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On extensions of families of operators

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Abstract. The strong closure of feasible states of families of operators is studied. The results are obtained for self-adjoint operators in reflexive Banach spaces and for more concrete case - families of elliptic systems encountered in the optimal layout of r materials. The results show when it is possible to parametrize the strong closure by the same type of operators. The results for systems of elliptic operators for the case when number of unknown functions m is less than the dimension n of the reference domain are well-known, but we present several different approaches in this paper to prove that parametrization of the strong closure of feasible states can be done by convexification. Also, a new approach is offered to prove result for the strong closure of cogradients. There are given counterexamples for the case $m \geq n$ when the parametrization by convexification is not possible. This extends the known result for the case $m = n = 2$.

Keywords: strong closure; feasible state; operator; elliptic system

Classification: 49J45, 49J20

1. Introduction

This paper is devoted to a problem which has its primary origin in the theory of optimal control problems for elliptic equations and systems. To clarify what the problem is and how it has arisen, let us give some insight on the theory developed in the last decades.

As is known, a very large part of optimal control problems does not possess optimal solutions in the classical sense. The first investigation of such problems was carried out by K. A. Lurie in [11] in the end of 1960s. Other counterexamples on the existence of solutions for optimal control problems governed by elliptic equations were constructed by F. Murat in [12].

The two features played the crucial role in the nonexistence of optimal solutions in the mentioned examples, namely, the non-convexity of sets of admissible controls and the dependence of the main part of partial differential equation on the controls.

After that, starting from the works by K. A. Lurie [11] and L. Tartar [21], it became apparent that the theory of G-convergence must be involved in the

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investigations of optimal control problems with distributed parameters. So the theory of G-convergence was rapidly developed and became a universal tool in the theory of optimal control problems. Such a universality is due to the fact that it allows to construct extensions (the so-called G-closures) of optimal control problems in such a way that extended problems possess optimal solutions which may be used for construction of minimizing sequences of the original problems.

The theory of G-convergence is mainly applicable to problems with weakly continuous cost functionals. Although it may be applied to problems with non-weakly continuous cost functionals, unfortunately it gives only the formal extensions of the original problems. There is also a disadvantage of the G-closure approach, namely, there are only few types of families of operators encountered in applications for which an analytic description of the G-closure exists, see V. V. Zhikov, S. M. Kozlov and O. A. Olejnik [29]. For this reason, there was proposed the so-called method of convexification for treating problems governed by elliptic equations whose main part depends on controls, see U. Raitums [16]. This method has its source and is very fruitful in the theory of optimal control problems governed by ordinary differential equations, see R. V. Gamkrelidze [7], J. Warga [25]. This approach leads to sets of admissible controls with a simple structure convenient for numerical evaluations. Also, it may be applied to problems with non-weakly continuous cost functionals (see the example optimal control problem (1)–(3) below).

There are two disadvantages of this approach. One of them is that it does not lead, in general, to problems with the existence of optimal solutions. The second disadvantage is connected to the fact that it is impossible to apply this approach to problems governed by systems of elliptic equations where the number of equations is greater or equal to the dimension of the reference domain Ω. In this case the convexification does not preserve the cost of the problem, see U. Raitums [17].

In L. Tartar [23], there is studied an optimal control problem of the form

$$I(u) := \int_\Omega (\mathrm{grad}(u-v))^2 \, \mathrm{d}x \to \min, \tag{1}$$

where $v \in H_0^1(\Omega)$ is fixed, u is a solution of a differential equation in a bounded open subset Ω of $\mathbb{R}^n$

$$-\mathrm{div}(a\, \mathrm{grad} u) = f \qquad \text{in } \Omega,\ u \in H_0^1(\Omega), \tag{2}$$

with $f \in H^{-1}(\Omega)$ and $a \in L^\infty(\Omega)$ satisfying

$$\begin{cases} 0 < \lambda_1 \le a(x) \le \lambda_2 \ \text{ for a.e. } \ x \in \Omega, \\ a(x) \in \{\mu_1, \dots, \mu_r\} \ \text{ for a.e. } \ x \in \Omega, \\ \mathrm{meas}\,\{x \in \Omega \colon\ a(x) = \mu_q\} \le k_q,\ \sum_{q=1}^r k_q \ge \mathrm{meas}\,\Omega, \end{cases} \tag{3}$$

where $\lambda_1, \lambda_2, \mu_1, \dots, \mu_r, k_1, \dots, k_r$ are positive constants. The functional $I = I(u)$ is being minimized with respect to u that is a unique solution of elliptic equation (2) with a control that is the elliptic equation coefficient $a(x)$ having values only in a finite set as it is given by (3). The optimal control problem (1)–(3) is called a *problem of optimal layout of r materials*, $r \geq 2$.

As noted in [23], the just described problems involving $\operatorname{grad} u$ nonlinearly are more realistic than those which contain only u. We refer to R. V. Kohn and G. Strangin [8], [9], [10], for other problems involving $\operatorname{grad} u$ nonlinearly.

Let S denote the set of all functions satisfying (3). In [23] the equality

$$\begin{aligned}\operatorname{cl}_s\{u \in H_0^1(\Omega)\colon\ -\operatorname{div}(a \operatorname{grad} u) = f \text{ in } \Omega,\ a \in S\}\\ = \{u \in H_0^1(\Omega)\colon\ -\operatorname{div}(b \operatorname{grad} u) = f \text{ in } \Omega,\ b \in \overline{\operatorname{co}}\, S\}\end{aligned} \tag{4}$$

for all $f \in H^{-1}(\Omega)$ is proved, where cl_s denotes the operation of the strong closure in $H_0^1(\Omega)$ and $\overline{\operatorname{co}}\, S$ stands for the symbol of the closed convex hull in $L^2(\Omega)$. Here we would like to refer to U. Raitums [19] for more general results, see also J. Dvořák, J. Haslinger and M. Miettinen [5], U. Raitums [18].

The equality (4) gives a good basis for further analysis. For instance, one can consider the extended problem

$$\begin{cases} I(u) := \int_\Omega (\operatorname{grad}(u-v))^2 \,\mathrm{d}x \to \min,\\ -\operatorname{div}(b \operatorname{grad} u) = f \ \text{ in}\, \Omega,\ u \in H_0^1(\Omega),\\ b \in \overline{\operatorname{co}}\, S,\end{cases}$$

with $\overline{\operatorname{co}}\, S$ instead of S and, by using the convexity of $\overline{\operatorname{co}}\, S$, one can derive the necessary conditions of optimality as it was done in [23]. In addition, the convex set $\overline{\operatorname{co}}\, S$ is more preferable from the point of view of further numerical evaluation.

It is important to know whether the equality of the type (4) holds for other families and, especially, for families of linear systems of elliptic operators in which the number m of unknown functions is greater than or equal to the number n of independent variables. More precisely, let $\mathfrak{U}$ be a family of all elliptic operators of the type

$$A = \operatorname{div} \mathcal{A}\, \nabla, \tag{5}$$

where $\mathcal{A}$ is a positive definite $(mn \times mn)$-matrix with entries from $L^\infty(\Omega)$ taking its values in a finite set $\{\mathcal{A}_1, \dots, \mathcal{A}_r\}$ of positive definite constant $(mn \times mn)$-matrices $\mathcal{A}_1, \dots, \mathcal{A}_r$. Here the matrices represent the properties of r different materials.

For $\tilde f \in [H^{-1}(\Omega)]^m$ and $\tilde u \in [H_0^1(\Omega)]^m$, define

$$
\begin{aligned}
&F(\mathfrak{U}, \tilde f) = \{u \in [H_0^1(\Omega)]^m \colon Au = \tilde f,\ A \in \mathfrak{U}\} \text{ (set of feasible states)},\\
&E(\mathfrak{U}, \tilde u) = \{A\tilde u \in [H^{-1}(\Omega)]^m \colon A \in \mathfrak{U}\}.
\end{aligned} \tag{6}
$$

Now the question we are interested in can be formulated as follows: does there exist a family $\mathfrak{B}$ of operators of the type (5) such that

$$
\mathrm{cl}_s F(\mathfrak{U}, f) = F(\mathfrak{B}, f) \qquad \text{for all } f \in [H^{-1}(\Omega)]^m\,? \tag{7}
$$

Since the operators in $\mathfrak{U}$ are uniformly continuous and coercive, the previous question is equivalent to the following: does there exist a family $\mathfrak{B}$ of operators of the type (5) such that

$$
\mathrm{cl}_s E(\mathfrak{U}, u) = E(\mathfrak{B}, u) \qquad \text{for all } u \in [H_0^1(\Omega)]^m\,? \tag{8}
$$

As we already mentioned before there are papers dealing with the question (7), namely, [5], [19], [23]. In these papers, it is shown that for the case where $m < n$ the answer gives the convexification of the family $\mathfrak{U}$. This result is similar to that which is known from the optimal control theory of ODEs.

In Section 3, we consider the general case of self-adjoint operators and obtain the new result on the description of the strong closure of feasible states using G-convergence theory. Section 4 reformulates the obtained description using the notion of Γ-convergence.

In the following sections, we consider systems of elliptic equations and study both cases $m < n$ and $m \geq n$. For the case $m < n$, in Section 5 we shall give different proofs of known results, see [5], [19], [23], for the sake of completeness. We present here our proofs in order to show that there are possible various approaches to the problem. In Section 6, there is given an application of N-condition known from the theory of G-convergence to the problem. Section 7 presents an analysis for the periodic case. In Section 8, for the sake of completeness, we present the result concerning strong closure of cogradients which was considered in [20] (we use different approach in our reasoning). For the case $m \geq n \geq 2$, we extend the result given in [27] where the counterexample was constructed for $m = n = 2$.

We would like to mention that these questions are connected to the theory of existence of continuous selections of multivalued mappings. Let us briefly outline this connection here.

A multivalued mapping $\mathcal{F}$ from a Banach space X into a Banach space Y is called *parametrized* if there exists a family $\mathfrak{U} = \{A_\chi\}_{\chi\in\Lambda}$ of continuous mappings $A_\chi \colon X \to Y$, $\chi \in \Lambda$, such that $\mathcal{F}(x) = \{A_\chi x \colon A_\chi \in \mathfrak{U}\}$ for all $x \in X$, see J.-P. Aubin and I. Ekeland [1, page 4], where Λ is a given set of parameters (controls). We shall say that the family $\mathfrak{U}$ is a *parametrization* of $\mathcal{F}$.

Assume that the family $\mathfrak{U} = \{A_\chi\}_{\chi\in\Lambda}$ is a subset of some set K of continuous mappings whose role is to select from all continuous mappings those which describe a given physical process (for instance, heat conduct). Denote by $\operatorname{graph}\mathcal{F}$ the graph of $\mathcal{F}$. Since $\operatorname{graph}\mathcal{F}$ is a subset of $X\times Y$, one can consider various closures of $\operatorname{graph}\mathcal{F}$ in the space $X\times Y$ when X,Y are endowed with weak or strong topologies. Let $\mathcal{G}$ be a multivalued mapping such that $\operatorname{graph}\mathcal{G} = \operatorname{cl}\operatorname{graph}\mathcal{F}$ where cl denotes the closure in some of these topologies. A question is to find a parametrization of $\mathcal{G}$ by the mappings from K. Now it is easy to see that the questions (7), (8) are simply a part of this general problem. Such general setting was investigated in [26].

2. Basic notation

n - the dimension of the reference space $\mathbb{R}^n$ of spatial variables
$\mathbb{R}^n$ - n-dimensional Euclidean space
$x = (x_1,\dots,x_n)$ - the element of $\mathbb{R}^n$
$\langle\cdot,\cdot\rangle$ - scalar product in Euclidean space
$|\cdot|$ - the euclidean norm in $\mathbb{R}^n$, i.e. $|x| \overset{\text{def}}{=} \sqrt{\langle x,x\rangle}$ for $x\in\mathbb{R}^n$
m - the number of equations in elliptic systems
Ω - bounded domain in $\mathbb{R}^n$
$\operatorname{meas} M$ or $|M|$ - the Lebesgue measure of a measurable set $M\subset\mathbb{R}^n$
$\operatorname{supp} h$ - the support of a function $h\colon\Omega\to\mathbb{R}$
$C_0^\infty(\Omega)$ - the set of functions with compact support in Ω having all derivatives of arbitrary order continuous in Ω
$L^p(\Omega),\ p=1,2$ - the space of classes of measurable functions f on Ω such that $x\to|f|^p(x)$ is integrable on Ω
$L^\infty(\Omega)$ - the space of classes of measurable functions f on Ω such that $x\to|f|(x)$ is essentially bounded on Ω
$H^s(\Omega)$ - the Sobolev space of functions $f\in L^2(\Omega)$ such that $\frac{\partial f^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}\in L^2(\Omega)$, $|\alpha|\le s,\ s\ge 1$, where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index, $|\alpha|=\alpha_1+\ldots+\alpha_n$ and $\alpha_1,\dots,\alpha_n$ are nonnegative integers
$H_0^1(\Omega)$ - the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, this space is equipped with the norm

$$\|v\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla v|^2\,\mathrm{d}x\right)^{1/2}$$

$H^{-1}(\Omega)$ - the dual space of $H_0^1(\Omega)$
$\mathbf{L}^2(\Omega) \overset{\text{def}}{=} [L^2(\Omega)]^{m\times n}$
$V \overset{\text{def}}{=} [H_0^1(\Omega)]^m$
$V^* \overset{\text{def}}{=} [H^{-1}(\Omega)]^m$

grad f or ∇f - the gradient of $f\colon \Omega \to \mathbb{R}$, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$
div f - the divergence of $f\colon \Omega \to \mathbb{R}^n$, $\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n}$
rot f - the curl of $f\colon \Omega \to \mathbb{R}^n$, $\operatorname{rot} f = \left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}\right)_{i,j=1,\dots,n}$
In what follows, $\mathcal{A}\xi$ is a matrix with the components $(\mathcal{A}\xi)_{ij} = a_{ijkl}\xi_{kl}$ (summation over repeated indices is assumed), $(\mathcal{A}\xi, \eta)$ (in short $\xi\,\mathcal{A}\eta$) is a bilinear form such that $(\mathcal{A}\xi, \eta) = a_{ijkl}\xi_{kl}\eta_{ij}$, $\xi, \eta \in \mathbb{R}^{m\times n}$.
Let $\nu_1, \nu_2 > 0$, $\mathcal{E}(\nu_1, \nu_2)$ be the set of all $(mn\times mn)$-matrices $\mathcal{A} = \{a_{ijkl}\}_{\substack{1 \le i,k \le m \\ 1 \le j,l \le n}}$ satisfying the conditions

$$(9) \qquad \begin{cases} \nu_1\xi_{ij}\xi_{ij} \le a_{ijkl}(x)\xi_{ij}\xi_{kl} \le \nu_2\xi_{ij}\xi_{ij} \text{ for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^{m\times n}, \\ a_{ijkl}(x) = a_{ilkj}(x) = a_{kjil}(x) \text{ for a.e. } x \in \Omega, \\ a_{ijkl} \in L^\infty(\Omega), \\ 1 \le i,k \le m;\ 1 \le j,l \le n, \end{cases}$$

and $\mathcal{A}$ acting on $\xi \in \mathbb{R}^{m\times n}$ as it is defined above, i.e. $(\mathcal{A}\xi)_{ij} = a_{ijkl}\xi_{kl}$, $1 \le i \le m$, $1 \le j \le n$. It can be written as follows:

$$\mathcal{A} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix},$$

where b_{ik}, $1 \le i,k \le m$ are $(m \times m)$-matrices.

We relate to $\mathcal{A}$ the elliptic operator $A\colon V \to V^*$ as

$$Au = \operatorname{div} \mathcal{A}\nabla u \qquad \text{for all } u \in V.$$

In this expression, the operators div and ∇ act in the usual manner.
Let $E(\nu_1, \nu_2)$ be the set of all operators of the form $A = \operatorname{div}\mathcal{A}\nabla$ with $\mathcal{A} \in \mathcal{E}(\nu_1, \nu_2)$.

We shall say that a family $\{\mathcal{A}_\varepsilon\} \subset \mathcal{E}(\nu_1, \nu_2)$ H-converges to a matrix $\mathcal{A}_0$ (we write $\mathcal{A}_\varepsilon \xrightarrow{H} \mathcal{A}_0$) if and only if for every $f \in V^*$ the solutions of the equations

$$\operatorname{div}\mathcal{A}_\varepsilon\nabla u_\varepsilon = f,$$
$$\operatorname{div}\mathcal{A}_0 u_0 = f$$

satisfy

$$(10) \qquad u_\varepsilon \rightharpoonup u_0 \qquad \text{in } V,$$

$$(11) \qquad \mathcal{A}_\varepsilon\nabla u_\varepsilon \rightharpoonup \mathcal{A}_0\nabla u_0 \qquad \text{in } \mathbf{L}^2(\Omega).$$

We shall call a matrix $\mathcal{A}^0$ homogenized for the family $\{\mathcal{A}_\varepsilon\}$ if (10) and (11) hold for any bounded domain $\Omega \subset \mathbb{R}^n$, see [29, page 20 and page 154].

In what follows, we shall consider homogenized matrices for the families defined as $\mathcal{A}_\varepsilon = \mathcal{A}(\varepsilon^{-1}x)$ and call the matrix $\mathcal{A}^0$ homogenized for $\mathcal{A}$. Denote by C the periodicity cell, i.e. the parallelepiped in $\mathbb{R}^n$ with sides $l_1, l_2, \ldots, l_n$ parallel to axes. Let $L^2(C)$ be the Lebesgue space of periodic measurable square integrable functions on C and $H^1_{\mathrm{per}}(C)$ the closure of the set of infinitely differentiable periodic functions in the Sobolev space $H^1(C)$. Detote by $H^{-1}(C)$ the dual space of $H^1_{\mathrm{per}}(C)$. Put $\langle h\rangle \stackrel{\mathrm{def}}{=} (1/|C|)\int_C h\,\mathrm{d}x$. The letters X, Y will always denote real normed spaces. The symbol "$\to$" ("$\rightharpoonup$" and "$\stackrel{*}{\rightharpoonup}$") means to be strongly convergent to (weakly convergent to and weakly $*$-convergent to, respectively). For $M \subset X$, $\mathrm{cl}_s M$ stands for the strong closure of M in X. Denote by $\mathcal{L}(X,Y)$ the set of all continuous linear operators from X into Y. Let "$\xrightarrow{w}$" be the sequential convergence of a sequence of operators in the weak topology of $\mathcal{L}(X,Y)$ and "$\xrightarrow{s}$" the sequential convergence of a sequence of operators in the strong topology of $\mathcal{L}(X,Y)$ (weak and strong topologies in $\mathcal{L}(X,Y)$ are meant in the sense of [4]). Denote by $(\cdot,\cdot)$ the duality pairing between a normed space X and its dual space X^* and $\|\cdot\|_X$ the norm in a normed space X and $M(\nu_1,\nu_2)$ the class of mappings $A\colon X \to X^*$ satisfying the conditions

$$
\begin{aligned}
(Ax - Ay, x - y) &\geq \nu_1 \|x-y\|_X^2,\\
\|Ax - Ay\|_{X^*} &\leq \nu_2 \|x-y\|_X.
\end{aligned} \tag{12}
$$

We shall say that a family $\{A_\varepsilon\} \subset M(\nu_1,\nu_2)$ G-converges to an operator A_0 (we write $A_\varepsilon \xrightarrow{G} A_0$) if and only if for every $f \in X^*$

$$
A_\varepsilon^{-1} f \rightharpoonup A_0^{-1} f \qquad \text{in } X.
$$

In [22], [29, page 167], it is proved that for elliptic operators the notion of H-convergence of their matrices is equivalent to the G-convergence of the corresponding operators provided that matrices belong to the class $\mathcal{E}(\nu_1,\nu_2)$.

3. General case: description of strong closure of feasible states using G-convergence

Let us describe the strong closure of sets of feasible states in the following situation. Let X be a separable reflexive Banach space. Let $\mathfrak{U} \subset M(\lambda_1,\lambda_2)$ for some $\lambda_1, \lambda_2 > 0$. Throughout this section we will assume that $\mathfrak{U}$ consists of self-adjoint linear operators $A\colon X \to X^*$, i.e.

$$
(Au, v) = (Av, u) \qquad \text{for all } u, v \in X.
$$

Proposition 3.1. *We have*

$$(13)\quad \mathrm{cl}_s F(\mathfrak{U}, f) = \left\{ u \in X \colon \exists\, A_+,\ A_0,\ \{A_k\} \subset \mathfrak{U} \colon \begin{array}{l} A_k \xrightarrow{G} A_0,\ A_k \xrightarrow{w} A_+, \\ A_+ u = A_0 u = f \end{array} \right\}$$

for all $f \in X^*$.

Proof: Denote by K the right-hand side of (13).

(i) Let $u \in \mathrm{cl}_s F(\mathfrak{U}, f)$. Then there exist $\{A_k\} \subset \mathfrak{U}$, $\{u_k\} \subset X$ such that $A_k u_k = f$ and $u_k \to u$ in X. It is clear (since X is separable reflexive Banach space, therefore we can use diagonal process) that for some subsequence of $\{A_k\}$ (still denoted by $\{A_k\}$) we have $A_k \xrightarrow{G} A_0$, $A_k \xrightarrow{w} A_+$ for some linear continuous operators A_0, A_+. Let us prove that $A_+ u = A_0 u = f$.

Since $u_k = A_k^{-1} f$, then $u = A_0^{-1} f$ due to $A_k \xrightarrow{G} A_0$ and $u_k \to u$ in X. Hence $A_0 u = f$. We can write ($|\cdot|$ denotes absolute value)

$$\begin{aligned} |(A_+ u - f, v)| &\le |(A_+ u - A_k u, v)| + |(A_k u - A_k u_k, v)| \\ &\le |(A_+ u - A_k u, v)| + \|A_k u - A_k u_k\|_{X^*} \|v\|_X \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

Therefore $A_+ u = f$. In other words, $u \in K$.

(ii) Let $u \in K$. Then there exists $\{A_k\} \subset \mathfrak{U}$ such that $A_k \xrightarrow{G} A_0$, $A_k \xrightarrow{w} A_+$, $A_+ u = A_0 u = f$ for some linear continuous operators A_+, A_0. Let $\{u_k\}$ be the sequence defined as $A_k u_k = f$. Since the sequence is bounded in X there is a subsequence $\{u_k\}$ (still denoted in the same way) which weakly converges to $v = A_0^{-1} f$ in X. From the formula $A_0 u = f$ we see that $v = u$.

Since

$$\lambda_1 \|u_k - u\|_X^2 \le (A_k(u_k - u), u_k - u)$$

it is needed to prove that the right-hand side of this inequality tends to 0. Indeed, we have

$$(14)\qquad (A_k(u_k - u), u_k - u) = (A_k u_k, u_k) - (A_k u_k, u) - (A_k u, u_k) + (A_k u, u).$$

Since the operators from $\mathfrak{U}$ are self-adjoint, we obtain that

$$(A_k u, u_k) = (A_k u_k, u).$$

Using the facts that $A_k u \rightharpoonup A_+ u$ and $u_k \rightharpoonup u$, we see that the right-hand side of (14) converges to $(f, u) - (f, u) - (f, u) + (A_+ u, u) = 0$ as $k \to \infty$. Hence, $u \in \mathrm{cl}_s F(\mathfrak{U}, f)$. $\square$

4. General case: description of strong closure of feasible states using Γ-convergence

In this section, we will reformulate results of the previous section using theory of Γ-convergence, see [29, pages 404–420] for the discussion on Γ-convergence.

Let X be a separable reflexive Banach space.

A functional $I\colon X \to \mathbb{R}$ is called a Γ-limit of a sequence of functionals $\{I_k\}$ and the sequence $\{I_k\}$ Γ-*converges to* I (we will write $I_k \xrightarrow{\Gamma} I$) if

(i) for every sequence $\{x_k\} \subset X$ weakly converging to x the condition

$$\underline{\lim}\, I_k(x_k) \geq I(x)$$

holds;

(ii) for every $x \in X$ there exists a sequence $\{x_k\} \subset X$ such that $x_k \rightharpoonup x$ and

$$\lim I_k(x_k) = I(x).$$

Let us relate to every operator $A \in M(\lambda_1, \lambda_2)$ the functional I_A as follows:

$$I_A(u) = (Au, u), \qquad u \in X.$$

As is known, the equation $Au = f$ is the Euler equation for the variational problem

$$E = \inf_{v \in X} J(v),$$
$$J(v) = \frac{1}{2}(Av, v) - (f, v).$$

It is also known that for $\{A_k\} \subset M(\lambda_1, \lambda_2)$, $A_k \xrightarrow{G} A_0$ if and only if $I_{A_k} \xrightarrow{\Gamma} I_{A_0}$. Using this fact, Proposition 3.1 can be rewritten in the following way:

Proposition 4.1. *We have*

$$\begin{aligned}\mathrm{cl}_s F(\mathfrak{U}, f) = \ &\{u \in X \colon \exists\, A_+,\ A_0,\ \{A_k\} \subset \mathfrak{U} \colon I_k \xrightarrow{\Gamma} I_0,\ I_k \to I_+ \ \textit{point-wisely} \\ &\textit{on } X;\ u \ \textit{is a solution to the problems } \inf\nolimits_{v \in X}[I_+(v) - 2(f, v)], \\ &\inf\nolimits_{v \in X}[I_0(v) - 2(f, v)]\}\end{aligned}$$

for all $f \in X^*$, where we denoted $I_k = I_{A_k}$, $I_+ = I_{A_+}$, $I_0 = I_{A_0}$.

Denoting by I^* the dual functional of I (we refer, for instance, to [6], [29, page 357] for the definition of the dual functional) and using the fact that $I_k \to I$ point-wisely on X if and only if $I_k^* \xrightarrow{\Gamma} I^*$ (provided that some conditions of regularity and coercivity are satisfied; see, for instance, [29, page 406]), one can obtain that

$$\begin{aligned}\mathrm{cl}_s F(\mathfrak{U}, f) = \; & \{u \in X\colon\, \exists\, A_+,\ A_0,\ \{A_k\} \subset \mathfrak{U}\colon\ I_k \xrightarrow{\Gamma} I_0,\ I_k^* \xrightarrow{\Gamma} I_+^*;\ u \text{ is a solution} \\ & \quad \text{to the problems } \inf_{v\in X}[I_+(v) - 2(f,v)],\ \inf_{v\in X}[I_0(v) - 2(f,v)]\}\end{aligned}$$

for all $f \in X^*$.

5. Elliptic operators, case $m < n$

In this section, the closures of the sets of feasible states for systems of elliptic equations in the strong topology of the Cartesian product $[H_0^1(\Omega)]^m$ of Sobolev spaces is considered for the case $m < n$.

Let us recall the compensated compactness principle, see [13], [29, page 12].

Proposition 5.1. *Let D be a bounded open set in $\mathbb{R}^n$. Let $\{h^\varepsilon\}$, $\{g^\varepsilon\} \subset \mathbf{L}^2(D)$ be such that*

$$h^\varepsilon \rightharpoonup h^0, \quad g^\varepsilon \rightharpoonup g^0 \qquad \text{in } \mathbf{L}^2(D).$$

If the families $\{\mathrm{rot}\, h^\varepsilon\}$ and $\{\mathrm{div}\, g^\varepsilon\}$ are compact in the corresponding Cartesian products of the Sobolev spaces $H^{-1}(D)$, then

$$h^\varepsilon \cdot g^\varepsilon \to h \cdot g \qquad \text{in } D \text{ in the sense of distributions.}$$

Further we shall use the following proposition where proof is obvious.

Proposition 5.2. *Let M be a subset of some normed space X. If for every $z_1, z_2 \in M$ it follows that $\alpha z_1 + (1-\alpha) z_2 \in \mathrm{cl}_s M$ for all $\alpha \in [0,1]$, then $\mathrm{cl}_s M$ is convex.*

As is known the space $\mathbf{L}^2(\Omega)$ admits orthogonal decomposition, see [29, page 370],

$$\begin{aligned}\mathbf{L}^2(\Omega) &= L^2_{\mathrm{sol}}(\Omega) \oplus \mathcal{V}^2_{\mathrm{pot}}(\Omega),\\ L^2_{\mathrm{sol}}(\Omega) &\stackrel{\mathrm{def}}{=} \{p \in \mathbf{L}^2(\Omega)\colon\ \mathrm{div}\, p = 0 \ \text{ in } \Omega\},\\ \mathcal{V}^2_{\mathrm{pot}}(\Omega) &\stackrel{\mathrm{def}}{=} \{\nabla u\colon\ u \in [H_0^1(\Omega)]^m\}.\end{aligned}$$

Denote by P the orthogonal projector from $\mathbf{L}^2(\Omega)$ on $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$. Let $\{h^\varepsilon\} \subset \mathbf{L}^2(\Omega)$.

Proposition 5.3. *Then $Ph^\varepsilon \to 0$ in $\mathbf{L}^2(\Omega)$ if and only if $\mathrm{div}\, h^\varepsilon \to 0$ in V^*.*

Proof: Vectors $\{h^\varepsilon\} \subset \mathbf{L}^2(\Omega)$ admit decomposition

$$h^\varepsilon = u^\varepsilon + Ph^\varepsilon$$

where $\{u^\varepsilon\} \subset L^2_{\mathrm{sol}}(\Omega)$. Taking the div of both sides gives

$$\mathrm{div}\, h^\varepsilon = \mathrm{div}\, Ph^\varepsilon.$$

So, if $Ph^\varepsilon \to 0$ in $\mathbf{L}^2(\Omega)$, then $\operatorname{div} h^\varepsilon \to 0$ in V^*.

The inverse conclusion follows from the equalities

$$\begin{aligned} Ph^\varepsilon &= \nabla\phi^\varepsilon \qquad \text{for some } \phi^\varepsilon \in [H_0^1(\Omega)]^m, \\ \operatorname{div} h^\varepsilon &= \operatorname{div} \nabla\phi^\varepsilon. \end{aligned}$$

So, if $\operatorname{div} h^\varepsilon \to 0$ in V^*, then $\phi^\varepsilon \to 0$ in $[H_0^1(\Omega)]^m$ due to the fact that the inverse of Laplacian is continuous. Hence, $Ph^\varepsilon \to 0$ in $\mathbf{L}^2(\Omega)$. □

We have

Proposition 5.4. *If $m < n, n \geq 2$, then for every $\alpha \in [0,1]$ and every $h \in \mathbf{L}^2(\Omega)$ there exists a family $\{\chi^\varepsilon\}$ of characteristic functions of measurable subsets of Ω such that*

$$\operatorname{div}(\chi^\varepsilon - \alpha)h \to 0 \qquad \textit{in } V^*.$$

Proof: Let $\alpha \in [0,1]$. Put

$$T_\alpha = \{h \in \mathbf{L}^2(\Omega) \colon \exists\, \{\chi^\varepsilon\} \colon \operatorname{div}(\chi^\varepsilon - \alpha)h \to 0 \text{ in } V^*\}.$$

This set has the following properties:

(i) If $h = c\chi_D$ where $c \in \mathbb{R}^{m\times n}$, D is an open subset of Ω, χ_D is a characteristic function of D, then $h \in T_\alpha$.

(ii) If $h_1, h_2 \in T_\alpha$ and $\operatorname{meas}(\operatorname{supp} h_1 \cap \operatorname{supp} h_2) = 0$, then $ah_1 + bh_2 \in T_\alpha$ for all $a, b \in \mathbb{R}$.

(iii) If $\{h^\varepsilon\} \subset T_\alpha$ and $h^\varepsilon \to h$ in $\mathbf{L}^2(\Omega)$, then $h \in T_\alpha$.

Let us prove these properties.

(i) Let $h = c\chi_D$ where $c \in \mathbb{R}^{m\times n}$. Then there exists $c^\perp \in \mathbb{R}^n$ such that $\langle c^\perp, c_i\rangle = 0$ for all $i = 1,\dots,m$, and $|c^\perp| = 1$ where c_i, $i = 1,\dots,m$, are the rows of c.

Let χ be the 1-periodic function on $\mathbb{R}$

$$\chi(t) = \begin{cases} 1, & t \in [0,\alpha), \\ 0, & t \in (\alpha, 1]. \end{cases} \tag{15}$$

By setting $\chi^\varepsilon = \chi(\varepsilon^{-1}\langle c^\perp, x\rangle)$ (we use so-called laminate, see, for instance, [3]) and $h^\varepsilon = (\chi^\varepsilon - \alpha)h$, we obtain

$$(Ph^\varepsilon, Ph^\varepsilon)_{\mathbf{L}^2(\Omega)} = (Ph^\varepsilon, h^\varepsilon)_{\mathbf{L}^2(\Omega)} = (Ph^\varepsilon, h^\varepsilon)_{\mathbf{L}^2(D)}. \tag{16}$$

Since

$$\operatorname{div} h^\varepsilon = \begin{pmatrix} \langle c^\perp, c_1\rangle \\ \vdots \\ \langle c^\perp, c_m\rangle \end{pmatrix} \frac{\mathrm{d}\chi}{\mathrm{d}t}(\varepsilon^{-1}\langle c^\perp, x\rangle) = 0, \qquad \operatorname{rot} Ph^\varepsilon = 0$$

in D in the sense of distributions and $h^\varepsilon \rightharpoonup 0$, $Ph^\varepsilon \rightharpoonup 0$ in $\mathbf{L}^2(D)$. Hence, using (16) and Proposition 5.1, we have $Ph^\varepsilon \to 0$ in $\mathbf{L}^2(\Omega)$. Then, from Proposition 5.3 we obtain

$$\operatorname{div}(\chi^\varepsilon - \alpha)h \to 0 \qquad \text{in } V^*.$$

(ii) Let $h_1, h_2 \in T_\alpha$ and $\operatorname{meas}(\operatorname{supp} h_1 \cap \operatorname{supp} h_2) = 0$. Then there exist families $\{\chi_1^\varepsilon\}, \{\chi_2^\varepsilon\}$ which satisfy

$$\left.\begin{array}{l} \operatorname{div}(\chi_1^\varepsilon - \alpha)h_1 \to 0 \\ \operatorname{div}(\chi_2^\varepsilon - \alpha)h_2 \to 0 \end{array}\right\} \qquad \text{in } V^*. \tag{17}$$

The $\{\chi^\varepsilon\}$ defined as $\chi^\varepsilon = \chi_1^\varepsilon \chi_{\operatorname{supp} h_1} + \chi_2^\varepsilon \chi_{\Omega\setminus \operatorname{supp} h_1}$ is required.

(iii) If $\{h^\varepsilon\} \subset T_\alpha$ and $h^\varepsilon \to h$ in $\mathbf{L}^2(\Omega)$, then using the diagonal process one can obtain the family $\{\chi^\varepsilon\}$ such that $\operatorname{div}(\chi^\varepsilon - \alpha)h \to 0$.

From (i)–(iii) it follows that $T_\alpha = \mathbf{L}^2(\Omega)$. □

In exactly the same way one can prove this proposition for $h \in \mathbf{L}^2(\Omega)$ such that $\operatorname{rank} h(x) \le n-1$ for a.e. $x \in \Omega$ without imposing the condition $m \le n-1$.

Let

$$\mathfrak{U} = \left\{ \operatorname{div} \sum_{p=1}^r \chi_p \mathcal{A}_p \nabla \right\}_{\chi \in \Lambda} \tag{18}$$

where

$$\Lambda = \left\{ (\chi_1, \dots, \chi_r) \in [L^\infty(\Omega)]^r \colon \ \sum_{p=1}^r \chi_p = 1,\ \chi_p(x) \in \{0,1\} \text{ for a.e. } x \in \Omega \right\}$$

and $\mathcal{A}_p$, $p = 1, \dots, r$, are matrices which belong to $\mathcal{E}(\lambda_1, \lambda_2)$. Let $\Theta = \overline{\mathrm{co}}\,\Lambda$ where $\overline{\mathrm{co}}$ is the symbol of the closed convex hull in $[L^2(\Omega)]^r$, i.e.

$$\Theta = \left\{ (\theta_1, \dots, \theta_r) \in [L^\infty(\Omega)]^r \colon \ \sum_{p=1}^r \theta_p = 1,\ 0 \le \theta_p(x) \le 1 \right\}. \tag{19}$$

Define

$$\mathfrak{B} = \left\{ \operatorname{div} \sum_{p=1}^r \theta_p \mathcal{A}_p \nabla \right\}_{\theta \in \Theta}. \tag{20}$$

Now we are able to formulate the following

Theorem 5.5. *Let $m \le n-1$. Then*

$$\mathrm{cl}_s \mathrm{F}(\mathfrak{U}, f) = \mathrm{F}(\mathfrak{B}, f) \qquad \text{for all } f \in V^*. \tag{21}$$

Proof: The proof immediately follows from Propositions 5.2 and 5.4 because they show that the set

$$\mathrm{cl}_s \mathrm{F}(\mathfrak{U}, f)$$

is convex in V for all $f \in V^*$. □

We can generalize this result using rank of matrices.

Theorem 5.6. *Let $g \in \mathbf{L}^2(\Omega)$. Suppose that*

$$\mathrm{rank}\,(\mathcal{A}_p(x) - \mathcal{A}_q(x))g(x) \le n-1 \qquad \text{for a.e. } x \in \Omega \text{ and all } p \neq q.$$

Then the set

$$\mathrm{cl}_s \bigcup_{\chi \in \Lambda} \left\{ \mathrm{div} \sum_{p=1}^{r} \chi_p \mathcal{A}_p g \right\}$$

is convex in V^*.

5.1 Application of H-convergence. Let $\mathfrak{U}, \mathfrak{B}$ be families defined by (18), (20), respectively. Using the similar reasoning as in Proposition 3.1, one can describe the strong closure of feasible states using the notion of H-convergence. For instance, for the case of two materials $r = 2$, we see that for every $f \in V^*$ the equality

$$\mathrm{cl}_s \mathrm{F}(\mathfrak{U}, f) = \left\{ u \in V : \exists\, \theta, \{\chi^\varepsilon\} : \begin{array}{l} \chi^\varepsilon \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega), \\ \mathcal{A}_{\chi^\varepsilon} \overset{H}{\longrightarrow} \mathcal{A}^0, \\ \mathcal{A}_\theta \nabla u = \mathcal{A}^0 \nabla u, \\ \mathrm{div}\, \mathcal{A}_\theta \nabla u = f \end{array} \right\}$$

holds, where $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{E}(\lambda_1, \lambda_2)$ and $\mathcal{A}_\theta = \theta \mathcal{A}_1 + (1-\theta)\mathcal{A}_2$.

In this section, we shall prove the analog of Theorem 5.5 using the theory of H-convergence of matrices for the case $r = 2$, $m \le n-1$, and when the matrices $\mathcal{A}_1, \mathcal{A}_2$ are isotropic constant matrices.

Proposition 5.7. *For every $h \in \mathbf{L}^2(\Omega)$ and for every $\theta \in \Theta$ there exists a family $\{\chi^\varepsilon\} \subset L^\infty(\Omega)$ of characteristic functions such that*

$$\chi^\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(\Omega), \qquad \mathcal{A}_{\chi^\varepsilon} \overset{H}{\longrightarrow} \mathcal{A}^0, \qquad \mathcal{A}_\theta h = \mathcal{A}^0 h.$$

Proof: It is clear that we can confine ourselves to the case where θ and h are constant since one can use approximation arguments to prove the result in general.

For the sake of brevity, we also assume that $m = 1$ as the case $n - 1 \geq m > 1$ is considered in the similar manner.

Let $\theta \in \mathbb{R}$, $0 \leq \theta \leq 1$. Let $\xi \in \mathbb{R}^{m\times n}$. From the below reasoning it will be seen that if

$$\operatorname{rank}(\mathcal{A}_1 - \mathcal{A}_2)\xi < n,$$

then there exists a family $\{\chi^\varepsilon\} \subset L^\infty(\Omega)$ of characteristic functions such that

$$\chi^\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(\Omega), \qquad \mathcal{A}_{\chi^\varepsilon} \xrightarrow{H} \mathcal{A}^0, \qquad \mathcal{A}_\theta \xi = \mathcal{A}^0\xi.$$

For a vector $e \in \mathbb{R}^n$ let us define

$$\chi^\varepsilon = \chi(\varepsilon^{-1}\langle e, x\rangle),$$

where χ is the 1-periodic function (15) with $\alpha = \theta$.

The homogenized matrix $\mathcal{A}^0$ can be calculated explicitly. From [28, page 71] it follows that

$$(\mathcal{A}^0 - \mathcal{A}_1)^{-1} = (1-\theta)^{-1}(\mathcal{A}_2 - \mathcal{A}_1)^{-1} + \theta(1-\theta)^{-1}S(e), \tag{22}$$

where $S(e) = (e \otimes e)/(e\mathcal{A}_1 e)$ and "$\otimes$" stands for the symbol of the tensor product. It can be readily seen that if

$$(\mathcal{A}^0 - \mathcal{A}_1)^{-1}(\mathcal{A}_\theta - \mathcal{A}_1)\xi = \xi,$$

then $\mathcal{A}_\theta\xi = \mathcal{A}^0\xi$. Indeed, by (22), we see that

$$(\mathcal{A}^0 - \mathcal{A}_1)^{-1}(\mathcal{A}_\theta - \mathcal{A}_1)\xi = \xi + \theta S(e)(\mathcal{A}_2 - \mathcal{A}_1)\xi.$$

Hence, if $S(e)(\mathcal{A}_2 - \mathcal{A}_1)\xi = 0$, then the equality $\mathcal{A}_\theta\xi = \mathcal{A}^0\xi$ follows. But, using the form of the matrix $S(e)$ and the assumption that $1 = m < n$, one can show that such a vector e exists. It can be chosen as perpendicular to the vector $(\mathcal{A}_2 - \mathcal{A}_1)\xi$. It is clear that this reasoning can be extended to the case $1 < m \leq n-1$, $n \geq 2$, and to the case when $\operatorname{rank}(\mathcal{A}_1 - \mathcal{A}_2)\xi < n$. We thus obtain the results stated by Theorems 5.5 and 5.6. $\square$

5.2 Application of Beran–Zhikov method. In this section, we shall apply a method introduced in [2] and developed in details in [28] for proving estimates for homogenized matrices.

Let $\mathcal{A}_\chi = \chi\mathcal{A}_1 + (1-\chi)\mathcal{A}_2$, where χ is a periodic characteristic function defined in the same way as in the previous section (it depends on the vector e).

Since the homogenized matrix $\mathcal{A}^0$ for $\mathcal{A}_\chi$ may be expressed as, see [29, page 22],

$$\mathcal{A}^0\xi = \langle \mathcal{A}_\chi(\xi + \nabla u)\rangle,$$

where $u \in [H^1_{\mathrm{per}}(C)]^m$ is a solution of the equation

$$\operatorname{div} \mathcal{A}_\chi(\xi + \nabla u) = 0 \qquad \text{in } [H^{-1}(C)]^m,$$

then

$$(\mathcal{A}^0 - \mathcal{A}_1)\xi = \langle(\mathcal{A}_\chi - \mathcal{A}_1)(\xi + \nabla u)\rangle = \langle(1-\chi)(\mathcal{A}_2 - \mathcal{A}_1)(\xi + \nabla u)\rangle.$$

The equation

$$\operatorname{div} \mathcal{A}_\chi(\xi + \nabla u) = 0$$

can be rewritten as

$$\operatorname{div}(\mathcal{A}_\chi - \mathcal{A}_1)(\xi + \nabla u) + \operatorname{div} \mathcal{A}_1 \nabla u = 0.$$

Hence,

$$\operatorname{div} \mathcal{A}_1 \nabla u = -\operatorname{div}(1-\chi)(\mathcal{A}_2 - \mathcal{A}_1)(\xi + \nabla u).$$

Define the operator $L\colon [L^2(C)]^{m\times n} \to [L^2(C)]^{m\times n}$ as follows:

$$Lh = \nabla A_1^{-1} \operatorname{div}(1-\chi)(\mathcal{A}_2 - \mathcal{A}_1)h, \qquad h \in [L^2(C)]^{m\times n},$$

where $A_1 = \operatorname{div} \mathcal{A}_1 \nabla$. Then

$$\begin{aligned} \nabla u &= -L(\xi + \nabla u), \\ \xi &= (\xi + \nabla u) - \nabla u = (\xi + \nabla u) + L(\xi + \nabla u) = (E + L)(\xi + \nabla u). \end{aligned}$$

Therefore

$$(\mathcal{A}^0 - \mathcal{A}_1)\xi = \langle(1-\chi)(\mathcal{A}_2 - \mathcal{A}_1)(E + L)^{-1}\xi\rangle,$$

where $E\colon [L^2(C)]^{m\times n} \to [L^2(C)]^{m\times n}$ is the identity operator.

Now, by choosing e to be perpendicular to all rows of matrix $(\mathcal{A}_2 - \mathcal{A}_1)\xi$, we obtain that

$$\operatorname{div}(1-\chi)(\mathcal{A}_2 - \mathcal{A}_1)\xi = 0 \qquad \text{in } [H^{-1}(C)]^m.$$

Hence $L\xi = 0$. This gives that

$$(\mathcal{A}^0 - \mathcal{A}_1)\xi = (1-\theta)(\mathcal{A}_2 - \mathcal{A}_1)\xi.$$

Therefore $\mathcal{A}_\theta \xi = \mathcal{A}^0 \xi$.

This gives another proof of Theorems 5.5 and 5.6.

6. Application of N-condition

Let $\tilde{A} = \operatorname{div} \tilde{\mathcal{A}} \nabla$, $A^\varepsilon = \operatorname{div} \mathcal{A}^\varepsilon \nabla$ where $\mathcal{A}^\varepsilon \in \mathcal{E}(\lambda_1, \lambda_2)$, $\varepsilon \in (0,1)$. We can rewrite $\tilde{A}, A^\varepsilon$ as follows:

$$
\begin{aligned}
A^\varepsilon &= \frac{\partial}{\partial x_i}\Big(a_\varepsilon^{ij} \frac{\partial}{\partial x_j}\Big), \\
\tilde{A} &= \frac{\partial}{\partial x_i}\Big(\tilde{a}^{ij} \frac{\partial}{\partial x_j}\Big),
\end{aligned} \tag{23}
$$

where $\tilde{a}^{ij}, a_\varepsilon^{ij}$, $i,j = 1, \dots, n$, are $(m \times m)$-matrices which correspond to $\tilde{\mathcal{A}}, \mathcal{A}^\varepsilon$. Let

$$
\begin{aligned}
A^\varepsilon u^\varepsilon &= f, \qquad u^\varepsilon \in V, \\
\tilde{A} u^0 &= f, \qquad u^0 \in V.
\end{aligned} \tag{24}
$$

We shall say that for the family $\{A^\varepsilon\}$ an N-*condition* is fulfilled, see [15, page 79] and [30, page 80], if there exist $(m \times m)$-matrices $N_\varepsilon^s \in [H^1(\Omega)]^{m\times m}$, $s = 1, \dots, n$, such that

(i) $N_\varepsilon^s \rightharpoonup 0$ in $[H^1(\Omega)]^{m\times m}$, $i, s = 1, \dots, n$;

(ii) $\tilde{a}_\varepsilon^{ij} \overset{\text{def}}{=} a_\varepsilon^{il} \frac{\partial N_\varepsilon^j}{\partial x_l} + a_\varepsilon^{ij} \rightharpoonup \tilde{a}^{ij}$ in $[L^2(\Omega)]^{m\times m}$, $i, j = 1, \dots, n$;

(iii) $\frac{\partial}{\partial x_i}(\tilde{a}_\varepsilon^{ij} - \tilde{a}^{ij}) \to 0$ in $[H^{-1}(\Omega)]^{m\times m}$, $j = 1, \dots, n$.

In [15] it is proved that $\mathcal{A}^\varepsilon \xrightarrow{H} \tilde{\mathcal{A}}$ if and only if the N-condition holds with some $N_\varepsilon^s \in [H^1(\Omega)]^{m\times m}$, $s = 1, \dots, n$, where $\tilde{\mathcal{A}}, \mathcal{A}^\varepsilon$ correspond to $\tilde{a}^{ij}, a_\varepsilon^{ij}$, $i, j = 1, \dots, n$. In [15] it is also proved that $\tilde{\mathcal{A}} \in \mathcal{E}(\lambda_1, \lambda_2)$.

Let us set

$$
\begin{aligned}
\alpha_\varepsilon &= \max_s \|N_\varepsilon^s\|_{[L^2(\Omega]^{m\times m}}, \\
\beta_\varepsilon &= \max_{i,j} \|\tilde{a}_\varepsilon^{ij} - \tilde{a}^{ij}\|_{[H^{-1}(\Omega)]^{m\times m}}, \\
\gamma_\varepsilon &= \max_j \Big\| \frac{\partial}{\partial x_i}(\tilde{a}_\varepsilon^{ij} - \tilde{a}^{ij}) \Big\|_{[H^{-1}(\Omega)]^{m\times m}}.
\end{aligned}
$$

Assuming that $u_0 \in [C_0^\infty(\Omega)]^m$ the following estimate holds:

$$
\Big\| A^\varepsilon \Big(u^\varepsilon - u^0 - N_\varepsilon^s \frac{\partial u^0}{\partial x_s} \Big) \Big\|_V \le C_1 r_\varepsilon, \tag{25}
$$

where $r_\varepsilon = (\alpha_\varepsilon + \beta_\varepsilon + \gamma_\varepsilon)\|u^0\|_{[H^2(\Omega)]^m}$ and C_1 does not depend on u^0. Indeed, we can calculate

$$A^\varepsilon\Big(u^\varepsilon - u^0 - N_\varepsilon^s\frac{\partial u^0}{\partial x_s}\Big) = \tilde{A}u^0 - \tilde{A}^\varepsilon u^0 + \tilde{A}^\varepsilon u^0 - A^\varepsilon u^0 - A^\varepsilon N_\varepsilon^s\frac{\partial u^0}{\partial x_s}$$
$$= \tilde{A}u^0 - \tilde{A}^\varepsilon u^0 - \frac{\partial}{\partial x_i}\Big(a^{il}N_\varepsilon^j\frac{\partial^2 u^0}{\partial x_i\partial x_j}\Big)$$
$$= \frac{\partial}{\partial x_i}(\tilde{a}^{ij} - \tilde{a}_\varepsilon^{ij})\frac{\partial u^0}{\partial x_j} + (\tilde{a}^{ij} - \tilde{a}_\varepsilon^{ij})\frac{\partial^2 u^0}{\partial x_i\partial x_j}$$
$$- \frac{\partial}{\partial x_i}\Big(a^{il}N_\varepsilon^j\frac{\partial^2 u^0}{\partial x_l\partial x_j}\Big),$$

where $\tilde{A}_\varepsilon\colon V \to V^*$ is the operator corresponding to $\tilde{a}_\varepsilon^{ij}$. From this we infer (25).

Proposition 6.1. *Let $u^0, \{u^\varepsilon\}$ be defined by (23). Suppose that N-condition is fulfilled and $u^0 \in [C_0^\infty(\Omega)]^m$. Then $u^\varepsilon \to u^0$ in V if and only if*

$$E_\varepsilon(u^0) = \Big\|N_\varepsilon^s\frac{\partial u^0}{\partial x_s}\Big\|_V \to 0,$$

where $\{N_\varepsilon^s\} \subset [H^1(\Omega)]^{m\times m}$, $s = 1,\dots,n$, are families of matrices from N-condition for the family $\{A^\varepsilon\}$.

Proof: The proof follows from the following inequality that is derived from (25):

$$\Big|\,\|A^\varepsilon(u^\varepsilon - u^0)\|_{V^*} - \Big\|A^\varepsilon N_\varepsilon^s\frac{\partial u^0}{\partial x_s}\Big\|_{V^*}\Big| \le \Big\|A^\varepsilon\Big(u^\varepsilon - u^0 - N_\varepsilon^s\frac{\partial u_0}{\partial x_s}\Big)\Big\|_{V^*} \le C_1 r_\varepsilon$$

and also from $r_\varepsilon \to 0$ because N-condition is fulfilled. □

Corollary 6.1.1. *If $\frac{\partial}{\partial x_i}(a_\varepsilon^{ij} - a_0^{ij}) \to 0$ in $[H^{-1}(\Omega)]^{m\times m}$, $j = 1,\dots,n$, and $a_\varepsilon^{ij} \rightharpoonup a_0^{ij}$ in $[L^2(\Omega)]^{m\times m}$, $i,j = 1,\dots,n$, then $A_\varepsilon^{-1} \xrightarrow{s} A_0^{-1}$.*

Proof: Let $A_\varepsilon u^\varepsilon = f$ and $A_0 u^0 = f$. One can see that N-condition is satisfied with $N_\varepsilon^s = 0$, $s = 1,\dots,n$, and $\tilde{a}^{ij} = a_0^{ij}$, $i,j = 1,\dots,n$. Using Proposition (6.1), we obtain $u^\varepsilon \to u^0$ in V provided that $u^0 \in [C_0^\infty(\Omega)]^m$. Since the set of such functions is dense in $[H_0^1(\Omega)]^m$, we receive the result. □

Corollary 6.1.2. *If $A_\varepsilon^{-1} \xrightarrow{s} A_0^{-1}$ and $a_\varepsilon^{ij} \rightharpoonup a_0^{ij}$ in $[L^2(\Omega)]^{m\times m}$, $i,j = 1,\dots,n$, and assume that the solutions u^0 of the equation $A_0 u^0 = f$ belong to $[C_0^\infty(\Omega)]^m$ for $f \in [C^\infty(\Omega)]^m$, then $\frac{\partial}{\partial x_i}(a_\varepsilon^{ij} - a_0^{ij}) \to 0$ in $[H^{-1}(\Omega)]^{m\times m}$, $j = 1,\dots,n$.*

Proof: Let $u^0 \in [C_0^\infty(\Omega)]^m$ and $A_\varepsilon u^\varepsilon = f$, $A_0 u^0 = f$, $u^\varepsilon \to u^0$ in $[H_0^1(\Omega)]^m$.

From the strong convergence of the inverse operators it follows that $\mathcal{A}_0$ is the H-limit of $\{\mathcal{A}_\varepsilon\}$. Hence, N-condition is fulfilled for some matrices $N_\varepsilon^s \in [H^1(\Omega)]^{m\times m}$, $s = 1,\dots,n$.

Taking into account the equality

$$A_\varepsilon\Big(u^\varepsilon - u^0 - N_\varepsilon^s \frac{\partial u^0}{\partial x_s}\Big) = A_\varepsilon(u^\varepsilon - u^0) - \frac{\partial}{\partial x_i}\Big(a_\varepsilon^{ij} \frac{\partial N_\varepsilon^s}{\partial x_j}\frac{\partial u^0}{\partial x_s}\Big) - \frac{\partial}{\partial x_i}\Big(a_\varepsilon^{ij} N_\varepsilon^s \frac{\partial^2 u^0}{\partial x_j \partial x_s}\Big)$$

we obtain that

$$\frac{\partial}{\partial x_i}\Big(a_\varepsilon^{ij}\frac{\partial N_\varepsilon^s}{\partial x_j}\Big) \to 0 \qquad \text{in } [H^{-1}(\Omega)]^{m\times m}$$

for all $s = 1, \dots, n$. Hence, from the points (ii), (iii) of the definition of N-condition it follows that $\frac{\partial}{\partial x_i}(a_\varepsilon^{ij} - a_0^{ij}) \to 0$ in $[H^{-1}(\Omega)]^{m\times m}$ for all $j = 1, \dots, n$. □

7. Periodic case

In this section, we shall asume that $m = 1$ since the general case can be considered in the similar manner.

Let $a^\varepsilon(x) = a(\varepsilon^{-1}x)$, $\varepsilon > 0$, where a is a positive definite periodic $(n \times n)$-matrix. Define $N_k \in H^1_{\mathrm{per}}(C)$, $k = 1, \dots, n$, by the equations

$$\operatorname{div} a(e^k + \nabla N_k) = 0, \qquad k = 1, \dots, n \ \text{ in } H^{-1}(C), \tag{26}$$

where e^k is the standard kth basis vector. Let $\nabla N = (\nabla N_1, \dots, \nabla N_n)$ (we stacked the columns).

Let $u^\varepsilon, u^0 \in H^1_0(\Omega)$ be the solutions of the equations

$$\operatorname{div} a^\varepsilon \nabla u^\varepsilon = f,$$
$$\operatorname{div} \tilde{a} \nabla u^0 = f,$$

with $f \in H^{-1}(\Omega)$ and $\tilde{a}$ is the homogenized matrix for a. The formula for $\tilde{a}$ is known, see, for instance, [29, page 26]:

$$\tilde{a} = \langle a(I + \nabla N)\rangle, \tag{27}$$

where I is an identity matrix.

In [15], it is proved that the following matrices satisfy N-condition in this case:

$$N_\varepsilon^s(x) = \varepsilon N_s\Big(\frac{x}{\varepsilon}\Big), \qquad s = 1, \dots, n,$$
$$\tilde{a} = \langle a(I + \nabla N)\rangle$$

for the family $\{a^\varepsilon\}$.

Proposition 7.1. *If* $\operatorname{div} a = 0$ *and* $u^0 \in C_0^\infty(\Omega)$, *then* $u^\varepsilon \to u^0$ *and* $\langle a \rangle = \tilde{a}$.

PROOF: Since $\operatorname{div} a = 0$, the solutions of (26) are $N_s = 0$, $s = 1, \dots, n$. Hence from the estimate (25) it follows that $u^\varepsilon \to u^0$ in $H_0^1(\Omega)$.

The homogenized matrix $\tilde{a}$ is defined by (27). Therefore $\tilde{a} = \langle a \rangle$. □

Proposition 7.2. *If* $\langle a \rangle = \tilde{a}$ *and* $u^0 \in C_0^\infty(\Omega)$, *then* $u^\varepsilon \to u^0$ *in* $H_0^1(\Omega)$ *and* $\operatorname{div} a = 0$.

PROOF: From N-condition points (ii), (iii) and that $\langle a \rangle = \tilde{a}$ it follows that $\operatorname{div} a^\varepsilon \rightharpoonup 0$ in $[H^{-1}(\Omega)]^n$. It means that $\operatorname{div} a = 0$. By Proposition 7.1, we see that $u^\varepsilon \to u^0$ in $H_0^1(\Omega)$. □

Proposition 7.3. *We have* $A_\varepsilon^{-1} \xrightarrow{s} B^{-1}$, *where* $B = \operatorname{div} b \nabla$ *if and only if* $b = \langle a \rangle = \tilde{a}$.

PROOF: (i) Let us assume that $A_\varepsilon^{-1} \xrightarrow{s} B^{-1}$. Hence, for every $u^0 \in H_0^1(\Omega)$ we obtain

$$b \nabla u^0 = \langle a \rangle \nabla u^0 = \tilde{a} \nabla u^0.$$

This means that the matrices are equal $b = \langle a \rangle = \tilde{a}$.

(ii) Let $b = \langle a \rangle = \tilde{a}$ and $\{u^\varepsilon\}$, u^0 be defined by the equations

$$\operatorname{div} a^\varepsilon \nabla u^\varepsilon = f,$$
$$\operatorname{div} b \nabla u^0 = f,$$

with $f \in H^{-1}(\Omega)$. Using Proposition 7.2, we obtain that $u^\varepsilon \to u^0$ in $H_0^1(\Omega)$ for $u^0 \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ we complete the proof. □

8. Strong closure of sets of cogradients

In this section, we shall study the strong closure of sets of cogradients (fluxes). This was considered in [5], [20]. We use a different approach to obtain the same result.

Let $\mathfrak{U}$ be the family defined by (18). Let us assume that $m = 1$, $r = 2$ and $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{E}(\lambda_1, \lambda_2)$ are constant matrices.

Let us prove that

$$\begin{aligned} \operatorname{cl}_s\{\mathcal{A}_\chi \nabla u \colon A_\chi u = f,\ A_\chi \in \mathfrak{U}\} = \big\{&\mathcal{A}_- \nabla u \colon \exists\, \mathcal{A}^0,\ \mathcal{A}_-,\ \{\mathcal{A}_{\chi_\varepsilon}\} \colon \mathcal{A}_{\chi_\varepsilon} \xrightarrow{H} \mathcal{A}^0, \\ &(\mathcal{A}_{\chi_\varepsilon})^{-1} \overset{*}{\rightharpoonup} (\mathcal{A}_-)^{-1} \text{ in } [L^\infty(\Omega)]^{mn \times mn}, \\ &\mathcal{A}_- \nabla u = \mathcal{A}^0 \nabla u,\ \operatorname{div} \mathcal{A}_- \nabla u = f\big\} \end{aligned} \tag{28}$$

for all $f \in V^*$. Indeed, let $\operatorname{div} \mathcal{A}_\varepsilon \nabla u_\varepsilon = f$, $\operatorname{div} \mathcal{A}^0 \nabla u = f$, where we set $\mathcal{A}_\varepsilon = \mathcal{A}_{\chi_\varepsilon}$. Since

$$\begin{aligned}\lambda_2^{-1} \|\mathcal{A}_\varepsilon \nabla u_\varepsilon - \mathcal{A}_- \nabla u\|_{\mathbf{L}^2(\Omega)} &\leq ((\mathcal{A}_\varepsilon)^{-1}(\mathcal{A}_\varepsilon \nabla u_\varepsilon - \mathcal{A}_- \nabla u), \mathcal{A}_\varepsilon \nabla u_\varepsilon - \mathcal{A}_- \nabla u)_{\mathbf{L}^2(\Omega)} \\ &\leq \lambda_1^{-1} \|\mathcal{A}_\varepsilon \nabla u_\varepsilon - \mathcal{A}_- \nabla u\|_{\mathbf{L}^2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\big((\mathcal{A}_\varepsilon)^{-1}&(\mathcal{A}_\varepsilon \nabla u_\varepsilon - \mathcal{A}_- \nabla u), \mathcal{A}_\varepsilon \nabla u_\varepsilon - \mathcal{A}_- \nabla u\big)_{\mathbf{L}^2(\Omega)} \\ &= (\nabla u_\varepsilon, \mathcal{A}_\varepsilon \nabla u_\varepsilon)_{\mathbf{L}^2(\Omega)} - (\nabla u_\varepsilon, \mathcal{A}_- \nabla u)_{\mathbf{L}^2(\Omega)} \\ &\quad - \big((\mathcal{A}_\varepsilon)^{-1} \mathcal{A}_- \nabla u, \mathcal{A}_\varepsilon \nabla u_\varepsilon\big)_{\mathbf{L}^2(\Omega)} + \big((\mathcal{A}_\varepsilon)^{-1} \mathcal{A}_- \nabla u, \mathcal{A}_- \nabla u\big)_{\mathbf{L}^2(\Omega)} \\ &\to (\mathcal{A}^0 \nabla u, \nabla u)_{\mathbf{L}^2(\Omega)} - (\mathcal{A}_- \nabla u, \nabla u)_{\mathbf{L}^2(\Omega)} \geq 0\end{aligned}$$

we conclude that $\mathcal{A}_\varepsilon \nabla u_\varepsilon \to \mathcal{A}_- \nabla u$ in $\mathbf{L}^2(\Omega)$ if and only if $(\mathcal{A}^0 - \mathcal{A}_-)\nabla u = 0$ because $\mathcal{A}^0 - \mathcal{A}_-$ is nonnegative definite due to the well known Reuss–Voigt estimate, see [29],

$$\big(\lim_{\varepsilon \to 0} (\mathcal{A}_\varepsilon)^{-1}\big)^{-1} \leq \mathcal{A}^0 \leq \lim_{\varepsilon \to 0} \mathcal{A}_\varepsilon, \tag{29}$$

where the limit is understood in the weak sense of $[L^2(\Omega)]^{mn \times mn}$ (we generalized the estimate for the systems of elliptic equations). Hence, we just proved the equality (28).

Now let us proceed as in Section 5.

Let $\xi \in \mathbb{R}^n$. Let us investigate when the equality

$$\mathcal{A}^0 \xi = \mathcal{A}_- \xi \tag{30}$$

holds.

For $\eta \in \mathbb{R}^n$, we have, see [29, page 43],

$$(\mathcal{A}^0)^{-1} \eta = \langle \mathcal{A}_\chi^{-1} p \rangle, \tag{31}$$

where $\mathcal{A}^0$ is the homogenized matrix for $\mathcal{A}_\chi$, $\langle p \rangle = \eta$, $p \in [L^2(C)]^n$, $\mathcal{A}_\chi^{-1} p$ is potential and p is solenoidal.

Let us show that for every $\xi \in \mathbb{R}^n$ such that there exists a vector e which is parallel to the vector $(\mathcal{A}_1^{-1} - \mathcal{A}_2^{-1})\mathcal{A}_- \xi$ the equality (30) holds, where $\mathcal{A}^0$ is the homogenized matrix for

$$\mathcal{A}_\chi = \chi(\langle e, x \rangle)\mathcal{A}_1 + (1 - \chi(\langle e, x \rangle))\mathcal{A}_2,$$

where χ is defined by (15) with $\alpha = \theta$, and

$$\mathcal{A}_- = (\theta \mathcal{A}_1^{-1} + (1 - \theta)\mathcal{A}_2^{-1})^{-1}.$$

To do this, let us change the variable as $\xi = \mathcal{A}_-^{-1}\eta$. Then we should verify that $(\mathcal{A}^0)^{-1}\eta = (\mathcal{A}_-)^{-1}\eta$. It can be readily seen that for $p = \eta$ the vector $(\chi(\langle e, x\rangle)\mathcal{A}_1^{-1} + (1-\chi(\langle e, x\rangle))\mathcal{A}_2^{-1})p$ is potential and p is solenoidal. Using (31), we obtain (30). It is clear that for $m = 1$ such a vector e can be always found.

Hence, using the approximation argument, we get that (28) becomes

$$\begin{aligned}\mathrm{cl}_s\{\mathcal{A}_\chi \nabla u\colon\ A_\chi u = f,\ A_\chi \in \mathfrak{U}\} = \big\{\mathcal{A}_-\nabla u\colon\ \exists\,\theta \in L^\infty(\Omega),\ \mathcal{A}_-\colon\ &\operatorname{div}\mathcal{A}_-\nabla u = f,\\ &\mathcal{A}_- = (\theta\mathcal{A}_1^{-1} + (1-\theta)\mathcal{A}_2^{-1})^{-1},\\ &0 \le \theta(x) \le 1 \text{ a.e. in } \Omega\big\}\end{aligned}$$

for all $f \in V^*$.

9. Elliptic operators, case $m \ge n$

In this section, we shall consider two examples for the cases $m = n = 1$ and $2 \le n \le m$. We shall show that the strong closure of the feasible states cannot be, in general, parametrized by the convexification of the original operators in this case.

Example 9.1. Let us first consider the case $m = n = 1$.

Let $\Omega = (0,1)$ and

$$a^\varepsilon(x) = a(\varepsilon^{-1}x),$$

where a is an 1-periodic function of the form $a(x) = \chi(x)a_1 + (1-\chi(x))a_2$ with χ being 1-periodic so that $\langle\chi\rangle = \theta$, $0 < \theta < 1$, is a constant and a_1, a_2 different positive constants.

The H-limit (the homogenization coefficient) is known in this case, it is $a^0 = \langle a^{-1}\rangle^{-1}$, see [29, page 21]. The formula for a^0 is as follows:

$$a^0 = \frac{1}{\theta a_1^{-1} + (1-\theta)a_2^{-1}}. \tag{32}$$

Assume that there exists $u \in H_0^1(0,1)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(a_\theta \frac{\mathrm{d}}{\mathrm{d}x}u\Big) = f, \qquad a_\theta = \theta a_1 + (1-\theta)a_2,$$

with $0 < \theta < 1$ and it belongs to the strong closure of the feasible states $\mathrm{cl}_s\mathrm{F}(\mathfrak{U}, f)$ with $f \in L^2(0,1)$, $f \ne 0$. Hence, there exist families $\{u^\varepsilon\}$, $\{a_{\chi^\varepsilon}\}$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big(a_{\chi^\varepsilon}\frac{\mathrm{d}}{\mathrm{d}x}u^\varepsilon\Big) &= f,\\ u^\varepsilon \to u \qquad &\text{in } H_0^1(0,1),\end{aligned}$$

where $a_{\chi^\varepsilon}(x) = \chi^\varepsilon(x)\, a_1 + (1 - \chi^\varepsilon(x))\, a_2$, and χ^ε are characteristic functions. The family $\{\chi^\varepsilon\}$ contains a subsequence that weakly-* converges to some $\theta_1(x)$ in $L^\infty(0,1)$ (we still denote it the same). It can be readily seen that $\theta_1(x) = \theta$ for all x such that $\mathrm{d}u(x)/\mathrm{d}x \neq 0$.

Since H-limit a^0 of the family $\{a^\varepsilon\}$ is the homogenized matrix of a, there exists a family $\{v^\varepsilon\}$ such that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big(a^\varepsilon \frac{\mathrm{d}}{\mathrm{d}x} v^\varepsilon\Big) &= f, && v^\varepsilon \in H_0^1(0,1),\\
v^\varepsilon &\rightharpoonup v && \text{in } H_0^1(0,1),\\
a^\varepsilon \frac{\mathrm{d}v^\varepsilon}{\mathrm{d}x} &\rightharpoonup a^0 \frac{\mathrm{d}v}{\mathrm{d}x} && \text{in } L^2(0,1),\\
\frac{\mathrm{d}}{\mathrm{d}x}\Big(a^0 \frac{\mathrm{d}}{\mathrm{d}x} v\Big) &= f, && v \in H_0^1(0,1),\\
a^0 &= \frac{1}{\theta a_1^{-1} + (1-\theta) a_2^{-1}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{\mathrm{d}(v-u)}{\mathrm{d}x} &= \Big(\frac{1}{a^0} - \frac{1}{a_\theta}\Big) \int_0^x f\,\mathrm{d}x,\\
\frac{\mathrm{d}(v^\varepsilon - u^\varepsilon)}{\mathrm{d}x} &= \Big(\frac{1}{a^\varepsilon} - \frac{1}{a_{\chi^\varepsilon}}\Big) \int_0^x f\,\mathrm{d}x.
\end{aligned}$$

Taking the weak limit in the last equality when $\varepsilon \to 0$ (the limit is 0 for a.e. x where $\theta_1(x) = \theta$) and comparing the limit to the first equality, we obtain

$$\frac{1}{\theta a_1^{-1} + (1-\theta) a_2^{-1}} = \theta a_1 + (1-\theta) a_2.$$

Simplifying this equality yields

$$2 = \frac{a_2}{a_1} + \frac{a_1}{a_2},$$

which is possible only if $a_1 = a_2$. This gives a contradiction with the assumption that u belongs to the strong closure of feasible states. The function $u \in H_0^1(0,1)$ is the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(a_\theta \frac{\mathrm{d}}{\mathrm{d}x} u\Big) = f$$

and is nonzero if $f \neq 0$. Hence, we found the nonzero element $u \in H_0^1(0,1)$ that belongs to the closed convex hull of the feasible states, but it does not belong to the corresponding strong closure.

Example 9.2. Now let us present the second counter-example for the case $m \geq n > 1$.

In what follows, we will need the following results:

Theorem 9.1 (G. Vainikko and K. Kunisch [24]). *Suppose that $n \geq 2$. Let a function $\sigma \in L^\infty(\Omega)$ and a function $v \in C_0^\infty(\Omega)$ satisfy the equation $\operatorname{div} \sigma \nabla v = 0$ in $H^{-1}(\Omega)$. Then $\sigma = 0$ on $\{x \in \Omega \colon \nabla v(x) \neq 0\}$.*

Theorem 9.2 (J. Nečas [14]). *Let D be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. Then there exists $c = c(D) > 0$ such that for every $g \in L^2(D)$*

$$\|g\|_{L^2(D)/\mathbb{R}} \leq c\|\nabla g\|_{[H^{-1}(D)]^n},$$

where $L^2(D)/\mathbb{R}$ is the factor space.

Let $\mathfrak{U}$ be a family of elliptic operators with matrices $\mathcal{A}$ of the form $\mathcal{A} = a(x)I$, where I is an identity matrix, $a(x) = \chi(x)a_1 + (1 - \chi(x))a_2$, and a_1, a_2 are different positive constants.

Assume that there exists $u \in V$ such that

$$\operatorname{div} a_\theta \nabla u = f, \quad a_\theta = \theta a_1 + (1 - \theta)a_2, \qquad \theta \in \mathbb{R},\ 0 < \theta < 1,$$

and it belongs to the strong closure of feasible states $\mathrm{cl}_s \mathrm{F}(\mathfrak{U}, f)$ with

$$f = \operatorname{div} a_\theta I\, \nabla \big(x_1\varphi \ldots x_n\varphi \ldots x_n\varphi\big)^{\mathrm{T}},$$

where φ is a bump function that equals to 1 in some ball B (we stacked functions $x_1\varphi, \ldots, x_n\varphi$ n times and the rest with $x_n\varphi$). Hence, there exist families $\{u^\varepsilon\}, \{a_{\chi^\varepsilon}\}$ such that

$$\operatorname{div} a_{\chi^\varepsilon} \nabla u^\varepsilon = f,$$
$$u^\varepsilon \to u \qquad \text{in } V,$$

where $a_{\chi^\varepsilon}(x) = \chi^\varepsilon(x)a_1 + (1 - \chi^\varepsilon(x))a_2$, χ^ε are characteristic functions. The family $\{\chi^\varepsilon\}$ contains a subsequence (we still denote it the same) that weakly-* converges to some $\theta_1(x)$ in $L^\infty(\Omega)$.

Using the explicit form of the function f, we get that

$$u = \big(x_1\varphi \ldots x_n\varphi \ldots x_n\varphi\big)^{\mathrm{T}}.$$

From Theorem 9.1 we obtain that $\theta_1(x) = \theta$ for all x such that $\nabla u(x) \neq 0$ because $u \in [C_0^\infty(\Omega)]^m$ (according to our choice of $f \in V$), i.e. $\theta_1(x) = \theta$ on a.e. $x \in B$.

Let us denote $\mathcal{A}^\varepsilon = a_{\chi^\varepsilon} I$. Let $\mathcal{A}^0$ be the H-limit matrix of some subsequence of $\{\mathcal{A}^\varepsilon\}$ (still denoted the same). The matrix $\mathcal{A}^0$ consists of m equal matrices a^0

corresponding to m equations. Those matrices are situated on the diagonal, and each a^0 is the H-limit of $a_{\chi^\varepsilon} I$, where I is an $n \times n$ identity matrix.

Since $\mathcal{A}^0 \nabla u = a_\theta \nabla u$ and $m \geq n$, we obtain that $\operatorname{tr} a^0(x) = n\, a_\theta$ in B. On the other hand, for the trace of a^0 the following inequality holds, see [29, page 199]:

$$\lim_{\varepsilon\to 0} a^\varepsilon - \frac{1}{\nu_1} \lim_{\varepsilon\to 0} \Big(a^\varepsilon - \lim_{\varepsilon\to 0} a^\varepsilon \Big)^2 \leq \frac{\operatorname{tr} a^0}{n} \leq \lim_{\varepsilon\to 0} a^\varepsilon - \frac{1}{\nu_2} \lim_{\varepsilon\to 0} \Big(a^\varepsilon - \lim_{\varepsilon\to 0} a^\varepsilon \Big)^2,$$

where $a^\varepsilon = a_{\chi^\varepsilon}$ and the constants ν_1, ν_2 are such that $0 < \nu_1 \leq a^\varepsilon \leq \nu_2$, the limits are understood in the weak sense of $L^2(\Omega)$.

Hence, we obtain that the family $\{\chi^\varepsilon\}$ converges strongly to the constant θ in $L^2(B)$. Now we can write this as the following expression:

$$\int_B (\chi^\varepsilon - \theta)^2 \, \mathrm{d}x \to 0.$$

Simplifying this expression and letting $\varepsilon \to 0$, we obtain the formula $\theta(1-\theta) = 0$. This means that θ is either 0 or 1. On the other hand, we chose θ to be $0 < \theta < 1$. This is a contradiction with our assumption on u.

Hence, there exists a nonzero $u \in [H_0^1(\Omega)]^m$ that is the solution of equation $\operatorname{div} \mathcal{A}_\theta \nabla u = f$, $\mathcal{A}_\theta = a_\theta I$, which does not belong to the strong closure of the feasible states $\mathrm{cl}_s \mathrm{F}(\mathfrak{U}, f)$.

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