Abstract. By properties of Cvetković-Kostić-Varga-type (or, for short, CKV-type) B-matrices, a new class of nonsingular matrices called CKV-type $\overline{\text{B}}$-matrices is given, and a new inclusion interval of the real eigenvalues of real matrices is presented. It is shown that the new inclusion interval is sharper than those provided by J. M. Peña (2003), and by H. B. Li et al. (2007). We also propose a direct algorithm for computing the new inclusion interval. Numerical examples are included to illustrate the effectiveness of the obtained results.

Keywords: CKV-type B-matrix; P-matrix; real eigenvalues localization

MSC 2020: 15A18, 65F15, 15B48

1. Introduction

A real square matrix is a P-matrix if all its principal minors are positive, see [6]. P-matrices are one class of the most important matrices from applications, such as error bound estimation for linear complementarity problems, Schur complement problems, structured tensors, and sub-direct sums, see for instance, [2], [5], [8], [10], [12]. In many applications, one is interested in locating the real eigenvalues of a real matrix by the properties of some subclasses of P-matrices.

The localization of the eigenvalues of matrices plays a key role in matrix theory and numerical analysis, see [9], [11],[13], [14], [15], [16], [18], [21]. Some well-known eigenvalue inclusion regions such as Geršgorin disks (see [7]), Brauer's ovals of Cassini

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(see [1]), and Brualdi's inclusion set (see [18]) have received extensive attention and application, see also [19]. Using B-matrices (see [13]), DB-matrices (see [14]), and SB-matrices (see [11]) (see the following definitions, as subclasses of P-matrices), alternatives to Geršgorin disks, Brauer's ovals of Cassini, and Cvetković's Geršgorin-type inclusion set (see [4]) for the localization of the real eigenvalues of a real matrix were presented in [11], [13], and [14], respectively. It is noted that the information on the real eigenvalues provided by these alternatives supplements the information given by the real intervals provided by the above well-known ones, and in some cases these alternatives are sharper than the later ones.

Very recently, the authors in [17] introduced a new subclass of P-matrices, called CKV-type B-matrices, which generalizes S-strictly dominant B-matrices (for short, SB-matrices). In the present paper, by making use of this new class of matrices, we aim to obtain a new inclusion interval for the real eigenvalues of real matrices. In Section 2, we give a new class of nonsingular matrices, CKV-type $\overline{\mathrm{B}}$-matrices, which can split into the product of a nonsingular diagonal matrix and a CKV-type B-matrix. Based on CKV-type $\overline{\mathrm{B}}$-matrices, a new inclusion interval called CKV-type $\overline{\mathrm{B}}$-interval for the real eigenvalues of real matrices is presented, and it is proved that the new interval improves the existing ones in [11], [13], [14]. In Section 3, a direct algorithm for computing CKV-type $\overline{\mathrm{B}}$-interval is put forward. Numerical examples show that the new inclusion interval efficiently locate the real eigenvalue of real matrices. Finally, Section 4 is a brief conclusion.

Throughout this paper, we will use the following notations and definitions. We denote by $\mathbb{R}^n$ the n-dimensional real vector space, and by $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) by the set of all n-order real matrices (complex matrices).

Definition 1.1 ([3]). A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a CKV-type matrix if $S_i^\star(A)$ is not empty for each $i \in N := \{1, \dots, n\}$, where

$$\begin{aligned} S_i^\star(A) := \{S \in \Sigma_i \colon\ |a_{ii}| > r_i^S(A) \text{ and } (|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\overline{S}}(A)) \\ > r_i^{\overline{S}}(A) r_j^S(A) \text{ for all } j \in \overline{S}\} \end{aligned}$$

with $\Sigma_i = \{S \subsetneq N \colon\ i \in S\}$ and $r_i^S(A) = \sum\limits_{j\in S\setminus\{i\}} |a_{ij}|$.

Definition 1.2 ([13]). A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a B-matrix if for each $i \in N$,

$$\sum_{k\in N} a_{ik} > 0 \quad \text{and} \quad \frac{1}{n}\left(\sum_{k\in N} a_{ik}\right) > a_{ij} \quad \text{for any } j \in N \text{ and } j \neq i.$$

Given a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, letting

$$r_i^+ := \max\{0, a_{ij} \colon\ j \neq i\} \quad \text{and} \quad r_i^- := \min\{0, a_{ij} \colon\ j \neq i\},$$

we can write A as $A = B^+ + C$ or $A = B^- + E$, where

$$(1.1) \qquad B^+ = \begin{pmatrix} a_{11} - r_1^+ & \dots & a_{1n} - r_1^+ \\ \vdots & & \vdots \\ a_{n1} - r_n^+ & \dots & a_{nn} - r_n^+ \end{pmatrix}, \quad C = \begin{pmatrix} r_1^+ & \dots & r_1^+ \\ \vdots & & \vdots \\ r_n^+ & \dots & r_n^+ \end{pmatrix},$$

and

$$(1.2) \qquad B^- = \begin{pmatrix} a_{11} - r_1^- & \dots & a_{1n} - r_1^- \\ \vdots & & \vdots \\ a_{n1} - r_n^- & \dots & a_{nn} - r_n^- \end{pmatrix}, \quad E = \begin{pmatrix} r_1^- & \dots & r_1^- \\ \vdots & & \vdots \\ r_n^- & \dots & r_n^- \end{pmatrix}.$$

Definition 1.3 ([14])**.** A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a doubly B-matrix (DB-matrix) if the matrix B^+ of the form (1.1) is a strictly doubly diagonally dominant (DSDD) matrix with positive diagonal entries, i.e.,

$$\begin{cases} a_{ii} > r_i^+ & \text{for all } i \in N, \\ (a_{ii} - r_i^+)(a_{jj} - r_j^+) > r_i(B^+) r_j(B^+) & \text{for all } j \neq i \text{ and } i, j \in N, \end{cases}$$

where $r_i(B^+) = \sum\limits_{j=1, j\neq i}^{n} |a_{ij} - r_i^+|$.

Definition 1.4 ([11])**.** A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is an S-strictly dominant B-matrix (SB-matrix) if the matrix B^+ of the form (1.1) is an S-SDD matrix with positive diagonal entries, that is,

$$\begin{cases} a_{ii} > r_i^+ & \text{for all } i \in N, \\ a_{ii} - r_i^+ > r_i^S(B^+) & \text{for all } i \in S, \\ (a_{ii} - r_i^+ - r_i^S(B^+))(a_{jj} - r_j^+ - r_j^{\overline{S}}(B^+)) > r_i^{\overline{S}}(B^+) r_j^S(B^+) & \text{for all } i \in S,\ j \in \overline{S}. \end{cases}$$

Definition 1.5 ([17])**.** A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a CKV-type B-matrix if the matrix B^+ of the form (1.1) is a CKV-type matrix with positive diagonal entries.

Lemma 1.1 ([17])**.** *If A is a CKV-type B-matrix, then A is a P-matrix.*

2. A new real eigenvalue inclusion interval for real matrices

In this section, using the CKV-type B-matrices, a new class of nonsingular matrices called CKV-type $\overline{\mathrm{B}}$-matrices is presented, and then a new real eigenvalue inclusion interval is obtained.

2.1. CKV-type $\overline{\mathrm{B}}$-matrices.

Definition 2.1. A matrix $A \in \mathbb{R}^{n\times n}$ is called a CKV-type $\overline{\mathrm{B}}$-matrix if A can be decomposed into $A = DB$, where B is a CKV-type B-matrix and $D = \operatorname{diag}(d_1, \dots, d_n)$ with $d_i \in \{1, -1\}$ for each $i \in N$.

By Lemma 1.1 and Definition 2.1, we easily get the following result.

Lemma 2.1. *If A is a CKV-type $\overline{\mathrm{B}}$-matrix, then it is nonsingular.*

The following results provide some equivalent characterization of CKV-type $\overline{\mathrm{B}}$-matrices, which will be used later.

Lemma 2.2. *A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a CKV-type $\overline{\mathrm{B}}$-matrix if and only if there exists a diagonal matrix $D \in \mathbb{D}$ such that DA is a CKV-type B-matrix, where*

$$\mathbb{D} := \{D = \operatorname{diag}(d_1, \dots, d_n) \colon d_i \in \{1, -1\}\}.$$

Proof. If A is a CKV-type $\overline{\mathrm{B}}$-matrix, then from Definition 2.1 it holds that $A = DB$, where $D \in \mathbb{D}$ and B is a CKV-type B-matrix. Since $D \in \mathbb{D}$ is invertible, it follows that $B = D^{-1}A$ is a CKV-type B-matrix. Therefore, the necessity is established. We next prove that sufficiency is true. If there exists a diagonal matrix $D \in \mathbb{D}$ such that DA is a CKV-type B-matrix, then from $A = D^{-1}DA$ and Definition 2.1 it holds that A is a CKV-type $\overline{\mathrm{B}}$-matrix. This completes the proof. $\square$

Theorem 2.1. *A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a CKV-type $\overline{\mathrm{B}}$-matrix if and only if for each $i \in N$,* (i) *$|a_{ii}| > |r_i|$, and* (ii) *there exists a set $S \in \Sigma_i$ such that $|a_{ii} - r_i| > r_i^S(B)$ and for all $j \in \overline{S}$,*

$$(|a_{ii} - r_i| - r_i^S(B))(|a_{jj} - r_j| - r_j^{\overline{S}}(B)) > r_i^{\overline{S}}(B) r_j^S(B),$$

where Σ_i is given by Definition 1.1,

$$r_i^S(B) = \begin{cases} r_i^S(B^+), & a_{ii} > 0, \\ r_i^S(B^-), & a_{ii} < 0, \end{cases} \qquad r_i^{\overline{S}}(B) = \begin{cases} r_i^{\overline{S}}(B^+), & a_{ii} > 0, \\ r_i^{\overline{S}}(B^-), & a_{ii} < 0, \end{cases} \tag{2.1}$$

and

$$r_i = \begin{cases} r_i^+, & a_{ii} > 0, \\ r_i^-, & a_{ii} < 0. \end{cases} \tag{2.2}$$

P r o o f. By Lemma 2.2, it follows that A is a CKV-type $\overline{\mathrm{B}}$-matrix if and only if there is a diagonal matrix $D \in \mathbb{D}$ such that DA is a CKV-type B-matrix, that is, $DA = \widehat{B} + \widehat{C}$, where $\widehat{B}$ defined as (1.1) is a CKV-type matrix with positive diagonal entries. Thus, it suffices to prove that $\widehat{B}$ is a CKV-type matrix with positive diagonal entries if and only if (i) and (ii) hold. Note that the ith row of DA is given by $(a_{i1}, \ldots, a_{in})$ if $a_{ii} > 0$ and by $(-a_{i1}, \ldots, -a_{in})$ if $a_{ii} < 0$. It follows that if $a_{ii} > 0$, then

$$r_i^+(DA) = \max\{0, a_{ij} \colon j \neq i\} = r_i^+,$$

and if $a_{ii} < 0$, then

$$r_i^+(DA) = \max\{0, -a_{ij} \colon j \neq i\} = -\min\{0, a_{ij} \colon j \neq i\} = -r_i^-.$$

Since

$$\widehat{B} = \begin{pmatrix} (DA)_{11} - r_1^+(DA) & \ldots & (DA)_{1n} - r_1^+(DA) \\ \vdots & & \vdots \\ (DA)_{n1} - r_n^+(DA) & \ldots & (DA)_{nn} - r_n^+(DA) \end{pmatrix}$$

and

$$\widehat{C} = \begin{pmatrix} r_1^+(DA) & \ldots & r_1^+(DA) \\ \vdots & & \vdots \\ r_n^+(DA) & \ldots & r_n^+(DA) \end{pmatrix},$$

it follows that the ith row of $\widehat{B}$ is given by $(a_{i1} - r_i^+, \ldots, a_{in} - r_i^+)$ if $a_{ii} > 0$, and by $(-a_{i1} - (-r_i^-), \ldots, -a_{in} - (-r_i^-))$ if $a_{ii} < 0$, and the ith row of $\widehat{C}$ is given by $(r_i^+, \ldots, r_i^+)$ if $a_{ii} > 0$, and by $(-r_i^-, \ldots, -r_i^-)$ if $a_{ii} < 0$.

Next, we prove that $\widehat{B}$ is a CKV-type matrix with positive diagonal entries if and only if (i) and (ii) hold. Matrix $\widehat{B}$ has positive diagonal entries implying that for each $i \in N$, $a_{ii} > r_i^+$ if $a_{ii} > 0$, and $a_{ii} < r_i^-$ if $a_{ii} < 0$, which are equivalent to $|a_{ii}| > |r_i|$ for each $i \in N$. Matrix $\widehat{B}$ is a CKV-type matrix meaning that for each $i \in N$, $S_i^\star(\widehat{B})$ is not empty, that is,

(a) if $a_{ii} > 0$, $a_{jj} > 0$, then $a_{ii} - r_i^+ > r_i^S(B^+)$, and for all $j \in \overline{S}$,

$$(a_{ii} - r_i^+ - r_i^S(B^+))(a_{jj} - r_j^+ - r_j^{\overline{S}}(B^+)) > r_i^{\overline{S}}(B^+) r_j^S(B^+);$$

(b) if $a_{ii} > 0$, $a_{jj} < 0$, then $a_{ii} - r_i^+ > r_i^S(B^+)$ and for all $j \in \overline{S}$,

$$(a_{ii} - r_i^+ - r_i^S(B^+))(-a_{jj} - (-r_j^-) - r_j^{\overline{S}}(B^-)) > r_i^{\overline{S}}(B^+) r_j^S(B^-);$$

(c) if $a_{ii} < 0$, $a_{jj} > 0$, then $-a_{ii} - (-r_i^-) > r_i^S(B^-)$, and for all $j \in \overline{S}$,

$$(-a_{ii} - (-r_i^-) - r_i^S(B^-))(a_{jj} - r_j^+ - r_j^{\overline{S}}(B^+)) > r_i^{\overline{S}}(B^-) r_j^S(B^+);$$

(d) if $a_{ii} < 0$, $a_{jj} < 0$, then $-a_{ii} - (-r_i^-) > r_i^S(B^-)$, and for all $j \in \overline{S}$,

$$(-a_{ii} - (-r_i^-) - r_i^S(B^-))(-a_{jj} - (-r_j^-) - r_j^{\overline{S}}(B^-)) > r_i^{\overline{S}}(B^-) r_j^S(B^-).$$

Obviously, (a)–(d) are equivalent to condition (ii). This completes the proof. □

2.2. CKV-type $\overline{B}$-interval. As is well-known, the nonsingularity of real matrices can generate the equivalent eigenvalue inclusion set in the real axis, see [11], [13], [14]. So, by the nonsingularity of CKV-type $\overline{\mathrm{B}}$-matrices, in this section we give a new real eigenvalue inclusion interval called CKV-type $\overline{B}$-interval for real matrices.

Theorem 2.2. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, λ be any real eigenvalue of A, B^+ and B^- be the matrices of* (1.1) *and* (1.2), *respectively. Then*

$$\lambda \in \Theta := \bigcup_{i=1}^{n}\Bigg(C_i \cup \Bigg(\bigcap_{S\in\Sigma_i}\Bigg(F_i^S \cup \Bigg(\bigcup_{j\in\overline{S}} H_{ij}\Bigg)\Bigg)\Bigg)\Bigg),$$

where Σ_i is given by Definition 1.1, $\overline{S} := N \setminus S$, $C_i = [a_{ii} - r_i^+, a_{ii} - r_i^-]$,

$$F_i^S = [a_{ii} - r_i^+ - r_i^S(B^+), a_{ii} - r_i^- + r_i^S(B^-)],$$

and

$$H_{ij} = \begin{cases} H_{ij}^1 \cup H_{ij}^2 \cup H_{ij}^3 & \text{if } a_{ii} \leqslant a_{jj}, \\ H_{ij}^1 \cup \widetilde{H}_{ij}^2 \cup H_{ij}^3 & \text{if } a_{ii} \geqslant a_{jj} \end{cases}$$

with

$$\begin{aligned} H_{ij}^1 = \{\lambda \in (-\infty, \min\{a_{ii}, a_{jj}\})\colon & \\ (|a_{ii} - r_i^+ - \lambda| - r_i^S(B^+))(|a_{jj} - r_j^+ - \lambda| - r_j^{\overline{S}}(B^+)) &\leqslant r_i^{\overline{S}}(B^+) r_j^S(B^+)\}, \\ H_{ij}^2 = \{\lambda \in (a_{ii}, a_{jj})\colon & \\ (|a_{ii} - r_i^- - \lambda| - r_i^S(B^-))(|a_{jj} - r_j^+ - \lambda| - r_j^{\overline{S}}(B^+)) &\leqslant r_i^{\overline{S}}(B^-) r_j^S(B^+)\}, \\ \widetilde{H}_{ij}^2 = \{\lambda \in (a_{jj}, a_{ii})\colon & \\ (|a_{ii} - r_i^+ - \lambda| - r_i^S(B^+))(|a_{jj} - r_j^- - \lambda| - r_j^{\overline{S}}(B^-)) &\leqslant r_i^{\overline{S}}(B^+) r_j^S(B^-)\}, \end{aligned}$$

and

$$\begin{aligned} H_{ij}^3 = \{\lambda \in (\max\{a_{ii}, a_{jj}\}, \infty)\colon & \\ (|a_{ii} - r_i^- - \lambda| - r_i^S(B^-))(|a_{jj} - r_j^- - \lambda| - r_j^{\overline{S}}(B^-)) &\leqslant r_i^{\overline{S}}(B^-) r_j^S(B^-)\}. \end{aligned}$$

P r o o f. Suppose, on the contrary, that $\lambda \notin \Theta$, that is

$$\lambda \notin \bigcup_{i=1}^{n} C_i \quad \text{and} \quad \lambda \notin \bigcup_{i\in N} \bigcap_{S\in\Sigma_i} \Bigl(F_i^S \cup \Bigl(\bigcup_{j\in\overline{S}} H_{ij}\Bigr)\Bigr).$$

Case I: $\lambda \notin \bigcup_{i=1}^{n} C_i$. By (2.2), it is obvious that $|a_{ii} - \lambda| > |r_i|$ for all $i \in N$.

Case II: $\lambda \notin \bigcup_{i\in N} \bigcap_{S\in\Sigma_i} \Bigl(F_i^S \cup \Bigl(\bigcup_{j\in\overline{S}} H_{ij}\Bigr)\Bigr)$, i.e., for all $i \in N$, there exists $S \in \Sigma_i$ such that $\lambda \notin F_i^S$ and $\lambda \notin H_{ij}$ for all $j \in \overline{S}$.

If $\lambda \notin F_i^S$, it follows from (2.1) and (2.2) that $|a_{ii} - r_i - \lambda| > r_i^S(B)$ for all $i \in S$.

If $\lambda \notin H_{ij}$, then $\lambda \notin H_{ij}^1$, $\lambda \notin H_{ij}^2$, $\lambda \notin \widetilde{H}_{ij}^2$, and $\lambda \notin H_{ij}^3$, that is, if $\lambda \in (-\infty, \min\{a_{ii}, a_{jj}\})$, then

$$(|a_{ii} - r_i^+ - \lambda| - r_i^S(B^+))(|a_{jj} - r_j^+ - \lambda| - r_j^{\overline{S}}(B^+)) > r_i^{\overline{S}}(B^+) r_j^S(B^+);$$

if $\lambda \in (a_{ii}, a_{jj})$, then

$$(|a_{ii} - r_i^- - \lambda| - r_i^S(B^-))(|a_{jj} - r_j^+ - \lambda| - r_j^{\overline{S}}(B^+)) > r_i^{\overline{S}}(B^-) r_j^S(B^+);$$

if $\lambda \in (a_{jj}, a_{ii})$, then

$$(|a_{ii} - r_i^+ - \lambda| - r_i^S(B^+))(|a_{jj} - r_j^- - \lambda| - r_j^{\overline{S}}(B^-)) > r_i^{\overline{S}}(B^+) r_j^S(B^-);$$

if $\lambda \in (\max\{a_{ii}, a_{jj}\}, \infty)$, then

$$(|a_{ii} - r_i^- - \lambda| - r_i^S(B^-))(|a_{jj} - r_j^- - \lambda| - r_j^{\overline{S}}(B^-)) > r_i^{\overline{S}}(B^-) r_j^S(B^-),$$

which together with (2.1) and (2.2) imply that

$$(|a_{ii} - r_i - \lambda| - r_i^S(B))(|a_{jj} - r_j - \lambda| - r_j^{\overline{S}}(B)) > r_i^{\overline{S}}(B) r_j^S(B).$$

We note that A and $A - \lambda I$ have the same off-diagonal elements. Then, from Case I, Case II, and Theorem 2.1, it follows that $A - \lambda I$ is a CKV-type $\overline{\mathrm{B}}$-matrix. So, $A - \lambda I$ is nonsingular, thus $|A - \lambda I| \neq 0$, which contradicts the assumption that λ is a real eigenvalue of A. Therefore, $\lambda \in \Theta$. The proof is complete. $\square$

2.3. Comparisons with some existing intervals. In this section, we will show that the CKV-type $\overline{\mathrm{B}}$-interval is better than some existing ones. Before that, some well-known results are listed.

Theorem 2.3 ([13], Theorem 3.5). *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, λ be any real eigenvalue of A, B^+ and B^- be the matrices of* (1.1) *and* (1.2), *respectively. Then*

$$\lambda \in \mathcal{S} := \bigcup_{i=1}^{n} [a_{ii} - r_i^+ - r_i(B^+), a_{ii} - r_i^- + r_i(B^-)].$$

Theorem 2.4 ([14], Theorem 3.3)**.** *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, λ be any real eigenvalue of A, B^+ and B^- be the matrices of* (1.1) *and* (1.2)*, respectively. Then*

$$\lambda \in \mathcal{B} := \left(\bigcup_{i=1}^{n} C_i\right) \cup \left(\bigcup_{j\neq i} B_{ij}\right),$$

where C_i is defined as in Theorem 2.2 *and assuming, without loss of generality, that $a_{ii} \leqslant a_{jj}$,*

$$B_{ij} := B_{ij}^1 \cup B_{ij}^2 \cup B_{ij}^3,$$

with

$$B_{ij}^1 := \{x \in (-\infty, a_{ii})\colon\ |a_{ii} - r_i^+ - x||a_{jj} - r_j^+ - x| \leqslant r_i(B^+)r_j(B^+)\},$$
$$B_{ij}^2 := \{x \in (a_{ii}, a_{jj})\colon\ |a_{ii} - r_i^- - x||a_{jj} - r_j^+ - x| \leqslant r_i(B^-)r_j(B^+)\},$$

and

$$B_{ij}^3 := \{x \in (a_{jj}, \infty)\colon\ |a_{ii} - r_i^- - x||a_{jj} - r_j^- - x| \leqslant r_i(B^-)r_j(B^-)\}.$$

Theorem 2.5 ([11], Theorem 3.7)**.** *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and λ be any real eigenvalue of A. Given any nonempty subset S of N, $\overline{S} := N \setminus S$. Then*

$$\lambda \in \varphi^S := \left(\bigcup_{i=1}^{n} C_i\right) \cup \left(\bigcup_{i\in S} F_i^S\right) \cup \left(\bigcup_{i\in S, j\in\overline{S}} H_{ij}\right),$$

where C_i, F_i^S, and H_{ij} are defined by Theorem 2.2*.*

As shown in [11], if S is a singleton, i.e., $S = \{i\}$, then

$$\bigcap_{i\in N} \varphi^{\{i\}} \subseteq \mathcal{B} \subseteq \mathcal{S}.$$

The following result shows that the CKV-type $\overline{\mathrm{B}}$-interval provided by Theorem 2.2 is better than that of Theorem 2.5, see [11], Theorem 3.7.

Theorem 2.6. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and λ be any real eigenvalue of A. Then*

$$\lambda \in \Theta \subseteq \varphi^S,$$

where Θ and φ^S are given by Theorems 2.2 *and* 2.5*, respectively.*

P r o o f. Suppose that $\lambda \in \Theta$. It follows from Theorem 2.2 that

$$\lambda \in \bigcup_{i\in N} C_i \quad \text{or} \quad \lambda \in \bigcup_{i\in N} \bigcap_{S\in\Sigma_i} \Bigl(F_i^S \cup \Bigl(\bigcup_{j\in\overline{S}} H_{ij}\Bigr)\Bigr).$$

If $\lambda \in \bigcup\limits_{i\in N} C_i$, then $\lambda \in \varphi^S$ holds directly. If $\lambda \in \bigcup\limits_{i\in N} \bigcap\limits_{S\in\Sigma_i} \Bigl(F_i^S \cup \Bigl(\bigcup\limits_{j\in\overline{S}} H_{ij}\Bigr)\Bigr)$, then there exists $i_0 \in N$ such that for any $S \in \Sigma_{i_0}$, $\lambda \in F_{i_0}^S$ or $\lambda \in H_{i_0,j}$ for some $j \in \overline{S}$. This implies that

$$\lambda \in \Bigl(\bigcup_{i\in S} F_i^S\Bigr) \cup \Bigl(\bigcup_{i\in S, j\in\overline{S}} H_{ij}\Bigr).$$

Hence, the conclusion follows. □

Remark here from Theorem 2.6 that $\Theta \subseteq \varphi^{\{i\}}$ for each $i \in N$. Therefore,

$$\Theta \subseteq \bigcap_{i\in N} \varphi^{\{i\}} \subseteq \mathcal{B} \subseteq \mathcal{S}.$$

3. A direct algorithm for computing CKV-type $\overline{\mathrm{B}}$-interval

In this section, a direct algorithm based on Theorem 2.2 for computing the CKV-type $\overline{\mathrm{B}}$-interval is put forward. Using this algorithm, some numerical examples are given to illustrate the effectiveness of the CKV-type $\overline{\mathrm{B}}$-interval.

Remark 3.1.

(i) Algorithm 1 is a direct method for computing the CKV-type $\overline{\mathrm{B}}$-interval, and the calculations only depend on the elements of the involved matrix and the subsets of N. Therefore, Algorithm 1 stops after finite steps.

(ii) CKV-type $\overline{\mathrm{B}}$-interval Θ seems to be difficult to calculate, since it involves an intersection over all possible nonempty proper subsets S of indices, and therefore requires lots of computations. But, sometimes it is worth investing in more calculations to obtain a better inclusion interval. Theorem 2.6 shows that CKV-type $\overline{\mathrm{B}}$-interval Θ improves the existing ones in [11], [13], [14], and Algorithm 1 can effectively compute the CKV-type $\overline{\mathrm{B}}$-interval Θ.

(iii) The MATLAB code for Algorithm 1 is freely available at [20], which facilitates the implementation of the CKV-type $\overline{\mathrm{B}}$-interval.

Algorithm 1: A direct method for computing CKV-type $\overline{\text{B}}$-interval Θ

Input: A matrix $A=[a_{ij}]\in\mathbb{R}^{n\times n}$,

for $i=1:n$ **do** $C_i=[a_{ii}-r_i^+,a_{ii}-r_i^-]$, $\Sigma_i=\{S\subsetneq N\colon\ i\in S\}$,

  for each $S\in\Sigma_i$

    $F_i^S=[a_{ii}-r_i^+-r_i^S(B^+),a_{ii}-r_i^-+r_i^S(B^-)]$,

    for $j\in\overline{S}$ **do**

      if $a_{ii}\leqslant a_{jj}$ then

      $H_{ij}=H_{ij}^1\cup H_{ij}^2\bigcup H_{ij}^3$,

      else

      $H_{ij}=H_{ij}^1\cup \widetilde{H}_{ij}^2\cup H_{ij}^3$,

      end if

    end for

    $H_i^S=\bigcup\limits_{j\in\overline{S}}H_{ij}$,

  end for

  $\bigcap\limits_{S\in\Sigma_i}(F_i^S\bigcup H_i^S)$,

end for

Output: $\Theta=\bigcup\limits_{i=1}^{n}\left(C_i\cup\left(\bigcap\limits_{S\in\Sigma_i}(F_i^S\cup H_i^S)\right)\right)$.

In the following, we implement Algorithm 1 to show that the localization of CKV-type $\overline{\text{B}}$-interval Θ is made efficiently. To make a comparison, we consider the following examples for numerical experiments with Geršgorin circles (see [7]), Cassini ovals (see [1]), $\overline{\text{B}}$ interval $\mathcal{S}$ (see [13], Theorem 3.5), D$\overline{\text{B}}$ interval $\mathcal{B}$ (see [14], Theorem 3.3), S$\overline{\text{B}}$ interval φ^S (see [11], Theorem 3.7). We implement all experiments in MATLAB version R2016 by using a PC with 3.40-GHz processors and 64 GB of memory.

Example 3.1. Consider the following matrices:

$$A_1=\begin{bmatrix}0.3112&0.2630&0.4505&0.1524\\0.5285&0.6541&0.0838&0.8258\\0.1656&0.6892&0.2290&0.5383\\0.6020&0.7482&0.9133&0.9961\end{bmatrix},\quad A_2=\begin{bmatrix}0.8147&0.6324&0.9575&0.9572\\0.9058&0.0975&0.9649&0.4854\\0.1270&0.2785&0.1576&0.8003\\0.9134&0.5469&0.9706&0.1419\end{bmatrix},$$

$$A_3=\begin{bmatrix}0.1829&0.4899&0.5005&0.0424\\0.2399&0.1679&0.4711&0.0714\\0.8865&0.9787&0.0596&0.5216\\0.0287&0.7127&0.6820&0.0967\end{bmatrix},\quad A_4=\begin{bmatrix}0.8181&0.6596&0.8003&0.0835\\0.8175&0.5186&0.4538&0.1332\\0.7224&0.9730&0.4324&0.1734\\0.1499&0.6490&0.8253&0.3909\end{bmatrix},$$

and

$$A_5 = \begin{bmatrix} 0.1125 & 0.9742 & 0.4021 & 0.5508 & 0.5047 & 0.6949 & 0.7395 & 0.8960 & 0.3320 \\ 0.5158 & 0.1973 & 0.2952 & 0.8709 & 0.4050 & 0.4114 & 0.5247 & 0.5154 & 0.8397 \\ 0.8378 & 0.1112 & 0.3065 & 0.0423 & 0.1736 & 0.0348 & 0.8045 & 0.5445 & 0.3717 \\ 0.9208 & 0.2974 & 0.1056 & 0.9047 & 0.5752 & 0.2928 & 0.8169 & 0.6064 & 0.8282 \\ 0.4982 & 0.3964 & 0.5938 & 0.1310 & 0.6062 & 0.8014 & 0.1895 & 0.7604 & 0.1765 \\ 0.2776 & 0.4208 & 0.2827 & 0.8337 & 0.2144 & 0.3465 & 0.1237 & 0.8553 & 0.1295 \\ 0.6525 & 0.3115 & 0.1552 & 0.8005 & 0.5199 & 0.0833 & 0.8210 & 0.3829 & 0.8799 \\ 0.9173 & 0.6938 & 0.0007 & 0.9179 & 0.9892 & 0.5111 & 0.6379 & 0.0846 & 0.0441 \\ 0.5098 & 0.0919 & 0.2836 & 0.1373 & 0.4899 & 0.3668 & 0.0161 & 0.7339 & 0.6867 \end{bmatrix}.$$

We list the numerical results in Table 1, where we compute $\bigcap_{i=1}^{n} \varphi^{\{i\}}$ for the $\mathrm{S\overline{B}}$ interval.

Inclusion sets	A_1	A_2
Geršgorin circles	$[-1.2674, 3.2596]$	$[-2.2890, 3.3618]$
Cassini ovals	$[-1.2041, 2.6374]$	$[-2.2736, 2.9893]$
$\overline{\mathrm{B}}$ interval $\mathcal{S}$	$[-1.2110, 3.2596]$	$[-1.8378, 3.3618]$
$\mathrm{D\overline{B}}$ interval $\mathcal{B}$	$[-1.1655, 2.6374]$	$[-1.5652, 2.9893]$
$\mathrm{S\overline{B}}$ interval φ^S	$[-1.0690, 2.5051]$	$[-1.3627, 2.8197]$
CKV-type $\overline{\mathbf{B}}$ interval Θ	$[-1.0109, 2.5051]$	$[-1.3544, 2.8197]$
Real eigenvalue values	2.1811,0.1833	-0.7158,-0.4400,-0.0346,2.4022
A_3	A_4	A_5
$[-2.3272, 2.4464]$	$[-1.4364, 2.3615]$	$[-4.9817, 5.3480]$
$[-1.7651, 1.9214]$	$[-1.3307, 2.3345]$	$[-4.8008, 5.2827]$
$[-1.4684, 2.4464]$	$[-1.5908, 2.3615]$	$[-4.3133, 5.3480]$
$[-1.4122, 1.9214]$	$[-1.4758, 2.3345]$	$[-4.2632, 5.2827]$
$[-1.3401, 1.7828]$	$[-1.4184, 2.2401]$	$[-4.1308, 5.0737]$
$[-1.3401, 1.6343]$	$[-1.3411, 2.2282]$	$[-3.7841, 4.8689]$
-0.8081,-0.2556,0.0844,1.4864	0.3621,2.1819	-0.6254,-0.3454,0.0404,0.3988,4.4280

Table 1. The real eigenvalue inclusion sets for matrices A_i, $i = 1, 2, 3, 4, 5$.

From Table 1, we see that CKV-type $\overline{\mathrm{B}}$-interval Θ provided by Theorem 2.2 is sharper than Geršgorin circles and Cassini ovals, and, especially, it is more accurate than the $\overline{\mathrm{B}}$ interval, $\mathrm{D\overline{B}}$ interval, and $\mathrm{S\overline{B}}$ interval. Therefore, Theorem 2.2 provides a sharper interval to localize the real eigenvalues for real matrices.

Moreover, the following example shows that the CKV-type $\overline{\mathrm{B}}$-interval is smaller than Geršgorin circles and Cassini ovals in most cases.

Example 3.2. Using MATLAB function $A = \text{rand}(n)$, we randomly generate 180 real matrices. By Algorithm 1, we list the numerical results in Table 2. In this table, m specifies the number of randomly generated real matrices. In the "k_1" column, we show the number of matrices for which the CKV-type $\overline{\text{B}}$-interval is smaller than Geršgorin circles and Cassini ovals. In the "k_2" column, we show the number of matrices, where the CKV-type $\overline{\text{B}}$-interval and Cassini ovals do not contain each other. In the "k_3" column, we give the number of matrices for which the Geršgorin circles and Cassini ovals are smaller than the CKV-type $\overline{\text{B}}$-interval.

n (order)	m	k_1	k_2	k_3
4	20	14	6	0
5	20	16	4	0
6	20	16	4	0
7	20	13	7	0
8	20	17	3	0
9	20	16	4	0
10	20	18	2	0
11	20	16	4	0
12	20	18	2	0

Table 2. The numerical results of Example 3.2.

The results reported in Table 2 show that in most cases the real eigenvalue location given by the CKV-type $\overline{\text{B}}$-interval is more accurate than Geršgorin circles and Cassini ovals.

At the end of this section, an example is given to compare the CKV-type $\overline{\text{B}}$-interval with the latest CKV-type set proposed by Cvetković et al. in [3] which is better than the Geršgorin circles and the Cassini ovals. We recall the CKV-type set as follows.

Theorem 3.1 ([3], Theorem 16)**.** *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then for every eigenvalue λ it holds that*

$$\lambda \in \mathcal{V} := \bigcup_{i\in N} \bigcap_{S\in\Sigma_i} \left(\Gamma_i^S(A) \cup \left(\bigcup_{j\in\overline{S}} V_{ij}^S(A)\right)\right),$$

where Σ_i is given by Definition 1.1, $\overline{S} := N \setminus S$,

$$\Gamma_i^S(A) = \{z \in \mathbb{C} \colon |z - a_{ii}| \leqslant r_i^S(A)\},$$

and

$$V_{ij}^S(A) = \{z \in \mathbb{C} \colon (|z - a_{ii}| - r_i^S(A))(|z - a_{jj}| - r_j^{\overline{S}}(A)) \leqslant r_i^{\overline{S}}(A) r_j^S(A)\}.$$

Example 3.3. Using MATLAB function $A = \mathrm{rand}(n)$, we randomly generate 180 real matrices of different orders such that the elements of each matrix satisfy $a_{ij} \in (0,1)$. The numerical results are reported in Table 3.

n (order)	m	m_1	m_2	m_3
4	20	13	0	7
5	20	11	0	9
6	20	9	0	11
7	20	11	0	9
8	20	10	0	10
9	20	11	0	9
10	20	9	0	11
11	20	11	0	9
12	20	13	0	7

Table 3. The numerical results of Example 3.3.

In Table 3, the "m_1" column gives the number of matrices for which the CKV-type $\overline{\mathrm{B}}$-interval Θ is smaller than CKV-type set $\mathcal{V}$ of [3]. The "m_2" column gives the number of matrices, where the CKV-type $\overline{\mathrm{B}}$-interval Θ and CKV-type set $\mathcal{V}$ do not contain each other. The "m_3" column gives the number of matrices for which the CKV-type set $\mathcal{V}$ is smaller than or equal to the CKV-type $\overline{\mathrm{B}}$-interval Θ. The results given in Table 3 show that the real eigenvalue location given by the CKV-type $\overline{\mathrm{B}}$-interval is smaller than CKV-type set $\mathcal{V}$ of [3] in some cases. This means that the information on the real eigenvalues provided by CKV-type $\overline{\mathrm{B}}$-interval Θ supplements the information given by the real intervals provided by CKV-type set $\mathcal{V}$ of [3].

4. Conclusions

In this paper, we first present a new class of nonsingular matrices based on CKV-type B-matrices. Then by giving some corresponding equivalent conditions of the new class of matrices, we give a new inclusion interval called the CKV-type $\overline{\mathrm{B}}$-interval for the real eigenvalues of real matrices, which improves some existing ones in [11], [13], [14]. Also, we propose a direct algorithm to compute the CKV-type $\overline{\mathrm{B}}$-interval. Numerical examples show that the proposed results are efficient in locating the real eigenvalues of real matrices.

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