Shao-Yuan Huang; Ping-Han Hsieh

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EXACT MULTIPLICITY AND BIFURCATION CURVES OF POSITIVE SOLUTIONS OF GENERALIZED LOGISTIC PROBLEMS

Shao-Yuan Huang, Ping-Han Hsieh, Taipei

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Abstract. We study the exact multiplicity and bifurcation curves of positive solutions of generalized logistic problems

$$\begin{cases} -[\varphi(u')]' = \lambda u^p\Big(1-\dfrac{u}{N}\Big) & \text{in } (-L,L),\\ u(-L)=u(L)=0, \end{cases}$$

where $p>1$, $N>0$, $\lambda>0$ is a bifurcation parameter, $L>0$ is an evolution parameter, and $\varphi(u)$ is either $\varphi(u)=u$ or $\varphi(u)=u/\sqrt{1-u^2}$. We prove that the corresponding bifurcation curve is $\subset$-shape. Thus, the exact multiplicity of positive solutions can be obtained.

Keywords: positive solution; bifurcation curve; Minkowski-curvature problem, logistic problem

MSC 2020: 34B15, 34B18, 34C23, 74G35

1. Introduction

In this paper, we study the exact multiplicity and bifurcation curves of positive solutions of generalized logistic problems

$$\begin{cases} -[\varphi(u')]' = \lambda u^p\Big(1-\dfrac{u}{N}\Big) & \text{in } (-L,L),\\ u(-L)=u(L)=0, \end{cases} \tag{1.1}$$

where $p>1$, u is the population density, $N>0$ is the carrying capacity of the environment, $\lambda>0$ is a bifurcation parameter and the intrinsic growth rate, $L>0$ is an evolution parameter, and either $\varphi(u)=u$ or $\varphi(u)=u/\sqrt{1-u^2}$. Let

$$f(u)\equiv u^p\Big(1-\frac{u}{N}\Big).$$

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Notice that $g(u) = f(u)/u$ satisfies that $g(0) = 0$, $g'(u) > 0$ on $(0, pN/(p+1))$, $g'(u) < 0$ on $(pN/(p+1), N)$, $g(N) = 0$ and $g(u) < 0$ on (N, ∞). Thus, we also say that problem (1.1) has *weak Allee effect type*, cf. [9].

Assume that $\varphi(u) = u$. The solutions of (1.1) are the steady state solutions of a reaction-diffusion population model in one space dimension. A typical form of reaction-diffusion population model equation is

$$\frac{\partial}{\partial t} u = d\Delta u + f(u),$$

where $u(x,t)$ is the population density, $d > 0$ is the diffusion constant and $f(u)/u$ is the growth rate per capita. We refer to the work of McCabe, Leach and Needham (see [14]), Shi and Shivaji (see [15]), Wang and Kot (see [18]), and Xin (see [19]) and the references therein. For $L > 0$, we define the bifurcation curve $\overline{S}_L$ of (1.1) on the $(\lambda, \|u\|_\infty)$-plane by

$$\overline{S}_L \equiv \{(\lambda, \|u_\lambda\|_\infty)\colon\ \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1) with } \varphi(u) = u\}.$$

Assume that $\varphi(u) = u/\sqrt{1-u^2}$. Problem (1.1) plays an important role in certain fundamental issues in differential geometry and in the special theory of relativity, see for example [3], [5]. We refer the readers, for motivations and results, to [1], [7], [8], [9], [10] and the references cited therein. For any $L > 0$, we define the bifurcation curve $\widehat{S}_L$ of (1.1) on the $(\lambda, \|u\|_\infty)$-plane by

$$\widehat{S}_L = \big\{(\lambda, \|u_\lambda\|_\infty)\colon\ \lambda > 0 \text{ and } u_\lambda \in C^2(-L,L) \cap C[-L,L] \\ \text{is a positive solution of (1.1) with } \varphi(u) = u/\sqrt{1-u^2}\big\}.$$

Next, we give some terminologies related to the shapes of bifurcation curves $\overline{S}_L$ on the $(\lambda, \|u\|_\infty)$-plane (while similar terminologies for $\widehat{S}_L$ also hold).

- ▷ Monotone incresing (see Figure 1 (i)): We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $\overline{S}_L$ is strictly increasing if $\overline{S}_L$ consists of a continuous curve and for each pair of points $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ and $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ of $\overline{S}_L$, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ implies $\lambda_1 \leqslant \lambda_2$.
- ▷ $\subset$-shaped (see Figure 1 (ii)): We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $\overline{S}_L$ is $\subset$-shaped if $\overline{S}_L$ consists of a continuous curve with exactly one turning point, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, and
  - (i) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation diagram $\overline{S}_L$ turns to the right,
  - (ii) $\overline{S}_L$ initially continues to the left and eventually continues to the right.
- ▷ S-shaped (see Figure 1 (iii)): We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $\overline{S}_L$ is S-shaped if $\overline{S}_L$ has *exactly two turning* points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, where $\lambda_* < \lambda^*$ are two positive numbers such that

(i) $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$,
(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve $\overline{S}_L$ turns to the left,
(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve $\overline{S}_L$ turns to the right.

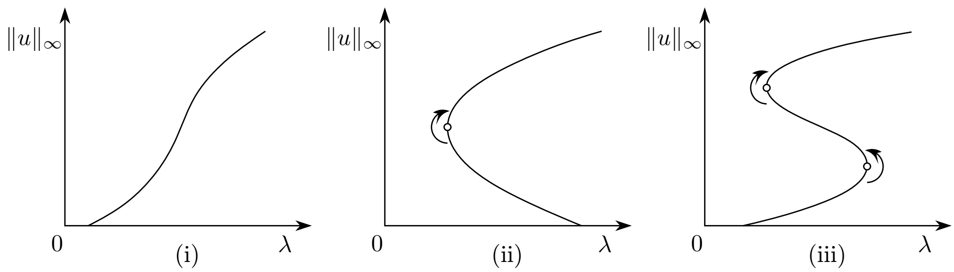

Figure 1. (i) Monotone increasing, (ii) ⊂-shaped, (iii) S-shaped.

In 1838, Verhulst proposed in [17] to study the logistic equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u\Big(1 - \frac{u}{N}\Big), \tag{1.2}$$

where u is the population density, λ is the intrinsic growth rate, and N is the carrying capacity of the environment. Obviously, every nontrivial solution of (1.2) converges monotonically to N as $t \to \infty$. Chafee and Infante in [2] studied the one-dimensional diffusive generalized logistic problem

$$\begin{cases} -u'' = \lambda u(1 - u^q) & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{1.3}$$

where $q > 0$, and they showed the global existence of positive solutions. Then Guedda and Veron in [6] extended the results of (1.3) to the problem

$$\begin{cases} -(\|u_x\|^{p-1}u_x)_x = \lambda u^p(1 - u^q) & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{1.4}$$

where $p, q > 0$. Then Takeuchi and Yamada in Theorem 3.3 of [16] obtained complete information on the global bifurcation structure of positive solutions of (1.4).

Hung and Wang in Theorem 2.1 of [12] studied the semilinear problem

$$\begin{cases} -u'' = \lambda(-au^3 + bu^2 + cu + d) & \text{in } (-L,L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.5}$$

where $a, b, d > 0$ and $c \geqslant 0$. They proved that there exists $a_0 > 0$ such that the corresponding bifurcation curve of (1.5) is S-shaped for $0 < a < a_0$ and monotone increasing for $a \geqslant a_0$. Huang in Theorem 3.2 of [9] studied the Minkowski-curvature problem

$$(1.6)\qquad \begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = \lambda(-au^3+bu^2+cu+d) & \text{in } (-L,L),\\ u(-L)=u(L)=0, \end{cases}$$

where $a,b,d>0$ and $c \geqslant 0$. He proved that

(i) if $a \geqslant a_0$, then the corresponding bifurcation curve of (1.6) is monotone increasing for all $L>0$, and

(ii) if $0 \leqslant a < a_0$, then there exists $L_0>0$ such that the corresponding bifurcation curve of (1.6) is monotone increasing for $0<L\leqslant L_0$ and S-like shape for $L>L_0$.

In this paper, the nonlinearity of (1.1) is a certain polynomial with a higher degree and without a constant term. As a comparison, the bifurcation curves of problems (1.5) and (1.6) are either monotone increasing or S-shaped but the bifurcation curve of problem (1.1) is $\subset$-shaped.

The paper is organized as follows. Section 2 contains statements of the main results. Section 3 contains several lemmas needed to prove the main results. Section 4 contains the proofs of the main results.

2. Main results

In this section, we present the shapes of bifurcation curve $\overline{S}_L$ of (1.1) with $\varphi(u)=u$, see Theorem 2.1, and the shapes of bifurcation curve $\widehat{S}_L$ of (1.1) with $\varphi(u)=u/\sqrt{1-u^2}$, see Theorem 2.2.

Theorem 2.1 (see Figure 2 (i)). *Consider* (1.1) *with* $\varphi(u)=u$. *Then the following statements hold:*

(i) *The bifurcation curve* $\overline{S}_L$ *is* $\subset$*-shaped on the plane* $(\lambda,\|u\|_\infty)$, *starts from* $(\infty,0)$ *and goes to* (∞,N) *for* $L>0$.

(ii) *For* $L>0$, *there exists* $\bar\lambda_L>0$ *such that* (1.1) *has no positive solutions for* $0<\lambda<\bar\lambda_L$, *exactly one positive solution* $\bar u_L$ *for* $\lambda=\bar\lambda_L$ *and exactly two positive solutions for* $\lambda>\bar\lambda_L$. *Furthermore,*

 (a) $\bar\lambda_L$ *is a continuous and strictly decreasing function with respect to* $L>0$.

 (b) $\kappa \equiv \|\bar u_L\|_\infty$ *is a constant for* $L>0$.

 (c)
$$\lim_{L\to 0^+}(\bar\lambda_L,\|\bar u_L\|_\infty)=(\infty,\kappa) \quad \textit{and} \quad \lim_{L\to\infty}(\bar\lambda_L,\|\bar u_L\|_\infty)=(0,\kappa).$$

Theorem 2.2 (see Figure 2 (ii)). *Consider* (1.1) *with* $\varphi(u)=u/\sqrt{1-u^2}$. *Then the following statements hold:*

(i) *The bifurcation curve* $\widehat{S}_L$ *is* $\subset$*-shaped on the plane* $(\lambda,\|u\|_\infty)$, *starts from* $(\infty,0)$ *and goes to* $(\infty,\min\{N,L\})$ *for* $L>0$.

(ii) *For* $L > 0$, *there exists* $\hat{\lambda}_L > 0$ *such that* (1.1) *has no positive solutions for* $0 < \lambda < \hat{\lambda}_L$, *exactly one positive solution* $\widehat{u}_L$ *for* $\lambda = \hat{\lambda}_L$ *and exactly two positive solutions for* $\lambda > \hat{\lambda}_L$. *Furthermore,*

   (a) $\hat{\lambda}_L$ *is a continuous and strictly decreasing function with respect to* $L > 0$.

   (b) $\|\widehat{u}_L\|_\infty$ *is a continuous function with respect to* $L > 0$. *In addition, when* $1 < p \leqslant 4$, $\|\widehat{u}_L\|_\infty$ *is strictly increasing with respect to* $L > 0$.

   (c) *Let* κ *be defined in Theorem* 2.1. *Then*

$$\lim_{L \to 0^+} (\hat{\lambda}_L, \|\widehat{u}_L\|_\infty) = (\infty, 0) \quad \text{and} \quad \lim_{L \to \infty} (\hat{\lambda}_L, \|\widehat{u}_L\|_\infty) = (0, \kappa).$$

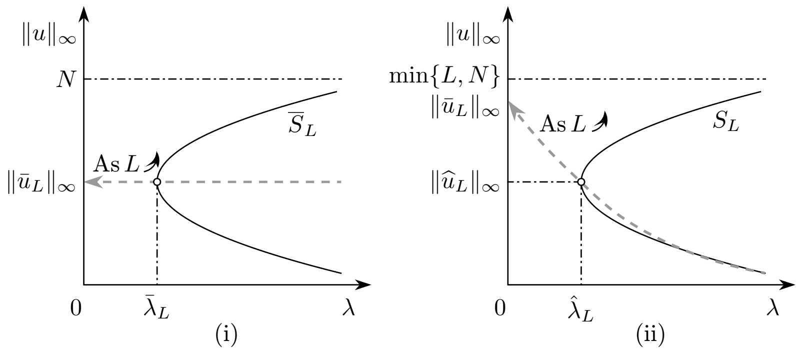

Figure 2. (i) The graph of bifurcation curve $\overline{S}_L$ of (1.1) with $\varphi(u) = u$ and the grey dashed line is the orbit of $(\bar{\lambda}_L, \|\bar{u}_L\|_\infty)$ for $L > 0$. (ii) The graph of bifurcation curve $\widehat{S}_L$ of (1.1) with $\varphi(u) = u/\sqrt{1-u^2}$ and the grey dashed line is the orbit of $(\hat{\lambda}_L, \|\widehat{u}_L\|_\infty)$ for $L > 0$.

Remark 2.1. In Theorem 2.2 (ii) (b), by numerical simulations, we conjecture that $\|\widehat{u}_L\|_\infty$ is strictly increasing with respect to $L > 0$ for all $p > 4$. Further investigation is needed.

3. Lemmas

To prove Theorems 2.1 and 2.2, we first introduce the time-map method used in [13] and [4], page 127. We define the time-map formula for (1.1) with $\varphi(u) = u$ by

$$\overline{T}_L(\alpha) \equiv \frac{1}{L} \int_0^\alpha \frac{1}{\sqrt{F(\alpha) - F(u)}} \, \mathrm{d}u \quad \text{for } 0 < \alpha < N, \tag{3.1}$$

and the time-map formula for (1.1) with $\varphi(u) = u/\sqrt{1-u^2}$ by

$$\widehat{T}_\lambda(\alpha) \equiv \int_0^\alpha \frac{\lambda[F(\alpha) - F(u)] + 1}{\sqrt{\{\lambda[F(\alpha) - F(u)] + 1\}^2 - 1}} \, \mathrm{d}u \quad \text{for } 0 < \alpha < N \text{ and } \lambda > 0, \tag{3.2}$$

where $F(u) \equiv \int_0^u f(t)\,\mathrm{d}t$. Observe that positive solutions $u_\lambda \in C^2(-L,L) \cap C[-L,L]$ for (1.1) with $\varphi(u) = u$ correspond to

$$\|u_\lambda\|_\infty = \alpha \quad \text{and} \quad \overline{T}_L(\alpha) = \sqrt{\lambda}, \tag{3.3}$$

and positive solutions $u_\lambda \in C^2(-L,L) \cap C[-L,L]$ for (1.1) with $\varphi(u) = u/\sqrt{1-u^2}$ correspond to

$$\|u_\lambda\|_\infty = \alpha \quad \text{and} \quad \widehat{T}_\lambda(\alpha) = L.$$

So by definitions of $\overline{S}_L$ and $\widehat{S}_L$, we have that

$$\overline{S}_L = \{(\lambda,\alpha)\colon\ \overline{T}_L(\alpha) = \sqrt{\lambda} \text{ for some } \alpha \in (0,N) \text{ and } \lambda > 0\} \tag{3.4}$$

and

$$\widehat{S}_L = \{(\lambda,\alpha)\colon\ \widehat{T}_\lambda(\alpha) = L \text{ for some } \alpha \in (0,N) \text{ and } \lambda > 0\}.$$

Thus, it is important to understand fundamental properties of the time-maps $\overline{T}_L(\alpha)$ and $\widehat{T}_\lambda(\alpha)$ on $(0,N)$ in order to study the shapes of the bifurcation curves $\overline{S}_L$ and $\widehat{S}_L$ of (1.1) for any fixed $L > 0$. Note that it can be proved that $\overline{T}_L(\alpha)$ and $\widehat{T}_\lambda(\alpha)$ are twice continuously differentiable functions of $\alpha \in (0,N)$ and $\lambda > 0$. The proofs are easy but tedious and hence we omit them.

Lemma 3.1. *Consider* (1.1). *Then the following statements hold:*

(i) $\lim\limits_{\alpha\to 0^+} \overline{T}_L(\alpha) = \lim\limits_{\alpha\to N^-} \overline{T}_L(\alpha) = \infty$ *for* $L > 0$.

(ii) *For* $L > 0$, *there exists* $\overline{\alpha} \in (0,N)$, *independent of* L, *such that*

$$\overline{T}'_L(\alpha) \begin{cases} < 0 & \text{for } 0 < \alpha < \overline{\alpha}, \\ = 0 & \text{for } \alpha = \overline{\alpha}, \\ > 0 & \text{for } \overline{\alpha} < \alpha < N. \end{cases} \tag{3.5}$$

Furthermore, $\overline{T}_L(\overline{\alpha})$ *is a continuous and strictly decreasing function with respect to* $L > 0$,

$$\lim_{L\to 0^+} \overline{T}_L(\overline{\alpha}) = \infty \quad \text{and} \quad \lim_{L\to\infty} \overline{T}_L(\overline{\alpha}) = 0.$$

Proof. (i) We let

$$T(\alpha) \equiv L\overline{T}_L(\alpha) = \int_0^\alpha \frac{1}{\sqrt{F(\alpha)-F(u)}}\,\mathrm{d}u \quad \text{for } 0 < \alpha < N. \tag{3.6}$$

Clearly, $T(\alpha)$ is independent of L. By [13], Theorems 2.6 and 2.9 we have

$$\lim_{\alpha\to 0^+} T(\alpha) = \lim_{\alpha\to N^-} T(\alpha) = \infty. \tag{3.7}$$

So statement (i) holds by (3.6).

(ii) By (3.6), we compute

(3.8)
$$T'(\alpha) = \frac{1}{2\sqrt{2}\alpha}\int_0^\alpha \frac{2B-A}{B^{3/2}}\,\mathrm{d}t \quad \text{and} \quad T''(\alpha) = \frac{1}{4\sqrt{2}\alpha^2}\int_0^\alpha \frac{3A^2-4AB-2BC}{B^{5/2}}\,\mathrm{d}u,$$

where

$$A \equiv \alpha f(\alpha) - uf(u), \quad B \equiv F(\alpha) - F(u), \quad C \equiv \alpha^2 f'(\alpha) - u^2 f'(u), \tag{3.9}$$

cf. [11], (14) and (16). Let $X = \alpha^{p+1} - u^{p+1}$ and $Y = \alpha^{p+2} - u^{p+2}$. Clearly, $X > 0$ and $Y > 0$ for $0 < u < \alpha$. Then it is easy to see that

$$A = X - \frac{Y}{N}, \quad B = \frac{X}{p+1} - \frac{Y}{(p+2)N} \quad \text{and} \quad C = pX - \frac{(p+1)Y}{N}. \tag{3.10}$$

Since $p > 1$, we see that

$$p(p+1) = (p+2)(p-1) + 2 > (p+2)(p-1). \tag{3.11}$$

Since $p > 1$, and by (3.10) and (3.11), we observe that for $0 < u < \alpha$,

(3.12)
$$\begin{aligned} 3A^2 - 4AB - 2BC &= \frac{(p+2)(p-1)N^2X^2 - 2(p+2)(p-1)NXY + p(p+1)Y^2}{(p+2)(p+1)N^2} \\ &> \frac{(p+2)(p-1)N^2X^2 - 2(p+2)(p-1)NXY + (p+2)(p-1)Y^2}{(p+2)(p+1)N^2} \\ &= \frac{(p-1)(NX-Y)^2}{(p+1)N^2} \geqslant 0. \end{aligned}$$

So by (3.8), we see that $T''(\alpha) > 0$ for $0 < \alpha < N$. Then by (3.7), there exists $\overline{\alpha} \in (0, N)$, independent of L, such that

$$T'(\alpha) \begin{cases} < 0 & \text{for } 0 < \alpha < \overline{\alpha}, \\ = 0 & \text{for } \alpha = \overline{\alpha}, \\ > 0 & \text{for } \overline{\alpha} < \alpha < N. \end{cases}$$

So by (3.6), (3.5) holds. Since $T(\overline{\alpha})$ is independent on L, we see that $\overline{T}_L(\overline{\alpha}) = T(\overline{\alpha})/L$ is a continuous and strictly decreasing function with respect to $L > 0$,

$$\lim_{L\to 0^+} \overline{T}_L(\overline{\alpha}) = \lim_{L\to 0^+} \frac{T(\overline{\alpha})}{L} = \infty \quad \text{and} \quad \lim_{L\to\infty} \overline{T}_L(\overline{\alpha}) = \lim_{L\to\infty} \frac{T(\overline{\alpha})}{L} = 0.$$

The proof is complete. □

Lemma 3.2. *Consider* (1.1). *Then the following statements hold:*

(i) $\lim_{\alpha\to 0^+} \widehat{T}_\lambda(\alpha) = \lim_{\alpha\to N^-} \widehat{T}_\lambda(\alpha) = \infty$.

(ii) *For* $\lambda > 0$, *there exists* $\alpha_\lambda \in (0, N)$ *such that*

$$\widehat{T}'_\lambda(\alpha) \begin{cases} < 0 & \text{for } 0 < \alpha < \alpha_\lambda, \\ = 0 & \text{for } \alpha = \alpha_\lambda, \\ > 0 & \text{for } \alpha_\lambda < \alpha < N, \end{cases} \quad \text{and} \quad \widehat{T}''_\lambda(\alpha_\lambda) > 0. \tag{3.13}$$

Proof. (i) Since

$$\lim_{u\to 0^+} \frac{F(u)}{u^2} = \lim_{u\to 0^+} \frac{f(u)}{2u} = \lim_{u\to 0^+} \frac{u^{p-1}}{2}\Big(1 - \frac{u}{N}\Big) = 0, \tag{3.14}$$

and by [9], Lemmas 4.1 and 4.2 we see that $\lim_{\alpha\to 0^+} \widehat{T}_\lambda(\alpha) = \lim_{\alpha\to N^-} \widehat{T}_\lambda(\alpha) = \infty$. Thus, statement (i) holds.

(ii) We compute

$$\widehat{T}'_\lambda(\alpha) = \frac{1}{\alpha} \int_0^\alpha \frac{\lambda^3 B^3 + 3\lambda^2 B^2 + \lambda[2B - A]}{(\lambda^2 B^2 + 2\lambda B)^{3/2}} \,\mathrm{d}u \tag{3.15}$$

and

$$\widehat{T}''_\lambda(\alpha) = \frac{1}{\alpha^2} \int_0^\alpha \frac{\lambda^3 B(3A^2 - 2AB - BC) + \lambda(3A^2 - 4AB - 2BC)}{(\lambda^2 B^2 + 2\lambda B)^{5/2}} \,\mathrm{d}u, \tag{3.16}$$

where A, B and C are defined by (3.9). Since $f(u) > 0$ on $(0, N)$, we see that

$$B = \int_u^\alpha f(t)\,\mathrm{d}t > 0 \quad \text{for } 0 < u < \alpha \text{ and } 0 < \alpha < N. \tag{3.17}$$

Clearly,

$$(2p+3)(p+1) = (p+2)(2p+1) + 1 > (p+2)(2p+1). \tag{3.18}$$

By (3.10) and (3.18), we observe that for $0 < u < \alpha$,

$$\begin{aligned} 3A^2 - 2AB - BC &= \frac{(p+2)(2p+1)N^2X^2 - 2(p+2)(2p+1)NXY + (2p+3)(p+1)Y^2}{(p+2)(p+1)N^2} \\ &> \frac{(p+2)(2p+1)N^2X^2 - 2(p+2)(2p+1)NXY + (p+2)(2p+1)Y^2}{(p+2)(p+1)N^2} \\ &= \frac{(2p+1)(NX - Y)^2}{(p+1)N^2} \geqslant 0. \end{aligned} \tag{3.19}$$

By (3.12), (3.16), (3.17) and (3.19), we obtain that $\widehat{T}''_\lambda(\alpha) > 0$ for $0 < \alpha < N$ and $\lambda > 0$. Then by statement (i), there exists $\alpha_\lambda \in (0, N)$ such that (3.13) holds. The proof is complete. □

Lemma 3.3. *Consider* (1.1). *Let* α_λ *be defined in Lemma* 3.2. *Then*

$$0 < \alpha_\lambda < \frac{(p+3)(p-1)N}{p(p+2)} \quad \textit{for } \lambda > 0. \tag{3.20}$$

Proof. We compute and find that

$$[2B-A]_{u=0} = 2F(\alpha) - \alpha f(\alpha) = \frac{p\alpha^{p+1}}{N(p+2)}\Big[\alpha - \frac{(p-1)(p+2)N}{p(p+1)}\Big] \tag{3.21}$$

and

$$\frac{\partial}{\partial u}[2B-A] = \frac{pu^p}{N}\Big[\frac{(p-1)N}{p} - u\Big]\begin{cases} >0 & \text{for } 0<u<\dfrac{(p-1)N}{p},\\ =0 & \text{for } u=\dfrac{(p-1)N}{p},\\ <0 & \text{for } u>\dfrac{(p-1)N}{p}.\end{cases} \tag{3.22}$$

In addition, it is easy to observe that

$$\frac{(p-1)N}{p} < \frac{(p+3)(p-1)N}{p(p+2)} < \frac{(p-1)(p+2)N}{p(p+1)}. \tag{3.23}$$

Since $[2B-A]_{u=\alpha}=0$, and by (3.21)–(3.23), we obtain that

$$2B-A>0 \quad \text{for } 0<u<\alpha \text{ and } \alpha \geqslant \frac{(p-1)(p+2)N}{p(p+1)}.$$

So by (3.15) and (3.17),

$$\widehat{T}'_\lambda(\alpha) > 0 \quad \text{for } \alpha \geqslant \frac{(p-1)(p+2)N}{p(p+1)} \text{ and } \lambda>0. \tag{3.24}$$

Next, we assume that

$$\frac{(p+3)(p-1)N}{p(p+2)} \leqslant \alpha < \frac{(p-1)(p+2)N}{p(p+1)}. \tag{3.25}$$

Since $[2B-A]_{u=\alpha}=0$, and by (3.21)–(3.23), there exists $\varepsilon_\alpha \in (0,\alpha)$ such that

$$2B-A\begin{cases} <0 & \text{for } 0<u<\varepsilon_\alpha,\\ =0 & \text{for } u=\varepsilon_\alpha,\\ >0 & \text{for } \varepsilon_\alpha<u<\alpha.\end{cases} \tag{3.26}$$

For the sake of convenience, let $B(\alpha,u)=B$. By (3.15), (3.17), (3.25) and (3.26), we observe that

$$\begin{aligned}\widehat{T}'_\lambda(\alpha) &\geqslant \frac{\lambda}{\alpha}\int_0^\alpha \frac{2B-A}{(\lambda^2B^2+2\lambda B)^{3/2}}\,\mathrm{d}u\\ &= \frac{\lambda}{\alpha}\Big[\int_0^{\varepsilon_\alpha} \frac{2B-A}{(\lambda^2B^2+2\lambda B)^{3/2}}\,\mathrm{d}u + \int_{\varepsilon_\alpha}^\alpha \frac{2B-A}{(\lambda^2B^2+2\lambda B)^{3/2}}\,\mathrm{d}u\Big]\\ &> \frac{\lambda}{\alpha}\Big[\int_0^{\varepsilon_\alpha} \frac{2B-A}{[\lambda^2B^2(\alpha,\varepsilon_\alpha)+2\lambda B(\alpha,\varepsilon_\alpha)]^{3/2}}\,\mathrm{d}u\\ &\quad + \int_{\varepsilon_\alpha}^\alpha \frac{2B-A}{[\lambda^2B^2(\alpha,\varepsilon_\alpha)+2\lambda B(\alpha,\varepsilon_\alpha)]^{3/2}}\,\mathrm{d}u\Big]\\ &= \frac{\lambda}{\alpha[\lambda^2B^2(\alpha,\varepsilon_\alpha)+2\lambda B(\alpha,\varepsilon_\alpha)]^{3/2}}\int_0^\alpha (2B-A)\,\mathrm{d}u\\ &= \frac{\lambda}{\alpha[\lambda^2B^2(\alpha,\varepsilon_\alpha)+2\lambda B(\alpha,\varepsilon_\alpha)]^{3/2}}\,\frac{p\alpha^{p+2}}{N(p+3)}\Big[\alpha-\frac{(p+3)(p-1)N}{p(p+2)}\Big] \geqslant 0.\end{aligned}$$

So by (3.24), we have

$$\widehat{T}'_\lambda(\alpha)>0 \quad \text{for } \alpha \geqslant \frac{(p+3)(p-1)N}{p(p+2)} \text{ and } \lambda>0.$$

It follows that (3.20) holds because $\widehat{T}'_\lambda(\alpha_\lambda)=0$ for $\lambda>0$. The proof is complete. □

Lemma 3.4. *Consider* (1.1). *Then*

(3.27)
$$\frac{\partial}{\partial\lambda}\big[\sqrt{\lambda}\widehat{T}'_\lambda(\alpha)\big] = \sqrt{\lambda}\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)+\frac{1}{2\sqrt{\lambda}}\widehat{T}'_\lambda(\alpha)>0 \quad \textit{for } 0<\alpha\leqslant\frac{(p+3)N}{p+4} \textit{ and } \lambda>0.$$

Furthermore, $\alpha_\lambda<(p+3)N/(p+4)$ *for sufficiently large* $\lambda>0$.

Proof. We compute

(3.28) $$\frac{\partial}{\partial u}(A+2B)=\frac{(p+4)u^p}{N}\Big[u-\frac{(p+3)N}{p+4}\Big]<0 \quad \text{for } 0<u<\alpha\leqslant\frac{(p+3)N}{p+4}.$$

Since $(A+2B)_{u=\alpha}=0$ and by (3.28), we see that

(3.29) $$A+2B>0 \quad \text{for } 0<u<\alpha\leqslant\frac{(p+3)N}{p+4}.$$

By (3.17) and (3.29), we further see that

$$\begin{aligned}\frac{\partial}{\partial\lambda}\big[\sqrt{\lambda}\widehat{T}'_\lambda(\alpha)\big] &= \sqrt{\lambda}\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)+\frac{1}{2\sqrt{\lambda}}\widehat{T}'_\lambda(\alpha)\\ &= \frac{1}{2\alpha\sqrt{\lambda}}\int_0^\alpha \frac{\lambda^3B^2(B^3\lambda^2+5B^2\lambda+3A+6B)}{(\lambda^2B^2+2\lambda B)^{5/2}}\,\mathrm{d}u\\ &>0 \quad \text{for } 0<\alpha\leqslant\frac{(p+3)N}{p+4}.\end{aligned}$$

It implies that (3.27) holds. By (3.15), we compute

$$\lim_{\lambda\to\infty} \widehat{T}'_\lambda\Big(\frac{(p+3)N}{p+4}\Big) = 1.$$

So by (3.27), there exists $\lambda_1 > 0$ such that

$$\widehat{T}'_\lambda\Big(\frac{(p+3)N}{p+4}\Big) > 0 \quad \text{for } \lambda \geqslant \lambda_1.$$

It follows that

$$\alpha_\lambda < \frac{(p+3)N}{p+4} \quad \text{for } \lambda \geqslant \lambda_1$$

by Lemma 3.2 (ii). The proof is complete. □

Lemma 3.5. *Consider* (1.1). *Let* $\overline{\alpha}$ *and* α_λ *be defined in Lemmas* 3.1 *and* 3.2, *respectively. Then the following statements hold:*

(i) α_λ *is a continuously differentiable function with respect to* $\lambda > 0$.

(ii) $\widehat{T}_\lambda(\alpha_\lambda)$ *is a continuous and strictly decreasing function with respect to* $\lambda > 0$. *Moreover,*

$$0 = \lim_{\lambda\to\infty} \widehat{T}_\lambda(\alpha_\lambda) < \lim_{\lambda\to 0^+} \widehat{T}_\lambda(\alpha_\lambda) = \infty.$$

(iii) $0 = \lim_{\lambda\to\infty} \alpha_\lambda < \lim_{\lambda\to 0^+} \alpha_\lambda = \overline{\alpha}$.

(iv) *If* $1 < p \leqslant 4$, *then* α_λ *is strictly decreasing for* $\lambda > 0$.

Proof. We divide this proof into the next five steps.

Step 1: We prove statement (i). Statement (i) follows immediately from Lemma 3.2 (ii) and the implicit function theorem.

Step 2: We prove statement (ii). By statement (i), it is easy to see that $\widehat{T}_\lambda(\alpha_\lambda)$ is a continuously differentiable function with respect to $\lambda > 0$. By (3.17), we compute and find that

$$\frac{\partial}{\partial\lambda}\widehat{T}_\lambda(\alpha) = \int_0^\alpha \frac{-B}{(\lambda^2 B^2 + 2\lambda B)^{3/2}}\,\mathrm{d}u < 0 \quad \text{for } 0 < \alpha < N \text{ and } \lambda > 0. \tag{3.30}$$

Since $\widehat{T}'_\lambda(\alpha_\lambda) = 0$ for $\lambda > 0$, and by (3.30), we see that

$$\frac{\partial}{\partial\lambda}\widehat{T}_\lambda(\alpha_\lambda) = \frac{\partial}{\partial\lambda}\widehat{T}_\lambda(\alpha)\Big|_{\alpha=\alpha_\lambda} + \widehat{T}'_\lambda(\alpha_\lambda)\frac{\partial\alpha_\lambda}{\partial\lambda} = \frac{\partial}{\partial\lambda}\widehat{T}_\lambda(\alpha)\Big|_{\alpha=\alpha_\lambda} < 0 \quad \text{for } \lambda > 0.$$

It implies that $\widehat{T}_\lambda(\alpha_\lambda)$ is strictly decreasing for $\lambda > 0$. By Lemma 3.2 (ii), we have $\widehat{T}_\lambda(\alpha) \geqslant \widehat{T}_\lambda(\alpha_\lambda)$ for $0 < \alpha < N$ and $\lambda > 0$. Then by (3.2), we observe that

$$0 \leqslant \lim_{\lambda\to\infty} \widehat{T}_\lambda(\alpha_\lambda) \leqslant \lim_{\lambda\to\infty} \widehat{T}_\lambda(\alpha) = \alpha \quad \text{for } 0 < \alpha < N.$$

Since α is arbitrary, we obtain $\lim\limits_{\lambda\to\infty}\widehat{T}_\lambda(\alpha_\lambda)=0$. In addition, by Lemma 3.3, there exists a convergent subsequence of $\{\alpha_\lambda\}$. We assume without loss of generality that

$$\lim_{\lambda\to 0^+}\alpha_\lambda=\eta\in\Big[0,\frac{(p+3)(p-1)N}{p(p+2)}\Big].$$

If $\eta>0$, then by (3.2),

$$\lim_{\lambda\to 0^+}\widehat{T}_\lambda(\alpha_\lambda)=\lim_{\lambda\to 0^+}\widehat{T}_\lambda(\eta)=\lim_{\lambda\to 0^+}\int_0^\eta\frac{\lambda B+1}{\sqrt{\lambda^2B^2+2\lambda B}}\,\mathrm{d}u=\infty.$$

Notice that by (3.2)

$$\lim_{\lambda\to 0^+}\sqrt{\lambda}\widehat{T}_\lambda(\alpha)=\lim_{\lambda\to 0^+}\int_0^\alpha\frac{\lambda B+1}{\sqrt{\lambda B^2+2B}}\,\mathrm{d}u=\frac{1}{\sqrt{2}}T(\alpha). \tag{3.31}$$

If $\eta=0$, then by (3.6), (3.7) and (3.31),

$$\lim_{\lambda\to 0^+}\sqrt{\lambda}\widehat{T}_\lambda(\alpha_\lambda)=\lim_{\lambda\to 0^+}\frac{1}{\sqrt{2}}T(\alpha_\lambda)=\lim_{\alpha\to 0^+}\frac{1}{\sqrt{2}}T(\alpha)=\infty,$$

which implies that $\lim\limits_{\lambda\to 0^+}\widehat{T}_\lambda(\alpha_\lambda)=\infty$. Thus, statement (ii) holds.

Step 3: We prove that α_λ is strictly decreasing for sufficiently large $\lambda>0$. Since $\widehat{T}'_\lambda(\alpha_\lambda)=0$ for $\lambda>0$, and by Lemma 3.4, we observe that

$$\begin{aligned}0&<\sqrt{\lambda}\Big[\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)\Big]_{\alpha=\alpha_\lambda}+\frac{1}{2\sqrt{\lambda}}\widehat{T}'_\lambda(\alpha_\lambda)\\&=\sqrt{\lambda}\Big[\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)\Big]_{\alpha=\alpha_\lambda}\quad\text{for sufficiently large }\lambda>0.\end{aligned} \tag{3.32}$$

Since $\widehat{T}'_\lambda(\alpha_\lambda)=0$ for $\lambda>0$, we see that

$$0=\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha_\lambda)=\Big[\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)\Big]_{\alpha=\alpha_\lambda}+\widehat{T}''_\lambda(\alpha_\lambda)\frac{\partial\alpha_\lambda}{\partial\lambda}.$$

So by (3.32), we obtain

$$\frac{\partial\alpha_\lambda}{\partial\lambda}=-\frac{[(\partial/\partial\lambda)\widehat{T}'_\lambda(\alpha)]_{\alpha=\alpha_\lambda}}{\widehat{T}''_\lambda(\alpha_\lambda)}<0\quad\text{for sufficiently large }\lambda>0.$$

Step 4: We prove statement (iii). Let

$$\eta_1=\lim_{\lambda\to\infty}\alpha_\lambda,\quad \check{\eta}_2=\liminf_{\lambda\to 0^+}\alpha_\lambda\quad\text{and}\quad\hat{\eta}_2=\limsup_{\lambda\to 0^+}\alpha_\lambda.$$

Assume that $\eta_1>0$. By Step 3, $\alpha_\lambda>\eta_1$ for sufficiently large $\lambda>0$. By Lemma 3.2 (ii), we see that $\widehat{T}'_\lambda(\eta_1)<0$ for sufficiently large $\lambda>0$. So by (3.15), we compute and find that

$$0\geqslant\lim_{\lambda\to\infty}\widehat{T}'_\lambda(\eta_1)=1,$$

which is a contradiction. Thus, $\eta_1 = 0$. Assume that $\check{\eta}_2 \neq \overline{\alpha}$. By (3.8) and (3.15), we observe that

$$\lim_{\lambda \to 0^+} \sqrt{\lambda}\widehat{T}'_\lambda(\alpha) = T'(\alpha). \tag{3.33}$$

Then we consider two cases.

Case 1: Assume that $\overline{\alpha} < \check{\eta}_2$. There exists $\{\lambda_{1,n}\}$ such that

$$\lim_{n\to\infty} \lambda_{1,n} = 0 \quad \text{and} \quad \overline{\alpha} < \check{\eta}_2 < \alpha_{\lambda_{1,n}} \quad \text{for } n \in \mathbb{N}.$$

So by (3.33), Lemmas 3.1 (ii) and 3.2 (ii), we see that

$$0 \geqslant \lim_{n\to\infty} \sqrt{\lambda_{1,n}}\widehat{T}'_{\lambda_{1,n}}(\check{\eta}_2) = T'(\check{\eta}_2) > 0,$$

which is a contradiction.

Case 2: Assume that $\check{\eta}_2 < \overline{\alpha}$. There exists $\{\lambda_{2,n}\}$ such that

$$\lim_{n\to\infty} \lambda_{2,n} = 0 \quad \text{and} \quad \alpha_{\lambda_{2,n}} < \frac{\overline{\alpha} + \check{\eta}_2}{2} < \overline{\alpha} \quad \text{for } n \in \mathbb{N}.$$

So by (3.33), Lemmas 3.1 (ii) and 3.2 (ii), we see that

$$0 \leqslant \lim_{n\to\infty} \sqrt{\lambda_{2,n}}\widehat{T}'_{\lambda_{2,n}}\Big(\frac{\overline{\alpha} + \check{\eta}_2}{2}\Big) = T'\Big(\frac{\overline{\alpha} + \check{\eta}_2}{2}\Big) < 0,$$

which is a contradiction.

Thus, by Cases 1–2, we obtain $\check{\eta}_2 = \overline{\alpha}$. Similarly, we obtain $\hat{\eta}_2 = \overline{\alpha}$. So $\lim_{\lambda\to 0^+} \alpha_\lambda = \overline{\alpha}$.

Step 5: We prove statement (iv). Assume that $1 < p \leqslant 4$. It is easy to see that

$$\frac{(p+3)(p-1)N}{p(p+2)} = \frac{(p+3)N}{p+4} + \frac{N(p+3)(p-4)}{p(p+4)(p+2)} \leqslant \frac{(p+3)N}{p+4}. \tag{3.34}$$

Since $\widehat{T}'_\lambda(\alpha_\lambda) = 0$, and by Lemma 3.3, (3.27) and (3.34), we see that

$$0 < \sqrt{\lambda}\Big[\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)\Big]_{\alpha=\alpha_\lambda} + \frac{1}{2\sqrt{\lambda}}\widehat{T}'_\lambda(\alpha_\lambda) = \sqrt{\lambda}\Big[\frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)\Big]_{\alpha=\alpha_\lambda} \quad \text{for } \lambda > 0. \tag{3.35}$$

Since $\widehat{T}'_\lambda(\alpha_\lambda) = 0$, we further see that

$$0 = \frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha_\lambda) = \frac{\partial}{\partial\lambda}\widehat{T}'_\lambda(\alpha)\Big|_{\alpha=\alpha_\lambda} + \widehat{T}''_\lambda(\alpha_\lambda)\frac{\partial\alpha_\lambda}{\partial\lambda}.$$

So by Lemma 3.2 and (3.35),

$$\frac{\partial\alpha_\lambda}{\partial\lambda} = -\frac{(\partial/\partial\lambda)\widehat{T}'_\lambda(\alpha)|_{\alpha=\alpha_\lambda}}{\widehat{T}''_\lambda(\alpha_\lambda)} < 0 \quad \text{for } \lambda > 0.$$

The proof is complete. □

The following Lemma 3.6 holds by (3.14) and Lemma 4.6 of [9].

Lemma 3.6. *Consider* (1.1) *with fixed* $L > 0$. *Let* $m_{L,N} \equiv \min\{L, N\}$. *Then the following statements hold:*

(i) *There exists a positive and continuously differentiable function* $\lambda_L(\alpha)$ *on* $(0, m_{L,N})$ *such that* $\widehat{T}_{\lambda_L(\alpha)}(\alpha) = L$. *Moreover, the bifurcation curve* $\widehat{S}_L = \{(\lambda_L(\alpha), \alpha) \colon \alpha \in (0, m_{L,N})\}$ *is continuous on the* $(\lambda, \|u\|_\infty)$*-plane.*

(ii) $\operatorname{sgn}(\lambda_L'(\alpha)) = \operatorname{sgn}(\widehat{T}'_{\lambda_L(\alpha)}(\alpha))$ *for* $\alpha \in (0, m_{L,N})$, *where* $\operatorname{sgn}(u)$ *is the signum function.*

(iii) $\lim\limits_{\alpha \to 0^+} \lambda_L(\alpha) = \lim\limits_{\alpha \to m_{L,N}^-} \lambda_L(\alpha) = \infty$. *Moreover, the bifurcation curve* $\widehat{S}_L$ *starts from the point* $(\infty, 0)$ *and goes to* $(\infty, m_{L,N})$.

4. Proof of main theorems

Proof of Theorem 2.1. Theorem 2.1 (i) holds immediately by (3.4) and Lemma 3.1. Let $\bar{\lambda}_L = \overline{T}_L^2(\overline{\alpha})$ for $L > 0$, where $\overline{\alpha}$ is defined in Lemma 3.1. By (3.3) and Lemma 3.1, we see that for $L > 0$, (1.1) has no positive solutions for $0 < \lambda < \bar{\lambda}_L$, exactly one positive solution $\bar{u}_L$ for $\lambda = \bar{\lambda}_L$ and exactly two positive solutions for $\lambda > \bar{\lambda}_L$.

By (3.3) and Lemma 3.1 (ii), $\kappa = \|\bar{u}_L\|_\infty = \overline{\alpha}$ is independent on L. So Theorem 2.1 (ii) (b) holds. Since $\bar{\lambda}_L = \overline{T}_L^2(\overline{\alpha})$, and by Lemma 3.1 (ii), we see that $\lim\limits_{L \to 0^+} \bar{\lambda}_L = \infty$ and $\lim\limits_{L \to \infty} \bar{\lambda}_L = 0$. Thus, Theorem 2.1 (ii) (a) (c) holds. The proof is complete. □

Proof of Theorem 2.2. By Lemma 3.6 (iii), $\lambda_L(\alpha)$ has at least one critical point on $(0, m_{L,N})$. Assume that $\lambda_L(\alpha)$ has two distinct critical points at α_1 and α_2 on $(0, m_{L,N})$. Let $\lambda_1 = \lambda_L(\alpha_1)$ and $\lambda_2 = \lambda_L(\alpha_2)$. By Lemma 3.6 (i) (ii), we obtain

$$\widehat{T}_{\lambda_1}(\alpha_1) = \widehat{T}_{\lambda_2}(\alpha_2) = L \quad \text{and} \quad \widehat{T}'_{\lambda_1}(\alpha_1) = \widehat{T}'_{\lambda_2}(\alpha_2) = 0.$$

So by Lemma 3.2 (ii), we have $\alpha_1 = \alpha_{\lambda_1}$ and $\alpha_2 = \alpha_{\lambda_2}$. Then

$$\widehat{T}_{\lambda_1}(\alpha_{\lambda_1}) = \widehat{T}_{\lambda_2}(\alpha_{\lambda_2}) = L.$$

So by Lemma 3.5 (ii), we find that $\alpha_1 = \alpha_{\lambda_1} = \alpha_{\lambda_2} = \alpha_2$. It is a contradiction. Thus, there exists $\widehat{\alpha}_L \in (0, m_{L,N})$ such that

$$\lambda_L'(\alpha) \begin{cases} < 0 & \text{for } 0 < \alpha < \widehat{\alpha}_L, \\ = 0 & \text{for } \alpha = \widehat{\alpha}_L, \\ > 0 & \text{for } \widehat{\alpha}_L < \alpha < m_{L,N}. \end{cases} \tag{4.1}$$

By Lemma 3.6 and (4.1), we obtain Theorem 2.2 (i). Moreover, for $L>0$, there exists $\hat{\lambda}_L>0$ such that (1.1) has no positive solutions for $0<\lambda<\hat{\lambda}_L$, exactly one positive solution $\widehat{u}_L$ for $\lambda=\hat{\lambda}_L$ and exactly two positive solutions for $\lambda>\hat{\lambda}_L$. Notice that

$$\hat{\lambda}_L=\lambda_L(\widehat{\alpha}_L) \quad \text{and} \quad \|\widehat{u}_L\|_\infty=\widehat{\alpha}_L. \tag{4.2}$$

In order to prove Theorem 2.2 (ii)(a)–(c), we divide the proof into the next four steps.

Step 1: We prove that $\hat{\lambda}_L$ is strictly decreasing for $L>0$, and if $1<p\leqslant 4$, then $\|\widehat{u}_L\|_\infty$ is strictly increasing for $L>0$. Let $L_2>L_1>0$. For the sake of convenience, we let $\lambda_1=\lambda_{L_1}(\widehat{\alpha}_{L_1})$ and $\lambda_2=\lambda_{L_2}(\widehat{\alpha}_{L_2})$. Since $\lambda'_{L_1}(\widehat{\alpha}_{L_1})=\lambda'_{L_2}(\widehat{\alpha}_{L_2})=0$, and by Lemmas 3.2 (ii) and 3.6 (ii), we see that

$$\alpha_{\lambda_1}=\widehat{\alpha}_{L_1} \quad \text{and} \quad \alpha_{\lambda_2}=\widehat{\alpha}_{L_2}.$$

Then by Lemma 3.6 (i), we further see that

$$\widehat{T}_{\lambda_2}(\alpha_{\lambda_2})=\widehat{T}_{\lambda_2}(\widehat{\alpha}_{L_2})=L_2>L_1=\widehat{T}_{\lambda_1}(\widehat{\alpha}_{L_1})=\widehat{T}_{\lambda_1}(\alpha_{\lambda_1}). \tag{4.3}$$

By (4.2), (4.3) and Lemma 3.5 (ii), then

$$\hat{\lambda}_{L_2}=\lambda_{L_2}(\widehat{\alpha}_{L_2})=\lambda_2<\lambda_1=\lambda_{L_1}(\widehat{\alpha}_{L_1})=\hat{\lambda}_{L_1}. \tag{4.4}$$

It implies that $\hat{\lambda}_L$ is strictly decreasing for $L>0$. Next, assume that $1<p\leqslant 4$. By (4.2), (4.4) and Lemma 3.5 (iv), we obtain

$$\|\widehat{u}_{L_2}\|_\infty=\widehat{\alpha}_{L_2}=\alpha_{\lambda_2}>\alpha_{\lambda_1}=\widehat{\alpha}_{L_1}=\|\widehat{u}_{L_1}\|_\infty. \tag{4.5}$$

It implies that $\|\widehat{u}_L\|_\infty$ is strictly increasing for $L>0$.

Step 2: We prove that $\lim\limits_{L\to0^+}\|\widehat{u}_L\|_\infty=0$ and $\lim\limits_{L\to\infty}\|\widehat{u}_L\|_\infty=\overline{\alpha}$. Let $L_1>0$ be given. Let $\lambda_1=\lambda_{L_1}(\widehat{\alpha}_{L_1})$. Since $\lambda'_{L_1}(\widehat{\alpha}_{L_1})=0$, and by Lemma 3.6 (i) (ii), we obtain

$$\widehat{T}_{\lambda_1}(\widehat{\alpha}_{L_1})=L_1 \quad \text{and} \quad \widehat{T}'_{\lambda_1}(\widehat{\alpha}_{L_1})=0.$$

So by Lemma 3.2 (ii), we have $\widehat{\alpha}_{L_1}=\alpha_{\lambda_1}$. It implies that $\widehat{T}_{\lambda_1}(\alpha_{\lambda_1})=L_1$. Then by Lemma 3.5 (ii), we observe that

(a) $L_1\to0^+$ if and only if $\lambda_1\to\infty$; and

(b) $L_1\to\infty$ if and only if $\lambda_1\to0^+$.

Then by Lemma 3.5 (iii),

$$\lim_{L_1\to0^+}\widehat{\alpha}_{L_1}=\lim_{\lambda_1\to\infty}\alpha_{\lambda_1}=0 \quad \text{and} \quad \lim_{L_1\to\infty}\widehat{\alpha}_{L_1}=\lim_{\lambda_1\to0^+}\alpha_{\lambda_1}=\overline{\alpha}.$$

Thus, by (4.2), we obtain $\lim\limits_{L\to0^+}\|\widehat{u}_L\|_\infty=0$ and $\lim\limits_{L\to\infty}\|\widehat{u}_L\|_\infty=\overline{\alpha}$.

Step 3: We prove that $\|\widehat{u}_L\|_\infty$ and $\hat{\lambda}_L$ are continuous functions with respect to L on $(0,\infty)$. Let $L_2, L_3 > 0$, $\lambda_2 = \lambda_{L_2}(\widehat{\alpha}_{L_2})$ and $\lambda_3 = \lambda_{L_3}(\widehat{\alpha}_{L_3})$. Since $\lambda'_{L_2}(\widehat{\alpha}_{L_2}) = \lambda'_{L_3}(\widehat{\alpha}_{L_3}) = 0$, and by Lemma 3.6 (i) (ii), we obtain

$$\widehat{T}_{\lambda_2}(\widehat{\alpha}_{L_2}) = L_2, \quad \widehat{T}'_{\lambda_2}(\widehat{\alpha}_{L_2}) = 0, \quad \widehat{T}_{\lambda_3}(\widehat{\alpha}_{L_3}) = L_3 \quad \text{and} \quad \widehat{T}'_{\lambda_3}(\widehat{\alpha}_{L_3}) = 0.$$

So by Lemma 3.2 (ii), we have

$$\widehat{\alpha}_{L_2} = \alpha_{\lambda_2} \quad \text{and} \quad \widehat{\alpha}_{L_3} = \alpha_{\lambda_3}. \tag{4.6}$$

It implies that

$$\widehat{T}_{\lambda_2}(\alpha_{\lambda_2}) = L_2 \quad \text{and} \quad \widehat{T}_{\lambda_3}(\alpha_{\lambda_3}) = L_3. \tag{4.7}$$

By (4.7) and Lemma 3.5 (ii), we see that

$$L_3 \to L_2 \text{ if and only if } \lambda_3 \to \lambda_2. \tag{4.8}$$

By Lemma 3.5 (i), (4.2), (4.6) and (4.8), we further see that

$$\lim_{L_3 \to L_2} \|\widehat{u}_{L_3}\|_\infty = \lim_{L_3 \to L_2} \widehat{\alpha}_{L_3} = \lim_{\lambda_3 \to \lambda_2} \alpha_{\lambda_3} = \alpha_{\lambda_2} = \widehat{\alpha}_{L_2} = \|\widehat{u}_{L_2}\|_\infty.$$

Thus, $\|\widehat{u}_L\|_\infty$ is a continuous function with respect to L on $(0,\infty)$.

Since $\widehat{T}_{\lambda_L(\alpha)}(\alpha) = L$ by Lemma 3.6 (i), and by (3.30) and the implicit function theorem, we see that $\lambda_L(\alpha)$ is a continuous function with respect to L on $(0,\infty)$. So $\hat{\lambda}_L = \lambda_L(\widehat{\alpha}_L)$ is also continuous function with respect to L on $(0,\infty)$.

Step 4: We prove that

$$0 = \lim_{L\to\infty} \hat{\lambda}_L < \lim_{L\to 0^+} \hat{\lambda}_L = \infty.$$

Let $\lambda_\infty = \lim_{L\to\infty} \hat{\lambda}_L$. Assume that $\lambda_\infty > 0$. By Step 1 and (4.2), we find that

$$\lambda_\infty < \hat{\lambda}_L = \lambda_L(\widehat{\alpha}_L) \quad \text{for } L > 0. \tag{4.9}$$

By Lemma 3.6 (i), (3.30), (4.9) and Step 2, we observe that

$$\infty = \lim_{L\to\infty} L = \lim_{L\to\infty} \widehat{T}_{\lambda_L(\widehat{\alpha}_L)}(\widehat{\alpha}_L) \leqslant \lim_{L\to\infty} \widehat{T}_{\lambda_\infty}(\widehat{\alpha}_L) = \widehat{T}_{\lambda_\infty}(\overline{\alpha}) < \infty.$$

It is a contradiction. So $\lambda_\infty = 0$. Let $\lambda_0 = \lim_{L\to 0^+} \hat{\lambda}_L$. Assume that $\lambda_0 < \infty$. By Step 1 and (4.2), we find that

$$\lambda_L(\widehat{\alpha}_L) = \hat{\lambda}_L < \lambda_0 < \infty \quad \text{for } L > 0.$$

So by Lemmas 3.2 (i), 3.6 (i), (3.30) and Step 2, we observe that

$$0 = \lim_{L\to 0^+} L = \lim_{L\to 0^+} \widehat{T}_{\hat{\lambda}_L}(\widehat{\alpha}_L) \geqslant \lim_{L\to 0^+} \widehat{T}_{\lambda_0}(\widehat{\alpha}_L) = \lim_{\alpha\to 0^+} \widehat{T}_{\lambda_0}(\alpha) = \infty.$$

It is a contradiction. So $\lambda_0 = \infty$. Thus, Theorem 2.2 (ii) (a)–(c) hold by Steps 1–4. The proof is complete. □

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Authors' address: Shao-Yuan Huang (corresponding author), Ping-Han Hsieh, Department of Mathematics and Information Education, College of Science, National Taipei University of Education, No. 134, Sec. 2, Heping E. Rd., Da-an District, Taipei City 106, Taiwan, e-mail: syhuang@mail.ntue.edu.tw, g110827008@grad.ntue.edu.tw.