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SATURATION NUMBERS FOR LINEAR FORESTS $P_6 + tP_2$

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Abstract. A graph G is H-saturated if it contains no H as a subgraph, but does contain H after the addition of any edge in the complement of G. The saturation number, $\mathrm{sat}(n, H)$, is the minimum number of edges of a graph in the set of all H-saturated graphs of order n. We determine the saturation number $\mathrm{sat}(n, P_6 + tP_2)$ for $n \geqslant \frac{10}{3}t + 10$ and characterize the extremal graphs for $n > \frac{10}{3}t + 20$.

Keywords: saturation number; saturated graph; linear forest

MSC 2020: 05C35, 05C38

1. Introduction

In this paper we consider only simple graphs. For the terminology and notations we follow the books, see [4], [18]. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. The order and size of a graph G, denoted $|G|$ and $|E(G)|$, are its number of vertices and edges, respectively. For a vertex $v \in V(G)$, $d_G(v)$ is the degree of v and $N_G(v)$ is the neighborhood of v, $N_G[v] = N_G(v) \cup \{v\}$. If the graph G is clear from the context, we omit it as the subscript. Further, $\overline{G}$ and $\delta(G)$ denote the complement and minimum degree of a graph G, respectively. Denote by $G[A]$ the subgraph of G induced by $A \subseteq V(G)$. Furthermore, P_n, K_n and S_n stand for the *path*, *complete graph* and *star* of order n, respectively.

Given graphs G and H, a *copy* of H in G is a subgraph of G that is isomorphic to H. The notation $G + H$ means the *disjoint union* of G and H. Then tG denotes the disjoint union of t copies of G. For graphs we use equality up to isomorphism, so $G = H$ means that G and H are isomorphic.

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A graph G is *H-saturated* if G contains no H as a subgraph but $G+e$ contains H for any edge $e \in E(\overline{G})$. The set of H-saturated graphs of order n is denoted by $\mathrm{SAT}(n,H)$. Further, $\overline{\mathrm{SAT}}(n,H)$ and $\underline{\mathrm{SAT}}(n,H)$ stand for the set of H-saturated graphs with the maximum number of edges and minimum number of edges, respectively. The number of edges in a graph in $\overline{\mathrm{SAT}}(n,H)$ is the Turán number (see [16]), denoted by $\mathrm{ex}(n,H)$. The number of edges in a graph in $\underline{\mathrm{SAT}}(n,H)$ is saturation number, denoted by $\mathrm{sat}(n,H)$.

The first result about the saturation number of a graph was introduced by Erdős, Hajnal, and Moon in [9] in which the authors proved $\mathrm{sat}(n,K_t) = \binom{t-2}{2}+(n-t+2)(t-2)$ and $\underline{\mathrm{SAT}}(n,K_t) = \{K_{t-2} \vee \overline{K}_{n-t+2}\}$, where $\vee$ denotes the *join* of K_{t-2} and $\overline{K}_{n-t+2}$, which is obtained from $K_{t-2}+\overline{K}_{n-t+2}$ by adding edges joining every vertex of K_{t-2} to every vertex of $\overline{K}_{n-t+2}$. In addition to cliques, some of the graphs for which the saturation number is known include unions of cliques (see [2], [13]), complete bipartite graphs (see [3], [8], [15]), forests (see [5], [10]), books (see [6]), small cycles (see [7], [17]) and trees, see [11], [14]. Readers interested in the article can be referred to [12].

In fact, both $\mathrm{sat}(n,tP_2)$ and $\underline{\mathrm{SAT}}(n,tP_2)$ are established by Kászonyi and Tuza in [14]. Chen et al. in [5] focused on the saturation numbers for P_k+tP_2 with $k \geqslant 3$. Fan and Wang in [10] determined the saturation number $\mathrm{sat}(n,P_5+tP_2)$ for $n \geqslant 3t+8$ and characterized the extremal graphs for $n > \frac{1}{5}(18t+76)$, such as the following results.

Theorem 1 ([14])**.** *For $n \geqslant 3t-3$, $\mathrm{sat}(n,tP_2) = 3t-3$ and $\underline{\mathrm{SAT}}(n,tP_2) = \{(t-1)K_3+\overline{K}_{n-3t+3}\}$ or $t=2$, $n=4$, $\underline{\mathrm{SAT}}(4,2P_2) = \{K_3+K_1, S_4\}$.*

Theorem 2 ([5])**.** *For n sufficiently large,*
(1) $\mathrm{sat}(n,P_3+tP_2) = 3t$ *and* $tK_3+\overline{K}_{n-3t} \in \underline{\mathrm{SAT}}(n,P_3+tP_2)$,
(2) $\mathrm{sat}(n,P_4+tP_2) = 3t+7$ *and* $K_5+(t-1)K_3+\overline{K}_{n-3t-2} \in \underline{\mathrm{SAT}}(n,P_4+tP_2)$.

Theorem 3 ([10])**.** *Let n and t be two positive integers with $n \geqslant 3t+8$. Then,*
(1) $\mathrm{sat}(n,P_5+tP_2) = \min\{\lceil\frac{5n-4}{6}\rceil, 3t+12\}$,
(2) $\underline{\mathrm{SAT}}(n,P_5+tP_2) = \{K_6+(t-1)K_3+\overline{K}_{n-3t-3}\}$ *for* $n > \frac{1}{5}(18t+76)$.

In this paper, we further consider saturation numbers of the linear forests P_6+tP_2 with $t \geqslant 1$. The integers t mentioned below all satisfy that $t \geqslant 1$. In addition, it is difficult and complex to determine saturation numbers of the linear forests P_k+tP_2 with $k \geqslant 7$ using the same method.

Theorem 4. *Let n and t be two positive integers with $n \geqslant \frac{10}{3}t+10$. Then,*
(1) $\mathrm{sat}(n,P_6+tP_2) = \min\{n-\lfloor\frac{n}{10}\rfloor, 3t+18\}$,
(2) $\underline{\mathrm{SAT}}(n,P_6+tP_2) = \{K_7+(t-1)K_3+\overline{K}_{n-3t-4}\}$ *for* $n > \frac{10}{3}t+20$.

2. Preliminaries

For an integer $i \geqslant 0$, let $V_i(G) = \{v \in V(G)\colon\ d(v) = i\}$. In other words, $|V_0(G)|$ represents the number of isolated vertices in G. In this section, we list several lemmas and give the result for saturation numbers for the linear forests $P_6 + tP_2$ with $|V_0(G)| \geqslant 2$.

Lemma 5 (Berge-Tutte formula [1]). *For a graph G,*

$$\alpha'(G) = \frac{1}{2}\min\{|G| + |S| - o(G - S)\colon\ S \subseteq V(G)\},$$

where $\alpha'(G)$ is the matching number of G and $o(G - S)$ is the number of odd components of $G - S$.

Lemma 6 ([5]). *Let $k_1, \ldots, k_m \geqslant 2$ be m integers and G be a $(P_{k_1}+P_{k_2}+\ldots+P_{k_m})$-saturated graph. If $d(x) = 2$ and $N(x) = \{u, v\}$, then $uv \in E(G)$.*

Lemma 7 ([10]). *Let G be a $(P_5 + tP_2)$-saturated graph. If $V_0(G) \neq \emptyset$, then $V_1(G) = \emptyset$. Moreover, for any $x \in V(G) \setminus V_0(G)$, we have*

$$N_G[x] \cup \{w\} \subseteq V(H),$$

where H is any copy of $P_5 + tP_2$ in $G + xw$ and w is a vertex in $V_0(G)$.

Using the same method as in Lemma 7, we can get a more general result, which is the content of Lemma 8.

Lemma 8. *Let G be a (P_k+tP_2)-saturated graph with $k \geqslant 2$, $t \geqslant 1$. If $V_0(G) \neq \emptyset$, then $V_1(G) = \emptyset$. Moreover, for any $x \in V(G) \setminus V_0(G)$, we have*

$$N_G[x] \cup \{w\} \subseteq V(H),$$

where H is any copy of $P_k + tP_2$ in $G + xw$ and w is a vertex in $V_0(G)$.

A *book* B_k consists of k triangles sharing one edge. A *k-fan* F_k consists of k triangles sharing one vertex. Moreover, G is *H-free* means G does not contain H as a subgraph.

Lemma 9. *Let G be a connected graph of order $n \geqslant 6$ and $\delta(G) \geqslant 2$. If G satisfies*
(1) *G is P_6-free and G contains P_4 as a subgraph, and*
(2) *if $d(x) = 2$ and $N(x) = \{u, v\}$, then $uv \in E(G)$,*
we get $G = B_i$, $i \geqslant 4$, or $G = F_j$, $j \geqslant 3$, with n odd.

P r o o f. Select the longest path P in G, say $P = x_1, x_2, \ldots, x_k$. As G satisfies the condition (1), we have $4 \leqslant k < 6$. It is easily verified that there exists $x \notin V(P)$ such that $N(x) \cap V(P) \neq \emptyset$, $N(x) \cap \{x_1, x_k\} = \emptyset$. We distinguish two cases.

Case 1: $k = 4$. Observe that if $|N(x) \cap \{x_2, x_3\}| = 2$, then G contains a path x_1, x_2, x, x_3, x_4, contradicting the fact that P is the longest path. We conclude that $|N(x) \cap \{x_2, x_3\}| = 1$. Because of the symmetry of x_2 and x_3, suppose x is adjacent to x_2. Since $\delta(G) \geqslant 2$, there is one vertex $y \in N(x)$ and $y \notin V(P)$. Thus, G contains a path y, x, x_2, x_3, x_4, contradicting $k = 4$.

Case 2: $k = 5$. If x is adjacent to x_2 or x_4, we assert that $N(x) \cap (V(G) \setminus V(P)) = \emptyset$ and $x_3 \notin N(x)$. Otherwise, G contains a path of order at least 6, contradicting $k = 5$. Since $\delta(G) \geqslant 2$, then $d(x) = 2$ and $N(x) = \{x_2, x_4\}$. If $d(x_3) > 2$, $y \in N(x_3) \setminus \{x_2, x_4\}$ (possibly $y = x_1$ or $y = x_5$), G contains a path y, x_3, x_2, x, x_4, x_5 or y, x_3, x_4, x, x_2, x_1, contradicting the fact that P is the longest path. Thus, $d(x_3) = 2$ and $N(x_3) = \{x_2, x_4\}$. As G satisfies the condition (2), x_2 is adjacent to x_4. Clearly, $N(x_1), N(x_5) \subseteq V(P)$. Since $\delta(G) \geqslant 2$, then $N(x_1) = \{x_2, x_4\}$ and $N(x_5) = \{x_2, x_4\}$. Hence, $G[x_1, x_2, x_3, x_4, x_5, x] = B_4$. For any vertex $y \in V(G) \setminus (V(P) \cup \{x\})$, y is adjacent to x_2 or x_4. Using the same method, we have $d(y) = 2$ and $N(y) = \{x_2, x_4\}$. Hence, $G = B_i$, $i \geqslant 4$.

If x is adjacent to x_3, it is easy to check that x is not adjacent to x_2 or x_4. Thus, there is a vertex $y \in N(x)$ and $y \notin V(P)$. Note that P is not the longest path if $N(y) \neq \{x, x_3\}$. If x_1 is adjacent to x_4, G contains a path x_4, x_1, x_2, x_3, x, y, contradicting $k = 5$. Thus, $d(x_1) = 2$ and $N(x_1) = \{x_2, x_3\}$. Similarly, $d(x_5) = 2$ and $N(x_5) = \{x_3, x_4\}$. Now we consider the degrees of vertices x, x_2 and x_4. If any vertex of $\{x, x_2, x_4\}$ has degree greater than two, G has a path of order at least 6. Hence, $G[x_1, x_2, x_3, x_4, x_5, x, y] = F_3$. For any vertex $z \in V(G) \setminus (V(P) \cup \{x, y\})$, z is adjacent to x_3. Using the same method, we have $G = F_j$, $j \geqslant 3$, with n odd. This completes the proof of Lemma 9. □

Theorem 10. *Let $G \in \mathrm{SAT}(n, P_6 + tP_2)$ and $Q = Q_1 + Q_2 + \ldots + Q_k$, where $Q_1, \ldots, Q_k$ are all the nontrivial components of G. If $|Q| \geqslant 2t + 6$, $\delta(Q) \geqslant 2$, $|Q_i| \geqslant 6$ and Q_i is not a book or fan, $1 \leqslant i \leqslant k$, then*

(1) *$G \in \mathrm{SAT}(n, P_4 + (t+1)P_2)$,*

(2) *if $V_0(G) \neq \emptyset$, then $|E(G)| > 3t + 18$.*

P r o o f. (1) Since $G \in \mathrm{SAT}(n, P_6 + tP_2)$, $G + e$ contains $P_6 + tP_2$ for any edge $e \in E(\overline{G})$. It follows that $G + e$ contains $P_4 + (t+1)P_2$ for any edge $e \in E(\overline{G})$.

If $G \notin \mathrm{SAT}(n, P_4 + (t+1)P_2)$, then G contains $P_4 + (t+1)P_2$. Without loss of generality, suppose that Q_1 contains P_4 as a subgraph. Since $|Q_1| \geqslant 6$, $\delta(Q) \geqslant 2$ and Q_1 is not a book or fan, by Lemmas 6 and 9, there exists P_6 in Q_1. Hence, G contains a copy of $P_6 + tP_2$, a contradiction.

(2) Suppose that $|E(G)| \leqslant 3t+18$. By (1), we have $Q \in \mathrm{SAT}(n, P_4+(t+1)P_2)$. Then, $\alpha'(Q) \geqslant t+2$. If $\alpha'(Q) \geqslant t+3$, G must contain a copy of $(t+3)P_2$. Since $\delta(Q) \geqslant 2$ and $|Q_i| \geqslant 6$ $(1 \leqslant i \leqslant k)$, it is clear that Q has a copy of $P_4+(t+1)P_2$, which contradicts $Q \in \mathrm{SAT}(n, P_4+(t+1)P_2)$. So, we have $\alpha'(Q) = t+2$. By Lemma 5, we have

$$t+2 = \frac{1}{2}\min\{|Q|+|X|-o(Q-X)\colon\ X \subseteq V(Q)\}.$$

Choose a subset $Y \subseteq V(Q)$ such that

$$t+2 = \frac{1}{2}(|Q|+|Y|-o(Q-Y)).$$

Let $Q-Y = Q'_1+Q'_2+\ldots+Q'_p$. We have two claims.

Claim 1: $Q[Y \cup V(Q'_i)]$ is a complete graph for $i \in \{1,2,\ldots,p\}$. To the contrary, suppose that there exist two vertices $u, v \in Y \cup V(Q'_i)$ such that $uv \notin E(Q)$. Let $Q' = Q+uv$. Since Q is $(P_4+(t+1)P_2)$-saturated, $\alpha'(Q') \geqslant t+3$. On the other hand, observe that $|Q'| = |Q|$ and $o(Q'-Y) = o(Q-Y)$. By Lemma 5, we have

$$\alpha'(Q') \leqslant t+2 = \frac{1}{2}(|Q'|+|Y|-o(Q'-Y)),$$

a contradiction.

Claim 2: $Y \neq \emptyset$. Suppose that $Y = \emptyset$. By Claim 1, $Q'_1,\ldots,Q'_p$ are all complete graphs of order at least 6. Hence, $\delta(Q) \geqslant 5$ and

$$2|E(Q)| = \sum_{x \in V(Q)} d_Q(x) = \sum_{j=1}^{p} |Q'_j|(|Q'_j|-1) \geqslant 5|Q| + |Q'_i|(|Q'_i|-6), \quad 1 \leqslant i \leqslant p.$$

Since $|Q| \geqslant 2t+6$ and $|E(Q)| = |E(G)| \leqslant 3t+18$, we have $|Q| = 2t+6$, $t = 1$ and $|Q'_i| = 6$ for $1 \leqslant i \leqslant p$. Thus, $8 = |Q| = 6p$, a contradiction. This completes the proof of Claim 2.

Let $x \in Y$ and $w \in V_0(G)$. By Lemma 8, we have $N_Q[x] \cup \{w\} \subseteq V(H)$, where H is a copy of P_6+tP_2 in $G+xw$. Hence, $|N_Q[x] \cup \{w\}| \leqslant |V(H)| = 2t+6$. On the other hand, by Claim 1, $|N_Q[x] \cup \{w\}| = |Q|+1 \geqslant 2t+6+1 = 2t+7$, a contradiction. This completes the proof of Theorem 10. □

Theorem 11. *Let $G \in \mathrm{SAT}(n, P_6+tP_2)$ with $n \geqslant 3t+6$. If $|V_0(G)| \geqslant 2$ and $|E(G)| \leqslant 3t+18$, then $|E(G)| = 3t+18$ and $G = K_7+(t-1)K_3+\overline{K}_{n-3t-4}$.*

P r o o f. Since $|V_0(G)| \geqslant 2$, $V_1(G) = \emptyset$ by Lemma 8. Note that all the components of order 3, 4 or 5 in G are complete. Let

$$G = G' + t_3K_3 + t_4K_4 + t_5K_5 + B + F,$$

where t_k is the number of components of G with order k, $k \in \{3,4,5\}$, B is the graph consisting of all the components B_i, $i \geqslant 4$, and F is the graph consisting of all the components F_j, $j \geqslant 3$. We denote B_c and F_c the number of B_i, $i \geqslant 4$, and F_j, $j \geqslant 3$, respectively. Since $|B_i| \geqslant 6$, we have $|B| \geqslant 6B_c$.

Clearly, $|V_0(G')| = |V_0(G)| \geqslant 2$. Note that joining two isolated vertices in $V_0(G')$ in G, we have a copy of $P_6 + tP_2$. Thus, G' contains P_6. As $G \in \mathrm{SAT}(n, P_6 + tP_2)$, we have $t_3 + 2t_4 + 2t_5 + 2B_c + \frac{1}{2}(|F| - F_c) \leqslant t - 1$. Let $t' = t - t_3 - 2t_4 - 2t_5 - 2B_c - \frac{1}{2}(|F| - F_c)$. Then, $t' \geqslant 1$. Since $G \in \mathrm{SAT}(n, P_6 + tP_2)$, we have $G' \in \mathrm{SAT}(n', P_6 + t'P_2)$, where $n' = n - 3t_3 - 4t_4 - 5t_5 - |B| - |F|$.

Consider the graph Q' obtained from G' by deleting all trivial components. Clearly, every component of Q' has order at least 6 and is neither book nor fan. Note that $\delta(Q') \geqslant 2$ and $G' \in \mathrm{SAT}(n, P_6 + t'P_2)$ with $V_0(G') \neq \emptyset$. Since

$$\begin{aligned} |E(G')| &= |E(G)| - 3t_3 - 6t_4 - 10t_5 - (2|B| - 3B_c) - \frac{3(|F| - F_c)}{2} \\ &\leqslant 3t' + 18 - 4t_5 - (2|B| - 9B_c) \leqslant 3t' + 18, \end{aligned}$$

by Theorem 10 we have $|Q'| \leqslant 2t' + 5$. Note that joining two non-adjacent vertices in Q', there is no copy of $P_6 + t'P_2$ in G'. Then Q' is a complete graph. As $|V_0(G')| \neq \emptyset$, $|Q'| \geqslant 2t' + 5$ and hence $Q' = K_{2t'+5}$. Moreover, $|E(Q')| = |E(G')| \leqslant 3t' + 18$. It follows that $t' = 1$ and $Q' = K_7$.

Since $G' = K_7 + (n' - 7)K_1$ with $|E(G')| = 3t' + 18$, we have $t_5 = 0$ and $|B| = 0$. Consequently,

$$G = K_7 + (n' - 7)K_1 + t_3K_3 + t_4K_4 + F.$$

Note that G contains P_6. It is easy to verify that if $t_4 > 0$, joining the vertices in K_4 with the vertices in K_7 does not increase the number of paths P_2 in G. Similarly, if $|F| > 0$, joining two non-adjacent vertices in F_j, $j \geqslant 3$, also does not increase the number of paths P_2 in G. Therefore, $t_4 = 0$, $|F| = 0$ and $t_3 = t - 1$. Hence, $G = K_7 + (t - 1)K_3 + \overline{K}_{n-3t-4}$. This completes the proof of Theorem 11. $\square$

So far, we have proved that when $n \geqslant 3t + 6$ and $|V_0(G)| \geqslant 2$, $\mathrm{sat}(n, P_6 + tP_2) = 3t + 18$ and $\underline{\mathrm{SAT}}(n, P_6 + tP_2) = \{K_7 + (t - 1)K_3 + \overline{K}_{n-3t-4}\}$.

3. Proof of Theorem 4

For a graph H, using the definition and notation in [10], $\mathrm{SAT}^*(n,H)$ and $\mathrm{sat}^*(n,H)$ denote the set of H-saturated graphs G of order n with $|V_0(G)| = 0$ and the minimum number of edges in a graph in $\mathrm{SAT}^*(n,H)$, respectively.

Let T be the tree of order 10 as shown in Figure 1. Let T^* be the tree of order $n = 10 + r$, $0 \leqslant r \leqslant 9$, obtained from $S_{4+\lfloor \frac{r}{3} \rfloor}$ by attaching two leaves to each of the $2 + \lfloor \frac{r}{3} \rfloor$ leaves of $S_{4+\lfloor \frac{r}{3} \rfloor}$ and attaching $n - (4 + \lfloor \frac{r}{3} \rfloor) - 2(2 + \lfloor \frac{r}{3} \rfloor)$ leaves to the remaining leaf of $S_{4+\lfloor \frac{r}{3} \rfloor}$.

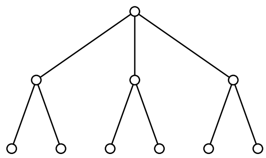

Figure 1. Tree T.

Lemma 12. *Let G be a $(P_6 + tP_2)$-saturated graph. If T_1 and T_2 are tree components of G, then $|T_1| \geqslant 10$, $|T_2| \geqslant 10$ and at least one of T_1 and T_2 contains T as a subgraph.*

P r o o f. Let v_i be a leaf of T_i with $N(v_i) = \{u_i\}$, $i \in \{1,2\}$. Since G is (P_6+tP_2)-saturated, $G + u_1u_2$ contains a copy of $P_6 + tP_2$. Let H be the copy. If u_1u_2 is not in P_6 of H, then $H - u_1u_2 + u_1v_1$ is a copy of $P_6 + tP_2$ in G, contrary to G is $(P_6 + tP_2)$-saturated. Thus, u_1u_2 is in P_6 of H. It follows that $T_1 + T_2$ contains P_4 starting from u_i for some $i = 1$ or 2 or $T_1 + T_2$ contains P_3 starting from u_i for $i = 1$ and $i = 2$. Now we discuss these two cases separately.

Case 1: $T_1 + T_2$ contains P_4 starting from u_i for some $i = 1$ or 2. Without loss of generality, assume $P_4 = u_1, x, y, z$. Clearly, $T_1[\{v_1, u_1, x, y, z\}]$ contains P_5. Let M be the copy of tP_2 in H. Note that any vertex of $\{u_1, v_1, u_2, v_2, x, y, z\}$ is not in M. As T_1 is a tree, by Lemma 6, T_1 has no vertex of degree 2. So, u_1, x and y all have neighbors not in $\{v_1, u_1, x, y, z\}$. Now we show that for any vertex $u_1' \in N(u_1) \setminus \{v_1, x\}$, it holds $d(u_1') = 1$. If $d(u_1') > 1$ and $u_1' \in V(M)$ then u_1' has a neighbor u_1'' such that $u_1'u_1''$ belongs to M. Clearly, $T_1[\{u_1'', u_1', u_1, x, y, z\}]$ contains P_6. Observe that tP_2 is in $M - u_1'u_1'' + u_2v_2$. Hence, G contains $P_6 + tP_2$, a contradiction. If $d(u_1') > 1$ and $u_1' \notin V(M)$, we also have that G contains $P_6 + tP_2$. Thus, $d(u_1') = 1$. Using the same method, for any vertex $y' \in N(y) \setminus \{x, z\}$, we have $d(y') = 1$. And the proof of $d(z) = 1$ is similar to the above, so we omit it. Assume that x has no neighbor x' with $d(x') > 1$, where x' is not equal to u_1 or y. The additional edge $e = u_1y$ in G does not increase the number of paths P_2 and T_1 does not contain P_6, contradicting $G \in \mathrm{SAT}(n, P_6 + tP_2)$. Hence, x has at least one neighbor of degree greater than 1. So, T_1 contains T.

Next we show that for any vertex $x' \in N(x) \setminus \{u_1, y\}$ with $d(x') > 1$, $N(x') \setminus \{x\}$ are leaves. We distinguish two cases.

Subcase 1.1: $x' \notin V(M)$. If there exists $x'' \in N(x') \setminus \{x\}$ with $d(x'') > 1$, we have two cases. One is $x'' \in V(M)$. Let x''' be the neighbor of x'' such that $x''x'''$ belongs to M. Then we have that $T_1[\{x''', x'', x', x, y, z\}]$ contains P_6 and uses one edge in M. By replacing $x''x'''$ with u_1v_1, we get a copy of $P_6 + tP_2$ in G. Another case is $x'' \notin V(M)$. Whether x''' belongs to $V(M)$ or not, using the same method, we always have that G contains $P_6 + tP_2$, a contradiction.

Subcase 1.2.: $x' \in V(M)$. If there exists $x'' \in N(x') \setminus \{x\}$ with $d(x'') > 1$, we can use the method of Subcase 1.1 to check that T_1 contains a copy of P_6 by using at most two edges of M. By replacing these two edges with u_1v_1 (or yz) and u_2v_2, we get a copy of $P_6 + tP_2$ in G, contrary to the claim that G is a $(P_6 + tP_2)$-saturated graph.

Recall that v_2 is a vertex of T_2 with $N(v_2) = \{u_2\}$. Since G is (P_6+tP_2)-saturated, there is $P_6 + tP_2$ in $G + xu_2$ containing the edge xu_2. Let H' be the copy and M' be the copy of tP_2 in H'. If xu_2 is not in P_6, by replacing xu_2 with u_2v_2, we have $P_6 + tP_2$ in G, a contradiction. Thus, xu_2 is in the copy of P_6. Since T_1 does not contain a path of length 3 with x as its endpoint, T_2 contains a path P' of length 2 with u_2 as its endpoint. Hence, $T_2[V(P') \cup \{v_2\}]$ contains a path P of length 3, $P = v_2, u_2, w_1, w_2$.

Now we show that T_2 contains T or $|T_2| \geqslant 10$. If $d(w_2) \neq 1$, it is easy to prove that there is one vertex in $N(w_2) \setminus \{w_1\}$ which is not in M'. Hence, T_2 contains P_4 starting from u_2. Using the same proof of the claim that T_1 contains P_4 starting from u_1, we have that T_2 contains T as a subgraph. Next suppose that $d(w_2) = 1$ and $N(w_2) = \{w_1\}$. As T_2 is a tree, by Lemma 6, T_2 has no vertex of degree 2. So, u_2 and w_1 both have neighbors not in $V(P)$. Let $U_2 = \{u_2' \in N(u_2) \setminus V(P) \colon d(u_2') > 1\}$ and $W_1 = \{w_1' \in N(w_1) \setminus V(P) \colon d(w_1') > 1\}$. Since G is a $(P_6 + tP_2)$-saturated graph, then $U_2 \cup W_1 \neq \emptyset$. If $U_2 \neq \emptyset$ and $W_1 \neq \emptyset$, by Lemma 6, we have $|T_2| \geqslant 10$. Obviously, if $|U_2| \geqslant 2$ or $|W_1| \geqslant 2$, we have that T_2 contains T. It remains the case of $U_2 = \emptyset$ and $|W_1| = 1$ (the proof of the case of $|U_2| = 1$ and $W_1 = \emptyset$ is similar). Let $w_1' \in W_1$. Joining w_1' with u_2 does not increase the numbers of paths P_2 and P_6, which contradicts that $G \in \mathrm{SAT}(n, P_6 + tP_2)$.

Case 2: $T_1 + T_2$ contains P_3 starting from u_i for $i = 1$ and $i = 2$. Denote by $P_3 = u_1, x, y$ in T_1 and $P_3 = u_2, w_1, w_2$ in T_2. Next, we only prove that T_1 contains T and T_2 contains T is similar. Clearly, $T_1[\{v_1, u_1, x, y\}]$ contains P_4. Let M'' be the copy of tP_2 in H. Note that any vertex of $\{u_1, v_1, u_2, v_2, x, y, w_1, w_2\}$ is not in M''. Then T_2 contains two copies of P_2 not in M''. For two cases $d(y) \neq 1$ and $d(y) = 1$, we can use a proof similar to Claim 1 to prove. So we omit it. This completes the proof of Lemma 12. □

Theorem 13. *For $n \geqslant \frac{10}{3}t + 10$, $\mathrm{sat}^*(n, P_6 + tP_2) = n - \lfloor \frac{n}{10} \rfloor$.*

Proof. Suppose $\mathrm{sat}^*(n, P_6 + tP_2) < n - \lfloor \frac{n}{10} \rfloor$, then there is a graph $G \in \mathrm{SAT}^*(n, P_6 + tP_2)$ with $|E(G)| < n - \lfloor \frac{n}{10} \rfloor$. Let $G = R + (T_1 + \ldots + T_k)$, where $T_1, \ldots, T_k$ are all the tree components of G. Hence,

$$|E(G)| = |E(R)| + \sum_{i=1}^{k} |E(T_i)| \geqslant |R| + \sum_{i=1}^{k} (|T_i| - 1) = |G| - k = n - k.$$

Since $|E(G)| < n - \lfloor \frac{n}{10} \rfloor$, we have $k > \lfloor \frac{n}{10} \rfloor$. As $k \geqslant 2$, by Lemma 12, $|T_i| \geqslant 10$ for $1 \leqslant i \leqslant k$. Hence, $n \geqslant 10k$, contrary to $k > \lfloor \frac{n}{10} \rfloor$. It follows that $\mathrm{sat}^*(n, P_6 + tP_2) \geqslant n - \lfloor \frac{n}{10} \rfloor$.

On the other hand, set $n = 10q + r$, where $q = \lfloor \frac{n}{10} \rfloor$, $0 \leqslant r \leqslant 9$. Since $n \geqslant \frac{10}{3}t + 10$, we have $10q + r \geqslant \frac{10}{3}t + 10$. Then

$$t \leqslant 3q + \left\lfloor \frac{3r}{10} \right\rfloor - 3 \leqslant 3q + \left\lfloor \frac{r}{3} \right\rfloor - 3.$$

Consider the graph

$$G^* = (q - 1)T + T^*.$$

Obviously G^* contains no copy of P_6 and $G^* + e$ contains a copy of $P_6 + (3q + \lfloor \frac{r}{3} \rfloor - 3)P_2$ for any $e \in E(\overline{G^*})$. This implies that G^* is $(P_6 + tP_2)$-saturated. Since $|V_0(G^*)| = 0$, $G^* \in \mathrm{SAT}^*(n, P_6 + tP_2)$. Hence, $\mathrm{sat}^*(n, P_6 + tP_2) = E(G^*) = n - \lfloor \frac{n}{10} \rfloor$. This completes the proof of Theorem 13. □

Finally, we show the proof of Theorem 4.

Proof of Theorem 4. (1) Suppose G is $(P_6 + tP_2)$-saturated. If $|V_0(G)| = 1$, by Lemma 8, $V_1(G) = \emptyset$. By degree-sum formula,

$$2|E(G)| = \sum_{x \in V(G)} d(x) \geqslant 2(|G| - 1).$$

For $n \geqslant \frac{10}{3}t + 10$, $|E(G)| \geqslant |G| - 1 = n - 1 > n - \lfloor \frac{n}{10} \rfloor \geqslant \min\{n - \lfloor \frac{n}{10} \rfloor, 3t + 18\}$. If $|V_0(G)| = 0$ or $|V_0(G)| \geqslant 2$, by Theorems 11 and 13, we have $\mathrm{sat}(n, P_6 + tP_2) = \min\{n - \lfloor \frac{n}{10} \rfloor, 3t + 18\}$ for $n \geqslant \frac{10}{3}t + 10$. This completes the proof.

(2) By $n > \frac{10}{3}t + 20$, we have $n - \lfloor \frac{n}{10} \rfloor > 3t + 18$. Consequently, $\mathrm{sat}(n, P_6 + tP_2) = 3t + 18$. Let $G \in \mathrm{SAT}(n, P_6 + tP_2)$ with $|E(G)| = 3t + 18$. By Theorem 13, we have $G \notin \mathrm{SAT}^*(n, P_6 + tP_2)$ and hence $|V_0(G)| \neq 0$. If $|V_0(G)| = 1$, we obtain that

$$|E(G)| \geqslant |G| - 1 > \frac{10t}{3} + 20 - 1 = \frac{10t}{3} + 19 > 3t + 18,$$

a contradiction. Thus, $|V_0(G)| \geqslant 2$. By Theorem 11, we have $\underline{\mathrm{SAT}}(n, P_6 + tP_2) = \{K_7 + (t - 1)K_3 + \overline{K}_{n-3t-4}\}$. This completes the proof of Theorem 4. □

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