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TIME REGULARITY OF GENERALIZED NAVIER-STOKES EQUATION WITH $p(x,t)$-POWER LAW

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Abstract. We show time regularity of weak solutions for unsteady motion equations of generalized Newtonian fluids described by $p(x,t)$-power law for $p(x,t)\geqslant(3n+2)/(n+2)$, $n\geqslant 2$, by using a higher integrability property and fractional difference method. Moreover, as its application we prove that every weak solution to the problem becomes a local in time strong solution and that it is unique.

Keywords: weak solution; time regularity; generalized Newtonian fluid, variable exponent

MSC 2020: 35D30, 35D35, 35K92, 76A05

1. Introduction

In this paper, we consider the initial-boundary value problem

$$\begin{cases} \partial_t u - \operatorname{div} S(p(x,t),\mathcal{D}u) + (u\cdot\nabla)u + \nabla\pi = -\operatorname{div} F, & \text{in } \Omega\times(0,T),\\ \operatorname{div} u = 0, & \text{in } \Omega\times(0,T),\\ u(0)=u_0, & \text{on } \Omega,\\ u=0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

where u is the velocity, π the pressure, F a prescribed symmetric $n\times n$ matrix-valued function and u_0 an initial data, $\mathcal{D}u := \frac{1}{2}(\nabla u + (\nabla u)^{\top})$ and Ω a bounded domain in $\mathbb{R}^n$. We assume that the extra stress tensor $S(p(x,t),\mathcal{D}u)$ satisfies the followings

$$|\nabla_A S(p(x,t),A)| \leqslant c_*(1+|A|^2)^{(p(x,t)-2)/2}, \tag{1.2}$$

$$\sum_{k,l,i,j=1}^{n} \frac{\partial S_{ij}(p(x,t),A)}{\partial A_{kl}} B_{ij}B_{kl} \geqslant c_*^{-1}(1+|A|^2)^{(p(x,t)-2)/2}|B|^2, \tag{1.3}$$

$$|\partial_p S(p(x,t),A)| \leqslant c_*(1+|A|^2)^{(p(x,t)-1)/2}\ln(1+|A|^2), \tag{1.4}$$

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where $c_* > 1$, (x,t) is a space-time point in $\Omega\times(0,T)\subset\mathbb{R}^{n+1}$, $p(x,t) > 1$ a prescribed function and A, B are the real symmetric $n\times n$ matrices.

The prototype of $S(p(x,t),\mathcal{D}u)$ satisfying (1.2)–(1.4) is the following:

$$S(p(x,t),\mathcal{D}u) = (1+|\mathcal{D}u|^2)^{(p(x,t)-2)/2}\mathcal{D}u.$$

System (1.1) arises from electrorheological flow (see [29]), thermo-rheological flow (see [4]), chemically reacting non-Newtonian flows (see [13]) and flows of generalized Newtonian fluids with concentration dependent power-law, see [1], [12]. We refer to [32], [33], [34], [35], [36] for recent results on the existence and regularity for a steady version of system (1.1). Note that regularity problems connected with the $p(x)$-Laplacian have been studied in the early 20th century by Acerbi-Mingione, see e.g., [2], [3].

The existence of weak solutions to system (1.1) was proved in [18] for $p = \text{const.} > 2n/(n+2)$ and in [15], [28], [29], [41] for $p(x,t)\neq \text{const.}$ under various assumptions on $p(x,t)$ and boundary conditions.

Time regularity of weak solution to the parabolic and Navier-Stokes equations is a well-known topic, see [19], [39], [40].

As mentioned in [10], for time regularity of weak solution to system (1.1), the main difficulty is that when using formally $\partial^2_{tt}u$ as a test function of the system, we cannot estimate the term $\int_0^T\int_\Omega |\nabla u||\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t$ in terms of known *a priori estimate*, at least without some additional information such as $p\geqslant\frac12(n+2)$. To overcome the difficulty, the authors in [10] proposed an idea that it would be possible to get some information about fractional derivatives in time of any weak solution using known information about the solution. Thus, this method needs iteration for improving of time regularity. Based on the idea, they proved in 3D that if for $p=\text{const.}>\frac{11}{5}$ the extra tensor $S(p,\cdot)$ satisfies (1.3), (1.2) and in addition

$$|S(x,t_1,A)-S(x,t_2,A)|\leqslant |t_1-t_2|^{\kappa}(1+|A|)^{p-2}|A|,\quad \exists\,\kappa\in(0,1], \tag{1.5}$$

and $\operatorname{div}F\in N^{\kappa,2}_{\rm loc}(I;L^2(\Omega))$, then every local in time weak solution u satisfies

$$u\in N^{\kappa,\infty}_{\rm loc}(I;L^2(\Omega))\cap N^{\kappa,2}_{\rm loc}(I;W^{1,2}(\Omega))\cap N^{2\kappa/p,p}_{\rm loc}(I;W^{1,p}(\Omega)), \tag{1.6}$$

where the spaces $N^{\kappa,q}_{\rm loc}(I;X)$ are Nikolskii ones; for its definition see Subsection 2.2.

Using this method with slight difference, the authors in [11] also showed for $p=\text{const.}\geqslant\frac{11}{5}$ in 3D that if $\operatorname{div}F\in N^{1/p',p'}(I,\mathcal{V}_p'(\Omega))$, then

$$u\in N^{1/2,\infty}(0,T;L^2(\Omega))\cap N^{1/2,2}(0,T;\mathcal{V}_2(\Omega))\cap N^{1/p,p}(0,T;\mathcal{V}_p(\Omega)) \tag{1.7}$$

provided that $u_0\in\mathcal{V}_p(\Omega)$. For the definition of $\mathcal{V}_p$, see Subsection 2.2.

This method was applied by several researchers to other problems. The authors in [14] used a similar iterative approach to get the interior regularity of time derivatives of local weak solution for a symmetric parabolic system of p-Laplace type. Novelty of the paper is to estimate mis-matching lower-order terms stemming from using a cut-off function in space due to localization. The method of differences in time was also used in [20] to get regularity for time derivatives of a weak solution for a symmetric p-Laplace system or models for non-Newtonian fluids like (1.1) with $p \equiv$ const. without convective term. In [21], [22] a similar approach was used to establish time regularity and uniqueness for Cahn-Hilliard-Navier-Stokes system with shear dependent viscosity, see also [27].

Time regularity of weak solution is not only independent of interest but used to get other interesting results.

The first example is weak-strong regularity. By combining (1.6) and $W^{2,q}$-regularity for steady problem corresponding to (1.1) with constant p, Beirão da Veiga, Kaplický and Růžička in [5] proved that every local in time weak solution to (1.1) with $p =$ const. $> (3n+2)/(n+2)$, $n \geqslant 2$ becomes a local in time strong solution.

The second example is uniqueness of weak solution. In [10] for $p =$ const. $> \frac{11}{5}$ and $n = 3$ there was proved uniqueness of local in time weak solution in the sense of trajectories in the sense that if u, v are weak solutions and $u = v$ on an interval $[t_1, t_2] \subset [0, T)$, then $u \equiv v$ on $[t_1, T)$. In [11] this was generalized to global in time one for $p =$ const. $\geqslant \frac{11}{5}$ provided that $u(0) \in \mathcal{V}_p(\Omega)$.

Besides, this is used in [23] to obtain full regularity of systems similar to (1.1) with $2 \leqslant p =$ const. < 4 in 2D and in [9] to compute bounds of dimension of attractor to system (1.1) with $p =$ const. $> \frac{12}{5}$ in 3D.

To our knowledge, there seems to be no work on time regularity of weak solution to problem (1.1) with nonconstant $p(x,t)$. So we first show time regularity like (1.6) and (1.7) of a weak solution to the problem in n-D, $n \geqslant 2$. This is achieved by combining the method in [10], [11], which is based on iteration for gradually improving time regularity of convective term, with a higher integrability condition.

Here it is worth noting that condition (1.5) from [10] is not sufficient in our problem in which $S(p(x,t),\cdot)$ depends on $p(x,t)$. Due to $p(x,t)$-dependence of $S(p(x,t),\mathcal{D}u)$, it seems to be impossible to show time regularity of any weak solution to problem (1.1) with $p \neq$ const. by means of time difference method without further assumption. In fact, using time difference yields

$$\begin{aligned} S(p(x,t+h),&\mathcal{D}u(x,t+h)) - S(p(x,t),\mathcal{D}u(x,t)) \\ &= (S(p(x,t+h),\mathcal{D}u(x,t+h)) - S(p(x,t+h),\mathcal{D}u(x,t))) \\ &\quad + (S(p(x,t+h),\mathcal{D}u(x,t)) - S(p(x,t),\mathcal{D}u(x,t))). \end{aligned}$$

The first term on the right-hand side of the previous identity is controlled by (1.2) or (1.3) as the case $p = \text{const.}$ The main obstacle arises from the second term on the right-hand side. By (1.4) the term is estimated as follows:

$$\begin{aligned}
&|S(p(x,t+h),\mathcal{D}u(x,t)) - S(p(x,t),\mathcal{D}u(x,t))|\\
&\qquad = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} S(p(x,r(t+h)+(1-r)t),\mathcal{D}u(x,t))\,\mathrm{d}r\\
&\qquad \leqslant ch\int_0^1 (1+|\mathcal{D}u(x,t)|^2)^{(p(x,r(t+h)+(1-r)t)-1)/2}\ln(1+|\mathcal{D}u(x,t)|^2)\,\mathrm{d}r.
\end{aligned}$$

However, this cannot be estimated by an *a priori estimate* (e.g., see (3.8) below) on a weak solution due to the logarithmic term. So, in order to estimate the term, we need higher integrability of weak solutions to problem (1.1). For more comprehension, see the estimate on the term J_2 in Subsection 3.4.

The rest of the argument is similar to the ones in [10], [11]. The difference is that we have to introduce localization argument such that the oscillation of $p(x,t)$ is small enough to use a higher integrability result and that we consider $n(\geqslant 2)$-D domain instead of 3D. In particular, for $n = 2$ we have to use slightly different methods from [10], [11] since in that papers the condition $p > 2$ was basically used, while the lower bound on $p(x,t)$ in this paper is $p = (3n+2)/(n+2) = 2$ for $n = 2$. For more detail, see Lemma 3.9 and Subsection 4.2.

As an application of the time regularity results we show uniqueness of weak solution similar to [10], [11] for $p(x,t) \geqslant (3n+2)/(n+2)$. Also we show weak-strong regularity in the sense that every weak solution to (1.1) becomes local in time strong. This is achieved by combining the above and space-$W^{2,q}$-regularity results from [35] for a steady problem corresponding to (1.1). Uniqueness of weak solution to system (1.1) for $p = \text{const.} \geqslant \frac{1}{2}(n+2)$ and $u_0 \in L^2(\Omega)$, $\operatorname{div} u_0 = 0$ is well-known, see [24], [25]. For regularity results of the problem with constant p we refer to [7], [17], [25], [26]. Růžička in [29] proved the existence of unique global strong solution for Dirichlet boundary condition when

$$\frac{9}{4} \leqslant p_- \leqslant p(x,t) \leqslant p_+ \leqslant \frac{3(3-p_+)}{2(5-2p_+)} \quad \text{and} \quad u_0 \in W^{1,p(x,0)}(\Omega), \quad \operatorname{div} u_0 = 0.$$

For a 3D-space periodic boundary condition, short time existence of unique strong solution for large data to system (1.1) is proved in [30] under the restriction

$$\frac{3}{2} < p_- \leqslant p(x,t) \leqslant p_+ \leqslant 2,$$

and in [17] under the restriction

$$\frac{7}{5} < p_- \leqslant p(x,t) \leqslant p_+ \leqslant 2.$$

The paper is organized as follows. In Section 2, we give the main result, notations and some properties of Nikolskii spaces and outline the strategy of the proof of the main result. Sections 3–6 are devoted to the proof of the main results. More precisely, in Section 3 we show auxiliary results such as an *a priori* estimate, improvement of regularity for time derivative of velocity, time regularity of the convective term and time regularity of velocity. In Section 4 we give the proof of the first statement of Theorem 2.1 with $\widehat{\kappa} < 1$, in Section 5, the second one and finally in Section 6 the proof of the first statement of Theorem 2.1 with $\widehat{\kappa} = 1$. In particular, the argument is divided into two categories: $p > (3n+2)/(n+2)$ and $p = (3n+2)/(n+2)$. Section 7 is devoted to the proof of uniqueness of weak solution and of the weak-strong regularity.

2. The main result and preliminaries

2.1. Notations. We denote the space-time points in $\Omega \times (0,T) \subset \mathbb{R}^{n+1}$ by $z = (x,t)$ and employ a shorthand notation $\mathrm{d}z = \mathrm{d}x\,\mathrm{d}t$. From now on, let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and $\Omega_T := \Omega \times (0,T)$.

For $n \times n$-matrices F, H, denote $F : H = \sum\limits_{i,j=1}^{n} F_{ij}H_{ij}$, $|F| \equiv (F : F)^{1/2}$. For vectors a and b, we denote their tensor product by $a \otimes b := (a_i b_j)_{n\times n}$ and their symmetric tensor product by $a \odot b := \frac{1}{2}(a \otimes b + (a \otimes b)^{\top})$.

In this paper, $A \subset\subset B$ means that A is bounded and $\overline{A} \subset B$.

We use universal constants c and C, the dependence on certain parameters of which is expressed, for example, by $c = c(n,p)$.

For $p \in L^{\infty}(\Omega_T)$, $p \geqslant 1$, define

$$p_- := \operatorname*{ess\,inf}_{\Omega_T} p(z), \quad p_+ := \operatorname*{ess\,sup}_{\Omega_T} p(z), \tag{2.1}$$

$$p^* := \begin{cases} \dfrac{np}{n-p} & \text{if } p = \text{const.} < n, \\ \forall\, q \in (1,\infty) & \text{if } p = \text{const.} = n, \\ \infty & \text{if } p = \text{const.} > n, \end{cases} \qquad p'(z) := \begin{cases} \dfrac{p(z)}{p(z)-1} & \text{if } p(z) > 1, \\ \infty, & \text{otherwise.} \end{cases}$$

We do not use different notation for scalar, vector- and tensor-valued functions (or spaces) as far as there will be no misunderstandings.

2.2. Function spaces. By $L^q(0,T;X)$ we denote the space of all Bochner measurable functions $f\colon (0,T) \mapsto X$ such that

$$\|f\|_{L^q(0,T;X)} := \left(\int_0^T \|f(t)\|_X^q \,\mathrm{d}t\right)^{1/q} < \infty \quad \text{if } 1 \leqslant q < \infty,$$
$$\|f\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\|_X < \infty \quad \text{if } q = \infty.$$

Next we recall the definition of Nikolskii space. Let $I \subset \mathbb{R}$ be an arbitrary time interval. For $h > 0$ we set

$$I_h := \{t \in I\colon\ t + h \in I\}, \quad d^h f(t) := f(t+h) - f(t), \quad t \in I_h.$$

Then for $q \in [1, \infty]$, $s \in (0, 1)$ we define Nikolskii space by

$$N^{s,q}(I; X) := \{f \in L^q(I; X)\colon\ \|f\|_{s,q} < \infty\},$$
$$\|f\|_{s,q} := \|f\|_{L^q(I;X)} + \sup_{h>0} h^{-s} \|d^h f\|_{L^q(I_h;X)}.$$

For general $\sigma = k+s$, where $k \in \mathbb{N}$ and $s \in (0,1)$, we define $N^{\sigma,q}(I;X)$ as the space of all functions with $(\mathrm{d}/\mathrm{d}t)^j f \in L^q(I;X)$ for $j = 0, \ldots, k-1$ and $(\mathrm{d}/\mathrm{d}t)^k f \in N^{s,q}(I;X)$. So the space with $s = 1$ is equivalent to Sobolev space $W^{1,q}(I;X)$.

Let us recall some properties of the Nikolskii space to be used later.

Proposition 2.1 ([6], [31]). *Let $s \in (0,1)$, $q \geqslant 1$. Then*

$$N^{s,q}(I;X) \hookrightarrow L^r(I;X) \qquad \text{if } \frac{1}{r} > \frac{1}{q} - s \geqslant 0, \tag{2.2}$$

$$N^{s,q}(I;X) \hookrightarrow C^{0,\alpha}(I;X) \quad \text{if } \alpha = s - \frac{1}{q} > 0. \tag{2.3}$$

Proposition 2.2 ([10], Lemma 2.3). *Let H be a Hilbert space and X a separable Banach space continuously and densely embedded into H, and X^* a dual space of X. Then for $s, r \geqslant 0$*

$$N^{s,q}(I;X) \cap N^{r,q'}(I;X^*) \hookrightarrow N^{(s+r)/2,2}(I;H). \tag{2.4}$$

For $p \in L^\infty(\Omega_T)$, $p \geqslant 1$, the variable exponent Lebesgue space $L^{p(z)}(\Omega_T)$ is defined by

$$L^{p(z)}(\Omega_T) := \left\{f\colon\ \Omega \to \mathbb{R}\colon\ \varrho_{p(z)}(f) := \int_{\Omega_T} |f|^{p(z)}\,\mathrm{d}z < \infty\right\}$$

endowed with the norm $\|f\|_{p(z),\Omega_T} := \inf\{\lambda > 0\colon\ \varrho_{p(z)}(f/\lambda) \leqslant 1\}$. We denote the usual Lebesgue and Sobolev spaces by $(L^p(\Omega), \|\cdot\|_{p,\Omega})$, $(W^{k,p}(\Omega), \|\cdot\|_{k,p,\Omega})$, respectively, for constant p. We define $W_0^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Let us define

$$\mathcal{H}_p(\Omega) := \{f \in L^p(\Omega)\colon\ f \cdot \nu|_{\partial\Omega} = 0,\ \operatorname{div} f = 0\},$$
$$\mathcal{V}_p(\Omega) := \{f \in W_0^{1,p}(\Omega)\colon\ \operatorname{div} f = 0\}.$$

Let $\mathcal{V}_p'(\Omega)$ be dual to $\mathcal{V}_p(\Omega)$ and denote the dual product between $\mathcal{V}_p(\Omega)$ and $\mathcal{V}_p'(\Omega)$ by $\langle\cdot,\cdot\rangle_{1,p}$.

2.3. The main results.

Definition 2.1. Assume that $S(\cdot,\cdot)$ satisfies (1.2) and (1.3) with $2n/(n+2) \leqslant p(z) < \infty$. Let $F \in L^1(\Omega_T)$ and $u(0) \in \mathcal{H}_2(\Omega)$. A vector-valued function $u \in L^\infty(0,T;\mathcal{H}_2(\Omega))$ with u, $\nabla u \in L^{p(z)}(\Omega_T)$ is called a *weak solution to* (1.1) if the identity

$$
\begin{aligned}
(2.5)\qquad -\int_{\Omega_T} u\cdot\partial_t\varphi\,\mathrm{d}z + \int_{\Omega_T} S(p(z),\mathcal{D}u) : \mathcal{D}\varphi\,\mathrm{d}z - \int_{\Omega_T}(u\otimes u) : \mathcal{D}\varphi\,\mathrm{d}z \\
= \int_{\Omega_T} F : \mathcal{D}\varphi\,\mathrm{d}z + \int_\Omega u_0\cdot\varphi(0)\,\mathrm{d}x
\end{aligned}
$$

holds for all $\varphi \in C^\infty(\Omega_T)$ with $\operatorname{div}\varphi = 0$ and $\operatorname{supp}\varphi \subset \Omega\times[0,T)$.

Before we state the main result, we give the assumptions on $p(z)$ for concise statement of the result: let function $p\colon \overline{\Omega}_T \mapsto (1,\infty)$ be such that

$$
(2.6)\qquad |p(x_1,t_1) - p(x_2,t_2)| \leqslant L(|t_1-t_2| + |x_1-x_2|)
$$

and in addition

$$
(2.7)\qquad p(z)\begin{cases} \geqslant \dfrac{3n+2}{n+2} & \text{if } n = 2,3,4,\\ > p_{1,1} & \text{if } n \geqslant 5,\end{cases}
$$

where

$$
(2.8)\qquad p_{1,1} := \frac{n^2+6n}{3(2n+4)} + \mathcal{R} + \frac{\mathcal{T}}{\mathcal{R}},
$$

$$
\mathcal{R} := \Big(\mathcal{P} - \frac{n}{2} + \Big(\Big(\mathcal{P} - \frac{n}{2} + \mathcal{Q}\Big)^2 - \mathcal{T}^3\Big)^{1/2} + \mathcal{Q}\Big)^{1/3}, \quad \mathcal{P} := \frac{(n^2+6n)^3}{27(2n+4)^3},
$$

$$
\mathcal{Q} := \frac{(-2n^2+6n)(n^2+6n)}{6(2n+4)^2}, \quad \mathcal{T} := \frac{-2n^2+6n}{3(2n+4)} + \frac{(n^2+6n)^2}{9(2n+4)^2}.
$$

Remark 2.1. In Subsection 4.1, we will show that for $n \geqslant 5$

$$
(2.9)\qquad \frac{3n+2}{n+2} < \frac{n}{2} < p_{1,1} < \frac{n+2}{2}.
$$

The main results are the followings.

Theorem 2.1. *Assume that*

(A1) *Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with $C^{0,1}$-boundary.*

(A2) *The extra tensor $S(p(z),\mathcal{D}u)$ satisfies* (1.2)–(1.4).

(A3) *A function u is a weak solution to system* (1.1) *such that for some $\delta > 0$*

$$
(2.10)\qquad \nabla u \in L^{p(z)(1+\delta)}(\Omega_T).
$$

(A4) *A function* $p\colon \overline{\Omega}_T \mapsto (1,\infty)$ *satisfies* (2.6), (2.7) *and in addition, for all* $t\in[0,T)$ *and* δ *from* (2.10)

(2.11)
$$\max_{x\in\Omega} p(z) - \min_{x\in\Omega} p(z) \leqslant \begin{cases} \dfrac{1}{2}\delta \min\limits_{x\in\Omega} p(z) \min\Big\{1, \dfrac{\min_{x\in\Omega} p(z)-2}{2}\Big\} & \text{if } \min\limits_{x\in\Omega} p(z) > 2,\\ \dfrac{1}{2}\delta \min\limits_{x\in\Omega} p(z) & \text{if } \min\limits_{x\in\Omega} p(z) = 2,\end{cases}$$
$$\max_{x\in\Omega} p'(z) \leqslant \min_{x\in\Omega} p'(z)(1+\delta).$$

Then we have:

(1) *If* $F\in N^{\widehat{\kappa},2}_{\rm loc}(0,T;L^2(\Omega))$ *with some* $\widehat{\kappa}\in(0,1]$, *then*

$$u\in N^{\widehat{\kappa},\infty}_{\rm loc}(0,T;\mathcal{H}_2(\Omega))\cap N^{\widehat{\kappa},2}_{\rm loc}(0,T;\mathcal{V}_2(\Omega)),\quad V_{p(z)}(\mathcal{D}u)\in N^{\widehat{\kappa},2}_{\rm loc}(0,T;L^2(\Omega)),\tag{2.12}$$

where
$$V_{p(z)}(\mathcal{D}u) := (1+|\mathcal{D}u|^2)^{(p(z)-2)/4}\mathcal{D}u.$$

(2) *If* $u_0\in\mathcal{V}_{p_+}(\Omega)$ *and* $F\in N^{\widehat{\kappa},2}(0,T;L^2(\Omega))$ *with some* $\widehat{\kappa}\in(0,\frac12]$ *and in addition,*

$$\sup_{h\in(0,T)}\frac{1}{h}\int_0^h \|F\|_2^2\,{\rm d}t<\infty\quad \textit{with } \widehat{\kappa}=\frac{1}{2},$$

then

$$u\in N^{\widehat{\tau},\infty}(0,T;\mathcal{H}_2(\Omega))\cap N^{\widehat{\tau},2}(0,T;\mathcal{V}_2(\Omega)),\quad V_{p(z)}(\mathcal{D}u)\in N^{\widehat{\tau},2}(0,T;L^2(\Omega)),\tag{2.13}$$

where $\widehat{\tau}$ *is arbitrarily close to* $\widehat{\kappa}$ *if* $\widehat{\kappa}<\frac12$ *and equals to* $\frac12$ *if* $\widehat{\kappa}=\frac12$.

Remark 2.2. The appearance of $\delta>0$ in Theorem 2.1, i.e., condition (2.10), is needed only for $p(z)\neq$ const. Hence, if $n=3$, then Theorem 2.1 coincides with the results from [10], [11].

Remark 2.3. It is not yet known how to get condition (2.10). In fact, in [37] we showed that the condition holds under some assumptions, but there is a serious mistake in the proof.

It is worth noting that in [16], [38], $C^{1,\alpha}$-continuity of solution to system (1.1) in 2D with periodic or Dirichlet boundary condition is shown. We also note that in [11], a higher integrability in time for system (1.1) with constant p is proved.

In particular, we would like to emphasise that recently the authors in [8] have shown a fractional (time) differentiability of the $p(x,t)$-Laplacian system. However, it is impossible to apply their method to our problem. The reason is that the nonlocal characteristic of pressure for unsteady flow problem prevents us to use localization argument for space used in the paper.

For system (1.1) even with constant p it seems to be hard or even impossible to get (2.10) by means of reverse Hölder's inequality though it is well known for steady versions, see e.g., [2], [35]. This is the reason why the nonlocal characteristic of pressure for unsteady flow problem also prevents us to use localization argument for space, which is essential in obtaining a reverse Hölder's inequality, see [29], Chapter 4, [38]. Within our knowledge, if we consider suitable weak solution for system (1.1), then it seems to be possible for us to get (2.10) via reverse Hölder's inequality.

Remark 2.4. To simplify the calculation, we need condition (2.11) that the oscillation of $p(z)$ on Ω is not very large. In fact, if not, it is necessary to introduce a localization argument on the space domain Ω such that the oscillation of $p(z)$ on Ω is small enough, for example, such as (3.9). However, the localization is impossible due to the same reason as in Remark 2.3.

Remark 2.5. Theorem 2.1 also holds for $F \in N^{2\widehat{\kappa}/p'_-,p'_-}(0,T;L^{p'_-}(\Omega))$ with $\widehat{\kappa} \in (0,\frac{1}{2}]$. Moreover, if $p = \text{const.}$, $u_0 \in \mathcal{V}_p(\Omega)$ and $F \in N^{1/p',p'}(0,T;L^{p'}(\Omega))$, $\sup\limits_{h\in(0,T)} h^{-1}\int_0^h \|F\|_{p'}^{p'}\,\mathrm{d}t < \infty$, then

$$
\begin{aligned}
(2.14) \qquad u &\in N^{1/2,\infty}(0,T;\mathcal{H}_2(\Omega)) \cap N^{1/2,2}(0,T;\mathcal{V}_2(\Omega)),\\
V_{p(z)}(\mathcal{D}u) &\in N^{1/2,2}(0,T;L^2(\Omega)),
\end{aligned}
$$

which is Corollary 3.2 from [11].

Corollary 2.1. *Let $n = 3$. Assume that the assumptions* (A2)–(A4) *in Theorem* 2.1 *hold and Ω is a bounded domain in $\mathbb{R}^3$ with $C^{2,1}$-boundary and $\widehat{\kappa} = 1$. In addition, let $\operatorname{div} F \in L^\infty(0,T;L^2(\Omega))$. Then for case* (1) *every weak solution u to problem* (1.1) *becomes a strong one in a short time and moreover*

$$
(2.15) \qquad u \in W^{1,2}(t_1,t_2;W^{2,2}_{\mathrm{loc}}(\Omega)), \quad V_{p(z)}(\mathcal{D}u) \in W^{1,2}((t_1,t_2)\times\Omega),
$$

$$
(2.16) \qquad u \in W^{1,2}(t_1,t_2;W^{1,\overline{p}(x,\cdot)}(\Omega) \cap W^{2,\overline{r}(x,\cdot)}(\Omega))
$$

for every small enough interval $(t_1,t_2) \subset (0,T)$ and

$$
\overline{p}(x,\cdot) := 2(p(x,\cdot)+1) - \mu, \quad \overline{r}(x,\cdot) := \frac{2\overline{p}(x,\cdot)}{p(x,\cdot)+\overline{p}(x,\cdot)-2},
$$

where μ is an arbitrary small positive real number if $p(x,\cdot)$ is a function in x and $\mu \equiv 0$ if $p(x,\cdot)$ is a constant in x.

Corollary 2.2. *Assume that all the assumptions of Theorem* 2.1 *except the ones of F hold. Assume that $F \in N^{\tau,2}(0,T;L^2(\Omega))$ with*

$$\tau > \tau_{\rm uni} := \max\Big\{0, \frac{n+2-2p_-}{4}\Big\}.$$

Let u_1, u_2 be two weak solutions to problem (1.1) *with $u_1(0) = u_2(0) \in \mathcal{V}_{p_+(1+\delta)}(\Omega)$. Then $u_1 \equiv u_2$ on $(0,T) \times \Omega$.*

If u_1, u_2 are two weak solutions to problem (1.1) *that coincide on some $[t_1,t_2] \times \Omega$ with $0 < t_1 < t_2 < T$, then $u_1 \equiv u_2$ on $[t_1,T) \times \Omega$.*

2.4. Strategy for the proof of Theorem 2.1. As mentioned in Section 1, our method is based on the following idea: it would be possible for us to use iteration for gradually improving regularity of fractional derivatives in time of the convective term and so of any weak solution satisfying the higher integrability condition (2.10).

To begin with, we introduce the localization argument on a time interval on which the oscillation of $p(z)$ is small enough to use the higher integrability condition. This enables us to show $u \in N^{1/2,2}(J;L^2(\Omega))$ (see (3.12)), which is a starting point.

Let us denote $p_1 = \inf\limits_{\Omega\times J} p(z)$ for a small $J = (t_1,t_2)$. Let I be such that $I \subset\subset J$ and vary from step to step.

Let us outline the strategy for the proof of the first statement of Theorem 2.1 with $p(z) > (3n+2)/(n+2)$.

We note that the weak solution u to system (1.1) satisfies condition (2.17) provided that $\sigma = \frac{1}{2}$, $\gamma = 0$, see Remark 3.1 for more details. Starting with it, we show that if for some $\sigma \in [\frac{1}{2},1)$, $\gamma \in [0,1)$,

$$u \in N^{\sigma,2}(J;\mathcal{H}_2(\Omega)) \cap L^{p_1/(1-\gamma)}(J;\mathcal{V}_{p_1}(\Omega)), \tag{2.17}$$

then for $\tau(\gamma) = \min\{\hat\kappa, \kappa(\gamma)\} \leqslant \frac{1}{2}$ with some $\kappa(\gamma) > 0$ (defined by (3.35)),

$$u \in N^{\tau(\gamma),\infty}(I;\mathcal{H}_2(\Omega)) \cap N^{\tau(\gamma),2}(I;\mathcal{V}_2(\Omega)) \cap N^{2\tau(\gamma)/p_1,p_1}(I;\mathcal{V}_{p_1}(\Omega)), \tag{2.18}$$

see Lemma 3.5. To prove (2.18) we obtain an explicit formula about fractional derivatives in time of any weak solution from the first equations of (1.1) and higher integrability condition (2.10), see Subsection 3.4, while showing: time regularity of the convective term under condition (2.17), see Subsection 3.5.

We next use the boot-strap argument. By the validity of (2.17) with $\sigma = \frac{1}{2}$, $\gamma = 0$, (2.12) holds if $\kappa(0) > \hat\kappa$. If $\kappa(0) \leqslant \hat\kappa$, then by (2.18) and Nikolskii embedding (2.2), the exponent γ from (2.17) is improved from 0 to $2\kappa(0) > 0$. This enables us to get (2.18) with $\gamma = 2\kappa(0)$. Furthermore, we can iterate the process above. If $\widehat\kappa \leqslant \kappa(\gamma_0)$ for the limit bound, $\kappa(\gamma_0)$, of improving $\kappa(\gamma)$ in the iterative process, then we arrive at (2.12) after a finite number of iterations.

Thus, it remains to prove (2.12) when $\widehat{\kappa} > \kappa(\gamma_0)$. To this end, we first show that under condition (2.7),

$$u \in L^{2p_1/(2p_1-n)}(I, \mathcal{V}_{p_1}(\Omega)). \tag{2.19}$$

For every weak solution u to problem (1.1) this holds true for $p_1 \geqslant \frac{1}{2}(n+2)$ since $2p_1/(2p_1-n) \leqslant p_1$. Note that the condition $p_1 = \frac{1}{2}(n+2)$ is the critical bound for uniqueness of the weak solution. But for $p_1 < \frac{1}{2}(n+2)$, this is not trivial. So we show (2.19) for $p_1 < \frac{1}{2}(n+2)$ under condition (2.7) by iteratively applying (2.18) and Nikolskii embedding (2.2).

Once we have got (2.19), we next prove that if for some $\sigma \in (0,1)$

$$u \in N^{\sigma,2}(I; \mathcal{H}_2(\Omega)) \cap L^{2p_1/(2p_1-n)}(I; \mathcal{V}_{p_1}(\Omega)), \tag{2.20}$$

then for $\tau := \min\{\sigma, \widehat{\kappa}\}$

$$u \in N^{\tau,\infty}(I; \mathcal{H}_2(\Omega)) \cap N^{\tau,2}(I; \mathcal{V}_2(\Omega)) \cap N^{2\tau/p_1,p_1}(I; \mathcal{V}_{p_1}(\Omega)), \tag{2.21}$$

see Lemma 3.7. Since $u \in N^{1/2,2}(J; L^2(\Omega))$, we can see from (2.21) that if $\widehat{\kappa} < \frac{1}{2}$, then there holds (2.12), while if $\widehat{\kappa} \geqslant \frac{1}{2}$, then

$$u \in N^{1/2,\infty}(I, \mathcal{H}_2(\Omega)) \cap N^{1/p_1,p_1}(I, \mathcal{V}_{p_1}(\Omega)), \quad V_{p(z)}(\mathcal{D}u) \in N^{1/2,2}(I, L^2(\Omega)). \tag{2.22}$$

To proceed the argument, in this point we show that if for some $\sigma \in (0,1)$

$$u \in N^{\sigma,\infty}(I; \mathcal{H}_2(\Omega)) \cap N^{2\sigma/p_1,p_1}(I; \mathcal{V}_{p_1}(\Omega)), \quad V_{p(z)}(\mathcal{D}u) \in N^{\sigma,2}(I; L^2(\Omega)), \tag{2.23}$$

then for some $\tau(\sigma) > 0$

$$\partial_t u \in N^{\tau(\sigma),p_1'}(I; \mathcal{V}_{p_1}'(\Omega)), \tag{2.24}$$

see Lemma 3.8. Since condition (2.23) with $\sigma = \frac{1}{2}$ is satisfied by (2.22), we get $\partial_t u \in N^{\tau(1/2),p_1'}(I; \mathcal{V}_{p_1}'(\Omega))$, which together with $u \in N^{1/p_1,p_1}(I, \mathcal{V}_{p_1}(\Omega))$ (see (2.22)) and Proposition 2.2 implies

$$u \in N^{1/2(1+\tau(1/2)+1/p_1),2}(I, \mathcal{H}_2(\Omega)).$$

It is clear that $\frac{1}{2}(1+\tau(\frac{1}{2})+1/p_1) > \frac{1}{2}$ and hence, this gives the improved regularity of u better than $u \in N^{1/2,2}(I; L^2(\Omega))$. Thus, for $\sigma = \frac{1}{2}(1+\tau(\frac{1}{2})+p_1^{-1})$, condition (2.20) holds and in turn we can get (2.21) and (2.24) if $\frac{1}{2}(1+\tau(\frac{1}{2})+1/p_1) < \widehat{\kappa}$.

Furthermore, by (2.24), (2.21) and Proposition 2.2 we have that if $u \in N^{\sigma,2}(I, \mathcal{H}_2(\Omega))$ with $\sigma \in [\frac{1}{2}, \widehat{\kappa}]$, then

$$u \in N^{1/2(1+\tau(\sigma)+2\sigma/p_1),2}(I, \mathcal{H}_2(\Omega)).$$

Based on this iterative argument, we can get (2.12) after a finite number of iterations by showing that

$$\frac{1}{2}\Big(1 + \tau(\sigma) + \frac{2\sigma}{p_1}\Big) - \sigma \geqslant c(p_1, \delta, n) > 0.$$

For the case $p_1 = (3n+2)/(n+2)$, the idea is the same as above but the proofs of (2.18) and (2.19) are different from before, relying essentially on the condition $p_1 = (3n+2)/(n+2)$. For more detail, see Lemmas 3.4, 3.6, 3.9 and Subsection 4.2.

The second statement in Theorem 2.1 is proved by the same method with the help of Lemma 5.1.

3. Auxiliary results

In this section, we provide some auxiliary tools to be used for the proof of the main result.

Since we are interested in time regularity, in the rest of the paper, we will omit the notation of dependence on the space variable, such as $p(t)$, $u(t)$, $S(p(t), \mathcal{D}u(t))$, $F(t)$, etc, if not differently specified or if something else is not clear from the context.

In this section, constants $\varepsilon_0 > 0$ are small enough and vary from line to line.

For simplicity we omit similar parts as in [10], [11] and emphasize different ones.

3.1. Some inequalities. We denote $f \cong g$ if there exist two positive constants c_1, c_2 such that $c_1 f \leqslant g \leqslant c_2 f$. We begin with introduction of the following properties on $S(p(t), A)$ which will be often used later.

Proposition 3.1 ([30]). *Let A, B be the real symmetric $n \times n$ matrices. Assume that S satisfies* (1.2), (1.3). *Then the following holds:*

$$|S(p(t), A)| \leqslant c(|A|^{p(t)-1} + 1), \tag{3.1}$$

$$S(p(t), A)\colon\ A \geqslant c(|A|^{p(t)} - 1), \tag{3.2}$$

$$(S(p(t), A) - S(p(t), B)) : (A - B) \cong (1 + |A|^2 + |B|^2)^{(p(t)-2)/2}|A - B|^2, \tag{3.3}$$

$$|S(p(t), A) - S(p(t), B)| \cong (1 + |A|^2 + |B|^2)^{(p(t)-2)/2}|A - B|, \tag{3.4}$$

$$|V_{p(t)}(A) - V_{p(t)}(B)| \cong (1 + |A|^2 + |B|^2)^{(p(t)-2)/4}|A - B|, \tag{3.5}$$

$$|V_{p(t)}(A)|^2 \cong |A|^{p(t)} + |A|^2, \quad \textit{if } p(t) \geqslant 2. \tag{3.6}$$

Proposition 3.2 ([25], Chapter 5, Lemma 4.35). *Let* $u \in W_0^{1,2}(\Omega)$ *and* $q \in [2, 2n/(n-2)]$ *for* $n \geqslant 3$ *and* $q \in (2, \infty)$ *for* $n = 2$. *Then there exists* $c > 0$ *such that*

$$\|u\|_q \leqslant c\|u\|_2^{\alpha}\|\nabla u\|_2^{1-\alpha} \tag{3.7}$$

with $\alpha := (2n - q(n-2))/(2q)$.

Indeed, this was proved for periodic boundary condition but is extended to our case without any difficulty.

Let $t \in [0, T)$. Then by standard method we obtain *a priori* estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + \int_\Omega (|V_{p(t)}(\mathcal{D}u(t))|^2 + |\mathcal{D}u(t)|^{p(t)} + |\mathcal{D}u(t)|^2)\,\mathrm{d}x \leqslant c\int_\Omega |F(t)|^{p'(t)}\,\mathrm{d}x + C. \tag{3.8}$$

3.2. Localization. Let $p \equiv$ const. Then since $u \in L^p(0, T; \mathcal{V}_p(\Omega))$ and $\partial_t u \in L^{p'}(0, T; \mathcal{V}_p'(\Omega))$ for every weak solution u to problem (1.1), it follows from Proposition 2.2 that $u \in N^{1/2,2}(0, T; L^2(\Omega))$.

But since $u \in L^{p_-}(0, T; \mathcal{V}_{p_-}(\Omega))$ and $\partial_t u \in L^{p'_+}(0, T; \mathcal{V}'_{p_+}(\Omega))$ (see [37]) for a function $p(t)$, it is not clear whether $u \in N^{1/2,2}(0, T; L^2(\Omega))$, which is a starting point for the proof of Theorem 2.1. Thus, to proceed the argument we introduce a suitable localization technique based on the higher integrability condition (2.10).

To begin with, we denote

$$p_1 := \inf_{\Omega\times(t_1,t_2)} p(z), \quad p_2 := \sup_{\Omega\times(t_1,t_2)} p(z) \quad \text{for } x \in \Omega,\ 0 \leqslant t_1 < t_2 < T$$

and fix an interval (t_1, t_2) such that for δ from (2.10)

$$p_1' \leqslant p_2'(1+\delta), \quad p_2 - p_1 \leqslant \begin{cases} \frac{1}{2}\delta p_1 \min\left\{1, \frac{p_1 - 2}{2}\right\} & \text{if } p_1 > 2, \\ \frac{1}{2}\delta p_1 & \text{if } p_1 = 2. \end{cases} \tag{3.9}$$

Due to (2.11), this means that oscillation of $p(z)$ on time interval (t_1, t_2) is not very large. This localization is always possible due to (2.6). In particular, by (3.9) and (2.10) we have

$$\nabla u \in L^{p_2}(\Omega \times (t_1, t_2)). \tag{3.10}$$

The following lemma is used in the proof of time regularity for time derivative of velocity.

Lemma 3.1. *Let $0 \leqslant t_1 < t_2 < T$. Let assumptions* (A1)–(A3) *in Theorem* 2.1 *hold and $p(z) \geqslant (3n+2)/(n+2)$. Then for almost all $t \in (t_1, t_2)$ there is a constant C_t, depending only on n, p_-, p_+, c_* and (essentially bounded) $\|u(t)\|_2$ such that*

$$\|\partial_t u(t)\|_{\mathcal{V}'_{p_1}(\Omega)} \leqslant C_t(1 + \||\mathcal{D}u|^{p(t)-1}\|_{p'_1} + \|u(t)\|_{1,p_1}^{p_1-1} + \|F(t)\|_{p'_1}). \tag{3.11}$$

Remark 3.1. In [37] we showed that $\partial_t u \in L^{p'_2}(0,T;\mathcal{V}'_{p_2}(\Omega))$ and

$$\|\partial_t u(t)\|_{\mathcal{V}'_{p_2}(\Omega)} \leqslant C_t(1 + \||\mathcal{D}u|^{p(t)-1}\|_{p'_2} + \|u(t)\|_{1,p_1}^{p_1-1} + \|F(t)\|_{p'_2}).$$

So Lemma 3.1 is its generalization in the sense of improvement of time regularity. In particular, since $u \in L^{p_1}(t_1,t_2;\mathcal{V}_{p_1}(\Omega))$ by the definition and $\partial_t u \in L^{p'_1}(0,T;\mathcal{V}'_{p_1}(\Omega))$ by Lemma 3.1, we obtain that by Proposition 2.2

$$u \in N^{1/2,2}(t_1,t_2;L^2(\Omega)). \tag{3.12}$$

P r o o f. The proof is very similar to [37]. The difference lies only in using the higher integrability condition (2.10) instead of *a priori estimate* (3.8). Multiplying the first equation in (1.1) by $\varphi \in \mathcal{V}_{p_1}(\Omega)$ with $\|\varphi\|_{\mathcal{V}_{p_1}(\Omega)} \leqslant 1$ and integrating over Ω, we obtain that for almost all $t \in (t_1, t_2)$

$$\begin{aligned} \langle \partial_t u(t), \varphi\rangle_{1,p_1} &= -\int_\Omega S(p(t),\mathcal{D}u) : \mathcal{D}\varphi \,\mathrm{d}x + \int_\Omega (u\otimes u) : \mathcal{D}\varphi \,\mathrm{d}x + \int_\Omega F : \mathcal{D}\varphi \,\mathrm{d}x \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.13}$$

It is easy to see that for almost all $t \in (t_1, t_2)$

$$|I_3| \leqslant \int_\Omega |F||\mathcal{D}\varphi| \,\mathrm{d}x \leqslant \|F(t)\|_{p'_1}. \tag{3.14}$$

By (3.1), we have

$$|I_1| \leqslant \int_\Omega (1+|\mathcal{D}u|)^{p(t)-1}|\mathcal{D}\varphi| \,\mathrm{d}x \leqslant \|(1+|\mathcal{D}u|)^{p(t)-1}\|_{p'_1}. \tag{3.15}$$

Here we note that by (2.10) and the first inequality of (3.9), the right-hand side (RHS) of (3.15) is bounded for almost all $t \in (t_1, t_2)$.

By the same argument as in [37], Lemma 3.1 the term I_2 can be estimated as follows:

$$|I_2| \leqslant c(1 + \|u(t)\|_{1,p_1}^{p_1-1}). \tag{3.16}$$

Gathering estimates (3.13)–(3.16), we get (3.11). □

In the rest of the paper, we denote $J := (t_1, t_2)$ and $J_h := (t_1, t_2 - h)$ for simplicity. From now on, constants c, C depend on p_-, p_+, n, c_* and in addition, C on u_0 and F via *a priori estimate* (3.8) as well as on $\|u\|_{L^{p(z)(1+\delta)}(\Omega_T)}$ via (3.10).

3.3. A fundamental estimate.

Lemma 3.2. *Let the assumptions* (A2), (A3) *and* (2.6) *of Theorem* 2.1 *hold and* $p(z) \geqslant (3n+2)/(n+2)$. *Then for almost all* $t \in J_h$ *and all* $h \in (0, t_2 - t_1)$ *we have*

(3.17)
$$\frac{\mathrm{d}}{\mathrm{d}t}\|d^h u(t)\|_2^2 + \int_\Omega (|d^h V_{p(t)}(\mathcal{D}u(t))|^2 + |d^h \mathcal{D}u(t)|^{p_1} + |d^h \mathcal{D}u(t)|^2)\,\mathrm{d}x \leqslant ch^2 \int_\Omega (|\mathcal{D}u(t)|^{p(t)(1+\delta)} + 1)\,\mathrm{d}x + c\int_\Omega (|d^h F(t)|^2 + |(d^h u \cdot \nabla)u \cdot d^h u(t)|)\,\mathrm{d}x.$$

P r o o f. Let $t \in J_h$ be such that

$$u(t) \in W^{1,p(t)(1+\delta)}(\Omega), \quad u(t+h) \in W^{1,p(t+h)(1+\delta)}(\Omega). \tag{3.18}$$

Recalling that $\partial_t u \in L^{p_1'}(t_1, t_2; \mathcal{V}_{p_1}'(\Omega))$ by Lemma 3.1, testing (1.1) with $d^{-h}(d^h u(t))$ at times t and $t+h$ and subtracting from each other yield that for all $h \in (0, t_2 - t_1)$

$$\begin{aligned}\langle \partial_t d^h u(t), d^h u(t)\rangle_{1,p_1} &= -\int_\Omega d^h S(p(t), \mathcal{D}u(t)) : d^h \mathcal{D}u(t)\,\mathrm{d}x \\ &\quad - \int_\Omega d^h F(t) : d^h \mathcal{D}u(t)\,\mathrm{d}x - \int_\Omega d^h[(u\cdot\nabla)u]\cdot d^h u(t)\,\mathrm{d}x \\ &=: I_4 + I_5 + I_6.\end{aligned} \tag{3.19}$$

Let us estimate all the terms on the RHS of (3.19).

To begin with, we introduce the following decomposition:

$$\begin{aligned}-d^h S(p(t), \mathcal{D}u(t)) &= S(p(t), \mathcal{D}u(t)) - S(p(t), \mathcal{D}u(t+h)) \\ &\quad + S(p(t), \mathcal{D}u(t+h)) - S(p(t+h), \mathcal{D}u(t+h)) \\ &=: J_1 + J_2.\end{aligned} \tag{3.20}$$

Then by (3.3) and (3.5)

$$-\int_\Omega J_1 : d^h \mathcal{D}u(t)\,\mathrm{d}x \geqslant c\int_\Omega |V_{p(t)}(\mathcal{D}u(t+h)) - V_{p(t)}(\mathcal{D}u(t))|^2\,\mathrm{d}x =: \mathcal{A}(t). \tag{3.21}$$

Here we note that $V_{p(t)}(\mathcal{D}u(t+h)) = (1 + |\mathcal{D}u(t+h)|^2)^{(p(t)-2)/4}\mathcal{D}u(t+h)$.

On the other hand, by (1.4),

$$\begin{aligned}J_2 &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} S(p(r(t+h) + (1-r)t), \mathcal{D}u(t+h))\,\mathrm{d}r \\ &\leqslant ch\int_0^1 (1 + |\mathcal{D}u(t+h)|^2)^{(p(r(t+h)+(1-r)t)-1)/2}\ln(1 + |\mathcal{D}u(t+h)|^2)\,\mathrm{d}r.\end{aligned} \tag{3.22}$$

Now fix $\beta > 0$ such that $5\beta = \delta p_1$ and let L be a constant from (2.6). Then taking h with $h \leqslant \beta/L$ yields that by (2.6),

$$p(r(t+h)+(1-r)t) \leqslant p(t) + Lh \leqslant p(t) + \beta \quad \forall\, r \in [0,1]. \tag{3.23}$$

Using the known formula

$$\ln a \leqslant c_\beta a^{\beta/2}, \quad \text{for } a \geqslant 1, \tag{3.24}$$

we obtain

$$\begin{aligned} J_2 &\overset{(3.22)}{\leqslant} ch\int_0^1 (1+|\mathcal{D}u(t+h)|^2)^{(p(r(t+h)+(1-r)t)-1+\beta)/2}\,\mathrm{d}r \\ &\overset{(3.23)}{\leqslant} ch[(1+|\mathcal{D}u(t+h)|^2)^{(p(t)-1+2\beta)/2}] \\ &\leqslant ch[(1+|\mathcal{D}u(t+h)|^2+|\mathcal{D}u(t)|^2)^{(p(t)-1+2\beta)/2}]. \end{aligned} \tag{3.25}$$

Hence, the decomposition $p(t)-1+2\beta = \frac{1}{2}(p(t)-2)+\frac{1}{2}(p(t)+4\beta)$ and the choice $5\beta = \delta p_1$ lead us to

$$\begin{aligned} &\int_\Omega J_2 : d^h\mathcal{D}u(t)\,\mathrm{d}x \\ &\quad\overset{(3.25)}{\leqslant} ch\int_\Omega (1+|\mathcal{D}u(t+h)|^2+|\mathcal{D}u(t)|^2)^{(p(t)-1+2\beta)/2}|d^h\mathcal{D}u|\,\mathrm{d}x \\ &\quad\leqslant \frac{\mathcal{A}(t)}{8} + ch^2\int_\Omega (|\mathcal{D}u(t+h)|^{p(t)+4\beta}+|\mathcal{D}u(t)|^{p(t)+4\beta}+1)\,\mathrm{d}x \\ &\quad\overset{(3.23)}{\leqslant} \frac{\mathcal{A}(t)}{8} + ch^2\int_\Omega (|\mathcal{D}u(t+h)|^{p(t+h)(1+\delta)}+|\mathcal{D}u(t)|^{p(t)(1+\delta)}+1)\,\mathrm{d}x. \end{aligned} \tag{3.26}$$

Connecting (3.21) and (3.26) via the definition of I_4, we obtain

$$\begin{aligned} \mathcal{A}(t) &\leqslant cI_4 + ch^2\int_\Omega (|\mathcal{D}u(t+h)|^{p(t+h)(1+\delta)}+|\mathcal{D}u(t)|^{p(t)(1+\delta)}+1)\,\mathrm{d}x \\ &\overset{(3.9)}{\leqslant} cI_4 + ch^2\int_\Omega (|\mathcal{D}u(t)|^{p(t)(1+\delta)}+1)\,\mathrm{d}x. \end{aligned} \tag{3.27}$$

Here we note that the second term on the RHS of (3.27) has meaning by (3.18).

Now we want to explain how to derive the second term on the left-hand side (LHS) of (3.17) from $\mathcal{A}(t)$. By (3.5) and the condition $p_1 \geqslant 2$,

$$\begin{aligned} &|V_{p(t)}(\mathcal{D}u(t+h)) - V_{p(t)}(\mathcal{D}u(t))|^2 \\ &\qquad\geqslant (1+|\mathcal{D}u(t+h)|^2+|\mathcal{D}u(t)|^2)^{(p(t)-2)/2}|\mathcal{D}u(t+h)-\mathcal{D}u(t)|^2 \\ &\qquad\geqslant (1+|\mathcal{D}u(t+h)|^2+|\mathcal{D}u(t)|^2)^{(p_1-2)/2}|d^h\mathcal{D}u(t)|^2 \\ &\qquad\overset{(3.6)}{\geqslant} |d^h\mathcal{D}u(t)|^2 + |d^h\mathcal{D}u(t)|^{p_1}. \end{aligned} \tag{3.28}$$

On the other hand, it is clear that

$$
\begin{aligned}
|d^h V_{p(t)}(\mathcal{D}u(t))|^2 \leqslant\ & 2|V_{p(t)}(\mathcal{D}u(t+h)) - V_{p(t)}(\mathcal{D}u(t))|^2 \\
& + \underbrace{2|V_{p(t+h)}(\mathcal{D}u(t+h)) - V_{p(t)}(\mathcal{D}u(t+h))|^2}_{=:J_3}.
\end{aligned} \tag{3.29}
$$

Since

$$
\begin{aligned}
J_3 =& 2|\mathcal{D}u(t+h)|^2 \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}(1+|\mathcal{D}u(t+h)|^2)^{(p(r(t+h)+(1-r)t)-2)/4}\,\mathrm{d}r \right|^2 \\
\leqslant& ch^2|\mathcal{D}u(t+h)|^2 \int_0^1 (1+|\mathcal{D}u(t+h)|^2)^{(p(r(t+h)+(1-r)t)-2)/2} \ln^2(1+|\mathcal{D}u(t+h)|^2)\,\mathrm{d}r,
\end{aligned}
$$

we follow the same argument as in the estimate of J_2 to get

$$
\int_\Omega J_3\,\mathrm{d}x \leqslant ch^2 \int_\Omega (|\mathcal{D}u(t)|^{p(t)(1+\delta)} + 1)\,\mathrm{d}x. \tag{3.30}
$$

Thus, connecting (3.28), (3.29) and (3.30) with (3.27) yields that

$$
\begin{aligned}
\int_\Omega (|d^h V_{p(t)}(\mathcal{D}u)|^2 + |d^h\mathcal{D}u(t)|^2 + |d^h\mathcal{D}u(t)|^{p_1})\,\mathrm{d}x \\
\leqslant cI_4 + ch^2 \int_\Omega (|\mathcal{D}u(t)|^{p(t)(1+\delta)} + 1)\,\mathrm{d}x.
\end{aligned} \tag{3.31}
$$

Next we use Young's inequality to get

$$
I_5 \leqslant c\int_\Omega |d^h F(t)|^2\,\mathrm{d}x + \frac{1}{4}\int_\Omega |d^h\mathcal{D}u(t)|^2\,\mathrm{d}x. \tag{3.32}
$$

By $\operatorname{div} d^h u(t) = 0$,

$$
I_6 = -\int_\Omega d^h[(u\cdot\nabla)u]\cdot d^h u(t)\,\mathrm{d}x = -\int_\Omega (d^h u\cdot\nabla)u\cdot d^h u(t)\,\mathrm{d}x. \tag{3.33}
$$

Inserting (3.31), (3.32) and (3.33) into (3.19), we conclude that the desired estimate (3.17) holds. This completes the proof. □

3.4. Time regularity of convective term. For our further argument, we will get a general information on time regularity of the convective term. Our argument is divided into two cases $p_1 > (3n+2)/(n+2)$ and $p_1 = (3n+2)/(n+2)$ since the methods in the cases are different. To begin with, we consider the case $p_1 > (3n+2)/(n+2)$.

Lemma 3.3. *Let the assumptions* (A1) *of Theorem* 2.1 *hold and supposed that* $p_1 > (3n+2)/(n+2)$. *Let* u *be a weak solution to* (1.1) *such that for some* $\sigma \in [0,1)$, $\gamma \in [0,1)$ *we have*

$$u \in N^{\sigma,2}(J;\mathcal{H}_2(\Omega)) \cap L^{p_1/(1-\gamma)}(J;\mathcal{V}_{p_1}(\Omega)) \tag{3.34}$$

and let

$$\kappa(\gamma) := \begin{cases} \sigma \dfrac{p_1[(n+2)p_1 - 3n - 2] + \gamma[(n+2)p_1 - 2n]}{(n+2)p_1^2 - (2n+2)p_1 - 2n + \gamma[(n+2)p_1 - 2n]} & \text{if } \gamma < \dfrac{2p_1}{(n+2)p_1 - 2n}, \\ \sigma \dfrac{p_1[(n+2)p_1 - 3n]}{(n+2)p_1^2 - 2np_1 - 2n} & \text{if } \gamma \geqslant \dfrac{2p_1}{(n+2)p_1 - 2n}. \end{cases} \tag{3.35}$$

We set

$$K := \int_\Omega |(d^h u \cdot \nabla) u \cdot d^h u(t)| \, \mathrm{d}x. \tag{3.36}$$

Then for a.e. $t \in J_h$ *we have*

$$\begin{aligned} h^{-2\kappa(\gamma)} K \leqslant \varepsilon_0 h^{-2\kappa(\gamma)} \|d^h u(t)\|_{p_1^*}^{p_1} + c(\varepsilon_0) \|\nabla u(t)\|_{p_1}^{p_1/(1-\gamma)} h^{-2\kappa(\gamma)} \|d^h u(t)\|_2^2 \\ + ch^{-2\sigma} \|d^h u(t)\|_2^2 + c\|\nabla u(t)\|_{p_1}^{p_1/(1-\gamma)}. \end{aligned} \tag{3.37}$$

Here p_1^* *is the Sobolev embedding exponent defined in* (2.1).

P r o o f. Following the argument on K_1 from [10], Lemma 10 and noting that $n \geqslant 2$, we obtain (3.37) and hence will omit the details. □

Remark 3.2. We note that $\kappa(\gamma)$ for $n = 3$ coincides with one from [10], Lemma 10. It is worth mentioning that $\kappa(\gamma)$ defined in (3.35) is less than σ.

Next we consider the case $p_1 = (3n+2)/(n+2)$. In this case, since $\kappa(0) = 0$ for $p_1 = (3n+2)/(n+2)$, we can not use Lemma 3.3 for further purpose. So we will use a different method using the higher integrability (2.10). This idea was first introduced in [11] for constant p.

Lemma 3.4. *Let the assumptions* (A1) *and* (A3) *of Theorem* 2.1 *hold and* $p_1 = (3n+2)/(n+2)$. *Let* u *be a weak solution to* (1.1) *such that for some* $\widetilde{\tau} \in [0, \frac{1}{2}]$ *and* $\widetilde{\sigma} \in [0, p_1^{-1})$,

$$u \in N^{1/2,2}(J;\mathcal{H}_2(\Omega)) \cap N^{\widetilde{\tau},\infty}(J;\mathcal{H}_2(\Omega)) \cap N^{\widetilde{\sigma},p_1}(J;\mathcal{V}_{p_1}(\Omega)). \tag{3.38}$$

For $\mu > 0$ arbitrarily close to 0 and δ from (2.10) we set

$$\widetilde{\kappa} := \begin{cases} \dfrac{2n}{n+2}\widetilde{\sigma} + \dfrac{2-2\widetilde{\sigma}n}{n+2}\widetilde{\tau} - \mu & \text{if } 0 < \widetilde{\sigma} < \dfrac{1}{n}, \quad \widetilde{\tau} > 0, \\ \dfrac{1}{n+2} + \dfrac{n}{n+2}\widetilde{\sigma} & \text{if } \dfrac{1}{n} \leqslant \widetilde{\sigma} < \dfrac{1}{p_1}, \quad \widetilde{\tau} > 0, \\ \dfrac{2n\delta}{(3n+2)(1+\delta)} & \text{if } \widetilde{\sigma} = 0, \quad \widetilde{\tau} = 0. \end{cases} \tag{3.39}$$

Then

$$(u\cdot\nabla)u \in N^{\widetilde{\kappa},p_1'}(J;\mathcal{V}_{p_1}'(\Omega)). \tag{3.40}$$

P r o o f. Let $h \in (0, t_2 - t_1)$ and $\psi \in L^{p_1}(J;\mathcal{V}_{p_1}(\Omega))$ with $\|\psi\|_{L^{p_1}(J;\mathcal{V}_{p_1}(\Omega))} \leqslant 1$.

Case 1: $\widetilde{\sigma} > 0$, $\widetilde{\tau} > 0$. In this case, following the argument from [11], Lemma 5.2 and noting that $n \geqslant 2$ and $p_1 = (3n+2)/(n+2)$, we obtain (3.40) for $\widetilde{\kappa}$ from the first two lines of (3.39) and hence will omit the details.

Case 2: $\widetilde{\sigma} = 0$, $\widetilde{\tau} = 0$. In this case, we also follow the same frame as in [11], Lemma 5.2 with a slight difference in using (2.10) due to $p_1 = (3n+2)/(n+2)$. Condition (2.10) implies $\nabla u \in L^{p_1(1+\delta)}(J\times\Omega)$ for some $\delta > 0$.

Let $a = n/(n+2)$. Let us define

$$\frac{1}{\widetilde{p}(0)} := 1 - \frac{1}{p_1} - \frac{2a}{p_1(1+\delta)}.$$

Hence, noting that $a = n/(n+2)$ and $p_1 = (3n+2)/(n+2)$ yields that

$$\frac{1}{\widetilde{p}(0)} = \frac{2np_1\delta}{(3n+2)p_1(1+\delta)} = \frac{2n\delta}{(3n+2)(1+\delta)}. \tag{3.41}$$

Now we are in a position to estimate the convective term. Using Hölder's inequality with a pair $(\widetilde{p}(0), p_1(1+\delta)/a, \infty, p_1(1+\delta)/a, p_1)$ and an interpolation yields that

$$\begin{aligned} \left|\int_{J_h}\int_\Omega d^h[(u\cdot\nabla)u]\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_{J_h}\int_\Omega d^h(u\otimes u) : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant c\int_{J_h} \|d^h u\|_2^{1-a}\|d^h u\|_{p_1^*}^a \|u\|_2^{1-a}\|u\|_{p_1^*}^a\|\nabla\psi\|_{p_1}\,\mathrm{d}t \\ &\leqslant c\biggl(\int_{J_h}\|d^h u\|_2^{(1-a)\widetilde{p}(0)}\,\mathrm{d}t\biggr)^{1/\widetilde{p}(0)}\biggl(\int_{J_h}\|d^h u\|_{p_1^*}^{p_1(1+\delta)}\,\mathrm{d}t\biggr)^{a/(p_1(1+\delta))} \\ &\quad\times \|u\|_{L^\infty(J;\mathcal{H}_2(\Omega))}^{1-a}\biggl(\int_{J_h}\|u\|_{p_1^*}^{p_1(1+\delta)}\,\mathrm{d}t\biggr)^{a/(p_1(1+\delta))} \\ &\leqslant C\biggl(\int_{J_h}\|d^h u\|_2^{(1-a)\widetilde{p}(0)}\,\mathrm{d}t\biggr)^{1/\widetilde{p}(0)}, \end{aligned} \tag{3.42}$$

where p_1^* is the Sobolev conjugate exponent to p_1 from (2.1) and in the last line we use that

$$\|u\|^{1-a}_{L^\infty(J;\mathcal{H}_2(\Omega))} \overset{(3.8)}{\leqslant} C,$$

$$\int_{J_h} (\|u(t+h)-u(t)\|^{p_1(1+\delta)}_{p_1^*} + \|u(t)\|^{p_1(1+\delta)}_{p_1^*})\,\mathrm{d}t \leqslant c\int_J \|u(t)\|^{p_1(1+\delta)}_{p_1^*}\,\mathrm{d}t \overset{(3.10)}{\leqslant} C.$$

Now we claim that $\widetilde{p}(0)(1-a) > 2$. Indeed, by the definition of $\widetilde{p}(0)$ and $a = n/(n+2)$ we have

$$\widetilde{p}(0)(1-a) = \frac{(3n+2)(1+\delta)}{n(n+2)\delta},$$

and direct calculation shows that a function $f(\delta) := (3n+2)(1+\delta) - 2n(n+2)\delta$ is decreasing in $\delta > 0$ and hence $f(\delta) > 0$ for $\delta \in (0, (3n+2)/(2n^2+n-2))$. This proves the claim since the exponent $\delta > 0$ can be taken small enough by (2.10).

Since $\widetilde{p}(0)(1-a) > 2$, we get

$$\text{(3.43)} \qquad \begin{aligned} \text{RHS of (3.42)} &= C\Big(\int_{J_h} \|d^h u\|_2^2 \|d^h u\|_2^{(1-a)\widetilde{p}(0)-2}\,\mathrm{d}t\Big)^{1/\widetilde{p}(0)} \\ &\leqslant C\Big(\int_{J_h} \|d^h u\|_2^2\,\mathrm{d}t\Big)^{1/\widetilde{p}(0)} \|d^h u\|^{((1-a)\widetilde{p}(0)-2)/\widetilde{p}(0)}_{L^\infty(J_h;\mathcal{H}_2)} \\ &\overset{(3.38)}{\leqslant} Ch^{1/\widetilde{p}(0)} =: Ch^{\widetilde{\kappa}(0)}. \end{aligned}$$

It is clear that

$$\widetilde{\kappa}(0) = \frac{2}{\widetilde{p}(0)} \overset{(3.41)}{=} \frac{4n\delta}{(3n+2)(1+\delta)}.$$

This is just the third line of (3.39). Moreover, joining (3.42) and (3.43) yields (3.40). Thus, Lemma 3.4 is completely proved. □

3.5. Time regularity of velocity. Hereafter, let $I \subset\subset J$, $I_h \subset\subset J_h$ and $\eta \in C^\infty(J_h)$ be a cut-off function such that $\eta \equiv 1$ in I_h and $\operatorname{supp}\eta \subset J_h$.

Lemma 3.5. *Let the assumptions* (A1)–(A3) *and* (2.6) *of Theorem* 2.1 *hold and* $p_1 > (3n+2)/(n+2)$. *Let* (3.34) *hold and* $\kappa(\gamma)$ *be from* (3.35). *Then we have:*

(1) *If* $J \subset\subset (0,T)$ *and* $F \in N^{\widehat{\kappa},2}(J; L^2(\Omega))$ *with some* $\widehat{\kappa} \in (0,1)$, *then for* $\tau(\gamma) := \min\{\kappa(\gamma), \widehat{\kappa}\}$

$$\text{(3.44)} \qquad \begin{aligned} &u \in N^{\tau(\gamma),\infty}(I;\mathcal{H}_2(\Omega)) \cap N^{\tau(\gamma),2}(I;\mathcal{V}_2(\Omega)) \cap N^{2\tau(\gamma)/p_1,p_1}(I;\mathcal{V}_{p_1}(\Omega)), \\ &V_{p(t)}(\mathcal{D}u) \in N^{\tau(\gamma),2}(I; L^2(\Omega)). \end{aligned}$$

(2) *If* $F \in N^{\widehat{\kappa},2}(0,T;L^2(\Omega))$ *with some* $\widehat{\kappa} \in (0,1)$, *and for sufficiently small* $h \in (0,1)$ *and for some* $\widehat{\tau} \in (0,\widehat{\kappa}]$

$$\|d^h u(0)\|_{2,\Omega} \leqslant ch^{\widehat{\tau}}, \tag{3.45}$$

then for $\tau(\gamma) := \min\{\kappa(\gamma),\widehat{\tau}\}$ *relations* (3.44) *hold with* J *instead of* I.

Proof. At first, we are concerned with the case $J \subset\subset (0,T)$. Multiplying (3.17) by η and integrating over J_h, we obtain

$$\begin{aligned}\sup_{t_1\leqslant t\leqslant t_2-h} \eta\|d^h u(t)\|_2^2 &+ \int_{\Omega\times J_h} \eta(|d^h V_{p(t)}(\mathcal{D}u)|^2 + |d^h\mathcal{D}u|^{p_1} + |d^h\mathcal{D}u|^2)\,\mathrm{d}z \\ &\leqslant ch^2\int_{\Omega\times J_h}(|\mathcal{D}u(t)|^{p(t)(1+\delta)}+1)\,\mathrm{d}z + c\int_{J_h}\|d^h u(t)\|_2^2\,\mathrm{d}t \\ &\quad + c\int_{\Omega\times J_h}|d^h F|^2\,\mathrm{d}z + c\int_{J_h} K\,\mathrm{d}t,\end{aligned} \tag{3.46}$$

where K is from (3.36).

By the condition $F \in N^{\widehat{\kappa},2}(J;L^2(\Omega))$ and Remark 3.2, we have

$$h^{-2\tau(\gamma)}\int_{\Omega\times J_h}|d^h F(t)|^2\,\mathrm{d}z \quad\leqslant\quad Ch^{-2\tau(\gamma)}h^{2\widehat{\kappa}} \leqslant C, \tag{3.47}$$

$$h^{-2\tau(\gamma)}\int_{J_h}\|d^h u(t)\|_2^2\,\mathrm{d}t \overset{(3.34)}{\leqslant} ch^{-2\tau(\gamma)}h^{2\sigma} \leqslant C. \tag{3.48}$$

From the definition of $\tau(\gamma)$, (3.37) and Sobolev's and Korn's inequalities it follows that

(3.49)

$$\begin{aligned}h^{-2\tau(\gamma)}\int_{J_h} K\,\mathrm{d}t \quad&\leqslant\quad \varepsilon_0 h^{-2\tau(\gamma)}\int_{J_h}\|d^h u(t)\|_{p_1^*}^{p_1}\,\mathrm{d}t \\ &\quad + c(\varepsilon_0)\int_{J_h}\|\nabla u(t)\|_{p_1}^{p_1/(1-\gamma)}(h^{-2\tau(\gamma)}\|d^h u(t)\|_2^2)\,\mathrm{d}t \\ &\quad + ch^{2(\kappa(\gamma)-\tau(\gamma))}\int_{J_h}(h^{-2\sigma}\|d^h u(t)\|_2^2 + \|\nabla u(t)\|_{p_1}^{p_1/(1-\gamma)})\,\mathrm{d}t \\ &\overset{(3.34)}{\leqslant} C + \varepsilon_0 h^{-2\tau(\gamma)}\int_{J_h}\|d^h\mathcal{D}u(t)\|_{p_1}^{p_1}\,\mathrm{d}t \\ &\quad + c(\varepsilon_0)\int_{J_h}\|\nabla u(t)\|_{p_1}^{p_1/(1-\gamma)}(h^{-2\tau(\gamma)}\|d^h u(t)\|_2^2)\,\mathrm{d}t.\end{aligned}$$

Inserting (3.47), (3.48) and (3.49) into (3.46) multiplied by $h^{-2\tau(\gamma)}$ yields that

$$\sup_{t\in I_h} h^{-2\tau(\gamma)}\|d^h u(t)\|_2^2 + h^{-2\tau(\gamma)}\int_{\Omega\times I_h}(|d^h V_{p(t)}(\mathcal{D}u)|^2 + |d^h\mathcal{D}u(t)|^{p_1})\,\mathrm{d}z \leqslant c\int_{J_h}\|\nabla u(t)\|_{p_1}^{p_1/(1-\gamma)}(h^{-2\tau(\gamma)}\|d^h u(t)\|_2^2)\,\mathrm{d}t + C,$$

which together with Gronwall's inequality implies (3.44).

Next we consider the case $t_1 = 0$. Integrating (3.17) over J_h yields that

$$\sup_{0\leqslant t\leqslant t_2-h}\|d^h u(t)\|_2^2 + \int_{\Omega\times J_h}(|d^h V_{p(t)}(\mathcal{D}u)|^2 + |d^h\mathcal{D}u|^{p_1} + |d^h\mathcal{D}u|^2)\,\mathrm{d}z \leqslant ch^2\int_{\Omega\times J_h}(|\mathcal{D}u(t)|^{p(t)(1+\delta)} + 1)\,\mathrm{d}z + \|d^h u_0\|_2^2 + c\int_{\Omega\times J_h}|d^h F|^2\,\mathrm{d}z + c\int_{J_h}K\,\mathrm{d}t. \tag{3.50}$$

From (3.45) it follows that

$$h^{-2\tau(\gamma)}\|d^h u_0\|_2^2 \leqslant Ch^{-2\tau(\gamma)}h^{2\widehat{\tau}} \leqslant C. \tag{3.51}$$

From (3.50) together with (3.47), (3.49) and (3.51) we arrive at (3.44) with J instead of I. □

Next we are concerned with the case $p_1 = (3n+2)/(n+2)$.

Lemma 3.6. *Let the assumptions* (A1)–(A3) *and* (2.6) *of Theorem* 2.1 *hold and* $p_1 = (3n+2)/(n+2)$. *Let* (3.38) *hold and* $\widetilde{\kappa}$ *be from* (3.39). *Then we have:*

(1) *If* $J \subset\subset (0,T)$ *and* $F \in N^{\widehat{\kappa},2}(J;L^2(\Omega))$ *with some* $\widehat{\kappa}\in(0,1)$, *then for* $\overline{\tau} := \min\{\widetilde{\kappa}p_1'/2, \widehat{\kappa}\}$

$$u \in N^{\overline{\tau},\infty}(I;\mathcal{H}_2(\Omega)) \cap N^{\overline{\tau},2}(I;\mathcal{V}_2(\Omega)) \cap N^{2\overline{\tau}/p_1,p_1}(I;\mathcal{V}_{p_1}(\Omega)), \quad V_{p(t)}(\mathcal{D}u) \in N^{\overline{\tau},2}(I;L^2(\Omega)). \tag{3.52}$$

(2) *If* $t_1 = 0$, $F \in N^{\widehat{\kappa},2}(J;L^2(\Omega))$ *with some* $\widehat{\kappa}\in(0,1)$ *and condition* (3.45) *for* $\widehat{\tau}\in(0,\widehat{\kappa}]$ *holds, then for* $\overline{\tau} := \min\{\widetilde{\kappa}p_1'/2, \widehat{\tau}\}$ *relations* (3.52) *hold with* J *instead of* I.

P r o o f. From Hölder's and Korn's inequalities and (3.40), it follows that

$$
\begin{aligned}
(3.53)\quad h^{-2\overline{\tau}} \int_{\Omega\times J_h} & |d^h((u\cdot\nabla)u)\cdot d^h u|\,\mathrm{d}z \\
&\leqslant \varepsilon_0 h^{-2\overline{\tau}} \int_{\Omega\times J_h} |d^h u|^{p_1}\,\mathrm{d}z + c(\varepsilon_0) h^{-2\overline{\tau}} \int_{J_h} \|d^h((u\cdot\nabla)u)\|_{\mathcal{V}'_{p_1}(\Omega)}^{p_1'}\,\mathrm{d}t \\
&\leqslant \varepsilon_0 h^{-2\overline{\tau}} \int_{\Omega\times J_h} |d^h \mathcal{D}u|^{p_1}\,\mathrm{d}z + c(\varepsilon_0) h^{-2\overline{\tau}} h^{\widetilde{\kappa} p_1'} \\
&\leqslant \varepsilon_0 h^{-2\overline{\tau}} \int_{\Omega\times J_h} |d^h \mathcal{D}u|^{p_1}\,\mathrm{d}z + C.
\end{aligned}
$$

The estimates for the rest of the terms in (3.53) are similar to the ones in the proof of Lemma 3.9 and so we will omit them.

Inserting (3.47), (3.48) with $\overline{\tau}$ instead of $\tau(\gamma)$ and (3.53) into (3.46) multiplied by $h^{-2\overline{\tau}}$ and taking into account the conditions of this lemma and (3.10) yield that

$$
\sup_{t\in I_h} h^{-2\overline{\tau}} \|d^h u(t)\|_2^2 + h^{-2\overline{\tau}} \int_{\Omega\times I_h} (|d^h V_{p(t)}(\mathcal{D}u)|^2 + |d^h \mathcal{D}u(t)|^{p_1})\,\mathrm{d}z \leqslant C,
$$

which implies (3.52).

Next we consider the case $t_1 = 0$. Then we can get (3.52) with J instead of I from (3.50) together with (3.47), (3.48) with $\overline{\tau}$ instead of $\tau(\gamma)$, (3.53) and (3.51). □

The following lemma will be used later.

Lemma 3.7. *Let the assumptions* (A1)–(A3) *and* (2.6) *of Theorem* 2.1 *hold and* $\sigma \in [0,1)$, $p_1 \geqslant (3n+2)/(n+2)$ *and* $p_1 > \frac{1}{2}n$. *Assume that*

$$
(3.54)\qquad u \in N^{\sigma,2}(J;\mathcal{H}_2(\Omega)) \cap L^{2p_1/(2p_1-n)}(J;\mathcal{V}_{p_1}(\Omega)).
$$

Then we have:

(1) *If* $J \subset\subset (0,T)$ *and* $F \in N^{\widehat{\kappa},2}(J;L^2(\Omega))$ *with some* $\widehat{\kappa} \in (0,1)$, *then for* $\tau := \min\{\sigma,\widehat{\kappa}\}$

$$
(3.55)\qquad \begin{gathered} u \in N^{\tau,\infty}(I;\mathcal{H}_2(\Omega)) \cap N^{\tau,2}(I;\mathcal{V}_2(\Omega)) \cap N^{2\tau/p_1,p_1}(I;\mathcal{V}_{p_1}(\Omega)), \\ V_{p(t)}(\mathcal{D}u) \in N^{\tau,2}(I;L^2(\Omega)). \end{gathered}
$$

(2) *If* $t_1 = 0$, $F \in N^{\widehat{\kappa},2}(J;L^2(\Omega))$ *with some* $\widehat{\kappa} \in (0,1)$ *and condition* (3.45) *for* $\widehat{\tau} \in (0,\widehat{\kappa}]$ *holds, then for* $\tau := \min\{\sigma,\widehat{\tau}\}$ *relation* (3.55) *hold with* J *instead of* I.

P r o o f. Let us again estimate the term K from (3.36). We use the interpolation inequality (3.7) with $q = 2p_1'$ (so $\alpha := (2p_1 - n)/(2p_1) \in (0,1)$) and Korn's inequality to get

$$K \leqslant \int_\Omega |\nabla u(t)||d^h u(t)|^2 \, \mathrm{d}x \leqslant \|\nabla u(t)\|_{p_1} \|d^h u(t)\|_{2p_1'}^2$$
$$\leqslant c\|\nabla u(t)\|_{p_1} \|d^h u(t)\|_2^{2\alpha} \|d^h \mathcal{D} u(t)\|_2^{2(1-\alpha)},$$

which implies that by Young's inequality

$$K \leqslant \varepsilon_0 \|d^h \mathcal{D} u(t)\|_2^2 + c(\varepsilon_0) \|\nabla u(t)\|_{p_1}^{2p_1/(2p_1-n)} \|d^h u(t)\|_2^2. \tag{3.56}$$

Here we note that the condition $\alpha \in (0,1)$ holds provided $2p_1 > n$.

Thus, inserting (3.47), (3.48) with $\tau(\gamma) = \tau$ and (3.56) into (3.46) multiplied by $h^{-2\tau}$, we conclude that

$$\sup_{t_1 \leqslant t \leqslant t_2 - h} h^{-2\tau} \eta \|d^h u(t)\|_2^2$$
$$+ h^{-2\tau} \int_{\Omega \times J_h} \eta (|d^h V_{p(t)}(\mathcal{D}u)|^2 \, \mathrm{d}z + |d^h \mathcal{D} u(t)|^2 + |d^h \mathcal{D} u(t)|^{p_1}) \, \mathrm{d}z$$
$$\leqslant c \int_{J_h} \|\nabla u(t)\|_{p_1}^{2p_1/(2p_1-n)} (h^{-2\tau} \eta \|d^h u(t)\|_2^2) \, \mathrm{d}t + C,$$

which together with (3.54) and Gronwall's inequality implies (3.55).

In the case $t_1 = 0$, the result follows from the same as above except that (3.51) is used instead of (3.48). □

3.6. Improvement of time regularity of time derivative of velocity.

Lemma 3.8. *Let the assumptions* (A1)–(A3) *and* (2.6) *of Theorem* 2.1 *hold. Let* $\sigma \in (0,1)$, $p_1 \geqslant (3n+2)/(n+2)$, $p_1 > 2$ *and* $F \in N^{\widehat{\kappa},2}(J; L^2(\Omega))$. *Assume that*

$$u \in N^{\sigma,\infty}(J; \mathcal{H}_2(\Omega)) \cap N^{2\sigma/p_1, p_1}(J; \mathcal{V}_{p_1}(\Omega)), \quad V_{p(t)}(\mathcal{D}u) \in N^{\sigma,2}(J; L^2(\Omega)). \tag{3.57}$$

Let $\tau_1 := \min\{\sigma, \widehat{\kappa}\}$ *and* $\tau_2 := \min\{\widehat{\alpha}\sigma, \widehat{\kappa}\}$, *where*

$$\widehat{\alpha} := \frac{2[(n+2)p_1^2 - (4n+2)p_1 + 4n]}{[(n+2)p_1 - 2n]p_1}. \tag{3.58}$$

Then we have

$$\partial_t u \in \begin{cases} N^{\tau_1, p_1'}(J; \mathcal{V}_{p_1}'(\Omega)) & \text{if } p_1 \geqslant \dfrac{4n}{n+2}, \\ N^{\tau_2, p_1'}(J; \mathcal{V}_{p_1}'(\Omega)) & \text{if } \dfrac{3n+2}{n+2} \leqslant p_1 < \dfrac{4n}{n+2}. \end{cases} \tag{3.59}$$

P r o o f. Let $\psi \in L^{p_1}(J;\mathcal{V}_{p_1}(\Omega))$ with $\|\psi\|_{L^{p_1}(J;\mathcal{V}_{p_1}(\Omega))} \leqslant 1$. Recalling that $\partial_t u \in L^{p_1'}(J;\mathcal{V}_{p_1}'(\Omega))$ by Lemma 3.1 and testing (1.1) with $d^h\psi(t)$, we obtain that for all $h \in (0, t_2 - t_1)$

$$
\begin{aligned}
(3.60)\qquad &\int_{J_h} \langle \partial_t d^h u(t), \psi(t)\rangle_{1,p_1}\,\mathrm{d}t \\
&= -\int_{J_h}\int_\Omega d^h S(p(t),\mathcal{D}u) : \mathcal{D}\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{J_h}\int_\Omega d^h F : \mathcal{D}\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{J_h}\int_\Omega d^h[(u\cdot\nabla)u]\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\
&=: I_7 + I_8 + I_9.
\end{aligned}
$$

At first, we estimate the term I_7. Note that $p_1 > 2$. By (3.4) and the same arguments as in (3.25) we can get

$$
\begin{aligned}
(3.61)\quad |d^h S(p(t),\mathcal{D}u(t))| &\leqslant c(1 + |\mathcal{D}u(t)|^2 + |\mathcal{D}u(t+h)|^2)^{(p(t)-2)/2}|d^h\mathcal{D}u(t)| \\
&\quad + ch[(1 + |\mathcal{D}u(t+h)|^2 + |\mathcal{D}u(t)|^2)^{(p(t)-1+\beta)/2}] \\
&\leqslant c\,\underbrace{(1 + |\mathcal{D}u(t)|^2 + |\mathcal{D}u(t+h)|^2)^{(p(t)-2)/4}}_{=:K_3}|d^h V_{p(t)}(\mathcal{D}u(t))| \\
&\quad + ch\,\underbrace{(1 + |\mathcal{D}u(t)|^{p(t)-1+2\beta})}_{=:K_4},
\end{aligned}
$$

where $\beta > 0$ is from (3.24). Now we choose $\beta > 0$ such that

$$
(3.62)\qquad p(t) - p_1 \leqslant p_1(\delta(p(t)-1) - 2\beta), \quad 2(p(t) - p_1) \leqslant p_1(\delta(p(t)-2) - \beta),
$$

which is possible due to (3.9). Indeed, taking $\beta = \min\{\frac14\delta, \frac14\delta(p_1-2)\}$ yields that

$$
\begin{aligned}
p_1(\delta(p(t)-1) - 2\beta) &\overset{p(t)>2}{>} p_1(\delta - 2\beta) \overset{(3.9)}{\geqslant} p_2 - p_1 \geqslant p(t) - p_1, \\
p_1(\delta(p(t)-2) - \beta) &\geqslant p_1\frac{3}{4}\delta(p_1 - 2) \geqslant 2(p_2 - p_1) \geqslant 2(p(t) - p_1).
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
(3.63)\quad K_3^{2p_1/(p_1-2)} &\leqslant c(1 + |\mathcal{D}u(t)|^{(p(t)-2)/2} + |\mathcal{D}u(t+h)|^{(p(t+h)-2+\beta)/2})^{2p_1/(p_1-2)} \\
&\leqslant c + c|\mathcal{D}u(t)|^{(p(t)-2+\beta)p_1/(p_1-2)} \leqslant c + c|\mathcal{D}u(t)|^{p(t)(1+\delta)},
\end{aligned}
$$

where for performing the last estimate we use the fact that by the second inequality of (3.62)

$$
\frac{p(t)-2+\beta}{2}\,\frac{2p_1}{p_1-2} < p(t)(1+\delta).
$$

We also have

$$
(3.64)\qquad K_4^{p_1/(p_1-1)} \leqslant c(1 + |\mathcal{D}u(t)|)^{p(t)(1+\delta)},
$$

where we use that by the first inequality of (3.62)

$$(p(t)-1+2\beta)\frac{p_1}{p_1-1} < p(t)(1+\delta).$$

Thus, combining (3.61) with (3.63) and (3.64) yields that

(3.65)

$$\begin{aligned} h^{-\sigma} I_7 &\leqslant ch^{-\sigma}\int_{J_h}\|d^h V_{p(t)}(\mathcal{D}u(t))\|_2\|K_3\|_{2p_1/(p_1-2)}\|\mathcal{D}\psi(t)\|_{p_1}\,\mathrm{d}t \\ &\quad + ch^{1-\sigma}\int_{J_h}\|K_4\|_{p_1'}\|\mathcal{D}\psi(t)\|_{p_1}\,\mathrm{d}t \\ &\leqslant c\|h^{-\sigma}d^h V_{p(t)}(\mathcal{D}u)\|_{L^2(J_h;L^2(\Omega))}\Bigl(c+\int_{\Omega\times J_h}|\mathcal{D}u(t)|^{p(t)(1+\delta)}\,\mathrm{d}z\Bigr)^{(p_1-2)/(2p_1)} \\ &\quad + ch^{1-\sigma}\Bigl(c+\int_{\Omega\times J_h}|\mathcal{D}u(t)|^{p(t)(1+\delta)}\,\mathrm{d}z\Bigr)^{(p_1-1)/p_1} \\ &\overset{(2.10),(3.57)}{\leqslant} C. \end{aligned}$$

Next we estimate the term I_8. By the condition $F\in N^{\widehat{\kappa},2}(J;L^2(\Omega))$ we have

$$h^{-\widehat{\kappa}} I_8 \leqslant h^{-\widehat{\kappa}}\|d^h F\|_{L^2(J_h;L^2(\Omega))} \leqslant C. \tag{3.66}$$

Now it remains to estimate the term I_9. Following the argument from [10], Lemma 11 and noting that $n\geqslant 2$ and $p_1\geqslant(3n+2)/(n+2)$, we arrive at

$$h^{-\sigma} I_9 \leqslant C \quad \text{if } p_1\geqslant \frac{4n}{n+2}, \tag{3.67}$$

while, for $\widehat{\alpha}$ from (3.58),

$$h^{-\widehat{\alpha}\sigma} I_9 \leqslant C \quad \text{if } \frac{3n+2}{n+2}\leqslant p_1 < \frac{4n}{n+2}. \tag{3.68}$$

Thus, from (3.60) with (3.65)–(3.68) we arrive at the desired result (3.59). $\square$

Next we consider the case $p_1=2$. This case is possible only if $n=2$ due to $p_1\geqslant(3n+2)/(n+2)$.

Lemma 3.9. *Let the assumptions* (A1)–(A3) *and* (2.6) *of Theorem* 2.1 *hold. Let* $\sigma\in(0,1)$, $p_1=n=2$ *and* $F\in N^{\widehat{\kappa},2}(J;L^2(\Omega))$. *Assume that* (3.57) *holds. Then we have*

$$\partial_t u\in N^{\tau_3,2}(J;\mathcal{V}_2'(\Omega)) \quad \textit{for } \tau_3:=\min\{\widetilde{\alpha}\sigma,\widehat{\kappa}\}, \tag{3.69}$$

where

$$\widetilde{\alpha} := \frac{2(1+\delta)-p_2}{p_2\delta}. \tag{3.70}$$

P r o o f. We will use the notations in Lemma 3.8. But we cannot estimate the term K_3 as above due to the condition $p_1 = 2$. So we use a peculiar method using the higher integrability: Without loss of generality we can assume that $p_2 > 2$ because otherwise $p(t) \equiv 2$. At first we observe that $p_2 < 2(1+\delta)$ by (3.9) and

$$K_3^{2p_2/(p_2-2)} \leqslant c(1+|\mathcal{D}u(t)|^2+|\mathcal{D}u(t+h)|^2)^{(p_2-2)p_2/(2(p_2-2))} \leqslant c(1+|\mathcal{D}u(t)|+|\mathcal{D}u(t+h)|)^{p_2} \tag{3.71}$$

and that by (3.6) and (2.10)

$$V_{p(t)}(\mathcal{D}u) \in L^{2(1+\delta)}(\Omega_T). \tag{3.72}$$

Hence, for an exponent $\widetilde{\alpha}$ from (3.70),

$$\|V_{p(t)}(\mathcal{D}u(t))\|_{p_2} \leqslant \|V_{p(t)}(\mathcal{D}u(t))\|_2^{\widetilde{\alpha}} \|V_{p(t)}(\mathcal{D}u(t))\|_{2(1+\delta)}^{1-\widetilde{\alpha}}. \tag{3.73}$$

We use Hölder's inequality with a triplet $(2p_2/(p_2-2), p_2, 2)$ and the interpolation between L^2 and $L^{2(1+\delta)}$ to get

$$\begin{aligned} &\int_{J_h}\int_\Omega K_3|d^hV_{p(t)}(\mathcal{D}u(t))||\mathcal{D}\psi(t)|\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant c\int_{J_h} \|d^hV_{p(t)}(\mathcal{D}u(t))\|_{p_2}\|K_3\|_{2p_2/(p_2-2)}\|\mathcal{D}\psi(t)\|_2\,\mathrm{d}t \\ &\quad\overset{(3.73)}{\leqslant} c\int_{J_h} \|d^hV_{p(t)}(\mathcal{D}u(t))\|_2^{\widetilde{\alpha}}\|d^hV_{p(t)}(\mathcal{D}u(t))\|_{2(1+\delta)}^{1-\widetilde{\alpha}} \\ &\qquad\times \|K_3\|_{2p_2/(p_2-2)}\|\mathcal{D}\psi(t)\|_2\,\mathrm{d}t \\ &\quad\overset{(3.71),(3.10),(3.72)}{\leqslant} Ch^{\sigma\widetilde{\alpha}}\|h^{-\sigma}d^hV_{p(t)}(\mathcal{D}u(t))\|_{L^2(J_h;L^2(\Omega))}^{\widetilde{\alpha}} \overset{(3.57)}{\leqslant} Ch^{\sigma\widetilde{\alpha}}. \end{aligned} \tag{3.74}$$

On the other hand, from (3.64) with $p_1 = 2$ it follows that

$$ch^{1-\sigma\widetilde{\alpha}}\int_{J_h}\|K_4\|_2\|\mathcal{D}\psi(t)\|_2\,\mathrm{d}t \leqslant ch^{1-\sigma\widetilde{\alpha}}\Big(\int_{\Omega\times J_h}(1+|\mathcal{D}u(t)|)^{p(t)(1+\delta)}\,\mathrm{d}z\Big)^{1/2} \overset{(2.10)}{\leqslant} C. \tag{3.75}$$

Joining (3.74) and (3.75) yields that

$$h^{-\sigma\widetilde{\alpha}}I_7 \leqslant C. \tag{3.76}$$

Now let us estimate the term I_9. Note that $4n/(n+2) = (3n+2)/(n+2) = p_1 = n = 2$. Following the argument as in (3.74), we have

$$h^{-\sigma\widetilde{\alpha}} I_9 \leqslant h^{-\sigma\widetilde{\alpha}} \int_{J_h}\int_\Omega |d^h u||u||\nabla\psi|\,\mathrm{d}x\,\mathrm{d}t \tag{3.77}$$
$$\leqslant h^{-\sigma\widetilde{\alpha}} \int_{J_h} \|d^h u\|_2^{\widetilde{\alpha}} \|d^h u\|_{2(1+\delta)}^{1-\widetilde{\alpha}} \|u\|_{2p_2/(p_2-2)} \|\nabla\psi\|_2\,\mathrm{d}t$$
$$\overset{(3.10),(2.10)}{\leqslant} C\|h^{-\sigma} d^h u\|_{L^\infty(J_h;L^2(\Omega))}^{\widetilde{\alpha}} \overset{(3.57)}{\leqslant} C,$$

where to control $\|d^h u\|_{2(1+\delta)}^{1-\widetilde{\alpha}}$ we use (2.10), $p_1 \geqslant 2$ and Poincare's inequality.

Thus, equation (3.60) together with (3.76), (3.66), (3.77) yields that

$$\partial_t u \in N^{\tau_3,2}(J;\mathcal{V}_2'(\Omega)) \quad \text{for } \tau_3 = \min\{\widetilde{\alpha}\sigma, \widehat{\kappa}\},$$

which is just (3.69). □

4. Proof of the first statement of Theorem 2.1: the case $\widehat{\kappa} < 1$

Let $I \subset\subset J$, $I_h \subset\subset J_h$ for interval J satisfying (3.9). We will use the same notation I in the iterative use of Lemmas 3.7, 3.5 since the number of uses is finite.

It is worth noting that throughout this section we use assumptions (A1)–(A4) and the condition $F \in N^{\widehat{\kappa},2}_{\mathrm{loc}}(0,T;L^2(\Omega))$ with some $\widehat{\kappa} \in (0,1)$.

Case 1: $p_1 > (3n+2)/(n+2)$.

Step 1: Iterative use of Lemma 3.5. We know that $u \in N^{1/2,2}(J, L^2(\Omega))$ by (3.12). When $0 \leqslant \gamma \leqslant 2p_1/((n+2)p_1 - 2n)$, we set $\sigma = \frac{1}{2}$ in Lemma 3.5. Then $\kappa(\gamma)$ is from the first line of (3.35) since $\kappa(\gamma)$ is continuous in $\gamma = 2p_1/((n+2)p_1 - 2n)$. So by Lemma 3.5 if

$$u \in L^{p_1/(1-\gamma)}(J;\mathcal{V}_{p_1}(\Omega)), \tag{4.1}$$

then for $\kappa(\gamma) \geqslant \widehat{\kappa}$ we have

$$u \in N^{\widehat{\kappa},\infty}(I,\mathcal{H}_2(\Omega)) \cap N^{\widehat{\kappa},2}(I,\mathcal{V}_2(\Omega)) \cap N^{2\widehat{\kappa}/p_1,p_1}(I,\mathcal{V}_{p_1}(\Omega)), \tag{4.2}$$
$$V_{p(t)}(\mathcal{D}u) \in N^{\widehat{\kappa},2}(I,L^2(\Omega)),$$

which implies (2.12), while for $\kappa(\gamma) < \widehat{\kappa}$

$$u \in N^{\kappa(\gamma),\infty}(I,\mathcal{H}_2(\Omega)) \cap N^{\kappa(\gamma),2}(I,\mathcal{V}_2(\Omega)) \cap N^{2\kappa(\gamma)/p_1,p_1}(I,\mathcal{V}_{p_1}(\Omega)), \tag{4.3}$$
$$V_{p(t)}(\mathcal{D}u) \in N^{\kappa(\gamma),2}(I,L^2(\Omega)).$$

Let us start iteration argument. It is clear that all the assumptions of Lemma 3.5 are satisfied for every weak solution with $\gamma = 0$, $\sigma = \frac{1}{2}$. So if $\kappa(0) \geqslant \widehat{\kappa}$, then we get (4.2), i.e., (2.12). If $\kappa(0) < \widehat{\kappa}$, then by (4.3) the exponent γ is improved from 0 to $2\kappa(0) > 0$. This allows us to use again Lemma 3.5. In order to calculate the bound of this improvement, we set $G(\gamma) := 2\kappa(\gamma) - \gamma$. It is clear that $G(0) = 2\kappa(0) = (p_1[(n+2)p_1 - 3n - 2])/((n+2)p_1^2 - (2n+2)p_1 - 2n) > 0$ and $G(2p_1/((n+2)p_1 - 2n)) > 0$ for $p_1 > (3n+2)/(n+2)$. In particular, $G(\gamma)$ is concave for $\gamma > 0$ provided $p_1 \geqslant 2$. Thus, we can easily see that $G(\gamma) \geqslant \min\{G(0), G(2p_1/((n+2)p_1 - 2n))\}$ for all $\gamma \in [0, 2p_1/((n+2)p_1 - 2n)]$.

If $\widehat{\kappa} \leqslant \kappa(2p_1/((n+2)p_1 - 2n))$, then this implies (4.2) after a finite number of iterations and hence we get (2.12). If $\widehat{\kappa} > \kappa(2p_1/((n+2)p_1 - 2n))$, then one can get only (4.3) for all $\gamma \in [0, 2p_1/((n+2)p_1 - 2n)]$ after a finite number of iterations. Furthermore, (4.3) holds for all $\gamma \in [0,1)$ because from the second line of (3.35) the values of $\kappa(\gamma)$ for all $\gamma > 2p_1/((n+2)p_1 - 2n)$ are fixed, that is, $\kappa(\gamma) = \kappa(2p_1/((n+2)p_1 - 2n))$. In this case our aim is to show

$$u \in L^{2p_1/(2p_1-n)}(I, \mathcal{V}_{p_1}(\Omega)). \tag{4.4}$$

This enables us to apply Lemma 3.7 to show (2.12). By Nikolskii embedding (2.2) this suffices to prove that $u \in N^{(2\kappa(\gamma))/p_1, p_1}(I, \mathcal{V}_{p_1}(\Omega))$ for γ satisfying

$$2\kappa(\gamma) > \frac{n+2-2p_1}{2}. \tag{4.5}$$

It is easily checked that for $p_1 > (3n+2)/(n+2)$

$$2\kappa\Big(\frac{2p_1}{(n+2)p_1 - 2n}\Big) = \frac{p_1[(n+2)p_1 - 3n]}{(n+2)p_1^2 - 2np_1 - 2n} > \frac{2p_1}{(n+2)p_1 - 2n}. \tag{4.6}$$

From (4.6) it follows that if

$$\frac{n+2-2p_1}{2} \leqslant \frac{2p_1}{(n+2)p_1 - 2n}, \tag{4.7}$$

then inequality (4.5) holds for $\gamma = 2p_1/((n+2)p_1 - 2n)$ and in turn by (4.3), (4.4) holds.

Now let us calculate the range of p_1 satisfying (4.7). It is clear that

$$\frac{n+2-2p_1}{2} \leqslant \frac{2p_1}{(n+2)p_1 - 2n} \leftrightarrows 2(n+2)p_1^2 - (n^2+8n)p_1 + 2n(n+2) \geqslant 0.$$

If $n \leqslant 3$, then since $D := (n^2+8n)^2 - 16n(n+2)^2 = n(n^3-64) < 0$, the inequality (4.7) always holds for all $p_1 \geqslant 1$. If $n = 4$, then since $D = 0$, the inequality always holds provided $p_1 \geqslant 2$. If $n \geqslant 5$, then the inequality holds provided

$$p_1 \geqslant \frac{n^2 + 8n + \sqrt{n^4 - 64n}}{4(n+2)}. \tag{4.8}$$

Remark 4.1. For $n \geqslant 5$, it is clear that

$$\frac{3n+2}{n+2} < \frac{n}{2} < \frac{n^2 + 8n + \sqrt{n^4 - 64n}}{4(n+2)} < \frac{n+2}{2}. \tag{4.9}$$

Up to now, we followed the argument from [10]. From now on, we are going to present our own calculations in order to improve the lower bound on $p(z)$ for $n \geqslant 5$.

Hence, the inverse inequality of (4.7), that is,

$$\frac{n+2-2p_1}{2} > \frac{2p_1}{(n+2)p_1 - 2n} \tag{4.10}$$

holds only for $n \geqslant 5$ and $p_1 < \frac{1}{4}(n^2 + 8n + \sqrt{n^4 - 64n})/(n+2)$. In this case, we need Lemma 3.5 for $\gamma > 2p_1/((n+2)p_1 - 2n)$, that is, for $\kappa(\gamma) = \kappa(2p_1/((n+2)p_1 - 2n))$. Thus, if

$$2\kappa\Big(\frac{2p_1}{(n+2)p_1 - 2n}\Big) = \frac{p_1[(n+2)p_1 - 3n]}{(n+2)p_1^2 - 2np_1 - 2n} > \frac{n+2-2p_1}{2}, \tag{4.11}$$

then the desired result (4.4) follows. So it remains to calculate the range of p_1 satisfying (4.11) and (4.10).

Remark 4.2. A necessary condition for validity of (4.11) is $p_1 > n/2$ since $\frac{1}{2}(n+2-2p_1) \geqslant 1$ for $p_1 \leqslant \frac{1}{2}n$.

By Remarks 4.1, 4.2 it suffices to consider p_1 in an interval

$$\Big(\frac{n}{2}, \frac{n^2 + 8n + \sqrt{n^4 - 64n}}{4(n+2)}\Big).$$

It is clear that (4.11) is equivalent to

$$2(n+2)p_1^3 - (n^2 + 6n)p_1^2 + (2n^2 - 6n)p_1 + 2n(n+2) > 0.$$

The three solutions for the corresponding equation are

$$\begin{aligned}
p_{1,1} &:= \frac{n^2 + 6n}{3(2n+4)} + \mathcal{R} + \frac{\mathcal{T}}{\mathcal{R}}, \\
p_{1,2} &:= \frac{n^2 + 6n}{3(2n+4)} - \frac{\mathcal{R}}{2} - \frac{\mathcal{T}}{2\mathcal{R}} + \frac{3^{1/2}}{2}\Big(\mathcal{R} - \frac{\mathcal{T}}{\mathcal{R}}\Big)i, \\
p_{1,3} &:= \frac{n^2 + 6n}{3(2n+4)} - \frac{\mathcal{R}}{2} - \frac{\mathcal{T}}{2\mathcal{R}} - \frac{3^{1/2}}{2}\Big(\mathcal{R} - \frac{\mathcal{T}}{\mathcal{R}}\Big)i,
\end{aligned}$$

where $\mathcal{R}$, $\mathcal{T}$ are defined in (2.8), that is,

$$\mathcal{R} := \Big(\mathcal{P} - \frac{n}{2} + \Big(\Big(\mathcal{P} - \frac{n}{2} + \mathcal{Q}\Big)^2 - \mathcal{T}^3\Big)^{1/2} + \mathcal{Q}\Big)^{1/3}, \quad \mathcal{P} := \frac{(n^2+6n)^3}{27(2n+4)^3},$$
$$\mathcal{Q} := \frac{(-2n^2+6n)(n^2+6n)}{6(2n+4)^2}, \quad \mathcal{T} := \frac{-2n^2+6n}{3(2n+4)} + \frac{(n^2+6n)^2}{9(2n+4)^2}.$$

These solutions can be calculated by symbolic operation from Matlab. Table 1 shows its values according to the space dimension.

n	$p_{1,1}$	$p_{1,2}$	$p_{1,3}$
5	2.7437	-0.8818	2.0667
6	3.2260	-0.8682	2.1422
7	3.7068	-0.8564	2.2051
8	4.1894	-0.8462	2.2568
9	4.6742	-0.8373	2.2995
10	5.1609	-0.8297	2.3354
50	25.0384	-0.7562	2.6408
100	50.0196	-0.7444	2.6856

Table 1. The values of $p_{1,1}$, $p_{1,2}$, $p_{1,3}$.

Thus, inequalities (4.11) and (4.10) hold for

$$p_1 \in \Big(p_{1,1}, \frac{n^2+8n+\sqrt{n^4-64n}}{4(n+2)}\Big)$$

and furthermore, (4.4) holds for $p(z)$ satisfying (2.7).

Remark 4.3. We note that here is the only point in the whole paper where we need the assumption $p_- > p_{1,1}$ for $n \geqslant 5$.

Step 2: Iterative use of Lemmas 3.7, 3.8. From the argument in Step 1 it suffices to prove Theorem 2.1 for $\widehat{\kappa} > \kappa(2p_1/((n+2)p_1 - 2n))$. We already proved the validity of (4.4) in this case. So all the assumptions of Lemma 3.7 are satisfied with $\sigma = \frac{1}{2}$ by (3.12). Thus, if $\widehat{\kappa} \in (0, \frac{1}{2})$, then

$$\text{(4.12)} \quad u \in N^{\widehat{\kappa},\infty}(I, \mathcal{H}_2(\Omega)) \cap N^{2\widehat{\kappa}/p_1,p_1}(I, \mathcal{V}_{p_1}(\Omega)), \quad V_{p(t)}(\mathcal{D}u) \in N^{\widehat{\kappa},2}(I, L^2(\Omega)),$$

which implies (2.12), while if $\widehat{\kappa} \geqslant \frac{1}{2}$, then

$$\text{(4.13)} \quad u \in N^{1/2,\infty}(I, \mathcal{H}_2(\Omega)) \cap N^{1/p_1,p_1}(I, \mathcal{V}_{p_1}(\Omega)), \quad V_{p(t)}(\mathcal{D}u) \in N^{1/2,2}(I, L^2(\Omega)),$$

which are exactly the assumptions of Lemma 3.8 with $\sigma = \frac{1}{2}$. This enables us to use Lemma 3.8.

If $p_1 > 4n/(n+2)$, then by Lemma 3.8 $\partial_t u \in N^{1/2,p_1'}(I, \mathcal{V}_{p_1}'(\Omega))$, so $u \in N^{3/2,p_1'}(I, \mathcal{V}_{p_1}'(\Omega))$ and in turn, by Proposition 2.2, $u \in N^{1/2(3/2+1/p_1),2}(I, \mathcal{H}_2(\Omega))$. Furthermore, one can see that if $u \in N^{\sigma,2}(I, \mathcal{H}_2(\Omega))$ with $\sigma \in [\frac{1}{2}, \widehat{\kappa}]$, then $u \in N^{1/2(1+\sigma+2\sigma/p_1),2}(I, \mathcal{H}_2(\Omega))$ by Lemmas 3.8, 3.7 and Proposition 2.2. Since for $\sigma < 1$ and $p_1 \geqslant 2$

$$\frac{1}{2}\Big(1 + \sigma + \frac{2\sigma}{p_1}\Big) - \sigma = \frac{1}{2} - \sigma\Big(\frac{1}{2} - \frac{1}{p_1}\Big) \geqslant \frac{1}{p_1},$$

we obtain $u \in N^{\widehat{\kappa},2}(I, \mathcal{H}_2(\Omega))$ after a finite number of iterations and again by Lemma 3.7 get (2.12).

Although in this subsection we consider only $p_1 > (3n+2)/(n+2)$, we here deal with the case $(3n+2)/(n+2) \leqslant p_1 \leqslant 4n/(n+2)$ in order to avoid overlap in next subsection. We will follow the same argument as above but need slightly more delicate attention. By Lemmas 3.7, 3.8 and Proposition 2.2 it follows that for $\sigma \in [\frac{1}{2}, \widehat{\kappa}]$

$$u \in N^{\sigma,2}(I, \mathcal{H}_2(\Omega)) \Rightarrow u \in N^{\sigma_{\widehat{\alpha}},2}(I, \mathcal{H}_2(\Omega)),$$

where

$$\sigma_{\widehat{\alpha}} := \frac{1}{2} + \sigma\Big(\frac{(n+2)p_1^2 - (4n+2)p_1 + 4n}{[(n+2)p_1 - 2n]p_1} + \frac{1}{p_1}\Big) = \frac{1}{2} + \sigma\frac{(n+2)p_1^2 - 3np_1 + 2n}{[(n+2)p_1 - 2n]p_1}.$$

Direct calculation shows that for $\sigma < 1$

$$\begin{aligned} \sigma_{\widehat{\alpha}} - \sigma &= \frac{1}{2} + \sigma\frac{-np_1 + 2n}{[(n+2)p_1 - 2n]p_1} \overset{\sigma<1,2-p_1\leqslant 0}{\geqslant} \frac{1}{2} + \frac{-np_1 + 2n}{[(n+2)p_1 - 2n]p_1} \\ &\geqslant \frac{(n+2)p_1^2 - 4np_1 + 4n}{[(n+2)p_1 - 2n]p_1} =: f(p_1). \end{aligned}$$

If $n \leqslant 6$, then $f(p_1)$ is decreasing in p_1 on the interval $[(3n+2)/(n+2), 4n/(n+2)]$. Moreover, the case $n > 6$ is excluded from the case $(3n+2)/(n+2) \leqslant p_1 \leqslant 4n/(n+2)$ since

$$p_1 > \frac{n}{2} > \frac{4n}{n+2} \quad \forall\, n > 6.$$

Thus, for $p_1 \in [(3n+2)/(n+2), 4n/(n+2)]$

$$\sigma_{\widehat{\alpha}} - \sigma \geqslant f\Big(\frac{4n}{n+2}\Big) = \frac{n+2}{4n}.$$

Finally, after a finite number of iterations we get $u \in N^{\widehat{\kappa},2}(I, \mathcal{H}_2(\Omega))$ and again by Lemma 3.7 the first statement of Theorem 2.1 is proved.

Case 2: $p_1 = (3n+2)/(n+2)$. In this case, we cannot use Lemma 3.5 and finally not follow Step 1 in Case 1 in Section 4 since

$$G(0) = G\Big(\frac{2p_1}{(n+2)p_1 - 2n}\Big) = 0 \quad \text{if } p_1 = \frac{3n+2}{n+2}.$$

So we use Lemma 3.6 and a slightly different method than before.

By Lemma 3.6 this is proved if $\widehat{\kappa} \leqslant \frac{1}{2}\widetilde{\kappa}p_1'$. So it remains to prove it for $\widehat{\kappa} > \frac{1}{2}\widetilde{\kappa}p_1'$, i.e., $2\overline{\tau} = \widetilde{\kappa}p_1'$.

As in Case 1 our first aim is to show (4.4). By (2.2) it suffices to prove that

$$u \in N^{\sigma_{\text{uni}},p_1}(I; \mathcal{V}_{p_1}(\Omega)) \tag{4.14}$$

for some

$$\sigma_{\text{uni}} > \frac{n+2-2p_1}{2p_1}.$$

Since $2\overline{\tau} = \widetilde{\kappa}p_1'$ and $p_1 - 1 = 2n/(n+2)$, we can easily see that by (3.39)

$$f(\widetilde{\sigma}) := \frac{2\overline{\tau}}{p_1} = \frac{\widetilde{\kappa}p_1'}{p_1} = \frac{\widetilde{\kappa}}{p_1 - 1} = \begin{cases} \widetilde{\sigma}(1-\widetilde{\tau}) + \dfrac{\widetilde{\tau}}{n} - \mu & \text{if } 0 < \widetilde{\sigma} < \dfrac{1}{n}, \quad \widetilde{\tau} > 0, \\ \dfrac{1}{2n} + \dfrac{1}{2}\widetilde{\sigma} & \text{if } \dfrac{1}{n} \leqslant \widetilde{\sigma} < \dfrac{1}{p_1}, \quad \widetilde{\tau} > 0, \\ \dfrac{1}{2n} & \text{if } \widetilde{\sigma} = 0, \quad \widetilde{\tau} = 0. \end{cases}$$

Here $\mu > 0$ is a real number arbitrarily close to 0. Then by Lemma 3.6 we get that if $u \in N^{\widetilde{\sigma},p_1}(I; \mathcal{V}_{p_1}(\Omega))$, then

$$u \in N^{f(\widetilde{\sigma}),p_1}(I; \mathcal{V}_{p_1}(\Omega)).$$

We can easily see that the mapping $\widetilde{\sigma} \mapsto f(\widetilde{\sigma})$ is a contraction on $(0, p_1^{-1})$. Moreover, it is clear that if $0 < \widetilde{\sigma} < n^{-1}$, then

$$f(\widetilde{\sigma}) - \widetilde{\sigma} = -\widetilde{\sigma}\widetilde{\tau} + \frac{\widetilde{\tau}}{n} - \mu = \widetilde{\tau}\Big(\frac{1}{n} - \widetilde{\sigma}\Big) - \mu > 0,$$

while if $n^{-1} \leqslant \widetilde{\sigma} < p_1^{-1}$, then

$$f(\widetilde{\sigma}) - \widetilde{\sigma} = \frac{1}{2n} + \frac{1}{2}\widetilde{\sigma} - \widetilde{\sigma} = \frac{1}{2}\Big(\frac{1}{n} - \widetilde{\sigma}\Big) \leqslant 0.$$

Hence, we can use Lemma 3.6 iteratively for $0 < \widetilde{\sigma} < n^{-1}$ and see that the fixed point of the mapping $\widetilde{\sigma} \mapsto f(\widetilde{\sigma})$ is arbitrarily close to n^{-1}.

Finally, we can get the desired result (4.14) if $(n+2-2p_1)/(2p_1) < n^{-1}$, which is possible since if $n \leqslant 4$, then

$$\frac{n+2}{2p_1} - 1 < \frac{1}{n} \leftrightarrows \frac{(n+2)^2}{2(3n+2)} < \frac{1+n}{n} \leftrightarrows n^3 - 4n^2 - 6n - 4 < 0$$

and if $n \geqslant 5$, then the case $p_1 = (3n+2)/(n+2)$ is excluded from consideration by (2.7) and Remark 4.1. Thus, we get (4.4).

So if $p_1 = (3n+2)/(n+2)$ and $p_1 > 2$, then we can prove the first statement of Theorem 2.1 by following the same argument as in Step 2 above because Lemmas 3.7, 3.8 hold even for $p_1 = (3n+2)/(n+2)$.

If $p_1 = 2$ and $n = 2$, then we will use Lemma 3.9 instead of Lemma 3.8. By Lemmas 3.9, 3.7 and Proposition 2.2 it follows that for $\sigma \in [\frac{1}{2}, \widehat{\kappa}]$

$$u \in N^{\sigma,2}(J, \mathcal{H}_2(\Omega)) \Rightarrow u \in N^{\sigma_{\widetilde{\alpha}},2}(I, \mathcal{H}_2(\Omega)),$$

where

$$\sigma_{\widetilde{\alpha}} := \frac{1}{2} + \sigma\Big(\frac{2(1+\delta)-p_2}{p_2\delta} + \frac{1}{2}\Big).$$

Direct calculation shows that for $\sigma < 1$

$$\sigma_{\widetilde{\alpha}} - \sigma = \frac{1}{2}(1-\sigma) + \sigma\frac{2(1+\delta)-p_2}{p_2\delta} \overset{\sigma<1}{\geqslant} \sigma\frac{2(1+\delta)-p_2}{p_2\delta} \overset{\sigma\geqslant 1/2}{\geqslant} \frac{2(1+\delta)-p_2}{2p_2\delta} > 0.$$

Finally, after a finite number of iterations we get $u \in N^{\widehat{\kappa},2}(I, \mathcal{H}_2(\Omega))$ and again by Lemma 3.7 the first statement of Theorem 2.1 is proved. □

5. Proof of the second statement of Theorem 2.1

To begin with, we claim that the second statement of Theorem 2.1 follows if we prove that for an interval $(0, t_2)$ satisfying (3.9)

(5.1) $u \in N^{\widehat{\kappa},\infty}(0, t_2; \mathcal{H}_2(\Omega)) \cap N^{\widehat{\kappa},2}(0, t_2; \mathcal{V}_2(\Omega)), \quad V_{p(z)}(\mathcal{D}u) \in N^{\widehat{\kappa},2}(0, t_2; L^2(\Omega)).$

Indeed, if (5.1) holds, then we can choose $t_1 \in (0, t_2 - h)$ such that

$$\|d^h u(t_1)\|_{2,\Omega} \leqslant ch^{\widehat{\kappa}}.$$

So we can consider t_1 as an initial time and hence, get (5.1) by using Lemmas 3.5, 3.6, 3.7 over a new interval (t_1, t_3) satisfying (3.9) instead of $(0, t_2)$. Thus, after a finite number of iterations we can get (2.13).

Furthermore, if (3.45) holds with $\widehat{\tau}$ arbitrary close to $\widehat{\kappa}$ for $\widehat{\kappa} < \frac{1}{2}$ and $\widehat{\tau} = \frac{1}{2}$ for $\widehat{\kappa} = \frac{1}{2}$, then from the arguments in Sections 3, 4 we can prove (5.1) without any difficulty. Thus, the rest of this section is devoted to proof of the validity of (3.45) for such $\widehat{\tau}$.

Lemma 5.1. *Let $p(z) \geqslant (3n+2)/(n+2)$, $t_1 = 0$ and $F \in N^{\widehat{\kappa},2}(0,t_2;L^2(\Omega))$ with $\widehat{\kappa} < \frac{1}{2}$. Let u be a weak solution to problem* (1.1) *with $u_0 \in \mathcal{V}_{p_+}$. Then we have for sufficiently small $h \in (0,t_2)$ and for all $\widehat{\tau} \in (0,\widehat{\kappa})$*

$$\|u(h) - u_0\|_2^2 \leqslant ch^{2\widehat{\tau}}, \tag{5.2}$$

where $c = c(n, p_-, p_+, c_*, \|F\|_{\widehat{\kappa},2}, \|\mathcal{D}u_0\|_{p_+(1+\delta)})$.

Moreover, if $F \in N^{1/2,2}(0,t_2;L^2(\Omega))$ and $\sup\limits_{h\in(0,T)} h^{-1}\int_0^h \|F\|_2^2 \,\mathrm{d}t < \infty$, then

$$\|u(h) - u_0\|_2^2 \leqslant ch. \tag{5.3}$$

P r o o f. We use basically the ideas in the proof of Lemma 4.1 of [11], with some modifications.

Recalling that u can be considered representative continuous in $[0,T]$ with value in $L^2(\Omega)$ yields that for sufficiently small $h \in (0,t_2)$

$$\begin{aligned}\|u(h) - u_0\|_2^2 &= (u(h)-u_0, u(h)-u_0) = \|u(h)\|_2^2 - \|u_0\|_2^2 - 2(u(h)-u_0,u_0) \\ &=: I_{10} + I_{11}.\end{aligned} \tag{5.4}$$

It follows from (3.2) and Young's inequality that

$$\begin{aligned}I_{10} &= \int_0^h \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 \,\mathrm{d}t = -\int_0^h\int_\Omega S(p(z),\mathcal{D}u(t)) : \mathcal{D}u(t)\,\mathrm{d}z + \int_0^h\int_\Omega F : \mathcal{D}u(t)\,\mathrm{d}z \\ &\overset{(3.2)}{\leqslant} -c\int_0^h\int_\Omega |\mathcal{D}u|^{p(t)}\,\mathrm{d}z + ch + c\int_0^h\int_\Omega |F|^{p'(t)}\,\mathrm{d}z.\end{aligned} \tag{5.5}$$

The term I_{11} can be rewritten as follows:

$$I_{11} = -2\int_0^h \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}u(t), u_0\Big\rangle_{1,p_+} \mathrm{d}t \leqslant c\int_0^h \|\partial_t u(t)\|_{\mathcal{V}'_{p_+}} \|u_0\|_{\mathcal{V}_{p_+}}\,\mathrm{d}t.$$

On the other hand, proceeding as in the derivation of (3.11) with p_+ instead of p_1, we arrive at

$$\int_0^h \|\partial_t u(t)\|_{\mathcal{V}'_{p_+}}\,\mathrm{d}t \leqslant C\int_0^h (1 + \||\mathcal{D}u|^{p(t)-1}\|_{p'_+} + \|u(t)\|_{1,p_1}^{p_1-1} + \|F(t)\|_{p'_+})\,\mathrm{d}t. \tag{5.6}$$

Here we note that the constant C in (5.6) depends on $\sup_{t\in(0,h)} \|u(t)\|_2$ by (3.11) and hence on $\|F(t)\|_{p_1'}$ via (3.8). Hence

$$(5.7)\quad I_{11} \leqslant c\|u_0\|_{\mathcal{V}_{p_+}} \int_0^h (1 + \||\mathcal{D}u|^{p(t)-1}\|_{p_+'} + \|u(t)\|_{1,p_1}^{p_1-1} + \|F(t)\|_{p_+'})\,\mathrm{d}t$$

$$\leqslant ch\|u_0\|_{\mathcal{V}_{p_+}}^{p_+} + \varepsilon \int_0^h (1 + \||\mathcal{D}u|^{p(t)-1}\|_{p_+'} + \|u(t)\|_{1,p_1}^{p_1-1} + \|F(t)\|_{p_+'})^{p_+'}\,\mathrm{d}t.$$

By Korn's inequality and the trivial inequality $(p_1-1)p_+' < p_1$ it is clear that

$$(5.8)\qquad \int_0^h \|u(t)\|_{1,p_1}^{(p_1-1)p_+'}\,\mathrm{d}t \leqslant c\int_0^h \int_\Omega |\mathcal{D}u(t)|^{p_1}\,\mathrm{d}z.$$

By the trivial inequality $(p(t)-1)p_+/(p_+-1) \leqslant p(t)$ we have

$$(5.9)\qquad \int_0^h \||\mathcal{D}u|^{p(t)-1}\|_{p_+',\Omega}^{p_+'}\,\mathrm{d}t \leqslant c(p_+)\int_0^h \int_\Omega (1+|\mathcal{D}u|)^{p(t)}\,\mathrm{d}z.$$

Joining (5.7) with (5.8) and (5.9) leads us to

$$(5.10)\qquad I_{11} \leqslant ch\|u_0\|_{\mathcal{V}_{p_+}}^{p_+} + \varepsilon\int_0^h \int_\Omega (1 + |\mathcal{D}u|^{p(t)} + |F(t)|^{p_+'})\,\mathrm{d}z.$$

Substituting (5.5) and (5.10) into (5.4) and using $p'(t) \leqslant 2$, we conclude that

$$(5.11)\qquad \|u(h)-u_0\|_2^2 \leqslant ch\|u_0\|_{\mathcal{V}_{p_1}}^{p_1} + c\int_0^h \int_\Omega (1+|F(t)|^2)\,\mathrm{d}z.$$

If $F \in N^{\widehat{\kappa},2}(0,t_2;L^2(\Omega))$, by embedding (2.2) and Hölder's inequality we obtain that $F \in L^{2/(1-2\widehat{\tau})}(0,t_2;L^2(\Omega))$ for any $\widehat{\tau} \in (0,\widehat{\kappa}]$ and

$$(5.12)\qquad \int_0^h \int_\Omega |F(t)|^2\,\mathrm{d}z \leqslant c\Big(\int_0^h \|F(t)\|_2^{2/(1-2\widehat{\tau})}\,\mathrm{d}t\Big)^{1-2\widehat{\tau}} h^{2\widehat{\tau}}.$$

It is clear that by $p_1 \leqslant p_+(1+\delta)$ and the condition $u_0 \in \mathcal{V}_{p_+(1+\delta)}$

$$(5.13)\qquad h\|u_0\|_{\mathcal{V}_{p_1}}^{p_1} \leqslant Ch.$$

Joining (5.11) with (5.12) and (5.13) we have the desired estimate (5.2).

Next let us prove estimate (5.3). We recall that by the assumption of Theorem 2.1

$$(5.14)\qquad \sup_{h\in(0,T)} \frac{1}{h}\int_0^h \|F\|_2^2\,\mathrm{d}t < \infty.$$

Joining (5.11) with (5.14) and (5.13) we have the desired estimate (5.3). □

6. Proof of Theorem 2.1: the case $\widehat{\kappa}=1$

By (2.12) with $\widehat{\kappa}<1$ we have $u\in N^{2\kappa/p_1,p_1}(J,\mathcal{V}_{p_1}(\Omega))$ for any $\frac{1}{2}<\kappa<1$ and so by Nikolskii embedding (2.3) it follows that

$$u\in L^{\infty}(J,\mathcal{V}_{p_1}(\Omega)). \tag{6.1}$$

On the other hand, we have $u\in N^{\kappa,p_1'}(J,\mathcal{V}_{p_1}'(\Omega))$ since $u\in N^{\kappa,\infty}(J,\mathcal{H}_2(\Omega))$ for any $\kappa\in(0,1)$ by (2.12) with $\widehat{\kappa}<1$ and $N^{\kappa,\infty}(J,\mathcal{H}_2(\Omega))\subset N^{\kappa,p_1'}(J,\mathcal{V}_{p_1}'(\Omega))$, while $\partial_t u\in N^{\kappa,p_1'}(J,\mathcal{V}_{p_1}'(\Omega))$ by Lemma 3.8 and (2.12) with $\widehat{\kappa}<1$. Hence, we get $u\in N^{1+\kappa,p_1'}(J,\mathcal{V}_{p_1}'(\Omega))$, which together with Proposition 2.2 implies that $u\in N^{\alpha,2}(J,\mathcal{H}_2(\Omega))$ for all $\alpha\in(1,1+p_1^{-1})$. Hence, we have

$$\partial_t u\in N^{\alpha-1,2}(I,\mathcal{H}_2(\Omega))\subset L^2(J,\mathcal{H}_2(\Omega)). \tag{6.2}$$

From (6.2) it follows that

$$\int_{J_h} h^{-2}\|d^h u(t)\|_2^2\,\mathrm{d}t\leqslant C. \tag{6.3}$$

Thus, from (3.17) it follows that

$$\begin{aligned}\sup_{t_1\in J_h} h^{-2}\eta\|d^h u(t)\|_2^2+h^{-2}\int_{\Omega\times J_h}\eta(|d^h V_{p(z)}(\mathcal{D}u)|^2\,\mathrm{d}z+|d^h\mathcal{D}u|^2)\,\mathrm{d}z\\ \overset{(3.56)}{\leqslant} C+ch^{-2}\int_{J_h}\|d^h F\|_2^2\,\mathrm{d}t+c\int_{J_h}\|\nabla u(t)\|_{p_1}^{p_1/(2p_1-n)}h^{-2}\|d^h u(t)\|_2^2\,\mathrm{d}t.\end{aligned}$$

This together with (6.1), (6.3) implies (2.12) with $\widehat{\kappa}=1$. □

7. Proof of Corollaries 2.1 and 2.2

Here the notations are the same as in the previous section.

Proof of Corollary 2.1. By Theorem 2.1 with $\widehat{\kappa}=1$ we have $\partial_t u\in L^{\infty}_{\mathrm{loc}}(0,T;L^2(\Omega))$ and $u\in L^{\infty}_{\mathrm{loc}}(0,T;W_0^{1,p(x,\cdot)}(\Omega))$ because of (3.6) and the embedding

$$W^{1,2}_{\mathrm{loc}}(0,T;L^2(\Omega))\subset L^{\infty}_{\mathrm{loc}}(0,T;L^2(\Omega)).$$

Let $J=(t_1,t_2)$ be an interval as before. Thus, we can consider that for a.e. $t\in J$, $u(t)$ is a weak solution to the steady problem

$$\begin{cases}-\mathrm{div}S(p(t),\mathcal{D}u(t))+(u(t)\cdot\nabla)u(t)+\nabla\pi(t)=\mathrm{div}\,F(t)-\partial_t u(t), & \text{in }\Omega,\\ \mathrm{div}\,u(t)=0, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega.\end{cases}$$

So we can apply the space-regularity result, Theorem 2.3 in [35] and hence get (2.16).

In particular, from Case 1 in Section 4 we get that $V_{p(z)}(\mathcal{D}u) \in W^{1,2}_{\rm loc}(\Omega)$ for a.e. $t \in J$ and hence (2.15). □

P r o o f of Corollary 2.2. It is well known that if $p_- \geqslant \frac{1}{2}(n+2)$, then problem (1.1) has a unique solution for $u_0 \in \mathcal{H}_2(\Omega)$, see [24]. So it suffices to prove this Corollary for $p_- < \frac{1}{2}(n+2)$. From Lemmas 3.5, 3.6 and Step 1 in Cases 1 and 2 in Section 4, we can see that if $F \in N^{\tau,2}(t_1,t_2;L^2(\Omega))$ with $\tau > \frac{1}{4}(n+2-2p_1)$, then

$$u_1, u_2 \in L^{2p_1/(2p_1-n)}(J, \mathcal{V}_{p_1}(\Omega)) \subset L^{2p_-/(2p_--n)}(J, \mathcal{V}_{p_-}(\Omega)). \tag{7.1}$$

In particular, we note that this continues to hold for $t_1 = 0$ if $u_1(0) = u_2(0) \in \mathcal{V}_{p_+(1+\delta)}(\Omega)$.

Since the rest is similar to the argument in [10], [11] we will omit it. □

Remark 7.1. Corollary 2.2 continues to hold for $F \in N^{\tau,p'_-}(0,T;L^{p'_-}(\Omega))$ with $\tau > \tau_{\rm uni} := \max\{0, (p_- - 1)((n+2)/(2p_-) - 1)\}$.

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