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1-PLANAR GRAPHS WITH GIRTH AT LEAST 6 ARE (1,1,1,1)-COLORABLE

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Abstract. A graph is 1-planar if it can be drawn in the Euclidean plane so that each edge is crossed by at most one other edge. A 1-planar graph on n vertices is optimal if it has $4n-8$ edges. We prove that 1-planar graphs with girth at least 6 are (1,1,1,1)-colorable (in the sense that each of the four color classes induces a subgraph of maximum degree one). Inspired by the decomposition of 1-planar graphs, we conjecture that every 1-planar graph is (2,2,2,0,0)-colorable.

Keywords: 1-planar graph; discharging

MSC 2020: 05C10, 05C15, 05C99

1. Introduction

The graphs considered in this paper are finite, simple and undirected. Let $d_1,\ldots,d_k$ be k nonnegative integers. A graph G is $(d_1,\ldots,d_k)$*-colorable* if the vertex set of G can be partitioned into k subsets $V_1,\ldots,V_k$, such that the maximum degree of the subgraph induced by V_i is at most d_i $(1\leqslant i\leqslant k)$. It is a *proper coloring* of G when $d_1=\ldots=d_k=0$; we also say that G is k*-colorable.* Particularly, when $d_1=\ldots=d_k=d\geqslant 1$, it is said that G has a d*-improper coloring* or d*-defective coloring.*

The coloring of planar graphs has been extensively investigated. In 1976–1977, the well-known Four Color Problem was proved by Appel and Haken using computer (see [1], [2], [3]), i.e., every plane graph is 4-colorable. Cowen et al. in [10] presented the classical result that every planar graph is (2,2,2)-colorable for improper coloring

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of plane graphs. Steinberg in 1976 in [17] asserts that every planar graph with no cycles of length four or five is 3-colorable. This problem has been attracting a substantial amount of attention among graph theorists. The conjecture was disproved in 2017 by Cohen-Addad et al. by constructing a counterexample, see [9]. Many other relaxations of the conjecture have been established, see [7], [10], [12], [19]. It has been proved that planar graphs without cycles of length 4 or 5 are (3,0,0)-colorable and planar graphs without cycles of length 4 and 6 or cycles of length 4, 5 and 9 are (1,0,0)-colorable. A 1-planar graph is a generalization of plane graph which was first considered by Ringel (see [14]) in connection with the simultaneous vertex-face coloring of plane graphs. A graph is 1-planar if it can be drawn in the Euclidean plane so that each edge is crossed by at most one other edge. It has been proved that 1-planar graphs can be colored with at most seven colors. Later, the precise number of colors needed to color 1-planar graphs was shown to be six, see [5]. That is to say, every 1-planar graph is 6-colorable. Fabrici and Madaras in [11] studied the existence of subgraphs of bounded degrees in 1-planar graphs. Zhang considered the edge coloring of 1-planar graphs in [21], [22] and [23]. Sun in [18] studied the total coloring of 1-planar graphs and showed that every 1-planar graph G with maximum degree $\Delta(G) \geqslant 12$ and girth at least five is totally $(\Delta(G)+1)$-colorable. For vertex coloring of 1-planar graphs, it is proved that every 1-planar graph without 4-cycles or adjacent 5-vertices is 5-colorable (see [15]), 1-planar graphs without 4-cycles or 5-cycles are 5-colorable, see [16]. Moreover, 1-planar graph with girth at least 7 is (1,1,1,0)-colorable, see [8]. It is conjectured that every 1-planar graph without 3-cycles is 5-colorable, see [6].

Inspired by the results above, we focus on the improper coloring of 1-planar graphs in this paper and prove that every 1-planar graph with girth at least 6 is (1,1,1,1)-colorable. We conjecture that all 1-planar graphs are (2,2,2,0,0)-colorable.

Now we introduce some basic definitions about graphs. A cycle in a graph is a nonempty trail in which only the first and last vertices are equal. A cycle containing k vertices and k edges is called a *k-cycle*, which is also referred to as a cycle of length k. The *girth* of a graph is the length of the shortest cycle contained in the graph. For an element $x \in V(G) \cup F(G)$, we use $d(x)$, $\delta(G)$, and $\Delta(G)$ to represent the degree of x, the minimum and maximum vertex degree of G, respectively. If $u_1, u_2, \ldots, u_n$ are vertices on the boundary of f in cyclic order, then we use $[u_1u_2\ldots u_n]$ to denote face f. A *k-vertex*, *k^+-vertex*, and *k^--vertex* is a vertex of degree k, at least k, and at most k, respectively. A similar notation can be applied to faces. For more details about graph theory, the reader is referred to classical textbooks, see [4], [20].

2. Preliminaries

Let G be a 1-planar graph. Assume that G has been drawn on a plane such that every edge is crossed at most once and the number of crossings is as small as possible. If z is a crossing formed by the intersection of two edges x_1y_1 and x_2y_2, the four vertices x_1, y_1, x_2, and y_2 are distinct. The *associated plane graph* G^* of G is a plane graph obtained by turning all crossings of G into new 4-vertices. The 4-vertices in G^* are called *cross vertices* if they are the crossings of G. If the vertices are both vertices of G and G^*, then they are called *true vertices*. One should note that cross vertices are not real vertices of G. In the same vein as the definition of cross vertices, the faces in G^* are called *cross faces* if there are some cross vertices on them. Otherwise, they are called *normal faces*. Particularly, the 4-faces with two non-adjacent cross vertices in G^* are called *bad* 4*-faces*. For the figures within this paper, the white and black dots will be used to represent the crossings and vertices of G (unless otherwise specified). We also use the following additional notation (here, v is a vertex and f is a face in G^*):

$n_{t4}(v)$: the number of true 4-vertices adjacent to v in G;
$n_{t4}(f)$: the number of true 4-vertices on f;
$n_6(f)$: the number of 6-vertices on f;
$n_c(f)$: the number of cross vertices on f;
$m_{3c}(v)$: the number of cross 3-faces incident to v;
$m_4(v)$: the number of 4-faces incident to v;
$m_{4c}(v)$: the number of cross 4-faces with one cross vertex incident to v;
$m_{4b}(v)$: the number of bad 4-faces incident to v;
$N_G(v)$: the neighbors of v in G;
$N_{G^*}(v)$: the neighbors of v in G^*.

3. 1-planar graphs with girth at least 6 are (1,1,1,1)-colorable

Theorem 3.1. *1-planar graphs with girth at least 6 are $(1,1,1,1)$-colorable.*

3.1. Structural properties. Let G be a 1-planar graph, $\mathcal{C} = \{1,2,3,4\}$ be a color set with four colors and φ be a coloring of G, where the color of vertex v is $\varphi(v) \in \mathcal{C}$. Theorem 3.1 is proved by contradiction. Assume that G is a counterexample with the minimum number of vertices and crossings satisfying that the girth of G is at least 6. Then G is not (1,1,1,1)-colorable. By the minimality of G, it is apparent that G is connected, and every subgraph of G with fewer vertices is (1,1,1,1)-colorable. Let G^* be the associated plane graph of G. The following properties hold.

Property 3.1. *The minimum degree of G satisfies $\delta(G) \geqslant 4$.*

Proof. By contradiction. Let G contain a 3^{-}-vertex v, and let $G^{'}$ be obtained from G by deleting v. Then, by minimality of G, $G^{'}$ is (1,1,1,1)-colorable with at most three colors used at neighbours of v. Hence, by assigning v the color which is not used in its neighbors, we can extend the (1,1,1,1)-coloring of $G^{'}$ to a (1,1,1,1)-coloring of G, a contradiction. □

Property 3.2. *No two 4-vertices are adjacent in G.*

Proof. Let v be a 4-vertex of G and $v_1, \ldots, v_4$ be its neighbors. Assume that v_1 is a 4-vertex. Consider the subgraph $G^{'}$ of G by deleting vertices v and v_1, i.e., $G^{'} = G - \{v, v_1\}$. By the minimality of G, we can obtain that $G^{'}$ has a (1,1,1,1)-coloring φ. Now we will show that the coloring φ of $G^{'}$ can be extended to a (1,1,1,1)-coloring of G. First, we assign a color in $\mathcal{C}$ to the vertex v_1 properly as v_1 is a 4-vertex adjacent to v, and there is at least one color available. Secondly, if $\{\varphi(v_1), \ldots, \varphi(v_4)\} \neq \mathcal{C}$, then the color to be assigned to v is $\varphi(v) = \mathcal{C} \setminus \{\varphi(v_1), \ldots, \varphi(v_4)\}$; otherwise, $\{\varphi(v_1), \ldots, \varphi(v_4)\} = \mathcal{C}$ and the vertex v can be colored with the same color as v_1. Then we obtain a (1,1,1,1)-coloring of G, which is in contradiction to the choice of G. □

Property 3.3. *Every 5-vertex is adjacent to at most two 4-vertices in G.*

Proof. Let v be a 5-vertex and $v_1, \ldots, v_5$ be the neighbors of v. Assume that v is adjacent to at least three 4-vertices in G. Without loss of generality, assume that $d(v_1) = d(v_2) = \ldots = d(v_i) = 4$ $(3 \leqslant i \leqslant 5)$. Consider the subgraph $G^{'} = G - \{v, v_1, v_2, \ldots, v_i\}(3 \leqslant i \leqslant 5)$. By the minimality of G, $G^{'}$ has (1,1,1,1)-coloring φ. Since there are four colors in $\mathcal{C}$ and the degree of each vertex of $v_1, v_2, \ldots, v_i$ is four; moreover, all of them are adjacent to vertex v. Hence, the vertices $v_1, v_2, \ldots, v_i$ can be colored properly. To color the vertex v, there are two cases. More precisely, if $\{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_5)\} \neq \mathcal{C}$, assign a color to vertex v as follows: $\varphi(v) = \mathcal{C} \setminus \{\varphi(v_1), \ldots, \varphi(v_5)\}$; otherwise, if $\{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_5)\} = \mathcal{C}$, then there is exactly one color used twice, we call it color k, we assign one of the following colors to vertex v in this case: $\varphi(v) = \{\varphi(v_1), \ldots, \varphi(v_i)\} \setminus \{k\}$ $(3 \leqslant i \leqslant 5)$. It is not difficult to verify that it is a (1,1,1,1)-coloring of G. Therefore, a 5-vertex cannot be adjacent to more than two 4-vertices in G. □

Property 3.4 ([21], Lemma 3)**.** *Let G^* be the associated plane graph of 1-planar graph G. Then for any two cross vertices u and $v, uv \notin E(G^*)$.*

Property 3.5. *Let v be a k-vertex in G^*. Then the number of cross 3-faces adjacent to v satisfies $m_{3c}(v) \leqslant L$, where*

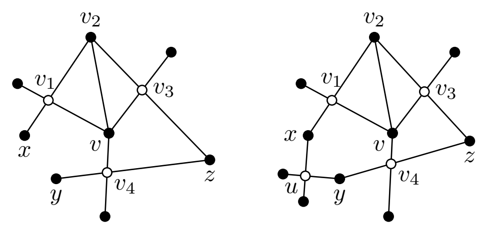
$$L = \begin{cases} \left\lfloor \dfrac{k}{3} \right\rfloor \times 2 + 1, & k \equiv 2 \pmod 3, \\ \left\lfloor \dfrac{k}{3} \right\rfloor \times 2, & \text{otherwise.} \end{cases}$$

Property 3.6. *Let v be a true 4-vertex incident to two cross 3-faces and one bad 4-face of G^*, see Figure 1. Then the following properties hold:*

(a) *The remaining face that v is incident to is a 6^+-face.*

(b) *If v is incident to a 6-face $[xv_1vv_4yux]$, then $[xv_1vv_4yux]$ has three cross vertices.*

(c) *If v is incident to a 7^+-face, then this face has at least two cross vertices.*



(1) (2)

Figure 1. $d(v) = 4$, $m_{3c}(v) = 2$, $m_{4b}(v) = 1$.

Proof. Suppose that v is a true 4-vertex incident to two cross 3-faces $[vv_1v_2]$, $[vv_2v_3]$, and one bad 4-face $[vv_3zv_4]$, see Figure 1.

(a) Note that the girth of G is at least 6. Hence, x, y are two different vertices and $xy \notin E(G)$, otherwise, $[xv_2zx]$ is a 3-cycle, or $[xv_2zyx]$ is a 4-cycle in G, see Figure 1 (2). Hence, the face $[xv_1vv_4yux]$ is a 6^+-face in this case.

(b) If $[xv_1vv_4yux]$ is a 6-face (see Figure 1 (2)), then u must be a cross vertex of G, otherwise, $[uxv_2zyu]$ is a 5-cycle of G^*, a contradiction.

(c) If $[xv_1vv_4yux]$ is a 7^+-face, where v_1 and v_4 are the cross vertices on it, then the 7^+-face has at least two cross vertices. □

It appears that analogous results hold also for a true 4-vertex incident to two cross 3-faces and one cross 4-face with exactly one cross vertex.

Property 3.7. *Let v be a true 4-vertex incident to two cross 3-faces and one cross 4-face with exactly one cross vertex of G^*, see Figure 2. Then the remaining face that v is incident to is a 6^+-face with at least two cross vertices.*

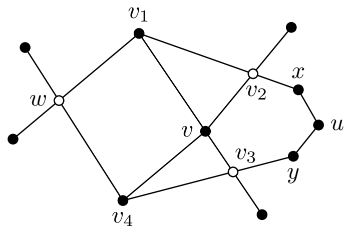


Figure 2. $d(v) = 4$, $m_{3c}(v) = 2$, $m_{4c}(v) = 1$.

Proof. Suppose that v is a true 4-vertex incident to two cross 3-faces $[vv_1v_2]$, $[vv_3v_4]$ and one cross 4-face with exactly one cross vertex $[wv_1vv_4]$, see Figure 2. Then x and y cannot be the same vertex, otherwise, $[xv_1vv_4y]$ is a 4-cycle of G. Moreover, $xy \notin E(G)$, otherwise $[xv_4vv_3yx]$ is a 5-cycle of G. Therefore, $[xv_2vv_3y\dots x]$ is a 6^+-face with at least two cross vertices v_2 and v_3 on it. $\square$

Property 3.8. *Let v be a 5-vertex. Suppose that v is adjacent to three cross 3-faces $[vv_1v_2]$, $[vv_2v_3]$, and $[vv_4v_5]$, see Figure 3. By the conditions on girth of G, y and z are different vertices, $yz \notin E(G)$, $xv_4 \notin E(G)$. Moreover, the following properties hold:*

(a) *The face $[v_1vv_5y\dots zv_1]$ is a cross 6^+-face with at least two cross vertices.*
(b) *If $[xv_3vv_4ux]$ is a 5-face, then it is a 5-face with two cross vertices.*
(c) *If $[xv_3vv_4\dots x]$ is a 6^+-face, then it is a cross 6^+-face with at least one cross vertex.*

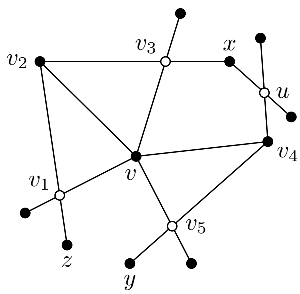


Figure 3. $d(v) = 5$, $m_{3c}(v) = 3$.

The proof is similar to the proof of Properties 3.6 and 3.7.

3.2. The discharging procedure. We prove Theorem 3.1 by contradiction. Assume that G is a 1-planar graph with girth at least 6, which is not (1,1,1,1)-colorable. Further G^* is the corresponding planar graph of G which is obtained by turning all crossings into cross vertices. Then G^* satisfies Euler's formula $|V(G^*)| - |E(G^*)| + |F(G^*)| = 2$. Moreover, the following relationship holds:

$$\sum_{v \in V(G^*)} d(v) = \sum_{f \in F(G^*)} d(f) = 2|E(G^*)|.$$

Taking now into account the associated plane graph G^* of the minimal counter-example G, we define a charge function ω on the vertex and face set of G^* as follows:

$$\omega(x) = d(x) - 4 \quad \forall x \in V(G^*) \cup F(G^*). \tag{3.1}$$

From Euler's formula, one can conclude that the total sum of charges of vertices and faces is equal to

$$\sum_{v \in V(G^*)} (d(v) - 4) + \sum_{f \in F(G^*)} (d(f) - 4) = -8. \tag{3.2}$$

Then appropriate discharging rules are defined to redistribute these charges in the way that the total sum of charges keeps fixed during the discharging process. These rules transform ω to a new charge function ω^*, for which it is shown that $\omega^*(x) \geqslant 0$ for every $x \in V(G^*) \cup F(G^*)$. This, however, leads to a contradiction, as

$$-8 = \sum_{x \in V(G^*) \cup F(G^*)} \omega(x) = \sum_{x \in V(G^*) \cup F(G^*)} \omega^*(x) \geqslant 0. \tag{3.3}$$

Let $\tau(x \to y)$ denote the charge that sends x to y for any $x, y \in V(G^*) \cup F(G^*)$. The discharge rules are defined as follows:

(R1) Let f be a cross 3-face, $f = [uvw]$, where w is a cross vertex, u and v are true vertices. Then $\tau(u \to f) = \frac{1}{2}$, $\tau(v \to f) = \frac{1}{2}$.

(R2) Let $u, v \in V(G)$, $uv \in E(G)$. If $d(v) = 4$, then $\tau(u \to v) = \frac{1}{8}$.

(R3) Let f be a face in G^* and v be a true 4-vertex on f:
- (R3.1) If f is a 5-face with two cross vertices, then $\tau(f \to v) = \frac{1}{4}$.
- (R3.2) If f is a 6-face with one cross vertex, then $\tau(f \to v) = \frac{1}{4}$.
- (R3.3) If f is a 6-face with at least two cross vertices, then $\tau(f \to v) = \frac{1}{2}$.
- (R3.4) If f is a cross 7^+-face, then $\tau(f \to v) = \frac{1}{2}$.

(R4) Let f be a face in G^*, v be a 5-vertex on f
- (R4.1) If f is a 5-face with two cross vertices, then $\tau(f \to v) = \frac{1}{4}$.
- (R4.2) If f is a 6^+-face with one cross vertex, then $\tau(f \to v) = \frac{1}{4}$.
- (R4.3) If f is a 6^+-face with at least two cross vertices, then $\tau(f \to v) = \frac{1}{2}$.

(R5) Let f be a face in G^*, v be a 6-vertex on f:
- (R5.1) If f is a 5-face with two cross vertices, then $\tau(f \to v) = \frac{1}{4}$.
- (R5.2) If f is a 6^+-face with one cross vertex, then $\tau(f \to v) = \frac{1}{4}$.
- (R5.3) If f is a 6^+-face with at least two cross vertices, then $\tau(f \to v) = \frac{3}{8}$.

In the following, we prove that $\omega^*(x) \geqslant 0$ for any $x \in V(G^*) \cup F(G^*)$. Let $v_1, v_2, \ldots, v_{d(v)}$ be the neighbors of v in G^* in cyclic order. Note that $\delta(G^*) \geqslant 4$ as $\delta(G) \geqslant 4$.

Final charge of vertex in G^*. Now consider the charge function of the vertices in G^*. Let v be a vertex of G^*. We consider the following cases:

(1) $d(v) = 4$.

Case 1: If v is a cross vertex in G^*, then, according to the discharge rules, v is not involved in discharging rules, so $\omega^*(v) = \omega(v) = 4 - 4 = 0$.

Case 2: If v is a true 4-vertex in G^*, v is incident to at most two 3-faces, see Figure 1 (1). By Property 3.2, the vertices in $N_G(v)$ are 5^+-vertices. According to discharge rule (R2), v can receive $\frac{1}{8}$ from each of its adjacent 5^+-vertices. We have three subcases according to the number of cross 3-faces that v is incident to.

Subcase 2.1: $d(v) = 4$, $m_{3c}(v) = 2$.

Subcase 2.1.1: If $d(v) = 4$, $m_{3c}(v) = 2$, $m_4(v) = 1$, Properties 3.6 and 3.7 give the details of bad 4-face or cross 4-face with exact one cross vertex that v is incident. We can conclude that whatever the 4-face is, v can always get $\frac{1}{2}$ from the remaining incident to 6^+-face by using discharge rules (R3.3) or (R3.4). By combining with discharge rules (R1) and (R2), the new charge function satisfies:

$$\omega^*(v) = \omega(v) + \frac{1}{8}d(v) - \frac{1}{2}m_{3c}(v) + \frac{1}{2} = 4 - 4 + \frac{1}{8} \times 4 - \frac{1}{2} \times 2 + \frac{1}{2} = 0.$$

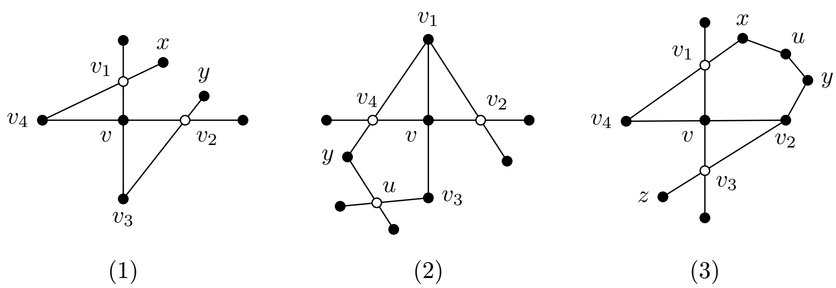


Figure 4. $d(v) = 4$, $m_{3c}(v) = 2$, $m_4(v) = 0$.

Subcase 2.1.2: If $d(v) = 4$, $m_{3c}(v) = 2$, $m_4(v) = 0$, see Figure 4. Suppose that the case shown in Figure 4 (1) happens. Then x, y should be different vertices, $xy \notin E(G^*)$, and the face $[xv_1vv_2y\dots x]$ is a 6^+-face with at least two cross vertices. So by discharge rule (R3.3) or (R3.4), v can receive $\frac{1}{2}$ from its incident 6^+-face $[xv_1vv_2y\dots x]$. If the cases shown in Figures 4 (2) and 4 (3) happen, then the other two faces v incident to are 5^+-faces. More precisely, if the face that v is incident to is a 5-face, it has two cross vertices (see Figure 4 (2)); then by discharging rule (R3.1), each 5-face with two cross vertices sends $\frac{1}{4}$ to v. Otherwise, the face is a cross 6^+-face with at least one cross vertex (see Figure 4 (3)), and by discharging rules (R3.2), (R3.3), or (R3.4), it will send at least $\frac{1}{4}$ to v. Therefore, the new charge function satisfies

$$\omega^*(v) \geqslant \omega(v) + \frac{1}{8}d(v) - \frac{1}{2}m_{3c}(v) + \frac{1}{2} = 4 - 4 + \frac{1}{8} \times 4 - \frac{1}{2} \times 2 + \frac{1}{2} = 0.$$

Subcase 2.2: $d(v) = 4$, $m_{3c}(v) \leqslant 1$. By discharging rules (R1) and (R2), v sends $\frac{1}{2}$ to at most one cross 3-face and receives $\frac{1}{8}$ from each of its adjacent vertices. So in this case, it is not necessary to consider received charge from its incident faces, hence

$$\omega^*(v) \geqslant \omega(v) + \frac{1}{8}d(v) - \frac{1}{2}m_{3c}(v) \geqslant 4 - 4 + \frac{1}{8} \times 4 - \frac{1}{2} = 0.$$

(2) $d(v) = 5$. If v is a vertex of degree 5 in G^*, we can conclude that v is incident to at most three cross 3-faces (by Property 3.5) and v is adjacent to at most two true 4-vertices (by Property 3.3), i.e., $n_{t4}(v) \leqslant 2$. According to the number of cross 3-faces that v is incident to, there are three subcases.

Case 1: $d(v) = 5$, $m_{3c}(v) = 3$. Suppose that v is a 5-vertex incident to three cross 3-faces, see Figure 3. According to discharge rules (R1) and (R2), v sends $\frac{1}{2}$ to each incident cross 3-face, $\frac{1}{8}$ to each adjacent 4-vertex. By discharge rule (R4.3) and Property 3.8 (a), v can always get $\frac{1}{2}$ from cross 6^+-face $[v_1vv_5y\dots zv_1]$. By discharge rules (R4.1), (R4.2) and Property 3.8 (b) (c), v can get at least $\frac{1}{4}$ from cross 5^+-face $[xv_3vv_4\dots x]$. Therefore,

$$\omega^*(v) \geqslant \omega(v) - \frac{1}{8}n_{t4}(v) - \frac{1}{2}m_{3c}(v) + \frac{1}{2} + \frac{1}{4} = 5 - 4 - \frac{1}{8} \times 2 - \frac{1}{2} \times 3 + \frac{1}{2} + \frac{1}{4} = 0.$$

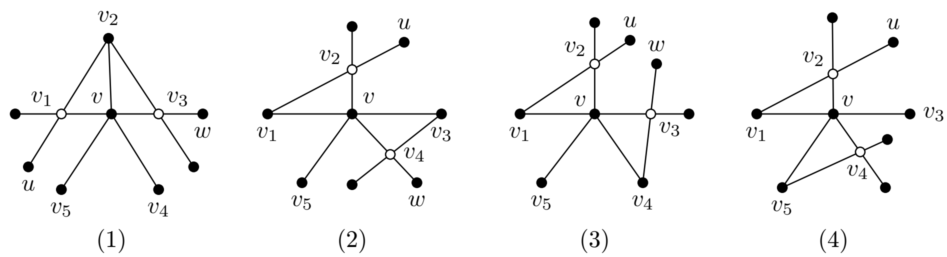


Figure 5. $d(v) = 5$, $m_{3c}(v) = 2$.

Case 2: $d(v) = 5$, $m_{3c}(v) = 2$. All the possibilities are shown in Figure 5. If the face $[uv_1vv_5\dots u]$ in Figure 5 (1), face $[uv_2vv_3\dots u]$ in Figure 5 (2) and face $[uv_2vv_3\dots u]$ in Figure 5 (4) are 5-faces, they must be 5-faces with two cross vertices. Hence, by discharge rule (R4.1), v can get $\frac{1}{4}$ from each of the cross 5-faces with two cross vertices. Otherwise, they are cross 6^+-faces with at least one cross vertex. Hence, by discharge rules (R4.2) and (R4.3), v can get $\frac{1}{4}$ from each of them. In addition, the face $[uv_2vv_3\dots wu]$ in Figure 5 (3) is a 6^+-face with at least two cross vertices; then v can get $\frac{1}{2}$ from this face. Hence, in each of these cases, v can always get at least $\frac{1}{4}$ from its incident cross 5^+-faces. Therefore,

$$\omega^*(v) \geqslant \omega(v) - \frac{1}{8}n_{t4}(v) - \frac{1}{2}m_{3c}(v) + \frac{1}{4} = 5 - 4 - \frac{1}{8} \times 2 - \frac{1}{2} \times 2 + \frac{1}{4} = 0.$$

Case 3: $d(v) = 5$, $m_{3c}(v) \leqslant 1$. In this case, it is not necessary to consider the charge received from the incident faces. According to discharge rules (R1) and (R2), we always have

$$\omega^*(v) \geqslant \omega(v) - \frac{1}{8}n_{t4}(v) - \frac{1}{2}m_{3c}(v) \geqslant 5 - 4 - \frac{1}{8} \times 2 - \frac{1}{2} \times 1 = \frac{1}{4} > 0.$$

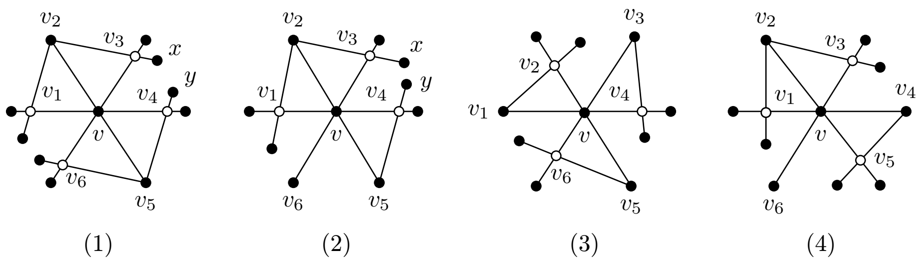


Figure 6. $d(v) = 6$.

(3) $d(v) = 6$. Assume that v is a 6-vertex in G. Then v is adjacent to at most six true 4-vertices, i.e., $n_{t4}(v) \leqslant 6$. Property 3.5 gives that v is incident to at most four cross 3-faces, see Figure 6 (1).

Case 1: $d(v) = 6$, $m_{3c}(v) = 4$. By observation, it is easy to conclude that x, y are two different vertices and $xy \notin E(G)$ in Figure 6 (1). Furthermore, the face $[xv_3vv_4y\dots x]$ that v is incident to is a 6^+-face with at least two cross vertices v_3 and v_4. In the same vein, $[v_1vv_6\dots v_1]$ is a 6^+-face with at least two cross vertices. According to discharge rules (R1), (R2) or (R5.3), we have that

$$\omega^*(v) \geqslant \omega(v) - \frac{1}{8}n_{t4}(v) - \frac{1}{2}m_{3c}(v) + \frac{3}{8} \times 2 = 6 - 4 - \frac{1}{8} \times 6 - \frac{1}{2} \times 4 + \frac{3}{4} = 0.$$

Case 2: $d(v) = 6$, $m_{3c}(v) = 3$. The possible cases are shown in Figure 6 (2) (3) (4). Suppose that the case in Figure 6 (2) happens, the face $[xv_3vv_4y\dots v_3]$ that v is incident to is a 6^+-face with at least two cross vertices v_3 and v_4, by discharge rule (R5.2), v can get $\frac{3}{8}$ from it. Suppose that one of the cases in Figure 6 (3) and Figure 6 (4) happens. Then either $[v_1vv_6\dots v_1]$ is a 5-face with two cross vertices, or a cross 6^+-face with at least one cross vertex. By discharge rules (R5.1), (R5.2) or (R5.3), we can conclude v can always get at least $\frac{1}{4}$ from its incident faces. Hence,

$$\omega^*(v) \geqslant \omega(v) - \frac{1}{8}n_{t4}(v) - \frac{1}{2}m_{3c}(v) + \frac{1}{4} = 6 - 4 - \frac{1}{8} \times 6 - \frac{1}{2} \times 3 + \frac{1}{4} = 0.$$

Case 3: $d(v) = 6$, $m_{3c}(v) \leqslant 2$. In this case, there is no need to consider receiving charge from its incident faces for v. According to discharge rules (R1), (R2) and (R5),

$$\omega^*(v) \geqslant \omega(v) - \frac{1}{8}n_{t4}(v) - \frac{1}{2}m_{3c}(v) \geqslant 6 - 4 - \frac{1}{8} \times 6 - \frac{1}{2} \times 2 = \frac{1}{4} > 0.$$

(4) $d(v) = k$ ($k \geqslant 7$), then $n_{t4}(v) \leqslant k$. According to discharge rules (R1) and (R2),

$$\omega^*(v) = \omega(v) - \frac{1}{8} \times n_{t4}(v) - \frac{1}{2}m_{3c}(v).$$

Now if $k = 7$, then $m_{3c}(v) \leqslant 4$ and $n_{t4}(v) \leqslant 7$, yielding

$$\omega^*(v) \geqslant 7 - 4 - \frac{1}{8} \times 7 - \frac{1}{2} \times 4 = \frac{1}{8} > 0.$$

If $k = 8$, then $m_{3c}(v) \leqslant 5$ and $n_{t4}(v) \leqslant 8$ with

$$\omega^*(v) \geqslant 8 - 4 - \frac{1}{8} \times 8 - \frac{1}{2} \times 5 = \frac{1}{2} > 0.$$

Finally, if $k \geqslant 9$, then $m_{3c}(v) \leqslant \lfloor \frac{k}{3} \rfloor \times 2 + 1$, $n_{t4}(v) \leqslant k$ and

$$\omega^*(v) \geqslant k - 4 - \frac{1}{8} \times k - \frac{1}{2} \times (\lfloor \frac{k}{3} \rfloor \times 2 + 1) \geqslant \frac{13k - 12 \times 9}{24} > 0.$$

So far, we have proved that $\omega^*(v) \geqslant 0$ for every vertex in G^*. In the following subsection, we consider the new charge function on the faces in planar graph G^*.

Final charge of faces in G^*. Note that all the 3-faces, 4-faces and 5-faces that appear in G^* are cross faces since the girth of G is at least 6. We start from cross 3-face and discuss the classification according to the degree of the face in G^*.

(1) $d(f) = 3$, $f = [uvw]$. If $d(f) = 3$, then by Property 3.4, there is exactly one cross vertex on f, say u. According to the discharge rule (R1), f receives $\frac{1}{2}$ from each of its true vertices v and w, therefore

$$\omega^*(f) = \omega(f) + \frac{1}{2} \times 2 = 3 - 4 + 1 = 0.$$

(2) $d(f) = 4$. According to the discharge rules, there is no flow in charge in this situation, so

$$\omega^*(f) = \omega(f) = 4 - 4 = 0.$$

(3) $d(f) = 5$. According to Property 3.4, no two cross vertices are adjacent. Hence, there are at most two cross vertices on f. We have the following two subcases according to the number of cross vertices on f.

Case 1: $n_c(f) = 2$. Then $n_{t4}(f) + n_5(f) + n_6(f) \leqslant 3$. According to discharge rules (R3.1), (R4.1) and (R5.1), f sends $\frac{1}{4}$ to each of the true 4-vertices, 5-vertices and 6-vertices on it. Hence,

$$\omega^*(f) = \omega(f) - \frac{1}{4}n_{t4}(f) - \frac{1}{4}n_5(f) - \frac{1}{4}n_6(f) = 5 - 4 - \frac{1}{4} \times [n_{t4}(f) + n_5(f) + n_6(f)] \geqslant \frac{1}{4} > 0.$$

Case 2: $n_c(f) < 2$. By discharge rules, there is no transfer of charge involving f. Hence,

$$\omega^*(f) = \omega(f) = 5 - 4 = 1 > 0.$$

(4) $d(f) = 6$. Then there are at most three cross vertices on f by Property 3.4. We discuss the new charge function in the following two subcases:

Case 1: $n_c(f) \geqslant 2$. Then $n_{t4}(f) + n_5(f) + n_6(f) \leqslant 4$. According to discharge rules (R3.3), (R4.3) and (R5.3), f sends $\frac{1}{2}$ to each of its true 4-vertices and 5-vertices and sends $\frac{3}{8}$ to each of its 6-vertices. Hence,

$$\begin{aligned}\omega^*(f) &= \omega(f) - \frac{1}{2}n_{t4}(f) - \frac{1}{2}n_5(f) - \frac{3}{8}n_6(f) \geqslant 6 - 4 - \frac{1}{2}[n_{t4}(f) + n_5(f) + n_6(f)] \\ &\geqslant 6 - 4 - \frac{1}{2} \times 4 = 0.\end{aligned}$$

Case 2: $n_c(f) = 1$. Then $n_{t4}(f) + n_5(f) + n_6(f) \leqslant 5$. According to discharge rules (R3.2), (R4.2) and (R5.2), f sends $\frac{1}{4}$ to each of the true 4-vertices, 5-vertices and 6-vertices on it. Hence,

$$\omega^*(f) = \omega(f) - \frac{1}{4}n_{t4}(f) - \frac{1}{4}n_5(f) - \frac{1}{4}n_6(f) = 6 - 4 - \frac{1}{4}[n_{t4}(f) + n_5(f) + n_6(f)] \geqslant \frac{3}{4} > 0.$$

Case 3: $n_c(f) = 0$. By discharge rules, there is no flow in charge for f. So

$$\omega^*(f) = \omega(f) = 6 - 4 = 2 > 0.$$

(5) $d(f) = k(k \geqslant 7)$.

Case 1: $n_c(f) \geqslant 1$. Then $n_{t4}(f) + n_5(f) + n_6(f) \leqslant k - 1$. According to discharge rules (R3.4), (R4.2), (R4.3), (R5.2) and (R5.3), f sends at most $\frac{1}{2}$ to each of true 4-vertices, 5-vertices, 6-vertices on it. Hence,

$$\begin{aligned}\omega^*(f) &\geqslant \omega(f) - \frac{1}{2}n_{t4}(f) - \frac{1}{2}n_5(f) - \frac{1}{2}n_6(f) \geqslant k - 4 - \frac{1}{2}[n_{t4}(f) + n_5(f) + n_6(f)] \\ &\geqslant \frac{k}{2} - \frac{7}{2} \geqslant 0.\end{aligned}$$

Case 2: $n_c(f) = 0$. According to the discharge rules, the new charge function satisfies the following inequality:

$$\omega^*(f) = \omega(f) = k - 4 > 0.$$

By now, we have proved that $\omega^*(f) \geqslant 0$ for every face in G^*.

The discussion above proves that

$$\sum_{x\in V(G^*)\cup F(G^*)} \omega^*(x) \geqslant 0.$$

This leads to the obvious contradiction

$$-8 = \sum_{x\in V(G^*)\cup F(G^*)} \omega(x) = \sum_{x\in V(G^*)\cup F(G^*)} \omega^*(x) \geqslant 0.$$

Hence, the counterexample does not exist and the proof of Theorem 3.1 is finished. □

In general, we believe that the following holds:

Conjecture 3.9. *Every* 1*-planar graph is* $(2, 2, 2, 0, 0)$*-colorable.*

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