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Farid Kourki; Rachid Tribak

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COMMUTATIVE RINGS WHOSE CERTAIN MODULES DECOMPOSE INTO DIRECT SUMS OF CYCLIC SUBMODULES

Farid Kourki, Rachid Tribak, Tangier

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Abstract. We provide some characterizations of rings R for which every (finitely generated) module belonging to a class $\mathcal{C}$ of R-modules is a direct sum of cyclic submodules. We focus on the cases, where the class $\mathcal{C}$ is one of the following classes of modules: semiartinian modules, semi-V-modules, V-modules, coperfect modules and locally supplemented modules.

Keywords: decomposition of a module; FGC-ring; Köthe ring; semiartinian module; (semi-)V-module; locally supplemented module

MSC 2020: 13C05, 13C13, 16D10, 16D80

1. Introduction

Throughout, all rings will be commutative with identities, all modules will be unitary modules, and R will always denote a ring. We will use the symbol Mod-R to denote the category of all R-modules. The set of all maximal ideals of R will be denoted by $\mathrm{Max}(R)$. Let M be an R-module. We set $\mathrm{Ann}_R(M) = \{r \in R\colon\ rx = 0$ for all $x \in M\}$ (the annihilator of M) and for any element $x \in M$, we set $\mathrm{Ann}_R(x) = \{r \in R\colon\ rx = 0\}$ (the annihilator of x). We will use the notation $N \subseteq M$ to denote that N is a subset of M. The notation $N \leqslant M$ means that N is a submodule of the module M. By $\mathrm{Rad}(M)$ we denote the Jacobson radical of M. Let $\mathbb{Z}$ denote the ring of integer numbers. For all undefined notions in this paper, we refer the reader to [2], [4], [27] and [38].

Many mathematicians have been attracted by the study of decomposition of modules into direct sums of cyclic submodules. A number of research papers which have been devoted to this subject can be found in the literature. Some of these papers focus on the following two questions:

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(i) *Which rings R have the property that every R-module is a direct sum of cyclic modules?*

(ii) *Which rings R have the property that every finitely generated R-module is a direct sum of cyclic modules?*

The aim of this paper is to provide some characterizations of rings R for which every (finitely generated) module belonging to a class $\mathcal{C}$ of R-modules is a direct sum of cyclic submodules. The investigations focus on the classes $\mathcal{C}$ of R-modules studied in the papers [22], [23], [24], [25], that is, semiartinian modules, semi-V-modules, V-modules, coperfect modules and locally supplemented modules.

2. Some generalizations of FGC-rings

A ring R is called an *FGC-ring* if every finitely generated R-module is a direct sum of cyclic submodules. It is well known that principal ideal domains are FGC-rings, see for example [33], Theorem 6.16, Corollary. In 1952 in [20] Kaplansky proved that almost maximal valuation domains are FGC-rings. A number of research papers have been devoted to the study of the converse of the last result for local rings. Among others, first in 1966 in [29] Matlis proved the converse for the domain case. Then in 1971 Gill in [16] and Lafon in [26] independently proved the general case. Other mathematicians including for example W. Brandal, T. Shores, A. I. Uzkov, P. Vámos, R. Wiegand and S. Wiegand have continued the study of FGC-rings. A combination of some of their results has lead to a full characterization of FGC-rings, see [4], Main Theorem 9.1.

In this section, we focus on the question of which commutative rings R have the property that every finitely generated R-module satisfying a property (P) is a direct sum of cyclic submodules. The next proposition will be useful in this study.

A nonempty class $\mathcal{C}$ of R-modules which is closed under submodules, homomorphic images and direct sums is called a *hereditary pretorsion* class. A module M is said to be a *DSC-module* if M is a direct sum of cyclic submodules.

Proposition 2.1. *Let R be a ring and let $\mathcal{C}$ be a hereditary pretorsion class in* Mod-R. *Then the following are equivalent:*

(i) *Any finitely generated R-module in $\mathcal{C}$ is a DSC-module.*

(ii) *R/I is an FGC-ring for any proper ideal I of R with $(R/I)_R \in \mathcal{C}$.*

P r o o f. (i) $\Rightarrow$ (ii) Let I be a proper ideal of R such that the R-module R/I belongs to $\mathcal{C}$. Let M be a finitely generated R/I-module. Then M can be regarded as an R-module with respect to the scalar multiplication: $ax = \bar{a}x$ for all $a \in R$ and $x \in M$, where $\bar{a} = a + I \in R/I$. Let $0 \neq x \in M$. It is clear that $I \subseteq \mathrm{Ann}_R(x)$. Moreover, we have $Rx \cong R/\mathrm{Ann}_R(x) \cong (R/I)/(\mathrm{Ann}_R(x)/I)$ (as R-modules). Since $R/I \in \mathcal{C}$

and $\mathcal{C}$ is closed under factor modules, it follows that $Rx \in \mathcal{C}$. Using the fact that $\mathcal{C}$ is a hereditary pretorsion class, we get $M = \sum_{x \in M} Rx \in \mathcal{C}$. Note that M is a finitely generated R-module. Then M is a direct sum of cyclic sub-R-modules by (i). Hence, M is also a direct sum of cyclic sub-R/I-modules. Therefore, R/I is an FGC-ring.

(ii) $\Rightarrow$ (i) Let $M = Rx_1 + \ldots + Rx_n$ be a finitely generated nonzero R-module belonging to $\mathcal{C}$. Then $R/\mathrm{Ann}_R(M)$ is isomorphic to a submodule of $Rx_1 \oplus \ldots \oplus Rx_n$. But $\mathcal{C}$ is a hereditary pretorsion class. So $R/\mathrm{Ann}_R(M) \in \mathcal{C}$. By hypothesis, $R/\mathrm{Ann}_R(M)$ is an FGC-ring. This implies that M is a direct sum of cyclic sub-$R/\mathrm{Ann}_R(M)$-modules. Therefore, M is also a direct sum of cyclic sub-R-modules. □

The next lemma is taken from [35], see also [37], page 171.

Lemma 2.2. *Let R be a commutative noetherian ring. Then the following are equivalent:*

(i) *R is an FGC-ring.*

(ii) *R is a principal ideal ring.*

An R-module M is called *locally noetherian* (*locally artinian*) if every finitely generated submodule of M is noetherian (artinian). It is not difficult to see that both the class of locally noetherian modules and the class of locally artinian modules are hereditary pretorsion classes.

Proposition 2.3. *The following statements are equivalent for a ring R:*

(i) *Every noetherian R-module is a DSC-module.*

(ii) *R/I is a principal ideal ring for every proper ideal I of R with R/I noetherian.*

Proof. Note that a finitely generated R-module M is locally noetherian if and only if M is noetherian.

(i) $\Rightarrow$ (ii) By Proposition 2.1, R/I is an FGC-ring for any proper ideal I of R with R/I noetherian. Hence, R/I is a principal ideal ring by Lemma 2.2.

(ii) $\Rightarrow$ (i) This follows by using again Proposition 2.1 and Lemma 2.2. □

Recall that a ring R is called a *Bézout ring* if any finitely generated ideal of R is principal. The following corollary is an immediate consequence of Proposition 2.3.

Corollary 2.4. *Let R be a Bézout ring. Then any noetherian R-module is a direct sum of cyclic submodules.*

A module M is called *coperfect* if the set of cyclic submodules of M satisfies the descending chain condition.

Proposition 2.5. *The following statements are equivalent for a ring R:*

(i) *Every finitely generated artinian R-module is a DSC-module.*

(ii) *Every finitely generated coperfect R-module is a DSC-module.*

(iii) *R/I is a principal ideal ring for every proper ideal I of R with R/I artinian.*

(iv) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for every* $\mathfrak{m} \in \operatorname{Max}(R)$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) These follow from [24], Corollary 4.10.

(i) $\Leftrightarrow$ (iii) Consider the following assertion:

(i′) Every finitely generated locally artinian R-module is a direct sum of cyclic submodules.

It is clear that the assertions (i) and (i′) are equivalent. Note that the class of locally artinian R-modules is a hereditary pretorsion class and any artinian ring is noetherian. Using Proposition 2.1 and Lemma 2.2, it follows that the assertions (i′) and (iii) are equivalent. This completes the proof. □

Recall that a module M is called *semiartinian* if each nonzero factor module of M has a simple submodule. The next lemma is needed to determine the class of rings R for which every finitely generated semiartinian R-module is a DSC-module.

Lemma 2.6. *The following are equivalent for a ring R:*

(i) *R is a perfect ring such that* $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for every* $\mathfrak{m} \in \operatorname{Max}(R)$.

(ii) *R is an artinian principal ideal ring.*

Proof. (i) $\Rightarrow$ (ii) First suppose R is a local ring with maximal ideal $\mathfrak{m}$. Then $\operatorname{Rad}(R) = \mathfrak{m}$. By hypothesis, we have $\mathfrak{m} = \mathfrak{m}^2 + aR$ for some $a \in R$. Since R is perfect, it follows that $\mathfrak{m}^2 = \operatorname{Rad}(R)\mathfrak{m} \ll \mathfrak{m}$ by [2], Remark 28.5 (3). Therefore, $\mathfrak{m} = aR$. But $\mathfrak{m}$ is the only prime ideal of R since $\mathfrak{m}$ is T-nilpotent. So R is an artinian principal ideal ring by Theorem 2.1 of [17]. Now suppose that R is not a local ring. Then R is a finite direct product of local rings each one of which satisfies the conditions stated in (i). Hence, R is a finite direct product of artinian principal ideal rings. Therefore, R itself is an artinian principal ideal ring.

(ii) $\Rightarrow$ (i) This is immediate. □

Given a simple module S, a module M is called *S-primary* if each nonzero factor module of M has a simple submodule isomorphic to S. Dickson in [11] called a ring R a *T-ring* if each semiartinian R-module decomposes into a direct sum of its primary components. It is shown in [11], Corollary 2.7 that noetherian rings and semilocal rings are T-rings. In [24], Theorem 3.4, we proved that a commutative ring R is a T-ring if and only if any semiartinian R-module is coperfect. As in [22], we call a module M *Π-semiartinian* if the direct product M^I is a semiartinian module for every nonempty set I.

Proposition 2.7. *The following statements are equivalent for a ring R:*

(i) *Every finitely generated semiartinian R-module is a DSC-module.*

(ii) *Every Π-semiartinian R-module is a DSC-module.*

(iii) *R/I is a principal ideal ring for every proper ideal I of R with R/I semiartinian.*

(iv) *R satisfies the following two conditions:*

  (a) *R is a T-ring;*

  (b) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for all* $\mathfrak{m} \in \mathrm{Max}(R)$.

P r o o f. (i) $\Leftrightarrow$ (iv) This follows from [34], Theorem 2.

(iv) $\Rightarrow$ (iii) Let I be a proper ideal of R such that R/I is a semiartinian ring. Since R is a T-ring, any semiartinian R-module is coperfect, see [24], Theorem 3.4. Therefore, R/I is a coperfect R-module and hence R/I is a perfect ring by [24], Proposition 2.5. Notice that condition (b) is closed under factor rings. By Lemma 2.6, we see that R/I is an artinian principal ideal ring.

(iii) $\Rightarrow$ (ii) Let M be a Π-semiartinian R-module. By [22], Proposition 3.2, $R/\mathrm{Ann}_R(M)$ is a semiartinian ring. By hypothesis, $R/\mathrm{Ann}_R(M)$ is an artinian principal ideal ring. By [33], Theorem 6.7, it follows that M, viewed as an $R/\mathrm{Ann}_R(M)$-module, is a direct sum of cyclic submodules. This clearly implies that the R-module M is also a direct sum of cyclic submodules.

(ii) $\Rightarrow$ (i) This follows from the fact that any finitely generated semiartinian R-module is Π-semiartinian, see [22], Corollary 3.3. $\square$

Following Hirano (see [19]) an R-module M is called a *V-module* (or *cosemisimple* as in [15]) if every proper submodule of M is an intersection of maximal submodules of M. Equivalently, $\mathrm{Rad}(N) = 0$ for every factor module N of M. The ring R is called a *V-ring* (or *cosemisimple*) if the R-module R is a V-module. A well-known result of Kaplansky shows that the commutative V-rings are exactly the von Neumann regular rings.

The next result is taken from [31], Corollary 21.7.

Lemma 2.8. *Let R be a commutative von Neumann regular ring. Then the following are equivalent:*

(i) *R is an FGC-ring.*

(ii) *R is a semisimple ring.*

In [25], we called a ring R a *VSA-ring* if any V-module in Mod-R is semiartinian. Clearly, any semiartinian ring is a VSA-ring. The class of VSA-rings also includes semilocal rings and noetherian rings, see [25], Example 3.15.

Proposition 2.9. *The following statements are equivalent for a ring R:*

(i) *Every finitely generated V-module in* Mod-R *is a DSC-module.*

(ii) *Every V-module in* Mod-R *is a DSC-module.*
(iii) R/I *is a semisimple ring for any proper ideal* I *of* R *with* R/I *von Neumann regular.*
(iv) *Every V-module in* Mod-R *is semisimple.*
(v) R *satisfies the following two conditions:*
  (a) R *is a* T*-ring;*
  (b) R *is a VSA-ring.*

P r o o f. Note that the class of V-modules is a hereditary pretorsion class by [19], Proposition 3.3.

(i) $\Leftrightarrow$ (iii) This follows from Proposition 2.1 and Lemma 2.8.

(iii) $\Rightarrow$ (iv) Let M be a V-module in Mod-R and let $0 \neq x \in M$. Then Rx is also a V-module and so $R/\mathrm{Ann}_R(x)$ is a von Neumann regular ring. Hence, $R/\mathrm{Ann}_R(x)$ is a semisimple ring by (iii). This implies that Rx is a semisimple R-module. Therefore, M is a semisimple R-module.

(iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

(iv) $\Leftrightarrow$ (v) This follows from [6], Theorem 4. □

Example 2.10. Assume that the ring R is noetherian or semilocal. Then R is a T-ring and a VSA-ring. From Proposition 2.9 it follows that every V-module in Mod-R is a direct sum of cyclic submodules.

Let M be an R-module. An R-module N is said to be *subgenerated by* M if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are all R-modules subgenerated by M, see [38], page 118. Following [7], 2.19, an R-module M is called a *Bass module* if every nonzero module in $\sigma[M]$ has a maximal submodule. It is clear that the R-module R is Bass if and only if every nonzero R-module has a maximal submodule; such a ring is called a *Bass ring*, see [7], page 18. Following [5], an R-module M is called a *semi-V-module* in Mod-R if every nonzero homomorphic image of M has a nonzero V-submodule. Any Bass module is a semi-V-module by [25], Corollary 3.5. In [25], we called an R-module M a Π-*Bass* module (Π-*semi-V-module*) if the direct product M^{Λ} is a Bass module (semi-V-module) for every nonempty set Λ. By [25], Proposition 4.2, a module M is Π-Bass if and only if it is a Π-semi-V-module.

Proposition 2.11. *The following statements are equivalent for a ring* R*:*
(i) *Every finitely generated semi-V-module in* Mod-R *is a DSC-module.*
(ii) *Every finitely generated Bass module in* Mod-R *is a DSC-module.*
(iii) *Every* Π*-semi-V-module in* Mod-R *is a DSC-module.*
(iv) *Every* Π*-Bass module in* Mod-R *is a DSC-module.*

(v) *R/I is an artinian principal ideal ring for any proper ideal I of R with R/I a Bass ring.*

(vi) *R satisfies the following three conditions:*

(a) *R is a T-ring;*

(b) *R is a VSA-ring;*

(c) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for all* $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof. (i) $\Rightarrow$ (vi) Note that any V-module is a semi-V-module. Hence, any finitely generated V-module in Mod-R is a direct sum of cyclic submodules. By Proposition 2.9, R is a T-ring and a VSA-ring. Moreover, any coperfect module is a semi-V-module by [25], Proposition 3.1. Using Proposition 2.5, it follows that R satisfies condition (c).

(vi) $\Rightarrow$ (v) Let I be a proper ideal of R such that R/I is a Bass ring. Since R is a T-ring and a VSA-ring, R/I is a coperfect R-module by [25], Proposition 3.17. Hence, R/I is a perfect ring by [24], Corollary 2.12. Moreover, it is not difficult to see that the ring R/I also satisfies condition (c). So by Lemma 2.6, it follows that R/I is an artinian principal ideal ring.

(v) $\Rightarrow$ (iv) Let M be a Π-Bass R-module. By [25], Proposition 4.2, $R/\mathrm{Ann}_R(M)$ is a Bass ring. Therefore, $R/\mathrm{Ann}_R(M)$ is an artinian principal ideal ring by (v). Using [33], Theorem 6.7, we conclude that M, viewed as an $R/\mathrm{Ann}_R(M)$-module or as an R-module, is a direct sum of cyclic submodules.

(iv) $\Rightarrow$ (ii) This follows from the fact that any finitely generated Bass module is Π-Bass, see [25], Corollary 4.3.

(ii) $\Rightarrow$ (i) This follows from the fact that a finitely generated R-module is a Bass module if and only if it is a semi-V-module, see [25], Corollary 3.8.

(iii) $\Leftrightarrow$ (iv) This follows from [25], Proposition 4.2. □

A family of sets is said to have the *finite intersection property* if the intersection of every finite subfamily is nonempty. An R-module M is *linearly compact* if whenever $\{m_i + M_i\}_{i\in I}$ is a family of cosets of submodules of M ($m_i \in M$ and $M_i \leqslant M$ for each $i \in I$) with the finite intersection property, then $\bigcap\limits_{i\in I}(m_i + M_i)$ is nonempty. A commutative ring R is a *maximal ring* if R is a linearly compact R-module and R is an *almost maximal ring* if R/I is a linearly compact R-module for all nonzero ideals I of R.

Let N be a submodule of a module M. Then N is called small in M (denoted by $N \ll M$) if $M \neq N + X$ for any proper submodule X of M. A submodule K of M is called a *supplement* of N in M if K is minimal with respect to the property $M = N+K$. Equivalently, $M = N+K$ and $N\cap K \ll K$. A module M is said to be *supplemented* if every submodule of M has a supplement in M. A module M is called *locally supplemented* if every finitely generated submodule of M is supplemented, see [23].

We need the following lemma to characterize the class of rings R having the property that finitely generated locally supplemented R-modules are DSC.

Lemma 2.12 ([4], Theorem 4.5). *Let R be a local ring. Then the following are equivalent:*
(i) *R is an FGC-ring.*
(ii) *R is an almost maximal valuation ring.*

Proposition 2.13. *The following statements are equivalent for a ring R:*
(i) *Every finitely generated supplemented R-module is DSC.*
(ii) *Every finitely generated locally supplemented R-module is DSC.*
(iii) *R/I is an almost maximal valuation ring for any proper ideal I of R with R/I a local ring.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Let I be a proper ideal of R such that R/I is a local ring. Then clearly R/I is a semiperfect ring. Thus, R/I is a locally supplemented R-module by [23], Lemma 2.5. Since the class of locally supplemented modules is a hereditary pretorsion class (see [23], Remark 2.8), it follows that R/I is an FGC-ring by Proposition 2.1. Using Lemma 2.12, we deduce that R/I is an almost maximal valuation ring.

(iii) $\Rightarrow$ (i) Let M be a finitely generated supplemented R-module with $I = \mathrm{Ann}_R(M)$. By Lemma 2.5 of [23], R/I is a semiperfect ring. Thus, $R/I \cong R/I_1 \times \ldots \times R/I_n$, where each I_i $(1 \leqslant i \leqslant n)$ is an ideal of R such that R/I_i is a local ring, see [28], Theorem 23.11. By hypothesis and using Lemma 2.12, we obtain that R/I_i is an FGC-ring for all $i = 1, \ldots, n$. Hence, R/I is an FGC-ring by [4], Lemma 4.1 (1). Therefore, M is a DSC-module. □

3. Generalized Köthe rings

The following question was raised by Köthe: *Which rings R (not necessarily commutative) have the property that every right R-module is a direct sum of cyclic submodules?*

Köthe proved that every R-module over a left artinian principal ideal ring is a DSC-module, see [21]. The converse for the case of commutative rings was proved by Cohen and Kaplansky, see [9].

A ring R is called a *Köthe ring* if any R-module is a direct sum of cyclic submodules. So we have the following result which can be also found in Theorem 6.7 of [33].

Theorem 3.1. *A commutative ring R is a Köthe ring if and only if R is an artinian principal ideal ring.*

In this section, we show some characterizations of rings R by some of their classes of R-modules which are DSC. We begin with the class of rings R for which every semiartinian R-module is a DSC-module. To do this, we need some lemmas. We first recall the following notions.

Following [18], the Krull dimension (denoted K-dim) of a module M is defined as follows: $K\text{-dim}(M) = -1$ when $M = 0$. Given an ordinal α, and assuming that the concept $K\text{-dim}(M) < \alpha$ is already defined, $K\text{-dim}(M)$ is defined to be α if $K\text{-dim}(M) \nless \alpha$ and there exists no descending sequence $M = M_0 \geqslant M_1 \geqslant \ldots$ of submodules of M with $K\text{-dim}(M_{n-1}/M_n) \nless \alpha$ for all $n \geqslant 1$. It is easy to check that $K\text{-dim}(M) = 0$ if and only if M is a nonzero artinian module. Also, note that every noetherian module has Krull dimension by Proposition 1.3 of [18].

Recall that a module M is called *tall* if it contains a submodule N such that both M/N and N are non-noetherian. A ring R is called *tall* if every non-noetherian R-module is tall. For example, every Bass ring R (i.e., every nonzero R-module has a maximal submodule) is tall by Corollary 1.2 of [30].

Let R be a ring, $\mathfrak{m}$ a maximal ideal in R and F a finitely generated free R-module. Following Facchini (see [13], Definition 2.2), a descending $\mathfrak{m}$-chain in F is a chain $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ of submodules of F such that:

(i) $F_n = \mathfrak{m}^n F + F_m$ for all $n \leqslant m$, and

(ii) F/F_n is a noetherian R-module for all $n \geqslant 1$.

The following characterization of tall rings will be of interest.

Lemma 3.2. *The following are equivalent for a ring R:*

(i) *R is a tall ring.*

(ii) *$R/\bigcap_{k\geqslant 1} \mathfrak{m}^k$ is tall for every maximal ideal $\mathfrak{m}$ of R.*

(iii) *Every R-module with Krull dimension is noetherian.*

(iv) *Every artinian R-module is noetherian.*

(v) *An R-module M is artinian if and only if M is noetherian.*

(vi) *For every maximal ideal $\mathfrak{m}$ in R, every descending $\mathfrak{m}$-chain in R is stationary.*

Proof. (i) ⇔ (ii) By [30], Corollary 2.7.

(i) ⇔ (iii) This follows from [32], Theorem 2.7.

(i) ⇔ (iv) By [30], Theorem 2.12.

(iv) ⇔ (vi) This follows from [13], Proposition 4.4.

(iv) ⇔ (v) By [13], Proposition 4.1. □

Lemma 3.3. *Let R be a ring such that $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) < \infty$ for every maximal ideal $\mathfrak{m}$ of R. Then the following are equivalent:*

(i) *R is a tall ring.*

(ii) *For every maximal ideal $\mathfrak{m}$ in R, there exists a positive integer n such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathfrak{m}$ be a maximal ideal of R. We claim that $R/\mathfrak{m}^n$ is a noetherian ring for every integer $n \geqslant 2$. Since $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) < \infty$, there exists a finitely generated ideal $I = a_1R+a_2R+\ldots+a_kR$ of R such that $\mathfrak{m} = \mathfrak{m}^2+I$. By induction on n, let us show that $\mathfrak{m} = \mathfrak{m}^n + I$ for every integer $n \geqslant 2$. Clearly, this property is true for $n = 2$. Now suppose that $\mathfrak{m} = \mathfrak{m}^n + I$ for some integer $n \geqslant 2$. Multiplying both sides of this equality by $\mathfrak{m}$, we get $\mathfrak{m}^2 = \mathfrak{m}^{n+1} + \mathfrak{m}I$. As $\mathfrak{m} = \mathfrak{m}^2 + I$, we obtain $\mathfrak{m} = \mathfrak{m}^{n+1} + \mathfrak{m}I + I$. Hence, $\mathfrak{m} = \mathfrak{m}^{n+1} + I$. Therefore, $\mathfrak{m} = \mathfrak{m}^n + I$ for any integer $n \geqslant 2$. It follows that for any integer $n \geqslant 2$, $\mathfrak{m}/\mathfrak{m}^n$ (which is the only prime ideal of $R/\mathfrak{m}^n$) is finitely generated. By a theorem of Cohen (see [8], Theorem 2), we conclude that each $R/\mathfrak{m}^n$ $(n \geqslant 2)$ is a noetherian ring. This implies that $\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \ldots$ is a descending $\mathfrak{m}$-chain in R. Since R is tall, this chain has to stabilize by Lemma 3.2.

(ii) $\Rightarrow$ (i) Let n be a positive integer such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. Then $R/\bigcap_{k\geqslant 1} \mathfrak{m}^k = R/\mathfrak{m}^n$ is a Bass ring by [14], Theorem A. This implies that $R/\bigcap_{k\geqslant 1} \mathfrak{m}^k$ is a tall ring, see [30], Corollary 1.2. Now using Lemma 3.2, it follows that R is a tall ring. □

Lemma 3.4. *Let R be a ring. Assume that any artinian R-module is DSC. Then R satisfies the following two conditions:*

(a) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for all* $\mathfrak{m} \in \operatorname{Max}(R)$*, and*

(b) *for each* $\mathfrak{m} \in \operatorname{Max}(R)$*, there exists a positive integer n such that* $\mathfrak{m}^n = \mathfrak{m}^{n+1}$.

Proof. First, note that condition (a) holds by Proposition 2.5. To show that R satisfies condition (b), take an artinian R-module M. By hypothesis, M is a finite direct sum of cyclic submodules. So M is finitely generated. Therefore, M is a noetherian module. By Lemma 3.2, it follows that R is a tall ring. Now application of Lemma 3.3 enables us to deduce that condition (b) holds. □

This brings us to the first main result of this section.

Theorem 3.5. *Let R be a ring with $\Omega = \operatorname{Max}(R)$. Then the following statements are equivalent:*

(i) *Any semiartinian R-module is a DSC-module.*

(ii) *For any semiartinian R-module M, we have $M \cong \bigoplus_{\mathfrak{m}\in\Omega} M_{\mathfrak{m}}$ and each $M_{\mathfrak{m}}$ $(\mathfrak{m} \in \Omega)$ is isomorphic to a direct sum of cyclic modules of the type $R/\mathfrak{m}^k$.*

(iii) *R satisfies the following three conditions:*

(a) *R is a T-ring;*

(b) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for all* $\mathfrak{m} \in \Omega$*;*

(c) *for each* $\mathfrak{m} \in \Omega$*, there exists a positive integer n such that* $\mathfrak{m}^n = \mathfrak{m}^{n+1}$.

Proof. (i) $\Rightarrow$ (iii) Since any finitely generated semiartinian R-module is DSC, the conditions (a) and (b) hold by Proposition 2.7. Moreover, since any artinian module is semiartinian, it follows from Lemma 3.4 that condition (c) holds.

(iii) $\Rightarrow$ (ii) By Proposition 2.7, every finitely generated semiartinian R-module is DSC. Hence, (ii) follows directly from [12], Theorem, page 17.

(ii) $\Rightarrow$ (i) This is obvious. □

Next, we present three examples to show that none of conditions (a), (b) and (c) in Theorem 3.5 can be omitted.

Example 3.6.

(i) Consider the ring $R = \mathbb{Z}$. Then R is a T-ring as R is a noetherian ring, see [11], Corollary 2.7. Moreover, it is clear that for all $\mathfrak{m} \in \operatorname{Max}(R)$, $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$. On the other hand, $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for every positive integer n.

(ii) Let K be a field and let V be an infinite dimensional vector space over K. Let $R = K\alpha V$ be the trivial extension of K by V. Then R is a local ring with maximal ideal $\mathfrak{m} = (0, V)$. Thus, R is a T-ring by [11], Corollary 2.7. Moreover, we have $\mathfrak{m}^2 = 0$. Hence, $\mathfrak{m}^2 = \mathfrak{m}^3 = 0$. However, it is clear that $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}$ is not finitely generated. So $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \geqslant 2$.

(iii) Let K be a field, set $K_n = K$ for all $n = 1, 2, \ldots$, and let S be the K-subalgebra of $\prod_{n\geqslant 1} K_n$ generated by 1 and $\bigoplus_{n\geqslant 1} K_n$. By [1], Remarks 3.14 (3), R is a von Neumann regular ring which is not a T-ring. Moreover, since R is von Neumann regular, we have $\mathfrak{m}^2 = \mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

Next, we will be concerned with the class of rings R for which every artinian R-module is DSC. We need the following lemma.

Lemma 3.7. *Let R be a ring and let* $\mathfrak{m} \in \operatorname{Max}(R)$*. Then the following hold:*

(i) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *if and only if* $R/\mathfrak{m}^2$ *is an artinian principal ideal ring.*

(ii) *For every positive integer n,* $R/\mathfrak{m}^n$ *and* $R_\mathfrak{m}/\mathfrak{m}^n R_\mathfrak{m}$ *are isomorphic as rings and as R-modules as well.*

(iii) *If* $R_\mathfrak{m}$ *is an artinian ring, then* $R_\mathfrak{m}$ *is a principal ideal ring if and only if* $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$.

P r o o f. (i) Assume that $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$. So $\mathfrak{m}/\mathfrak{m}^2$ is a principal ideal of $R/\mathfrak{m}^2$. Moreover, $\mathfrak{m}/\mathfrak{m}^2$ is the only prime ideal of $R/\mathfrak{m}^2$. Now using [17], Theorem 2.1, we conclude that $R/\mathfrak{m}^2$ is a principal ideal ring. Since any prime ideal of $R/\mathfrak{m}^2$ is maximal, it follows that $R/\mathfrak{m}^2$ is an artinian ring. The converse is clear.

(ii) A trivial verification shows that the map $f\colon R/\mathfrak{m}^n \to R_{\mathfrak{m}}/\mathfrak{m}^n R_{\mathfrak{m}}$ defined by $f(a+\mathfrak{m}^n) = a/1 + \mathfrak{m}^n R_{\mathfrak{m}}$ is an isomorphism of rings (and also of R-modules).

(iii) This follows by combining (i), (ii) and Lemma 2.6. □

Let M be an R-module and let $\mathfrak{m}$ be a maximal ideal of R. As in [24], page 536, $\mathrm{Cop}_{\mathfrak{m}}(M) = \{x \in M\colon\ x = 0$ or $R/\mathrm{Ann}_R(x)$ is a perfect local ring with $\mathrm{Ann}_R(x) \subseteq \mathfrak{m}\}$. It is easily seen that $\mathrm{Cop}_{\mathfrak{m}}(M)$ can be regarded as an $R_{\mathfrak{m}}$-module by the following operation: $(r/s)x := rx'$ with $x = sx'$ $(r \in R, s \in R \setminus \mathfrak{m})$. Moreover, the submodules of $\mathrm{Cop}_{\mathfrak{m}}(M)$ over R and over $R_{\mathfrak{m}}$ are identical.

Lemma 3.8. *Let R be a ring with $\Omega = \mathrm{Max}(R)$. Assume that the following conditions are satisfied:*

(a) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for all* $\mathfrak{m} \in \Omega$, *and*

(b) *for each* $\mathfrak{m} \in \Omega$, *there exists a positive integer* n *such that* $\mathfrak{m}^n = \mathfrak{m}^{n+1}$.

If M is a coperfect R-module, then $M \cong \bigoplus_{\mathfrak{m}\in\Omega} M_{\mathfrak{m}}$ and each $M_{\mathfrak{m}}$ $(\mathfrak{m} \in \Omega)$ is isomorphic to a direct sum of cyclic modules of the type $R/\mathfrak{m}^k$.

P r o o f. Let M be a coperfect R-module. By [24], Theorem 2.15 we have $M = \bigoplus_{\mathfrak{m}\in\Omega} \mathrm{Cop}_{\mathfrak{m}}(M)$ and $\mathrm{Cop}_{\mathfrak{m}}(M)$ is a coperfect $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \Omega$. Since R satisfies conditions (a) and (b), so is each $R_{\mathfrak{m}}$ by Lemma 3.7. Moreover, since each $R_{\mathfrak{m}}$ is a local ring, it follows that $R_{\mathfrak{m}}$ is a T-ring by [11], Corollary 2.7. Note that any coperfect module is semiartinian, see [24], Theorem 2.9. Using Theorem 3.5, we deduce that each $\mathrm{Cop}_{\mathfrak{m}}(M)$ is isomorphic to a direct sum of cyclic $R_{\mathfrak{m}}$-modules of the type $R_{\mathfrak{m}}/\mathfrak{m}^k R_{\mathfrak{m}}$. Again by Lemma 3.7, it follows that each $\mathrm{Cop}_{\mathfrak{m}}(M)$ is isomorphic to a direct sum of cyclic R-modules of the type $R/\mathfrak{m}^k$. Also, note that $\mathrm{Cop}_{\mathfrak{m}}(M) \cong M_{\mathfrak{m}}$ for all $\mathfrak{m} \in \Omega$, see [24], Theorem 2.15. This completes the proof. □

Theorem 3.9. *Let R be a ring with $\Omega = \mathrm{Max}(R)$. Then the following are equivalent:*

(i) *Any artinian R-module is a DSC-module.*

(ii) *Any R-module with Krull dimension is a DSC-module.*

(iii) *Any locally artinian R-module is a DSC-module.*

(iv) *Any locally noetherian R-module is a DSC-module.*

(v) *Any coperfect R-module is a DSC-module.*

(vi) *R satisfies the following two conditions:*

(a) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for all* $\mathfrak{m} \in \Omega$, *and*

(b) *for each* $\mathfrak{m} \in \Omega$, *there exists a positive integer* n *such that* $\mathfrak{m}^n = \mathfrak{m}^{n+1}$.

Moreover, if a ring R satisfies the above conditions, then any R-module M of type (i)–(v) *has the following form:*

(α) $M \cong \bigoplus_{\mathfrak{m}\in\Omega} M_{\mathfrak{m}}$ *and each* $M_{\mathfrak{m}}$ ($\mathfrak{m} \in \Omega$) *is isomorphic to a direct sum of cyclic modules of the type* $R/\mathfrak{m}^k$.

Proof. (i) $\Rightarrow$ (vi) This follows from Lemma 3.4.

(vi) $\Rightarrow$ (v) This follows from Lemma 3.8.

(v) $\Rightarrow$ (iii) $\Rightarrow$ (i) This is clear since any locally artinian module is coperfect and any artinian module is locally artinian.

So we have the equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi).

(iii) $\Rightarrow$ (iv) First, note that R is a tall ring by the equivalence (iii) $\Leftrightarrow$ (vi) and using Lemma 3.3. Therefore, any noetherian R-module is artinian by Lemma 3.2. This implies that any locally noetherian R-module is locally artinian. So any locally noetherian R-module is DSC.

(iv) $\Rightarrow$ (i) It is well known that any artinian ring is noetherian. So any artinian R-module is locally noetherian.

(i) $\Rightarrow$ (ii) Let M be an R-module with Krull dimension. From the equivalence (i) $\Leftrightarrow$ (vi) and Lemma 3.3, we conclude that R is a tall ring. Using Lemma 3.2 twice, we obtain that M is an artinian module. Therefore, M is DSC.

(ii) $\Rightarrow$ (i) This follows from the fact that any artinian module has Krull dimension.

Now assume that R satisfies conditions (i)–(vi). We proved above that any R-module M which is artinian or locally artinian or locally noetherian or M has Krull dimension, is coperfect and hence M has the form (α) by Lemma 3.8. □

In the following remark, we provide a ring R which satisfies the statements of Theorems 3.5 and 3.9, but R is not a Köthe ring.

Remark 3.10. Let K be a field and let $R = \prod_{i\geqslant 1} K_i$, where $K_i = K$ for all $i \geqslant 1$. Then R is a von Neumann regular ring. So $\mathfrak{m}^2 = \mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$. Moreover, R is a T-ring, see [6], page 320. By [6], Theorem 5, any semiartinian R-module is semisimple. On the other hand, R is neither an artinian ring nor a principal ideal ring.

In the next result, we show that to obtain a characterization of the class of rings R having the property that every semi-V-module in Mod-R is a DSC-module, we need to add a necessary condition to conditions (a), (b) and (c) in Proposition 2.11.

Theorem 3.11. *Let R be a ring with $\Omega = \mathrm{Max}(R)$. Then the following are equivalent:*

(i) *Any semi-V-module in* Mod-R *is a DSC-module.*

(ii) *For any semi-V-module M in* Mod-R*, we have $M \cong \bigoplus_{\mathfrak{m}\in\Omega} M_{\mathfrak{m}}$ and each $M_{\mathfrak{m}}$ ($\mathfrak{m} \in \Omega$) is isomorphic to a direct sum of cyclic modules of the type $R/\mathfrak{m}^k$.*

(iii) *R satisfies the following four conditions:*

 (a) *R is a T-ring;*

 (b) $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ *for all* $\mathfrak{m} \in \Omega$*;*

 (c) *For each $\mathfrak{m}$ in Ω, there exists a positive integer n such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$;*

 (d) *R is a VSA-ring.*

Proof. (i) $\Rightarrow$ (iii) Since any V-module is a semi-V-module, it follows that any V-module in Mod-R is DSC. By Proposition 2.9, conditions (a) and (d) hold. Moreover, note that any semiartinian module is a semi-V-module by [25], Proposition 3.1. Hence, any semiartinian R-module is DSC. By Theorem 3.5, conditions (b) and (c) also hold.

(iii) $\Rightarrow$ (ii) Let M be a semi-V-module in Mod-R. Since R is a VSA-ring, M is semiartinian by [25], Proposition 3.16. Using Theorem 3.5, we get the desired decomposition of M.

(ii) $\Rightarrow$ (i) This is immediate. □

Next, we present four examples which show that none of the conditions (a), (b), (c) and (d) in Theorem 3.11 can be dropped.

Example 3.12.

(i) Consider the ring $R = \mathbb{Z}$. By Example 3.6, R satisfies (a) and (b) but not (c). Moreover, since R is noetherian, R is a VSA-ring by [25], Example 3.15 (iii).

(ii) Let K be a field and let V be an infinite vector space over K. Let $R = K\alpha V$ be the trivial extension of K by V. By Example 3.6, R satisfies (a) and (c) but R does not satisfy (b). Moreover, since R is a local ring, R is a VSA-ring by [25], Example 3.15 (ii).

(iii) Consider the ring R given in Example 3.6 (iii). Then R satisfies (b) and (c) but R does not satisfy (a). Also, by [1], Remark 3.14 (3), R is a semiartinian ring and so R is a VSA-ring.

(iv) Let K be a field and let $R = \prod_{i\geqslant 1} K_i$, where $K_i = K$ for all $i \geqslant 1$. Then clearly R is a von Neumann regular ring. Hence, R satisfies (b) and (c). Moreover, R is a T-ring, see [6], page 320. However, R is not semiartinian by [3], Proposition 4.5. It follows that the R-module R is a V-module which is not semiartinian. Therefore, R is not a VSA-ring.

Remark 3.13. It is easily seen that

$$\{\text{Rings in Theorem 3.5}\} \subseteq \{\text{Rings in Proposition 2.7}\},$$
$$\{\text{Rings in Theorem 3.9}\} \subseteq \{\text{Rings in Proposition 2.5}\},$$
$$\{\text{Rings in Theorem 3.11}\} \subseteq \{\text{Rings in Proposition 2.11}\}.$$

Using Example 3.12 (i), we see that all these inclusions are strict.

Recall that a module M is called *finitely embedded* if there exist finitely many simple submodules S_i $(1 \leqslant i \leqslant n)$ of M such that $E(M) = E(S_1) \oplus \ldots \oplus E(S_n)$. It is well known that every artinian module is finitely embedded, see [36], Proposition 2*. A nonzero module M is said to have *finite uniform* (or *Goldie*) *dimension* if M contains no infinite direct sum of nonzero submodules. Equivalently, $E(M)$ is a finite direct sum of indecomposable injective submodules, see [27], Proposition 6.12. Hence, any finitely embedded module has finite uniform dimension.

Theorem 3.14. *Let R be a ring with $\Omega = \operatorname{Max}(R)$. Then the following are equivalent:*

(i) *Any finitely embedded R-module is a DSC-module.*

(ii) *For any finitely embedded R-module M we have $M \cong \bigoplus_{\mathfrak{m}\in\Omega} M_{\mathfrak{m}}$ and each $M_{\mathfrak{m}}$ is isomorphic to a direct sum of cyclic submodules of the type $R/\mathfrak{m}^k$.*

(iii) *Any R-module having finite uniform dimension is a DSC-module.*

(iv) *For any R-module having finite uniform dimension M we have $M \cong \bigoplus_{\mathfrak{m}\in\Omega} M_{\mathfrak{m}}$ and each $M_{\mathfrak{m}}$ is isomorphic to a direct sum of cyclic submodules of the type $R/\mathfrak{m}^k$.*

(v) *$R_{\mathfrak{m}}$ is an artinian principal ideal ring for all $\mathfrak{m} \in \Omega$.*

P r o o f. The implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) are clear.

(i) $\Rightarrow$ (v) Let M be a finitely embedded R-module. By hypothesis, M is a finite direct sum of cyclic submodules since M has finite uniform dimension. So M is finitely generated. By [36], Theorem 3, it follows that $R_{\mathfrak{m}}$ is an artinian ring for every $\mathfrak{m} \in \Omega$. Since artinian modules are finitely embedded, using Theorem 3.9, we conclude that $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ for all $\mathfrak{m} \in \Omega$. Therefore, $R_{\mathfrak{m}}$ is a principal ideal ring for every $\mathfrak{m} \in \Omega$ by Lemma 3.7 (iii).

(v) $\Rightarrow$ (iv) By Lemma 6.6 of [33], $R_{\mathfrak{m}}$ is an artinian valuation ring for all $\mathfrak{m} \in \Omega$. So any indecomposable R-module is artinian by [10], Theorem III.4. Let M be an R-module of finite uniform dimension. So $E(M)$ is a finite direct sum of artinian modules by [27], Proposition 6.12. Therefore, M itself is an artinian R-module.

Moreover, R satisfies condition (vi) (b) of Theorem 3.9 by (v). Also, R satisfies condition (a) of Theorem 3.9 by Lemma 3.7 (iii). From Theorem 3.9, it follows that $M \cong \bigoplus_{\mathfrak{m}\in\Omega} M_{\mathfrak{m}}$ and each $M_{\mathfrak{m}}$ is isomorphic to a direct sum of cyclic submodules of the type $R/\mathfrak{m}^k$.

The implications (iv) ⇒ (iii) ⇒ (i) are clear. □

We conclude this paper by characterizing the class of rings R for which every locally supplemented R-module is a DSC-module. The following lemma will be useful.

Lemma 3.15. *Let R be a ring and let $\mathcal{C}$ be a hereditary pretorsion class in* Mod-R. *Consider the following two conditions:*

(i) *Any R-module in $\mathcal{C}$ is a DSC-module.*

(ii) *R/I is an artinian principal ideal ring for any proper ideal I of R with $(R/I)_R \in \mathcal{C}$.*

Then (i) ⇒ (ii).

Proof. Let I be a proper ideal of R such that $R/I \in \mathcal{C}$ and let M be an R/I-module. By the same method as in the proof of Proposition 2.1((i) ⇒ (ii)), it follows that M is a DSC R/I-module. Hence, R/I is an artinian principal ideal ring by Theorem 3.1. □

The implication (ii) ⇒ (i) in Lemma 3.15 it not true, in general, as shown in the following example.

Example 3.16. Let $R = \mathbb{Z}$ and consider the class $\mathcal{C}$ of locally artinian $\mathbb{Z}$-modules. It is clear that R satisfies condition (ii) in Lemma 3.15. However, R does not satisfy condition (vi) (b) of Theorem 3.9. So R does not satisfy condition (i) in Lemma 3.15.

Proposition 3.17. *Let R be a ring with $\Omega = \mathrm{Max}(R)$. Then the following statements are equivalent:*

(i) *Every locally supplemented R-module is DSC.*

(ii) *R satisfies the following two conditions:*

  (a) *R/I is an artinian principal ideal ring for any proper ideal I of R with R/I a local ring;*

  (b) *for each $\mathfrak{m}$ in Ω, there exists a positive integer n such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$.*

(iii) *R satisfies the following two conditions:*

  (c) *R/I is a perfect ring for any proper ideal I of R with R/I a local ring;*

  (d) *for each $\mathfrak{m}$ in Ω, we have $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$ and there exists a positive integer n such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$.*

P r o o f. (i) $\Rightarrow$ (ii) To prove (a), let I be a proper ideal of R such that R/I is a local ring. Then R/I is a supplemented R-module. Therefore, R/I is a locally supplemented R-module by [23], Lemma 2.5. But the class of locally supplemented modules is a hereditary pretorsion class, see [23], Remark 2.8. Using Lemma 3.15, it follows that R/I is an artinian principal ideal ring.

To show (b), note that any locally artinian R-module is locally supplemented. Hence, any locally artinian R-module is a DSC-module by (i). Now use Theorem 3.9.

(ii) $\Rightarrow$ (iii) (c) is clear.

(d) Let $\mathfrak{m}$ be a maximal ideal of R. Since $R/\mathfrak{m}^2$ is a local ring, $R/\mathfrak{m}^2$ is an artinian principal ideal ring. By Lemma 3.7 (i), we have $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1$.

(iii) $\Rightarrow$ (i) Let M be a locally supplemented R-module and let $0 \neq x \in M$. Then $R/\mathrm{Ann}_R(x)$ is a semiperfect ring by [23], Lemma 2.5. Hence, $R/\mathrm{Ann}_R(x) \cong R/I_1 \times \ldots \times R/I_n$, where each I_i $(1 \leqslant i \leqslant n)$ is an ideal of R such that R/I_i is a local ring, see [28], Theorem 23.11. By (c), each R/I_i $(1 \leqslant i \leqslant n)$ is a perfect ring. So $R/\mathrm{Ann}_R(x)$ is a perfect ring by [28], Theorem 23.24. Using Proposition 2.5 of [24], it follows that M is a coperfect R-module. Combining (d) and Theorem 3.9, we deduce that M is a DSC-module. $\square$

Remark 3.18. To compare Propositions 2.13 and 3.17, it is clear that

$$\{\text{Rings in Proposition 3.17}\} \subseteq \{\text{Rings in Proposition 2.13}\}.$$

But this inclusion is strict. Indeed, it is well known that every finitely generated $\mathbb{Z}$-module is a DSC-module. On the other hand, the ring $\mathbb{Z}$ does not satisfy condition (b) in Proposition 3.17.

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Authors' address: Farid Kourki, Rachid Tribak (corresponding author), Equipe de Recherche: Algèbre & Didactique des Mathématiques, Centre Régional des Métiers de l'Education et de la Formation de la Région Tanger-Tétouan-Al Hoceima, (CRMEF-TTH)-Tanger, Avenue My Abdelaziz, Souani, BP: 3117, Tangier, Morocco, e-mail: `kourkifarid@hotmail.com`, `tribak12@yahoo.com`.