Jozef Sumec; Mária Minárová; Ľuboš Hruštinec

Anisotropic viscoelastic body subjected to the pulsating load

*Applications of Mathematics*, Vol. 68 (2023), No. 6, 829–844

Persistent URL: `http://dml.cz/dmlcz/151941`

# ANISOTROPIC VISCOELASTIC BODY SUBJECTED TO THE PULSATING LOAD

Jozef Sumec, Mária Minárová, Ľuboš Hruštinec, Bratislava

Received November 4, 2022. Published online September 14, 2023.

*Abstract.* Constitutive equations of continuum mechanics of the solid phase of anisotropic material is focused in the paper. First, a synoptic one-dimensional Maxwell model is explored, subjected to arbitrary deformation load. The explicit form is derived for stress on strain dependence. Further, the analogous explicit constitutive equation is taken in three spatial dimensions and treated mathematically. Later on, a simply supported straight concrete beam reinforced by the steel fibres is taken as an investigated domain. The reinforcement is considered and dealt as scattered within the beam. Material characteristics are determined in line with the theory of the reinforcement. Sinusoidal load is taken as the action, stress reaction function is observed. By exploitation of the Fourier transform within the stress-strain relation analysis, both time and frequency interpretation of the constitutive relation can be performed.

*Keywords*: linear viscoleasticity theory; constitutive equation; Duhamel hereditary integral; convolution; complex relaxation modulus of structural element; Fourier integral transform

*MSC 2020*: 42A38, 74-10

## 1. Introduction

As it is apparent directly from the sound of the word itself, viscoelasticity is a property of a material that preforms both viscous and elastic behaviour when undergoing mechanical load. Viscous materials exhibit a time-depended strain, viscosity being the result of the diffusion of atoms or molecules inside an amorphous material. Newton's viscous liquid, see (N)-Newtonian matter in Fig. 1, subjected to the load of non-hydrostatic pressure, is unable to accumulate mechanical energy due to its

The research has been supported by grants APVV-18-0052, APVV-17-0066, VEGA 1/0006/19, VEGA 1/0522/20 and VEGA 10036/23.

DOI: 10.21136/AM.2023.0256-22

dissipation. On the other hand, elasticity is usually the result of bond stretching along crystallographic planes in a crystalline solid, [3], [13]. The elastic material, see (H)-Hookean matter in Fig. 1 are capable of accumulating mechanical energy without its dissipation.

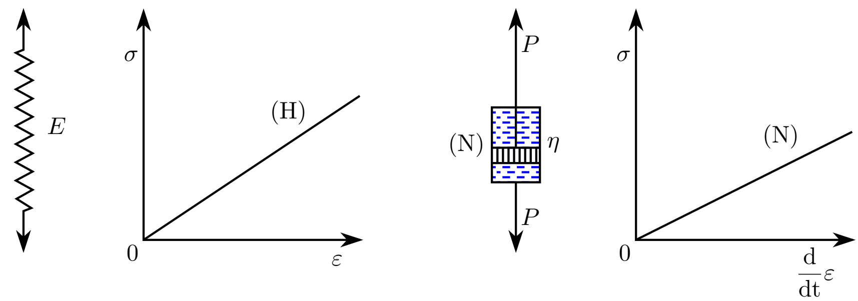



Figure 1. Hookean (H) and Newtonian (N) matters. Pictogram and stress-strain diagrams.

**Physical equations.** In one-dimensional case the physical equations, i.e., stress $\sigma(t)$—strain $\varepsilon(t)$ relationship have the form

$$\text{(H)}: \ \sigma = E\varepsilon, \tag{1.1}$$

$$\text{(N)}: \ \sigma = \eta \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon \tag{1.2}$$

with $E[Pa]$ being the Young elastic modulus and $\eta[Pa.s]$ the Newtonian viscosity coefficient. The physical equation can be graphically interpreted as in Fig. 1. In a one-dimensional case stress and strain are scalar functions of time, in two dimensions the stress and strain are tensor functions of time of the second order and material characteristics are 4th order tensors, etc.

R e m a r k 1.1. Alongside the text of Chapter 2, the deformation and stress variables, the time-dependent functions, will be shortened and written without the independent variable $t$ when possible; e.g. $\sigma, \varepsilon, \varepsilon_H$ instead of $\sigma(t)$, $\varepsilon(t)$, $\varepsilon_H(t)$, etc.

**Geometrical equations.** Having the base matters (H) and (N) at hand, by using parallel or serial connections, we can create various less or more configurations of viscoelastic components resulting in complex models, see the models creation in Fig. 2. If the parallel connection of two components is realised, both (or more) parallel legs move together, perpendicularly to their longitude axis, hence the deformations on each leg of the connection are the same, equal to the global deformation, as well; and the global stress is equal to the sum of both stress values on both legs of the parallel connection. In the case of serial connection the sum of both (or more) submodels connected serially yields the global deformation and the stress is equal in each element, being equal to the global stress value as well.

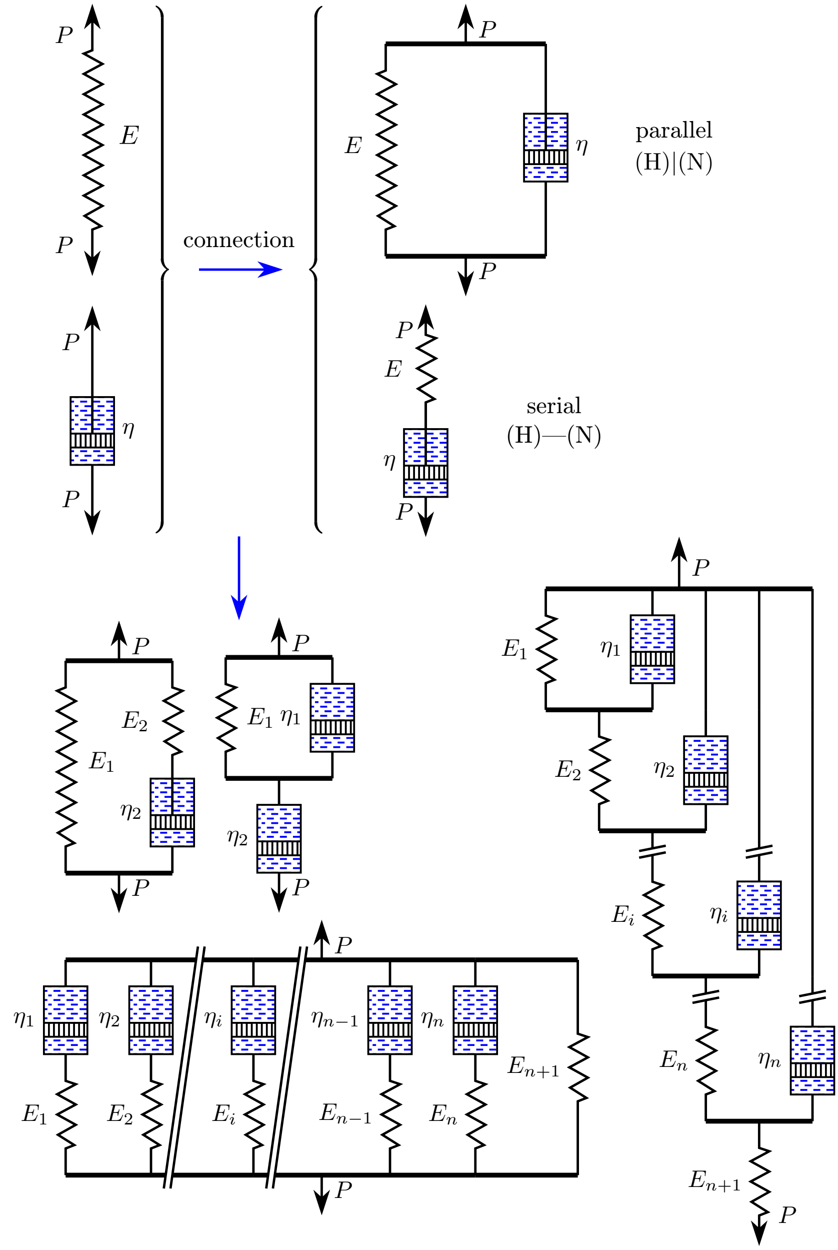



Figure 2. Basic matters, parallel and serial connections, simple and more complex viscoelastic models creation, [8], [9], [11].

Strictly speaking, any matter is neither purely elastic nor purely viscous. Even though a material is labelled as elastic, it still exhibits, be it a negligibly small, viscous behaviour. Accordingly, a part of externally added energy (even though a negligibly small amount) is dissipated. And, vice versa, each viscous material in practice exhibits also elastic behaviour-even though almost invisible; i.e., not all energy is ever dissipated. We can say that the ideal elastic body and the ideal viscous fluid are just some marginal theoretical stages in the theory of modelling. The real properties of each material are always somewhere between these margins. And, the received mechanical energy by a viscoelastic body is always partially accumulated, partially dissipated. Moreover, there are other influencing factors, for example thermal radiation, humidity effects, the effect of chemical or electrical fields quantified by their physical characteristics that affect the mechanical behaviour of materials, as well as the structure (homogeneity or inhomogeneity, isotropy or anisotropy, etc.) of the modelled material, see [22].

Recently, the importance of the viscoelasticity theory increases due to the current rising significance and rapid progress of the material science. Polymer-based materials, composite materials based on silicate reinforced with high-strength fibres, biocompatible materials and others used in industrial practice, medicine, immunology, sport, e.g., [10]—these and other newly invented materials are investigated. The study and the prediction of their mechanical response to various kind of load is focused in many research groups worldwide.

The motivation of our investigation is first of all in the application of the phenomenological theory of modelling of constitutive equations for the dynamic response to a stationary pulsed continuous load of the structural element made from an anisotropic viscoelastic material.

In this paper the response of a viscoelastic anisotropic material under a sinusoidal load with constant frequency is studied. The dynamic behaviour is of our interest since viscoelastic materials are used in situations in which the damping of vibration or the absorption of sound of a material is important. The frequency of the sinusoidal load of the object is sufficiently slow as no inertial terms appear. Analysed material is made of concrete of high strength, reinforced by the steel fibres and main steel reinforcement, i.e., anisotropy is considered, see Fig. 3. Indeed, prior to this investigation and for the sake of better understanding, Chapter 2 is included as a preliminary for customizing the notation and derivation of Duhamel hereditary integral in one-dimension.

Within the research we apply the phenomenological theory of modelling of constitutive equations.

Assumptions of the solution:

- ▷ Material of analysed body is homogeneous and anisotropic. It is deformed continuously under external oscillatory load.
- ▷ The isothermal state of the infinitesimal theory of liner dynamic viscoelasticity is supposed.

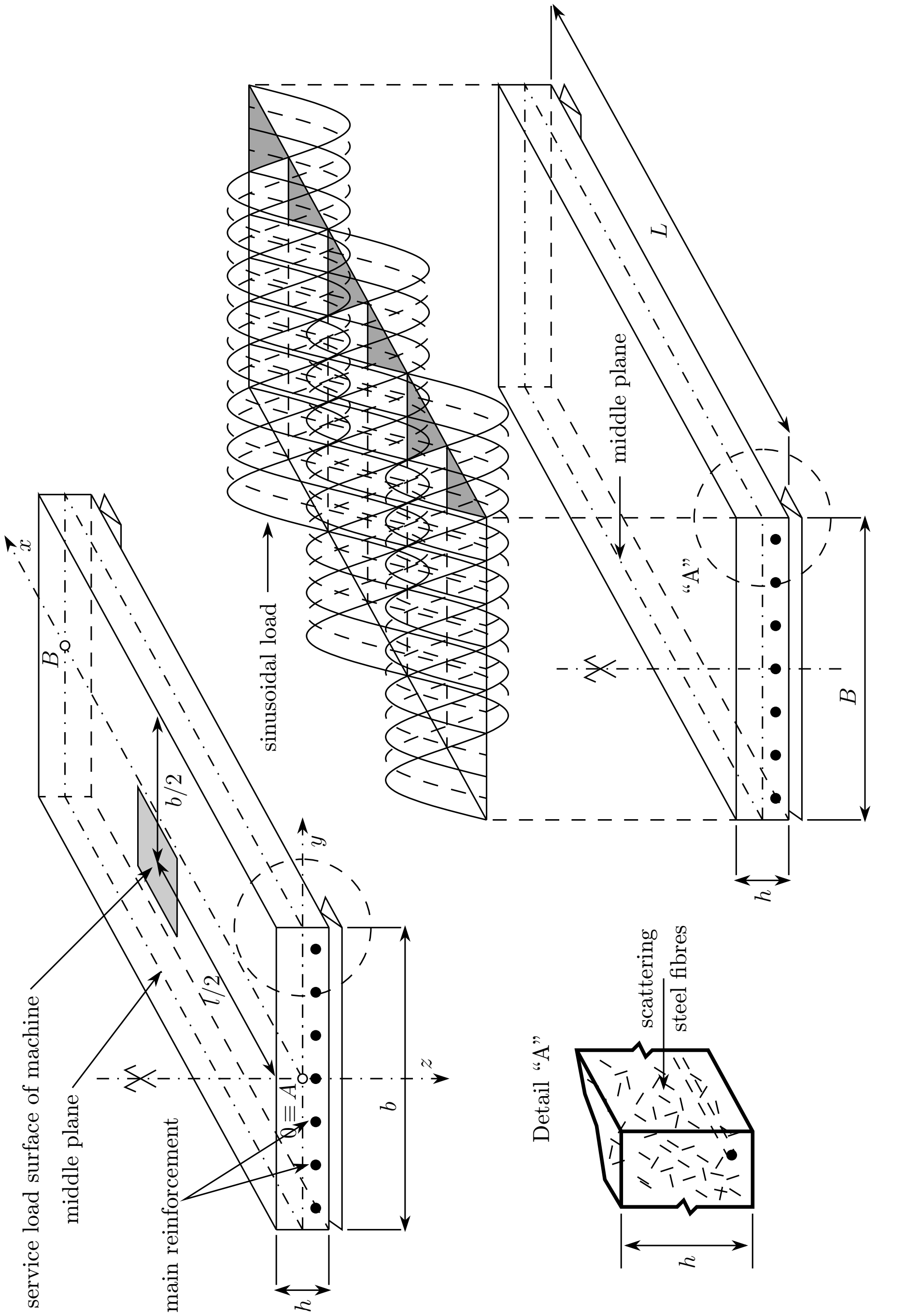
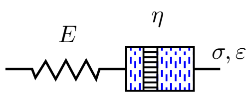



Figure 3. Scheme of analysed structural element, [20].

- ▷ Boltzmann principle of superposition is considered.
- ▷ Influence of cracks propagation is not taken into account.
- ▷ No mutual penetration of materials occurs.
- ▷ The hypothesis of fading material memory is taken into account.

## 2. Constitutive relation and Duhamel hereditary integral in a one-dimensional case

In general, the hereditary integral is an integro-differential relation explicitly expressing the reaction dependence on the action. Herein, either stress is action and strain is reaction or vice versa. Hereditary integral involves all load (action) history which affects the resulting response (reaction).




(M) = (H)−(N)

Figure 4. Maxwell model-serial connection of Hookean and Newtonian matter, [20].

For the sake of better clarity, we start with the Maxwell model, Fig. 4. We derive the general explicit constitutive equation (Duhamel integral) from its fundamental physical and geometrical equations. As pointed out above, in the case of a serial connection the summation of both Hookean and Newtonian partial deformations $\varepsilon_H$ and $\varepsilon_N$ are summed up to the entire deformation and equality of entire stress magnitude to partial stress magnitudes of both involved elements is ensured. Accordingly, the geometrical equations of Maxwell model are

$$\begin{aligned} \varepsilon &= \varepsilon_H + \varepsilon_N, \\ \sigma &= \sigma_H = \sigma_N. \end{aligned} \tag{2.1}$$

And, the physical equations corresponding to this model are then taken from (1.1) and (1.2), the affiliation to the particular matter being given by sub-indexes $H$ and $N$.

$$\begin{aligned} \sigma_H &= E\varepsilon_H, \\ \sigma_N &= \eta\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_N. \end{aligned} \tag{2.2}$$

Next, for the sake of deriving the relation between the stress and the strain by means of physical characteristics only, we have to eliminate $\varepsilon_H$, $\varepsilon_N$, $\sigma_H$, $\sigma_N$ from the system of equations (2.1) and (2.2). Prior to substituting to the first equation of (2.1) we differentiate it. Then we substitute for $\mathrm{d}\varepsilon_H/\mathrm{d}t$ and $\mathrm{d}\varepsilon_N/\mathrm{d}t$ expressed from (2.2). Finally, due to the second equation of (2.1) we can omit the indexes in stress variables obtaining the global physical relation of the Maxwell model

$$(2.3) \qquad \text{(M)}: \ \frac{1}{E}\frac{\mathrm{d}}{\mathrm{d}t}\sigma + \frac{\sigma}{\eta} = \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon.$$

The linear differential equation (2.3) stands as an implicit relation between variables $\varepsilon$ and $\sigma$; up to now it is not specified which of them is the action and which the reaction. It is called the *constitutive equation of the Maxwell model.* Taking $\varepsilon$ as the action, we are supposed to express the reaction $\sigma$ explicitly, by using the hereditary integral. We rearrange (2.3) by using the integrating factor method, e.g., [5]; with integrating factor $\mathrm{e}^{Et/\eta}$ we get

$$\mathrm{e}^{Et/\eta}\frac{\mathrm{d}}{\mathrm{d}t}\sigma + \mathrm{e}^{Et/\eta}\frac{E}{\eta}\sigma = \mathrm{e}^{Et/\eta}E\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{Et/\eta}\sigma(t)) = \mathrm{e}^{Et/\eta}E\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon.$$

Then

$$[\mathrm{e}^{E\tau/\eta}\sigma(\tau)]_{-\infty}^{t} = \int_{-\infty}^{t} \mathrm{e}^{E\tau/\eta}E\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}\,\mathrm{d}\tau \Rightarrow \sigma(\varepsilon(t)) = \int_{-\infty}^{t} \mathrm{e}^{-E(t-\tau)/\eta}E\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}\,\mathrm{d}\tau.$$

Finally, considering zero initial stress, we get the desired explicit expression of the reaction depending on the action in the integro-differential form

$$(2.4) \qquad \sigma(t) = \int_{-\infty}^{t} H(t-\tau)\frac{\mathrm{d}\varepsilon(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau$$

of (M), where we have used the *relaxation modulus*

$$(2.5) \qquad H(t) = E\mathrm{e}^{-Et/\eta}.$$

R e m a r k 2.1. The relaxation modulus $H(t)$ is a scalar function and it is unique for each particular one-dimensional model. It reflects both the configuration of the viscoelastic model and the material characteristics of the participating materials' physical characteristics. It involves all geometrical and physical relations within the configuration, while expression (2.4) in such a form stands as general explicit constitutive equation for all one-dimensional viscoelastic models.

As far as the fluidity of a material is concerned, it is worth mentioning the so-called relaxation time herein. Fluidity can be quantified by its length. From proportionality $t_{rlx} = \eta/E$ it directly follows: The longer relaxation time, the more "liquid like" behaviour of a material, the shorter relaxation time, the more "solid like" behaviour of a material. Relaxation time can be measured in a lab; and mathematically it is yielded from the characteristic equation corresponding to the constitutive equation with the load imposed and maintained on a constant level, $\varepsilon^* = \varepsilon(t)$ in (2.3). Relaxation time is then equal to a negative reciprocal root of that characteristic equation.

## 3. Anisotropic structural material under the pulsating load. Three dimensional study

Up to now, in accordance with the linear viscoelasticity theory, engineers obviously consider a simplified isotropic homogeneous material. In such a case, the relaxation tensor can be expressed, [18] as

$$H_{ijkl}(t) = \frac{1}{3}[\mathcal{H}_2(t) - \mathcal{H}_1(t)]\delta_{ij}\delta_{kl} + \frac{1}{2}[\mathcal{H}_2(t)](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{3.1}$$

which is the most general form of an isotropic 4th order tensor. Here, $\mathcal{H}_1(t)$ and $\mathcal{H}_2(t)$ are independent relaxation functions and $\delta_{ij}$ is the Kronecker operator

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Hence, the explicit constitutive relation $\sigma$-$\varepsilon$ for isotropic homogeneous material acquires the form

$$\sigma_{ij}(t) = \int_{-\infty}^{t} \frac{1}{3}[\mathcal{H}_2(t-\tau) - \mathcal{H}_1(t-\tau)]\delta_{ij}\delta_{kl} + \frac{1}{2}[\mathcal{H}_2(t-\tau)](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\mathrm{d}\frac{\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{3.2}$$

However, for more complex engineering problems, it is often inevitable to deal with a physically more precise approach involving anisotropy.

Likewise expression (2.4) for one-dimension, the three dimensional explicit constitutive equation (Duhamel hereditary integral) can be derived as well, where an anisotropic material is considered in general. Written component-wisely, it acquires the form, [1]

$$\sigma_{ij}(t) = \int_{-\infty}^{t} H_{ijkl}(t-\tau)\,\mathrm{d}\frac{\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{3.3}$$

with $\sigma_{ij}(t)$ and $\varepsilon_{kl}(t)$ being the stress and strain tensors of the $2^{nd}$ and $H_{ijkl}(t)$ the 4th order relaxation tensor acquiring zero value on the time interval $(-\infty, 0\rangle$. The relation (3.3) can be regarded as convolution $\sigma_{ij}(t) = H_{ijkl}(t-\tau) \star \varepsilon_{kl}(t)$ as well. Having an assumption of the periodic sinusoidal strain load as an external action, we observe the reaction, the stress tensor function. The strain is a harmonic function of time. It can be expressed as

$$\varepsilon_{ij}(t) = \overset{\circ}{\varepsilon}_{ij} \mathrm{e}^{\mathrm{i}\omega t} \tag{3.4}$$

with the imaginary unit $\mathrm{i} = \sqrt{-1}$, $\overset{\circ}{\varepsilon}_{ij}$ the amplitude of deformation, and $\omega$[rad/s] the angular frequency. Whenever needed, the angular frequency $\omega$ can be expressed by means of ordinary frequency $\nu$[Hz]: $\omega = 2\pi\nu$, too.

Each component of the relaxation modulus tensor $H_{ijkl}(t)$ has a bounded variation on an arbitrary closed subinterval of $(-\infty, \infty)$, see, e.g., [2]. From the symmetry of both stress and strain tensors, also the symmetry of relaxation modulus follows in all directions, [19]

$$H_{ijkl}(t) = H_{jikl}(t) = H_{ijlk}(t). \tag{3.5}$$

Relaxation modulus is a function of time and it can be expressed as a sum of the non-negative tensor of constant values $\overset{\infty}{H}_{ijkl}$-*equilibrium modulus* and the time dependent *residual modulus* $\widehat{H}_{ijkl}(t)$:

$$H_{ijkl}(t) = \overset{\infty}{H}_{ijkl} + \widehat{H}_{ijkl}(t). \tag{3.6}$$

Each component of the equilibrium modulus tensor is always positive for solid bodies and equals zero for liquids, e.g., [12]. Let us recall that under a stationary load each process of the system tends towards the equilibrium stage. It means

$$\widehat{H}_{ijkl}(t) \to 0 \quad \text{for } t \to \infty. \tag{3.7}$$

When substituting (3.4) and (3.6) to (3.3), we have the explicit constitutive equation for the pulsating load in the form

$$\sigma_{ij}(t) = \int_{-\infty}^{t} (\overset{\infty}{H}_{ijkl} + \widehat{H}_{ijkl}(t-\tau))\, \mathrm{d}\frac{\overset{\circ}{\varepsilon}_{kl}\mathrm{e}^{\mathrm{i}\omega\tau}}{\mathrm{d}\tau}\, \mathrm{d}\tau$$

and this yields

$$\sigma_{ij}(t) = \overset{\infty}{H}_{ijkl}\overset{\circ}{\varepsilon}_{ij}\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{i}\omega\overset{\circ}{\varepsilon}_{ij} \int_{-\infty}^{t} \widehat{H}_{ijkl}(t-\tau)\mathrm{e}^{\mathrm{i}\omega\tau}\, \mathrm{d}\tau. \tag{3.8}$$

Let us next proceed with (3.8) by using the substitution $\chi = t - \tau$. We obtain

$$\sigma_{ij}(t) = \overset{\circ}{\varepsilon}_{ij}\mathrm{e}^{\mathrm{i}\omega t}\left[\overset{\infty}{H}_{ijkl} - \mathrm{i}\omega \int_{0}^{\infty} \widehat{H}_{ijkl}(\chi)\mathrm{e}^{\mathrm{i}\omega\chi}\, \mathrm{d}\chi\right].$$

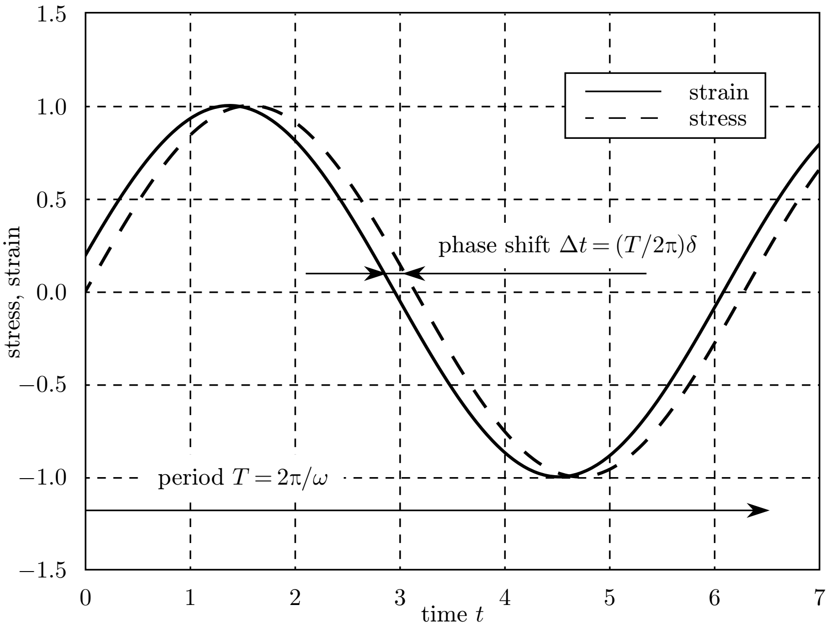



Figure 5. Stress and stain in time $t$ with the phase shift; $T$ being the period of action and reaction, $\omega$[rad/s] angular frequency, $\delta$[rad] phase angle between the stress and strain sinusoid, [15].

Now, when employing the Euler form $e^{-i\omega\tau} = \cos(\omega\tau) - i\sin(\omega\tau)$ we get

(3.9)
$$\sigma_{ij}(t) = \overset{\circ}{\varepsilon}_{ij} e^{i\omega t}\left[\overset{\infty}{H}_{ijkl} + \omega\int_0^\infty \widehat{H}_{ijkl}(\chi)\sin(\omega\chi)\,d\chi + i\omega\int_0^\infty \widehat{H}_{ijkl}(\chi)\cos(\omega\chi)\,d\chi\right].$$

Physically speaking, if we do not consider the damping in the model, then the stress response corresponding to a certain stationary pulsating deformation load acquires the same periodical character as the strain action. Hence, also the amplitude stays at the same value in time, [14]. Indeed, the response delay after the load, expressed as the time shift is

$$\Delta t = T\frac{\alpha}{2\pi}$$

with $T = 2\pi/\omega$ being the period and $\alpha$ the phase angle, see Fig. 5 and 6. It means that for this special kind of load the general stress-strain relation (3.3) takes the form

$$\sigma_{ij}(t) = H^*_{ijkl}(\omega)\overset{\circ}{\varepsilon}_{ij} e^{i\omega t}. \tag{3.10}$$

Herein, the expression in square brackets $H^*_{ijkl}(\omega)$ in (3.9), is known in the viscoelasticity theory as the *dynamical modulus*. Since $\sigma_{ij}$ is a complex function of frequency $\omega$, it is natural to split it to the real and imaginary parts

$$H^*_{ijkl}(\omega) = \bar{H}_{ijkl}(\omega) + i\bar{\bar{H}}_{ijkl}(\omega). \tag{3.11}$$

Moreover, the real and the imaginary parts of a dynamical modulus are called *storage modulus* and *loss modulus*, respectively, which are reasoned physically, too [6], [7]. Accordingly, we see from (3.9) that in our special load type, the pulsating load, the storage and loss moduli are

$$\bar{H}_{ijkl}(\omega) = \overset{\infty}{H}_{ijkl} + \omega \int_0^\infty \widehat{H}_{ijkl}(\chi) \sin(\omega\chi)\, \mathrm{d}\chi, \tag{3.12}$$

$$\bar{\bar{H}}_{ijkl}(\omega) = \omega \int_0^\infty \widehat{H}_{ijkl}(\chi) \cos(\omega\chi)\, \mathrm{d}\chi. \tag{3.13}$$

Moreover, the ratio

$$\frac{\bar{\bar{H}}(\omega)}{\bar{H}(\omega)} = \tan\alpha(\omega) \tag{3.14}$$

represents the so-called *loss tangent*, [16].

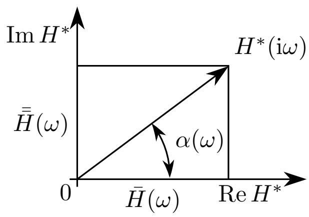



Figure 6. Loss tangent.

R e m a r k 3.1. The author of publication [2] points out some misunderstandings in terminology as far as the term dynamical modulus $H^*_{ijkl}(\omega)$ is concerned. Namely, the term "dynamical" does not indicate explicitly whether the inertial members are retained or not in the momentum equations. Maybe the name "undamped pulsation modulus" would be more appropriate in this case as suggested by [4].

For the sake of physical accomplishment we are supposed to examine the impact of the possible limiting values of the action frequency to the resulting dynamical modulus. For this purpose we temporarily rearrange the integrals in (3.12) and (3.13) by integrating by parts, getting

$$\bar{H}_{ijkl}(\omega) = \overset{\infty}{H}_{ijkl} + \widehat{H}_{ijkl}(0) + \int_0^\infty \frac{\mathrm{d}\widehat{H}_{ijkl}(\chi)}{\mathrm{d}\chi} \cos(\omega\chi)\, \mathrm{d}\chi, \tag{3.15}$$

$$\bar{\bar{H}}_{ijkl}(\omega) = -\int_0^\infty \frac{\mathrm{d}\widehat{H}_{ijkl}(\chi)}{\mathrm{d}\chi} \sin(\omega\chi)\, \mathrm{d}\chi. \tag{3.16}$$

After such a rearrangement, we can conveniently follow the limiting frequency values impact:

▷ For $\omega \to 0$ we have the static load

$$\bar{H}_{ijkl}(0) = \overset{\infty}{H}_{ijkl} = H_{ijkl}(t)|_{t\to\infty}, \tag{3.17}$$

$$\bar{\bar{H}}_{ijkl}(0) = 0. \tag{3.18}$$

▷ For a very high frequency $\omega \to \infty$ we get an impact load. After substitution $\omega\chi = \tau$ we get

$$\lim_{b\to\infty} \bar{H}_{ijkl}(b) = \overset{\infty}{H}_{ijkl} + \widehat{H}_{ijkl}(0) = H_{ijkl}(t)|_{t\to 0}, \tag{3.19}$$

$$\lim_{b\to\infty} \bar{\bar{H}}_{ijkl}(b) = 0. \tag{3.20}$$

Formulas (3.17) and (3.20) represent the limit values of the real and imaginary part of the complex relaxation modulus. Since zero frequency corresponds to a static load, it is apparent that the imaginary part (3.17) has to be zero. Relations (3.19) and (3.20) document the fact that in the case of a very high frequency of excitation, the imaginary part of the complex module converges to zero. By its mechanical response, the system behaves like an elastic anisotropic body.

It is worth emphasising that forms (3.12) and (3.13) physically perform the dependence between the frequency and the mechanical properties of the body expressed in the form of relaxation moduli. Therein, tensor $\bar{H}_{ijkl}(\omega)$ is a component of the stress-strain ratio in the direction of the deformation phase, see [17], whereas the tensor $\bar{\bar{H}}_{ijkl}(\omega)$ is declined from that direction by the angle of 90°.

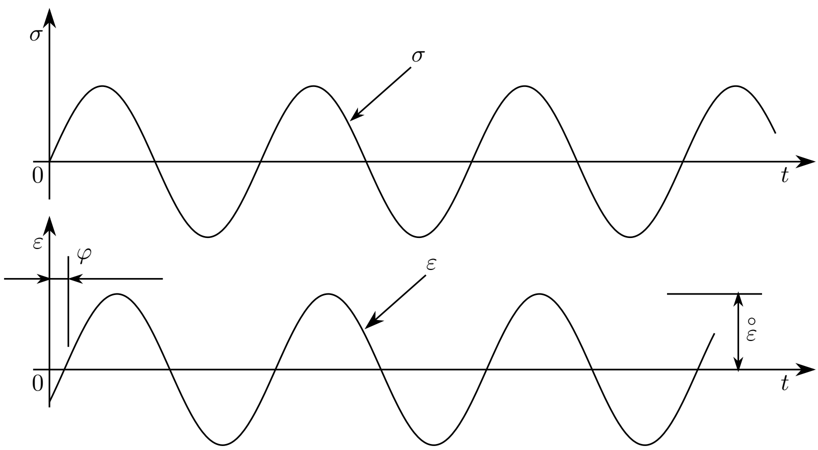



Figure 7. Periodical course of stress $\sigma$ and strain $\varepsilon$ of a viscoelastic material, [17].

Sometimes there is a need to express the constitutive equation (3.10) with phase shift $\varphi$ directly involved, [6]: $\sigma_{ij}(t) = H^*_{ijkl}(\omega)\mathring{\varepsilon}_{kl}\mathrm{e}^{\mathrm{i}(\omega t+\varphi)}$. In such a case, it is explicitly given that compared to the periodical deformation load, the stress response is delayed by $\varphi$, see Fig. 7, the delay can be computed, measured and recorded. Nevertheless, we are mostly focused on the mechanical behaviour of a device with regard to the danger of its damage. Thus, there is no need of tracing the time shift between action and reaction. Relations (3.17)–(3.20) document the fact that in both zero and infinity limiting cases of frequency, behaviour of the anisotropic body is almost purely elastic.

### 3.1. Pulsating load imposed on a viscoelastic body and the mechanical response of a viscoelastic body. The Fourier transform.

*Integral transform.* Integral transforms are used for solving a wide range of science and technology tasks. The Fourier integral transform is commonly used in linear tasks of continuum mechanics.

The direct Fourier transform, [6]:

$$\mathcal{F}(f(t)) = \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{3.21}$$

and its inverse form

$$\mathcal{F}^{-1}(\tilde{f}(\omega)) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{3.22}$$

can be used in its cosine and sine split as well:

$$\mathcal{F}_c(f(t)) = \tilde{f}(\omega) = \int_{0}^{\infty} f(t)\cos(\omega t)\,\mathrm{d}t, \tag{3.23}$$

$$\mathcal{F}_c^{-1}(\tilde{f}(\omega)) = f(t) = \frac{2}{\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\cos(\omega t)\,\mathrm{d}\omega, \tag{3.24}$$

$$\mathcal{F}_s(f(t)) = \tilde{f}(\omega) = \int_{0}^{\infty} f(t)\sin(\omega t)\,\mathrm{d}t, \tag{3.25}$$

$$\mathcal{F}_s^{-1}(\tilde{f}(\omega)) = f(t) = \frac{2}{\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\sin(\omega t)\,\mathrm{d}\omega. \tag{3.26}$$

In our investigation we use the sine and cosine Fourier transform for switching between the constitutive relations written in the sense of time and those written in the sense of frequency. It is a known fact that both sine and cosine Fourier transforms and its inverse is yielded by splitting (3.21), as well as (3.22) when the Euler form is used.

The storage and loss moduli (3.12) and (3.13) can be regarded as Fourier images, hence the originals can be derived as

$$\widehat{H}_{ijkl}(t) = \frac{2}{\pi}\int_0^\infty \frac{\bar{H}_{ijkl}(\omega) - \overset{\infty}{H}_{ijkl}}{\omega}\sin(\omega t)\,\mathrm{d}\omega, \tag{3.27}$$

$$\widehat{H}_{ijkl}(t) = \frac{2}{\pi}\int_0^\infty \frac{\bar{\bar{H}}_{ijkl}(\omega)}{\omega}\cos(\omega t)\,\mathrm{d}\omega \tag{3.28}$$

and for $t > 0$, we can rearrange (3.27) as

$$\begin{aligned}\widehat{H}_{ijkl}(t) &= \frac{2}{\pi}\int_0^\infty \frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin(\omega t)\,\mathrm{d}\omega - \frac{2}{\pi}\overset{\infty}{H}_{ijkl}\int_0^\infty \frac{\sin(\omega t)}{\omega}\,\mathrm{d}\omega \\ &= \frac{2}{\pi}\int_0^\infty \frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin(\omega t)\,\mathrm{d}\omega - \overset{\infty}{H}_{ijkl},\end{aligned}$$

where we have taken

$$\int_0^\infty \frac{\sin(\omega t)}{\omega}\,\mathrm{d}\omega = \frac{\pi}{2}$$

into account. So, we have

$$\widehat{H}_{ijkl}(t) + \overset{\infty}{H}_{ijkl} = H_{ijkl}(t) = \frac{2}{\pi}\int_0^\infty \frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin(\omega t)\,\mathrm{d}\omega. \tag{3.29}$$

And, ready-to-use relaxation coefficient (3.29) substituted in (3.3) yields the resulting explicit constitutive equation for an anisotropic material subjected to a periodic pulsating strain load

$$\sigma_{ij}(t) = \frac{2}{\pi}\int_{-\infty}^t\int_0^\infty \frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin(\omega(t-\tau))\,\mathrm{d}\frac{\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\omega\,\mathrm{d}\tau. \tag{3.30}$$

Consequently, by using the Fourier transform we get the Fourier image for stress tensor function

$$\widetilde{\sigma}_{ij}(\omega) = \int_{-\infty}^\infty\left[\int_{-\infty}^t H_{ijkl}(t-\tau)\frac{\mathrm{d}\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau\right]\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{3.31}$$

Moreover, whenever having $\widetilde{\sigma}_{ijkl}(t)$, by using the inverse Fourier transform we get the Fourier original $\sigma_{ij}(\omega)$. When we couple transforms (3.22) with (3.11)–(3.13), then the constitutive equation (3.31) can be expressed in the shortened form

$$\widetilde{\sigma}_{ij}(\omega) = H^*_{ijkl}(\omega)\widetilde{\varepsilon}_{kl}(\omega). \tag{3.32}$$

With regard to the type of solved problems, we can now use constitutive relations in the sence of time $t$, see (3.10), or frequency $\omega$, see (3.32).

There are two independent relaxation times in the case of an orthotropic material, see Fig. 3. The corresponding values can be measured in a lab, each one separately, by imposing the load in the corresponding direction. The resulting more "solid like" behaviour is performed alongside the main reinforcement ($x$ axis in figure) and more "liquid like" behaviour in all perpendicular directions.

## 4. Conclusions

The paper deals with deriving of an explicit constitutive equation describing the viscoelastic response of an anisotropic body to a specified periodic load. The investigation is carried out for boundary tasks of linear viscoelasticity, with the relaxation coefficient emphasised. The mechanical response of the viscoelastic model representing a simple reinforced concrete beam to a pulsating periodical external strain load is studied. Both the main and the steel fibre dispersed reinforcement are considered in the structural element. For this reason the anisotropic case is studied. As the mathematical tool the Fourier transform is used for dealing with constitutive relations, i.e., the stress dependence on the deformation load. This paper is a continuation and extension of the works [8], [21], [23] of the authors.

### *References*


[1] *J. Brilla*: Linear viscoelastic bending of anisotropic plates. Z. Angew. Math. Mech. *48* (1968), T128–T131. zbl

[2] *R. M. Christensen*: Theory of Viscoelasticity: An Introduction. Mir, Moscow, 1974. (In Russian.)

[3] *E. H. Dill*: Continuum Mechanics: Elasticity, Plasticity, Viscoelasticity. CRC Press, Boca Raton, 2007. doi

[4] *J. D. Ferry*: Viscoelastic Properties of Polymers. John Wiley, New York, 1961.

[5] *J. Kopfová, M. Minárová, J. Sumec*: Visco-elasto-plastic modeling. Proceedings of Equadiff Conference 2017. Slovak University of Technology, SPEKTRUM STU Publishing, Bratislava, 2017, pp. 173–180.

[6] *R. S. Lakes*: Viscoelastic Solids. CRC Press, Boca Raton, 1998. doi

[7] *B. J. Lazan*: Damping of Materials and Members in Structural Mechanics. Pergamon Press, New York, 1968.

[8] *M. Minárová*: Mathematical modeling of phenomenological material properties: Differential operator forms of constitutive equations. Slovak J. Civil Eng. *22* (2014), 19–24. doi

[9] *M. Minárová*: Rheology: Viscoelastic and Viscoelastoplastic Modelling. Habilitation Thesis. Slovak University of Technology, Bratislava, 2018.

[10] *M. Minárová, J. Sumec*: Constitutive equations for selected rheological models in linear viscoelasticity. Advanced and Trends in Engineering Science and Technologies. II. CRC Press, Boca Raton, 2016, pp. 207–212.

[11] *M. Minárová, J. Sumec*: Stress-strain response of the human spine intervertebral disc as an anisotropic body mathematical modeling and computation. Open Phys. *14* (2016), 426–435. doi

[12] *W. Nowacki*: Theory of Creep. Arcady, Warsaw, 1963. (In Polish.)

[13] *R. Plunkett*: Damping analysis: An historical perspective. Mechanical and Mechanism of Material Damping. ASTM: Philadelphia, 1992, pp. 562–569. doi
[14] *J. N. Rabotnov*: Creep Problems in Structural Members. North-Holland Series in Applied Mathematics and Mechanics 7. North-Holland, Amsterdam, 1969. zbl
[15] *D. Roylance*: Engineering Viscoelasticity. Available at `https://www.researchgate.net/publication/268295291_Engineering_Viscoelasticity` (2001), 37 pages.
[16] *I. N. Sneddon*: Fourier Transforms. McGraw Hill, New York, 1951. zbl MR
[17] *Z. Sobotka*: Rheology of Materials and Engineering Structures. Academia, Prague, 1984.
[18] *I. S. Sokolnikoff*: Mathematical Theory of Elasticity. McGraw Hill, New York, 1956. zbl MR
[19] *L. Šuklje*: Rheological Aspects of Soil Mechanics. John Wiley, London, 1969.
[20] *J. Sumec*: Mechanics-mathematical modeling of materials whose physical properties are time-dependent. Internal Research Report No. III-3-4/9.4-ISAR-SAS, Bratislava, Slovakia (1983).
[21] *J. Sumec, Ľ. Hruštinec*: Modeling of some effects in the viscoelastic selected type of materials. Proceedings of the 13th International Conference on New Trends in Statics and Dynamics of Buildings. Slovak University of Technology, Bratislava, 2017, pp. 61–78.
[22] *C. Truesdell, W. Noll*: The Nonlinear Field Theories of Mechanics. Handbuch der Physics. Band 3, No 3. Springer, Berlin, 1965.
[23] *I. Véghová, J. Sumec*: Basic rheological parameters of solid viscoelastic body under periodical loading. Proceedings of the 15th International Conference on New Trends in Statics and Dynamics of Buildings. Slovak University of Technology, Bratislava, 2017, pp. 19–26.



*Authors' address*: *Jozef Sumec*, *Mária Minárová* (corresponding author), *Ľuboš Hruštinec*, Slovak University of Technology, Department of Mathematics and Descriptive Geometry, Radlinského 11, 810 05 Bratislava, Slovakia, e-mail: `jozef.sumec@stuba.sk`, `maria.minarova@stuba.sk`, `lubos.hrustinec@stuba.sk`.