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KERNELS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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Abstract. A Coburn theorem says that a nonzero Toeplitz operator on the Hardy space is one-to-one or its adjoint operator is one-to-one. We study the corresponding problem for certain Toeplitz operators on the Bergman space.

Keywords: Toeplitz operator; Bergman space

MSC 2020: 47B35, 32A36

1. Introduction

Let D denote the unit disk of the complex plane $\mathbb{C}$ and $\mathbb{T}$ be the boundary of D. The Bergman space L^2_a is the closed subspace of the usual Lebesgue space $L^2 := L^2(D, A)$ consisting of all analytic functions on D, where the measure A is the normalized area measure on D. Let P be the Hilbert space orthogonal projection from L^2 onto L^2_a. For a bounded measurable function u on D, the *Toeplitz operator* T_u with the symbol u is defined by

$$T_u f = P(uf)$$

for functions $f \in L^2_a$. Clearly, T_u is a bounded linear operator on L^2_a. See [10] for details and more information on Toeplitz operators.

For Toeplitz operators acting on the Hardy space of unit disk, a celebrated theorem of Coburn asserts that a nonzero Toeplitz operator is one-to-one or its adjoint operator is one-to-one; see Theorem 4.1 of [3]. Coburns result can be rephrased as a nonzero Hardy space Toeplitz operator has either trivial kernel or dense range.

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Later, Vukotić in [9] reproved the theorem by showing that if a nonzero Hardy space Toeplitz operator is not injective then its range contains all polynomials, which is due to the simple fact that the commutator of the shift and any Hardy space Toeplitz operator have rank at most one. Also, the corresponding problem has been studied on the Hardy space of polydisk or Dirichlet space as in [1] and [5].

In this paper we consider the corresponding natural problem for Toeplitz operators acting on the Bergman space of the unit disk. First of all, we should mention that the Coburn type theorem fails generally on the Bergman space. Indeed, using (2.1) in Section 2, one can see that

$$T_{|z|^{2n}-1/(n+1)}(1) = 0$$

and moreover

$$\ker T_{|z|^{2n}-1/(n+1)} = \mathbb{C}\cdot 1$$

for all $n = 0, 1, \ldots$ Here, $\ker S$ denotes the kernel of operator S as usual. Since $T_\psi^* = T_{\overline{\psi}}$ for any bounded function ψ, the above shows that the Coburn type theorem fails for Toeplitz operators with symbol $|z|^{2n} - 1/(n+1)$ for every n.

In this paper, given a bounded analytic function f on D, we study the problem of when the Coburn type theorem holds for the Toeplitz operator with symbol $f + \overline{z}$. More explicitly, we consider analytic functions f on D continuous up to $\mathbb{T}$ with a certain boundary condition and then give a characterization in terms of integrability and the number of zeros of a certain function induced by f; see Theorem 2.1 of Section 2.

In Section 3, we specially take f to be cz^N, where $N \geqslant 0$ is an integer and $c \in \mathbb{C}$ is a constant. We then give complete descriptions for kernels of $T_{cz^N+\overline{z}}$ and $T^*_{cz^N+\overline{z}}$, respectively. Our result shows that the characterizations depend on the absolute value of the constant c; see Theorems 3.1 and 3.2. As an immediate consequence, we show that the Coburn type theorem holds for the Toeplitz operator $T_{cz^N+\overline{z}}$ for all N and c; see Corollary 3.1. But, we were not able to characterize a general symbol for which the Coburn type theorem holds for the corresponding Toeplitz operator.

2. The kernel of $T_{f+\bar{z}}$

As is well known, the projection P is the Bergman projection whose explicit formula can be given by

$$P\psi(z) = \int_D \psi(w)\overline{K_z(w)}\,\mathrm{d}A(w), \quad z \in D,$$

for functions $\psi \in L^2$. Here, function K_z is the Bergman kernel given by

$$K_z(w) = \frac{1}{(1 - w\overline{z})^2}, \quad w \in D.$$

Using this formula for P, one can see that for integers $n, m \geqslant 0$

$$P(z^n \overline{z^m}) = \begin{cases} \dfrac{n-m+1}{n+1} z^{n-m} & \text{if } n \geqslant m, \\ 0 & \text{if } n < m. \end{cases} \tag{2.1}$$

Also, given a function $h \in L_a^2$ with $h(0) = 0$, we have

$$P(\overline{w}h)(z) = \frac{1}{z} h(z) - \frac{1}{z^2} \int_0^z h \, d\zeta \tag{2.2}$$

for all $z \in D$; see Lemma 2 of [2] or Lemma 2.1 of [8] for details.

We start with an equivalent condition for the kernel of the Toeplitz operator $T_{f+\overline{z}}$, where $f \in H^\infty$. Here, the notation H^∞ stands for the space of all bounded analytic functions on D.

Lemma 2.1. *Let $f \in H^\infty$ and $h \in L_a^2$. Write*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad h(z) = \sum_{n=0}^{\infty} b_n z^n$$

for power series expansions of f and h, respectively. Then all the following statements are equivalent.

(a) $h \in \ker T_{f+\overline{z}}$.

(b) $(zf + 1)h' + (zf' + 2f)h = 0$.

(c) $b_{n+1} = -(n+2)/(n+1) \sum\limits_{k=0}^{n} a_k b_{n-k}$ *for all* $n = 0, 1, 2, \ldots$

Proof. By (2.1), we note $P(\overline{w}) = 0$. It follows from (2.2) that

$$\begin{aligned} T_{f+\overline{w}} h(z) &= f(z)h(z) + P(\overline{w}h)(z) = f(z)h(z) + P[\overline{w}(h - h(0)](z) + h(0)P(\overline{w})(z) \\ &= f(z)h(z) + P[\overline{w}(h - h(0)](z) \\ &= f(z)h(z) + \frac{1}{z}[h(z) - h(0)] - \frac{1}{z^2} \int_0^z [h(\zeta) - h(0)] \, d\zeta, \quad z \in D. \end{aligned}$$

Thus, (a) holds if and only if

$$z^2 f(z)h(z) = -z[h(z) - h(0)] + \int_0^z [h(\zeta) - h(0)] \, d\zeta, \quad z \in D,$$

which is equivalent to (b) by differentiating both sides. Hence, (a) $\Leftrightarrow$ (b) holds. Also, if we assume (b), by differentiating both sides of (b), we see that

$$(n+2)(fh)^{(n)} + z(fh)^{(n+1)} = -h^{(n+1)}$$

for all $n = 0, 1, 2, \ldots$ Evaluating at $z = 0$, we have

$$(n+2)(fh)^{(n)}(0) = -h^{(n+1)}(0)$$

and thus

$$b_{n+1}=\frac{h^{(n+1)}(0)}{(n+1)!}=-\frac{(n+2)}{(n+1)!}\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}f^{(k)}(0)h^{(n-k)}(0)=-\frac{n+2}{n+1}\sum_{k=0}^{n}a_kb_{n-k}$$

for all $n=0,1,\ldots$, so (c) holds. Finally, if we assume (c), we have

$$\frac{h^{(n+1)}(0)}{(n+1)!}=-\frac{n+2}{n+1}\frac{(fh)^{(n)}(0)}{n!}$$

for all $n=0,1,\ldots$ It follows that

$$\begin{aligned}2f(z)h(z)+z(fh)'(z)&=\sum_{n=0}^{\infty}(n+2)\frac{(fh)^{(n)}(0)}{n!}z^n=-\sum_{n=0}^{\infty}\frac{h^{(n+1)}(0)}{(n+1)!}(n+1)z^n\\&=-\sum_{n=0}^{\infty}(n+1)b_{n+1}z^n=-h'(z),\quad z\in D.\end{aligned}$$

So (b) holds and (b) $\Leftrightarrow$ (c) follows. The proof is complete. □

Lemma 2.1 shows that $h\in\ker T_{f+\bar z}$ if and only if $h\in L^2_a$ is a solution of the first order differential equation

$$(zf+1)H'+(zf'+2f)H=0. \tag{2.3}$$

As is well known, the space of all solutions of (2.3) is at most one dimensional. Thus, for $f\in H^\infty$, the dimension of $\ker T_{f+\bar z}$ is 0 or 1.

Proposition 2.1. *Let $f\in H^\infty$ be such that the function*

$$\psi:=\frac{2f+zf'}{1+zf}$$

is well defined as an analytic function on D. Put

$$F(z):=\exp\left[-\int_0^z\psi\,\mathrm{d}\zeta\right],\quad z\in D.$$

Then, the dimension of $\ker T_{f+\bar z}$ is 0 or 1, and the following statements hold.

(a) $\dim\ker T_{f+\bar z}=1$ *if and only if* $F\in L^2_a$ *and* $\ker T_{f+\bar z}=\mathbb{C}\cdot F$.

(b) $\dim\ker T_{f+\bar z}=0$ *if and only if* $F\notin L^2_a$.

P r o o f. First one can see that F satisfies the differential equation (2.3). Since the dimension of $\ker T_{f+\bar z}$ is 0 or 1 as mentioned above, (a) and (b) hold. The proof is complete. □

Let $\mathcal{B}$ denote the C^*-algebra consisting of all bounded linear operators on L^2_a. Also, let $\mathcal{K}$ be the algebra of all compact operators on L^2_a. An operator $L \in \mathcal{B}$ is said to be Fredholm if $L + \mathcal{K}$ is invertible in the Calkin algebra $\mathcal{B}/\mathcal{K}$. It turns out that for $L \in \mathcal{B}$, L is Fredholm if and only if the range of L is closed and both the dimensions of L and L^* are finite. For a Fredholm operator L on L^2_a, the index $\operatorname{ind}(L)$ of L is defined by

$$\operatorname{ind}(L) = \dim\ker L - \dim\ker L^*.$$

For a function $\varphi \in C(\overline{D})$, it is known that T_φ is Fredholm on L^2_a if and only if $\varphi(\zeta) \neq 0$ for all $\zeta \in \mathbb{T}$, in which case $\operatorname{ind}(T_\varphi)$ is given by $-\mathscr{W}(\varphi(\mathbb{T}), 0)$, where $\mathscr{W}(\gamma, a)$ is the winding number of a closed curve γ in $\mathbb{C}$ with respect to $a \notin \mathbb{C} \setminus \gamma$. Thus, we have

$$\mathscr{W}(\varphi(\mathbb{T}), 0) = -\dim\ker T_\varphi + \dim\ker T^*_\varphi,$$

see [7] and references therein for details and related facts.

The following result will be useful in our characterizations. Given a function ψ on D, let $\mathcal{Z}(\psi)$ denote the number of zeros of ψ in D.

Proposition 2.2. *Let $f \in H^\infty \cap C(\overline{D})$ and put $g := zf + 1$. Assume $g(\zeta) \neq 0$ for all $\zeta \in \mathbb{T}$. Then, the dimension of $\ker T_{f+\overline{z}}$ is 0 or 1, and the following statements hold.*

(a) $\dim\ker T_{f+\overline{z}} = 0$ *if and only if* $\mathcal{Z}(g) = \dim\ker T^*_{f+\overline{z}} + 1$.

(b) $\dim\ker T_{f+\overline{z}} = 1$ *if and only if* $\mathcal{Z}(g) = \dim\ker T^*_{f+\overline{z}}$.

Proof. Let $\varphi = f + \overline{z}$. For $|z| = 1$, noting that

$$\varphi = f + \frac{1}{z} = \frac{1 + zf}{z},$$

we obtain $g = z\varphi$ on $\mathbb{T}$. It follows from the argument principle that

$$\mathcal{Z}(g) = \mathscr{W}(g(\mathbb{T}), 0) = \mathscr{W}(\varphi(\mathbb{T}), 0) + 1. \tag{2.4}$$

Since $g = z\varphi$ on $\mathbb{T}$, we see that $\varphi(\zeta) \neq 0$ for all $\zeta \in \mathbb{T}$. Hence, T_φ is Fredholm and (2.4) gives

$$\mathcal{Z}(g) = -\dim\ker T_\varphi + \dim\ker T^*_\varphi + 1.$$

Since the dimension of $\ker T_\varphi$ is 0 or 1 as mentioned before, we have (a) and (b). The proof is complete. □

If $\dim\ker T_{f+\overline{z}} = 1$ and g has a zero in D in Proposition 2.2, we remark in passing that g has only finite many simple zeros. This is an immediate consequence of Theorem 2.4 of [4], where the authors investigate structure of spectrum of the Toeplitz operator $T_{f+\overline{z}}$ when f is a polynomial. See also [6] or [8] for more information on the study of spectrum of a Bergman space Toeplitz operator.

The following theorem is the main result of this section.

Theorem 2.1. *Let $f \in H^\infty \cap C(\overline{D})$. Suppose that $\zeta f(\zeta)+1 \neq 0$ for all $\zeta \in \mathbb{T}$ and the function $\psi := (2f+zf')/(1+zf)$ is well defined as an analytic function on D. Put*

$$F(z) = \exp\Big[-\int_0^z \psi \,\mathrm{d}\zeta\Big], \quad z \in D.$$

Then the the following statements are equivalent.

(a) *The Coburn type theorem holds for the Toeplitz operator $T_{f+\overline{z}}$.*

(b) *Either $F \notin L^2_a$ or $\mathcal{Z}(1+zf) = 0$.*

P r o o f. By Proposition 2.2, we see that $\dim\ker T_{f+\overline{z}} = 1$ and $\dim\ker T^*_{f+\overline{z}} = 0$ if and only if $\mathcal{Z}(1+zf) = 0$. Thus, the result follows from Proposition 2.1. The proof is complete. □

The following shows that an analytic function f on D with certain boundedness properties induces a Toeplitz operator $T_{f+\overline{z}}$ satisfying the Coburn type theorem. In the following, for $1 \leqslant p \leqslant \infty$, the notation $\|f\|_p$ stands for the usual L^p-norm for $f \in L^p(D, A)$.

Corollary 2.1. *Let $f \in H^\infty$. Suppose f' is bounded and $\|f\|_\infty < 1$. Then $\dim\ker T_{f+\overline{z}} = 1$ and $\dim\ker T^*_{f+\overline{z}} = 0$. Moreover,*

$$\ker T_{f+\overline{z}} = \mathbb{C}\cdot\exp\Big[-\int_0^z \frac{2f+\zeta f'}{1+\zeta f}\,\mathrm{d}\zeta\Big].$$

P r o o f. Since f, f' are bounded, f can be extended to a continuous function on $\overline{D}$ denoted also by f. Since $\|f\|_\infty < 1$ by the assumption, the maximum modulus principle implies that $|f(z)| < 1$ for all $z \in \overline{D}$, hence, the function $zf+1$ has no zeros on $\overline{D}$. By Theorem 2.1, $T_{f+\overline{z}}$ meets the Coburn type theorem and we have the first part by Proposition 2.2. Also, by the assumptions, one can check that the function F introduced at Proposition 2.1 is nonzero and contained in L^2_a. Thus, the kernel description follows from Proposition 2.1. The proof is complete. □

We remark in passing that the Coburn type theorem fails even for harmonic symbols generally. For example, Sundberg and Zheng in [8] constructed a function $f \in H^\infty$ satisfying the conditions of Theorem 2.1 such that $\mathcal{Z}(fz+1) = 1$ and the function F in Proposition 2.1 belongs to L^2_a. By Propositions 2.1 and 2.2, we see that both $\ker T_{f+\overline{z}}$ and $\ker T^*_{f+\overline{z}}$ have dimension 1.

3. Kernels of $T_{cz^N+\bar{z}}$ and $T^*_{cz^N+\bar{z}}$

Consider $f = cz^N$, where $N \geqslant 0$ is an integer and $c \in \mathbb{C}$ is a constant. Put $\varphi = cz^N + \overline{z}$. If $|c| < 1$, Corollary 2.1 shows that $\dim\ker T_\varphi = 1$ and $\dim\ker T^*_\varphi = 0$. Also, if $|c| > 1$, then $zf(z)+1 \neq 0$ for every $z \in \mathbb{T}$ and $\mathcal{Z}(zf+1) = N+1$. Moreover, $\dim\ker T_\varphi = 0$ (see the proof of Theorem 3.1 below) and hence, $\dim\ker T^*_\varphi = N$ by Proposition 2.2.

In this section, we provide complete descriptions for kernels of $T_{cz^N+\overline{z}}$ and $T^*_{cz^N+\overline{z}}$ for every integer $N \geqslant 0$ and constant $c \in \mathbb{C}$. Our result shows that the characterizations depend on the absolute value of the constant c.

We first consider Toeplitz operators $T_{cz^N+\overline{z}}$.

Theorem 3.1. *Let $N \geqslant 0$ be an integer and $c \in \mathbb{C}$ be a constant. Then, the following statements hold.*

(a) $\ker T_{cz^N+\overline{z}} = \{0\}$ *if and only if* $|c| \geqslant 1$.

(b) $|c| < 1$ *if and only if*

$$\ker T_{cz^N+\overline{z}} = \mathbb{C}\cdot\sum_{j=0}^{\infty}(-c)^j\frac{\Gamma(j+1+(N+1)^{-1})}{j!}z^{j(N+1)}.$$

Proof. First we prove (a). Corollary 2.1 shows that $\ker T_{cz^N+\overline{z}} = \{0\}$ implies $|c| \geqslant 1$. For the converse implication, assume $|c| \geqslant 1$ and $h \in \ker T_{cz^N+\overline{z}}$ for some $h \in L^2_a$. Write

$$h(z) = \sum_{n=0}^{\infty} b_n z^n$$

for the power series expansion of h. By Lemma 2.1, we see that $b_1 = b_2 = \ldots = b_N = 0$ and

$$b_{N+1} = -\frac{N+2}{N+1}cb_0, \quad b_{N+j} = -\frac{N+j+1}{N+j}cb_{j-1}$$

for all $j = 1, 2, \ldots$ Hence, $b_k = 0$ for all $k \notin (N+1)\mathbb{N}$ and

$$b_{j(N+1)} = -c\frac{j(N+1)+1}{j(N+1)}b_{(j-1)(N+1)}$$

for all $j = 1, 2, \dots$ and thus,

$$h(z) = \sum_{j=0}^{\infty} b_{j(N+1)} z^{j(N+1)}, \quad z \in D. \tag{3.1}$$

Note that

$$\begin{aligned} b_{j(N+1)} &= (-c)^j b_0 \prod_{k=1}^{j} \frac{(N+1)k+1}{(N+1)k} = (-c)^j b_0 \prod_{k=1}^{j} \frac{k+(N+1)^{-1}}{k} \\ &= (-c)^j b_0 \frac{\Gamma(j+1+(N+1)^{-1})}{j!\Gamma(1+(N+1)^{-1})} \end{aligned} \tag{3.2}$$

for all $j = 1, 2, \dots$ Since $|c| \geqslant 1$ by the assumption, (3.2) shows that $|b_{j(N+1)}| \geqslant |b_0|$ for all $j = 0, 1, 2, \dots$ It follows from (3.1) that

$$\begin{aligned} \|h\|_2^2 &= \sum_{j=0}^{\infty} \frac{|b_{j(N+1)}|^2}{j(N+1)+1} \geqslant |b_0|^2 \sum_{j=1}^{\infty} \frac{1}{j(N+1)+1} \\ &\geqslant |b_0|^2 \sum_{j=1}^{\infty} \frac{1}{2j(N+1)} = \frac{|b_0|^2}{2(N+1)} \sum_{j=1}^{\infty} \frac{1}{j}. \end{aligned}$$

Since $h \in L_a^2$, the above shows that $b_0 = 0$ and then $h = 0$ by (3.2) and (3.1). Thus, $\ker T_{cz^N+\overline{z}} = \{0\}$ and (a) holds.

Now we prove (b). Assume $|c| < 1$ and let

$$h(z) := \sum_{n=0}^{\infty} b_n z^n \in \ker T_{cz^N+\overline{z}}.$$

By the proof of (a), we see that

$$h(z) = \frac{b_0}{\Gamma(1+(N+1)^{-1})} \sum_{j=0}^{\infty} (-c)^j \frac{\Gamma(j+1+(N+1)^{-1})}{j!} z^{j(N+1)}, \quad z \in D.$$

Hence,

$$\ker T_{cz^N+\overline{z}} \subset \mathbb{C} \cdot \sum_{j=0}^{\infty} (-c)^j \frac{\Gamma(j+1+(N+1)^{-1})}{j!} z^{j(N+1)}.$$

To prove the reverse inclusion, let

$$g := \sum_{j=0}^{\infty} (-c)^j \frac{\Gamma(j+1+(N+1)^{-1})}{j!} z^{j(N+1)}.$$

We first show $g \in L_a^2$. Recall that

$$\frac{\Gamma(j+a)}{\Gamma(j+b)} \approx j^{a-b}$$

and hence,

$$\frac{\Gamma(j+1+(N+1)^{-1})^2}{(j!)^2[j(N+1)+1]} \approx \frac{j^{2/(N+1)}}{j} \approx \Big(\frac{1}{j+1}\Big)^{(N-1)/(N+1)}$$

as $j \to \infty$. Thus, we see that $g \in L_a^2$ if and only if

$$\sum_{j=0}^{\infty} |c|^{2j} \Big(\frac{1}{j+1}\Big)^{(N-1)/(N+1)} < \infty. \tag{3.3}$$

Note that the power series defined by

$$z \mapsto \sum_{j=0}^{\infty} \Big(\frac{1}{j+1}\Big)^{(N-1)/(N+1)} z^j$$

has the radius of convergence 1. Since $|c| < 1$ by assumption, we see that (3.3) holds and then $g \in L_a^2$. Also, by a simple calculation, we can see that g satisfies (c) of Lemma 2.1. Hence, $g \in \ker T_{cz^N+\overline{z}}$.

Also, the converse implication follows from (a). The proof is complete. □

Next, the following is about the kernel of $T^*_{cz^N+\overline{z}}$. Recall $T^*_{cz^N+\overline{z}} = T_{\overline{cz^N}+z}$. If $c = 0$, then $T^*_{cz^N+\overline{z}} = \ker T_z = \{0\}$ for all N. Also, if $N = 0$, we see $T^*_{cz^N+\overline{z}} = \ker T_{\overline{c}+z} = \{0\}$ for any constant c. Thus, we assume the cases $N \geqslant 1$ and $c \neq 0$.

Theorem 3.2. *Let $N \geqslant 1$ be an integer and $c \neq 0$ be a constant. Then, the following statements hold.*

(a) $\ker T^*_{cz^N+\overline{z}} = \{0\}$ *if and only if* $|c| \leqslant 1$.

(b) $|c| > 1$ *if and only if*

$$\ker T^*_{cz^N+\overline{z}} = \sum_{j=0}^{N-1} \mathbb{C} \cdot \sum_{k=0}^{\infty} \Big(\frac{-1}{\overline{c}}\Big)^k \frac{\Gamma(k+\beta_j)\Gamma(\alpha_j)}{\Gamma(k+\alpha_j)\Gamma(\beta_j)} z^{(N+1)k+j}, \tag{3.4}$$

where $\alpha_j := (j+2)/(N+1)$ and $\beta_j := (j+N+2)/(N+1)$ for simplicity.

Proof. We first prove (a). When $N = 1$, noting

$$\ker T^*_{cz+\overline{z}} = \ker T_{\overline{cz}+z} = \ker T_{z/\overline{c}+\overline{z}},$$

we see that $\ker T^*_{cz+\overline{z}} = \{0\}$ if and only if $1/|c| \geqslant 1$ by Theorem 3.1. Hence, the result holds for $N = 1$ and we furthermore assume $N > 1$.

First assume $|c| \leqslant 1$. Suppose there exists a nonzero function $h \in L_a^2$ such that $h \in \ker T^*_{cz^N+\overline{z}}$. Since $zh + \bar{c}P(\overline{z^N}h) = 0$, one can see that

$$\begin{aligned}\|zh\|_2^2 = |c|^2\|P(\overline{z^N}h)\|_2^2 \leqslant \int_D |z^N h|^2 \,\mathrm{d}A = \int_D |z|^{2(N-1)}|zh|^2 \,\mathrm{d}A \\ < \int_D |zh|^2 \,\mathrm{d}A = \|zh\|_2^2,\end{aligned}$$

which is impossible, so $h = 0$ and hence, $\ker T^*_{cz^N+\overline{z}} = \{0\}$ as desired.

For the converse, assume $|c| > 1$. We derive a contradiction by taking a nonzero function $h \in L_a^2$ such that $h \in \ker T^*_{cz^N+\overline{z}}$. First, let $a_0 = a_1 = \ldots = a_{N-1} = 1$ and $a_N = 0$. Also, define a_n inductively by

$$a_{(N+1)k+j} = \frac{-1}{\bar{c}} \frac{(N+1)k+j+1}{(N+1)(k-1)+j+2} a_{(N+1)(k-1)+j}$$

for $k = 1, 2, \ldots$ and $j = 0, 1, \ldots, N$. Define a nonzero analytic function h by $h = \sum_{n=0}^{\infty} a_n z^n$. Note $a_{(N+1)k+N} = 0$ for all $k = 0, 1, 2, \ldots$ and

$$\begin{aligned}a_{(N+1)k+j} &= \Big(\frac{-1}{\bar{c}}\Big)^k \prod_{l=0}^{k-1} \frac{(N+1)l+j+N+2}{(N+1)l+j+2} = \Big(\frac{-1}{\bar{c}}\Big)^k \prod_{l=0}^{k-1} \frac{l+\beta_j}{l+\alpha_j} \\ &= \Big(\frac{-1}{\bar{c}}\Big)^k \frac{\Gamma(k+\beta_j)\Gamma(\alpha_j)}{\Gamma(k+\alpha_j)\Gamma(\beta_j)}\end{aligned}$$

for all $k = 1, 2, \ldots$ and $j = 0, 1, \ldots, N-1$. Now, since

$$h = \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} a_{(N+1)k+j} z^{(N+1)k+j},$$

we have

$$\begin{aligned}\|h\|_2^2 &= \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} \frac{|a_{(N+1)k+j}|^2}{(N+1)k+j+1} \\ &= \sum_{j=0}^{N-1} \frac{\Gamma(\alpha_j)^2}{\Gamma(\beta_j)^2} \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta_j)^2}{|c|^{2k}[(N+1)k+j+1]\Gamma(k+\alpha_j)^2} \\ &\leqslant \sum_{j=0}^{N-1} \frac{\Gamma(\alpha_j)^2}{\Gamma(\beta_j)^2} \sum_{k=0}^{\infty} \frac{1}{|c|^{2k}} \frac{\Gamma(k+\beta_j)^2}{\Gamma(k+\alpha_j)^2}.\end{aligned}$$

Note that the power series defined by

$$z \mapsto \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta_j)^2}{\Gamma(k+\alpha_j)^2} z^k$$

has the radius of convergence 1. Since $|c| > 1$ by assumption, we see that

$$\sum_{k=0}^{\infty} \frac{1}{|c|^{2k}} \frac{\Gamma(k+\beta_j)^2}{\Gamma(k+\alpha_j)^2} < \infty$$

for each j and hence, $h \in L_a^2$. Moreover, by using (2.1) and the definition of h, one can see that $h \in \ker T^*_{cz^N+\overline{z}}$ and hence, (a) holds.

Now we prove (b). First, assume $|c| > 1$ and let $\mathcal{N}$ be the set on the right side of (3.1) and

$$f := \sum_{n=0}^{\infty} b_n z^n \in \ker T^*_{cz^N+\overline{z}}.$$

Then, $fz + P(\overline{cz^N} f) = 0$. By (2.1), we have

$$\bar{c} b_N + \sum_{j=0}^{\infty} \Big[b_j + \bar{c} b_{N+1+j} \frac{j+2}{N+j+2} \Big] z^{j+1} = 0.$$

Hence, $b_N = 0$ and

$$b_{N+1+j} = -\frac{1}{\bar{c}} \frac{N+j+2}{j+2} b_j$$

for all $j = 0, 1, 2, \ldots$ It follows that $b_{k(N+1)+N} = 0$ for all $k = 0, 1, 2, \ldots$ and

$$b_{(N+1)k+j} = \frac{-1}{\bar{c}} \frac{(N+1)k+j+1}{(N+1)(k-1)+j+2} b_{(N+1)(k-1)+j} = \Big(\frac{-1}{\bar{c}}\Big)^k \frac{\Gamma(k+\beta_j)\Gamma(\alpha_j)}{\Gamma(k+\alpha_j)\Gamma(\beta_j)} b_j$$

for $k = 1, 2, \ldots$ and $j = 0, 1, \ldots, N-1$. Hence,

$$f(z) = \sum_{j=0}^{N-1} b_j \sum_{k=0}^{\infty} \Big(\frac{-1}{\bar{c}}\Big)^k \frac{\Gamma(k+\beta_j)\Gamma(\alpha_j)}{\Gamma(k+\alpha_j)\Gamma(\beta_j)} z^{(N+1)k+j},$$

which shows that $\ker T^*_{cz^N+\overline{z}} \subset \mathcal{N}$. Also, by using (2.1) and the similar argument as in the proof of Theorem 3.1, one can see that every function in $\mathcal{N}$ belongs to both L_a^2 and $\ker T^*_{cz^N+\overline{z}}$. Hence, $\ker T^*_{cz^N+\overline{z}} = \mathcal{N}$. Since the converse implication follows from (a), we have (b). □

Finally, as an immediate consequence of Theorems 3.1 and 3.2, we have the following.

Corollary 3.1. *The Coburn type theorem holds for the Toeplitz operator $T_{cz^N+\overline{z}}$ for any integer $N \geqslant 0$ and $c \in \mathbb{C}$.*

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