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BINOMIAL SUMS VIA BAILEY'S CUBIC TRANSFORMATION

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Abstract. By employing one of the cubic transformations (due to W. N. Bailey (1928)) for the ${}_3F_2(x)$-series, we examine a class of ${}_3F_2(4)$-series. Several closed formulae are established by means of differentiation, integration and contiguous relations. As applications, some remarkable binomial sums are explicitly evaluated, including one proposed recently as an open problem.

Keywords: hypergeometric series; Bailey's cubic transformation; contiguous relation; reversal series; binomial coefficient

MSC 2020: 33C20, 05A19, 11B65

1. Introduction and outline

In a recent paper (see [10]), the following conjectured binomial identity was proposed as an open problem (here we have corrected a typo by exchanging the odd and even cases):

$$(1)\qquad \sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{2k} = \binom{\lfloor\frac{2n-1}{3}\rfloor+1}{\lfloor\frac{n}{3}\rfloor}\binom{2\left(\lfloor\frac{2n-1}{3}\rfloor\right)+1}{\lfloor\frac{2n-1}{3}\rfloor} \times \frac{(-1)^{\lfloor n/3\rfloor}}{n(n+1)}\begin{cases} 2+4\lfloor\frac{n+1}{6}\rfloor & \text{if } n \text{ is odd},\\ 4+4\lfloor\frac{n-1}{6}\rfloor & \text{if } n \text{ is even}.\end{cases}$$

By making use of a computer algebra system, Campbell in [3], Theorems 1 and 2 provided a verification proof via nontrivial applications of Zeilberger's algorithm

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and produced another similar identity. To achieve this, the binomial sum displayed in (1) was split into six individual target sums (see [3], Theorem 1) according to the residues of n modulo 6. Moreover, the most complicated aspect in the proofs given by Campbell (see [3], Theorems 1 and 2) is that they involve very large "certificates" and horrible polynomials, that are unreadable by human beings.

Taking into account the elegance and simplicity of (1), the author believes that there should exist a classical proof by human beings. In fact, by converting the binomial sum in (1) into a hypergeometric series, we find that the resulting expression has close ties with one of Bailey's cubic transformations, see [1]. The objective of the present paper is to illustrate how to derive binomial identity (1) and similar ones by the hypergeometric series approach, especially, Bailey's cubic transformation. Unlike the WZ-method to deal with binomial sums individually, the advantage of our method is that it can be utilized to find systematically a large number of binomial identities, but with none of them being verifiable via the WZ-method.

The rest of the paper is organized as follows. By differentiating and integrating the afore-mentioned Bailey's cubic transformation, we prove, in the next section, two important summation theorems for the terminating ${}_3F_2(4)$-series that lead us to eight closed formulae. Then in Section 3, further summation identities are derived through contiguous relations. The reversal series of argument $\frac{1}{4}$ is briefly reviewed in Section 4. Finally, the paper ends with Section 5, where nine remarkable binomial identities are proved, including (1).

Throughout the paper, the following notations are utilized. For a real number x, the greatest integer not exceeding x is denoted by $\lfloor x \rfloor$. When m is a natural number, $a \equiv_m b$ stands for that "a is congruent to b modulo m". The logical function χ is defined by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. For a nonnegative integer $n \in \mathbb{N}_0$ and an indeterminate x, the rising factorial is defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\dots(x+n-1) \quad \text{for } n \in \mathbb{N}.$$

The quotient form with multiparameters is abbreviated to

$$\left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix}\right]_n = \frac{(\alpha)_n(\beta)_n \dots (\gamma)_n}{(A)_n(B)_n \dots (C)_n}.$$

Following Bailey (see [2], Section 2.1), the classical hypergeometric series is defined by

$$(2) \qquad {}_{1+p}F_p \left[\begin{matrix} a_0, a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix} \middle| \, z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \dots (a_p)_k}{k!(b_1)_k \dots (b_p)_k} z^k.$$

The above series is terminating if one of the numerator parameters is a nonpositive integer. Otherwise, the series is nonterminating, which is convergent for all the complex z with $|z| < 1$.

Finally, we point out that not only (1) is not verifiable directly by the WZ-method, as Campbell in [3] claimed, but also the following slightly simpler binomial identity (36) is not verifiable either:

$$(3)\qquad \sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{2k} = 2(-1)^n\chi(n\equiv_3 0)\binom{\left\lfloor\frac{2n-1}{3}\right\rfloor+1}{\left\lfloor\frac{n}{3}\right\rfloor}\binom{2\left(\left\lfloor\frac{2n-1}{3}\right\rfloor\right)+1}{\left\lfloor\frac{2n-1}{3}\right\rfloor}.$$

Furthermore, the WZ-method cannot prove the identities for the reversal series of (3) and (1) as recorded below:

$$(4)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-k}{n} = 2\chi(n\equiv_3 0)\binom{\left\lfloor\frac{2n-1}{3}\right\rfloor+1}{\left\lfloor\frac{n}{3}\right\rfloor}\binom{2\left(\left\lfloor\frac{2n-1}{3}\right\rfloor\right)+1}{\left\lfloor\frac{2n-1}{3}\right\rfloor},$$

$$(5)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-k}{n-1} = \binom{\left\lfloor\frac{2n-1}{3}\right\rfloor+1}{\left\lfloor\frac{n}{3}\right\rfloor}\binom{2\left(\left\lfloor\frac{2n-1}{3}\right\rfloor\right)+1}{\left\lfloor\frac{2n-1}{3}\right\rfloor} \times \frac{(-1)^{n+\lfloor n/3\rfloor}}{n+1}\begin{cases} 2+4\left\lfloor\frac{n+1}{6}\right\rfloor & \text{if } n \text{ is odd,}\\ 4+4\left\lfloor\frac{n-1}{6}\right\rfloor & \text{if } n \text{ is even.}\end{cases}$$

In 1928, Bailey in [1] found the cubic transformation formula

$$(6)\quad {}_3F_2\left[\begin{matrix} u+v-\frac{1}{2}, u, v \\ 2u, 2v \end{matrix} \middle| 4x\right] = (1-x)^{1/2-u-v}{}_3F_2\left[\begin{matrix} \frac{u+v}{3}+\frac{1}{2}, \frac{u+v}{3}+\frac{1}{6}, \frac{u+v}{3}-\frac{1}{6} \\ \frac{1}{2}+u, \frac{1}{2}+v \end{matrix} \middle| \frac{27x^2}{4(1-x)^3}\right].$$

The terminating form of (6) under $u \to \frac{1}{2}$ and $v \to -m$ with $m \in \mathbb{N}_0$ is given in the following lemma.

Lemma 1. *Let $m \in \mathbb{N}_0$. Then*

$$(7)\quad {}_3F_2\left[\begin{matrix} \frac{1}{2}, -m, -m \\ 1, -2m \end{matrix} \middle| 4x\right] = (1-x)^m{}_3F_2\left[\begin{matrix} \frac{-m}{3}, \frac{1-m}{3}, \frac{2-m}{3} \\ 1, \frac{1}{2}-m \end{matrix} \middle| \frac{27x^2}{4(1-x)^3}\right].$$

When $x \to 1$, only the end term from ${}_3F_2$-series on the right-hand side survives. We can therefore deduce the closed formula

$$(8)\quad {}_3F_2\left[\begin{matrix} \frac{1}{2}, -m, -m \\ 1, -2m \end{matrix} \middle| 4\right] = \Delta_m \times \begin{cases} 1, & m \equiv_3 0, \\ 0, & m \not\equiv_3 0, \end{cases}$$

where Δ_m is the factorial quotient defined by

$$\Delta_m = \frac{(3\lfloor m/3 \rfloor)!}{(\frac{1}{2})_{3\lfloor m/3 \rfloor}} \times \left[\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1, 1 \end{matrix}\right]_{\lfloor m/3 \rfloor} = \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4} \\ 1, \frac{1}{2}, \frac{1}{6}, \frac{5}{6} \end{matrix}\right]_{\lfloor m/3 \rfloor}.$$

We remark that (8) can be rewritten as the following binomial identity:

$$(9)\quad \sum_{k=0}^{m}(-1)^k \binom{2k}{k}\binom{m}{k}^2 \Big/ \binom{2m}{k} = \Delta_m \times \begin{cases} 1, & m \equiv_3 0, \\ 0, & m \not\equiv_3 0. \end{cases}$$

Even though there exist numerous hypergeometric series identities in the mathematical literature (see [4], [5], [6], [7], [11], [12], [14] for example), the summation formula (8) has not appeared previously.

2.1. Differentiation. For an indeterminate λ, by considering the combination

$$\text{Equation (7)} + \frac{x}{\lambda}\mathcal{D}_x \text{ Equation (7)},$$

we can establish another transformation formula

$$(10)\quad {}_4F_3\left[\begin{matrix}\frac{1}{2}, 1+\lambda, -m, -m\\ \lambda, 1, -2m\end{matrix}\;\middle|\; 4x\right]$$
$$= \frac{\lambda-\lambda x - mx}{\lambda(1-x)^{1-m}}\,{}_4F_3\left[\begin{matrix}1+\frac{\lambda-\lambda x-mx}{2+x}, \frac{-m}{3}, \frac{1-m}{3}, \frac{2-m}{3}\\ \frac{\lambda-\lambda x-mx}{2+x}, 1, \frac{1}{2}-m\end{matrix}\;\middle|\; \frac{27x^2}{4(1-x)^3}\right].$$

Its limiting case as $x \to 1$ leads to the identity with a free parameter λ.

Theorem 2. *Let* $m \in \mathbb{N}_0$. *Then*

$${}_4F_3\left[\begin{matrix}\frac{1}{2}, 1+\lambda, -m, -m\\ \lambda, 1, -2m\end{matrix}\;\middle|\; 4\right] = \Delta_m \times \begin{cases}\frac{3\lambda+2m}{3\lambda}, & m \equiv_3 0;\\ \frac{-m(4m-1)}{3\lambda(2m-1)}, & m \equiv_3 1;\\ 0, & m \equiv_3 2.\end{cases}$$

For the sake of brevity, we introduce the notation

$$(11)\qquad \Phi_m\left[\begin{matrix}a, b\\ c, d\end{matrix}\right] = {}_3F_2\left[\begin{matrix}-m, -m-a, b+\frac{1}{2}\\ 1+c, d-2m\end{matrix}\;\middle|\; 4\right],$$

where $a, c \in \mathbb{N}_0$ and $b, d \in \mathbb{Z}$. Then (8) can be restated as

$$(12)\qquad \Phi_m\left[\begin{matrix}0, 0\\ 0, 0\end{matrix}\right] = \Delta_m \times \chi(m \equiv_3 0) = \frac{3\chi(m\equiv_3 0)}{4m+3}\nabla_m.$$

Henceforth, ∇_m denotes another factorial quotient

$$\nabla_m = \frac{m!}{(\frac{1}{2})_m} \times \left[\begin{matrix}\frac{3}{4}, \frac{5}{4}\\ 1, 1\end{matrix}\right]_{\lfloor m/3\rfloor},$$

which is related to Δ_m by

$$\Delta_m = \nabla_m \times \begin{cases} \dfrac{3}{4m+3}, & m \equiv_3 0, \\ \dfrac{3(2m-1)}{2m(4m-1)}, & m \equiv_3 1, \\ \dfrac{3(2m-1)(2m-3)}{4m(m-1)(4m-5)}, & m \equiv_3 2. \end{cases}$$

Both symbols Δ_m and ∇_m are alternatively employed in order to shorten most of evaluating expressions.

By assigning particular values for λ, we derive from Theorem 2 four identities below:

▷ Let $\lambda = \frac{1}{2}$. The first identity:

$$\Phi_m \begin{bmatrix} 0,1 \\ 0,0 \end{bmatrix} = \nabla_m \times \begin{cases} 1, & m \equiv_3 0, \\ -1, & m \equiv_3 1, \\ 0, & m \equiv_3 2. \end{cases} \tag{13}$$

▷ Let $\lambda = -1 - 2m$. The second identity:

$$\Phi_m \begin{bmatrix} 0,0 \\ 0,-1 \end{bmatrix} = \nabla_m \times \begin{cases} \dfrac{1}{2m+1}, & m \equiv_3 0, \\ \dfrac{1}{4m+2}, & m \equiv_3 1, \\ 0, & m \equiv_3 2. \end{cases} \tag{14}$$

▷ Let $\lambda = 0$. Multiply across the equation in Theorem 2 by λ and then let $\lambda \to 0$:

$$\Phi_m \begin{bmatrix} 0,1 \\ 1,-1 \end{bmatrix} = \nabla_m \times \begin{cases} \dfrac{1}{2m+1}, & m \equiv_3 0, \\ 0, & m \equiv_3 1, \\ \dfrac{-(4m+1)}{(2m+1)(4m+4)}, & m \equiv_3 2. \end{cases} \tag{15}$$

▷ Let $\lambda = -m$ and then reindex $m \to m+1$:

$$\Phi_m \begin{bmatrix} 1,0 \\ 0,-2 \end{bmatrix} = \nabla_m \times \begin{cases} \dfrac{1}{2m+1}, & m \equiv_3 0, \\ 0, & m \equiv_3 1, \\ \dfrac{4m+1}{8(m+1)(2m+1)}, & m \equiv_3 2. \end{cases} \tag{16}$$

2.2. Integration. For an indeterminate λ, multiplying across (7) by $x^{\lambda-1}$ and then integrating the resulting expression with respect to x over $[0,x]$, we have the following expression for the exotic ${}_4F_3$-series:

$$\text{(17)} \qquad \frac{x^\lambda}{\lambda}\,{}_4F_3\left[\begin{array}{c|c} \frac{1}{2},\lambda,-m,-m & \\ & 4x \\ 1+\lambda,1,-2m & \end{array}\right] = \sum_{j\geqslant 0}\frac{(-4)^{-j}\langle m\rangle_{3j}}{(j!)^2(\frac{1}{2}-m)_j}\int_0^x x^{\lambda-1+2j}(1-x)^{m-3j}\,\mathrm{d}x.$$

For $m,n\in\mathbb{N}_0$, by means of the integration by parts, it is not hard to verify that

$$\int_0^x x^n(1-x)^m\,\mathrm{d}x = \sum_{k=0}^{m}\frac{\langle m\rangle_k}{(n+1)_{k+1}}x^{n+k+1}(1-x)^{m-k}.$$

Then the right-hand side of (17) can be expressed as

$$\begin{aligned}\mathrm{RHS}(17) &= \sum_{j\geqslant 0}\frac{(-4)^{-j}\langle m\rangle_{3j}}{(j!)^2(\frac{1}{2}-m)_j}\sum_{k=0}^{m-3j}\frac{\langle m-3j\rangle_k}{(\lambda+2j)_{k+1}}x^{\lambda+2j+k}(1-x)^{m-3j-k}\\ &= \sum_{k=0}^{m}\frac{\langle m\rangle_k}{(\lambda)_{k+1}}\sum_{j\geqslant 0}\frac{(-4)^{-j}\langle m-k\rangle_{3j}(\lambda)_{2j}}{(j!)^2(\frac{1}{2}-m)_j(\lambda+k+1)_{2j}}\frac{x^{\lambda+k+2j}}{(1-x)^{3j+k-m}}.\end{aligned}$$

When $x\to 1$, the inner sum on the right reduces to the end term

$$\chi(m\equiv_3 k)\frac{(-4)^{-\lfloor (m-k)/3\rfloor}\langle m-k\rangle_{m-k}(\lambda)_{2\lfloor (m-k)/3\rfloor}}{(\lfloor (m-k)/3\rfloor !)^2(\frac{1}{2}-m)_{\lfloor (m-k)/3\rfloor}(\lambda+k+1)_{2\lfloor (m-k)/3\rfloor}}.$$

Replacing k by $m-3k$, we can further determine the limit

$$\begin{aligned}\lim_{x\to 1}\mathrm{RHS}(17) &= \sum_{k=0}^{\lfloor m/3\rfloor}\frac{\langle m\rangle_{m-3k}}{(\lambda)_{m-3k+1}}\times\frac{(-4)^{-k}\langle 3k\rangle_{3k}(\lambda)_{2k}}{(k!)^2(\frac{1}{2}-m)_k(\lambda+m-3k+1)_{2k}}\\ &= \frac{m!}{(\lambda)_{m+1}}\sum_{k=0}^{\lfloor m/3\rfloor}\frac{(\lambda)_{2k}(-\lambda-m)_k}{4^k(k!)^2(\frac{1}{2}-m)_k}.\end{aligned}$$

Therefore, the limit of (17) as $x\to 1$ results in the following identity.

Theorem 3. *Let $m\in\mathbb{N}_0$. Then*

$${}_4F_3\left[\begin{array}{c|c} \frac{1}{2},\lambda,-m,-m & \\ & 4 \\ 1+\lambda,1,-2m & \end{array}\right] = \frac{m!}{(1+\lambda)_m}\sum_{k=0}^{\lfloor m/3\rfloor}\frac{(\lambda)_{2k}(-\lambda-m)_k}{4^k(k!)^2(\frac{1}{2}-m)_k}.$$

By examining particular cases of the above identity and then making use of the Pfaff-Saalschütz theorem (see [2], Section 2.2) for the balanced ${}_3F_2$-series

$$(18)\qquad {}_3F_2\left[\begin{matrix}-n, a, b\\ c, 1+a+b-c-n\end{matrix}\,\middle|\,1\right]=\left[\begin{matrix}c-a, c-b\\ c, c-a-b\end{matrix}\right]_n,$$

we find the following four closed formulae for the terminating Φ_m-series.

▷ Let $\lambda=-\frac{1}{2}$. The first identity:

$$\Phi_m\left[\begin{matrix}0,-1\\ 0,0\end{matrix}\right]=\frac{m!}{(\frac{1}{2})_m}\sum_{k=0}^{\lfloor m/3\rfloor}\binom{\frac{1}{4}}{k}\binom{-\frac{1}{4}}{k}=\nabla_m=\frac{m!}{(\frac{1}{2})_m}\left[\begin{matrix}\frac{3}{4},\frac{5}{4}\\ 1,1\end{matrix}\right]_{\lfloor m/3\rfloor},$$

where the above sum with respect to k is evaluated by (18) as

$${}_3F_2\left[\begin{matrix}\frac{1}{4},-\frac{1}{4},-\left\lfloor\frac{m}{3}\right\rfloor\\ 1,-\left\lfloor\frac{m}{3}\right\rfloor\end{matrix}\,\middle|\,1\right]=\left[\begin{matrix}\frac{3}{4},\frac{5}{4}\\ 1,1\end{matrix}\right]_{\lfloor m/3\rfloor}.$$

▷ Let $\lambda=-1-m$. The second identity:

$$\Phi_m\left[\begin{matrix}1,0\\ 0,0\end{matrix}\right]=(-1)^m\sum_{k=0}^{\lfloor m/3\rfloor}\frac{(-m/2)_k((-1-m)/2)_k}{k!(\frac{1}{2}-m)_k}=(-1)^m\left[\begin{matrix}\frac{1-m}{2},\frac{2-m}{2}\\ 1,\frac{1}{2}-m\end{matrix}\right]_{\lfloor m/3\rfloor},$$

where the above sum with respect to k is evaluated by (18) as

$${}_3F_2\left[\begin{matrix}\frac{-m}{2},\frac{-1-m}{2},-\left\lfloor\frac{m}{3}\right\rfloor\\ \frac{1}{2}-m,-\left\lfloor\frac{m}{3}\right\rfloor\end{matrix}\,\middle|\,1\right]=\left[\begin{matrix}\frac{1-m}{2},\frac{2-m}{2}\\ 1,\frac{1}{2}-m\end{matrix}\right]_{\lfloor m/3\rfloor}.$$

There is also an equivalent expression:

$$(19)\qquad \Phi_m\left[\begin{matrix}1,0\\ 0,0\end{matrix}\right]=\nabla_m\times\begin{cases}\dfrac{1}{3+4m}, & m\equiv_3 0 \text{ and } m\neq 0,\\[2mm] \dfrac{-1}{2m}, & m\equiv_3 1,\\[2mm] \dfrac{4m+1}{4m(m+1)}, & m\equiv_3 2.\end{cases}$$

▷ Let $\lambda = 1$. The third identity:

$$\Phi_m\begin{bmatrix}0,0\\1,0\end{bmatrix} = \frac{1}{m+1}\sum_{k=0}^{\lfloor m/3\rfloor}\frac{(\frac{1}{2})_k(-1-m)_k}{k!(\frac{1}{2}-m)_k} = \frac{1}{m+1}\begin{bmatrix}\frac{3}{2}, 1+\left\lfloor\frac{2m+2}{3}\right\rfloor\\ 1, \frac{1}{2}+\left\lfloor\frac{2m+2}{3}\right\rfloor\end{bmatrix}_{\lfloor m/3\rfloor},$$

where the above sum with respect to k is evaluated by (18) as

$$\begin{aligned}{}_3F_2\left[\begin{matrix}\frac{1}{2}, -1-m, -\left\lfloor\frac{m}{3}\right\rfloor\\ \frac{1}{2}-m, -\left\lfloor\frac{m}{3}\right\rfloor\end{matrix}\;\middle|\; 1\right] &= \begin{bmatrix}\frac{3}{2}, -m\\ 1, \frac{1}{2}-m\end{bmatrix}_{\lfloor m/3\rfloor}\\ &= \begin{bmatrix}\frac{3}{2}, 1+\left\lfloor\frac{2m+2}{3}\right\rfloor\\ 1, \frac{1}{2}+\left\lfloor\frac{2m+2}{3}\right\rfloor\end{bmatrix}_{\lfloor m/3\rfloor}.\end{aligned}$$

There is also an equivalent expression:

$$\Phi_m\begin{bmatrix}0,0\\1,0\end{bmatrix} = \nabla_m \times \begin{cases}\dfrac{2m+3}{(m+1)(4m+3)}, & m \equiv_3 0,\\[2mm] \dfrac{1}{2(m+1)}, & m \equiv_3 1,\\[2mm] \dfrac{4m+1}{8(m+1)^2}, & m \equiv_3 2.\end{cases} \tag{20}$$

▷ Let $\lambda = -2m$. The fourth identity:

$$\Phi_m\begin{bmatrix}0,0\\0,1\end{bmatrix} = \frac{(-1)^m m!}{(m)_m}\sum_{k=0}^{\lfloor m/3\rfloor}\binom{m}{k}\binom{-m}{k} = \frac{(-1)^m m!}{(m)_m}\begin{bmatrix}1+m, 1-m\\ 1, 1\end{bmatrix}_{\lfloor m/3\rfloor},$$

where the above sum with respect to k is evaluated by (18) as

$${}_3F_2\left[\begin{matrix}m, -m, -\left\lfloor\frac{m}{3}\right\rfloor\\ 1, -\left\lfloor\frac{m}{3}\right\rfloor\end{matrix}\;\middle|\; 1\right] = \begin{bmatrix}1+m, 1-m\\ 1, 1\end{bmatrix}_{\lfloor m/3\rfloor}.$$

There is also an equivalent expression as below:

$$\Phi_m\begin{bmatrix}0,0\\0,1\end{bmatrix} = \nabla_m \times \begin{cases}\dfrac{4}{4m+3}, & m \equiv_3 0 \text{ and } m \neq 0,\\[2mm] \dfrac{-1}{2m}, & m \equiv_3 1,\\[2mm] \dfrac{1}{4m}, & m \equiv_3 2.\end{cases} \tag{21}$$

3. Contiguous relations

There exist numerous contiguous relations for the series $\Phi_m\left[\begin{smallmatrix}a,b\\c,d\end{smallmatrix}\right]$. Some of them will be recorded and then employed to evaluate further series in closed form.

3.1. The first contiguous relation.

Proposition 4. *Let $m \in \mathbb{N}_0$. Then*

$$\Phi_m\begin{bmatrix}a,b\\c,d\end{bmatrix} = \frac{2(a+c+m+1)(b+\frac{1}{2})}{(a+m+1)(2b-2c+1)}\Phi_m\begin{bmatrix}a+1,b+1\\c,d\end{bmatrix} - \frac{(2a+2b+2m+3)(c)}{(a+m+1)(2b-2c+1)}\Phi_m\begin{bmatrix}a+1,b\\c-1,d\end{bmatrix}.$$

In particular, when $a=b=d=0$ and $c=1$, we have the equation

$$\Phi_m\begin{bmatrix}0,0\\1,0\end{bmatrix} = \frac{2m+3}{m+1}\Phi_m\begin{bmatrix}1,0\\0,0\end{bmatrix} - \frac{m+2}{m+1}\Phi_m\begin{bmatrix}1,1\\1,0\end{bmatrix}.$$

Form this we deduce the summation formula

$$\Phi_m\begin{bmatrix}1,1\\1,0\end{bmatrix} = \frac{2m+3}{m+2}\Phi_m\begin{bmatrix}1,0\\0,0\end{bmatrix} - \frac{m+1}{m+2}\Phi_m\begin{bmatrix}0,0\\1,0\end{bmatrix} = \nabla_m \times \begin{cases} 0, & m \equiv_3 0 \text{ and } m \neq 0, \\ \dfrac{-3(m+1)}{2m(m+2)}, & m \equiv_3 1, \\ \dfrac{3(4m+1)}{8m(m+1)}, & m \equiv_3 2. \end{cases} \tag{22}$$

3.2. The second contiguous relation.

Proposition 5. *Let $m \in \mathbb{N}_0$. Then*

$$\Phi_m\begin{bmatrix}a,b\\c,d\end{bmatrix} = \frac{(c-d+2m+2)(2b+1)}{(c+1)(2b-2d+4m+3)}\Phi_m\begin{bmatrix}a,b+1\\c+1,d\end{bmatrix} + \frac{(2b-2c-1)(d-1-2m)}{(c+1)(2b-2d+4m+3)}\Phi_m\begin{bmatrix}a,b\\c+1,d-1\end{bmatrix}.$$

In particular, when $b = c = d = 0$ and $a = 1$, we have the equation

$$\Phi_m \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} = \frac{2m+2}{4m+3}\Phi_m \begin{bmatrix} 1,1 \\ 1,0 \end{bmatrix} + \frac{2m+1}{4m+3}\Phi_m \begin{bmatrix} 1,0 \\ 1,-1 \end{bmatrix}.$$

Form this we deduce the summation formula

$$\Phi_m \begin{bmatrix} 1,0 \\ 1,-1 \end{bmatrix} = \frac{4m+3}{2m+1}\Phi_m \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} - \frac{2m+2}{2m+1}\Phi_m \begin{bmatrix} 1,1 \\ 1,0 \end{bmatrix} = \nabla_m \times \begin{cases} \frac{1}{2m+1}, & m \equiv_3 0, \\ \frac{1}{2m+4}, & m \equiv_3 1, \\ \frac{4m+1}{4(m+1)(2m+1)}, & m \equiv_3 2. \end{cases} \tag{23}$$

3.3. The third contiguous relation.

Proposition 6. *Let $m \in \mathbb{N}_0$. Then*

$$\Phi_m \begin{bmatrix} a,b \\ c,d \end{bmatrix} = \frac{(1+2b-2d+4m)(a+m)}{(1-2b)(1-a-d+m)}\Phi_m \begin{bmatrix} a-1,b-1 \\ c,d \end{bmatrix} + \frac{(2a+2b-1+2m)(d-1-2m)}{(1-2b)(1-a-d+m)}\Phi_m \begin{bmatrix} a,b-1 \\ c,d-1 \end{bmatrix}.$$

In particular, when $a = b = d = 1$ and $c = 0$, we derive the summation formula

$$\Phi_m \begin{bmatrix} 1,1 \\ 0,1 \end{bmatrix} = \frac{2m(2m+3)}{m-1}\Phi_m \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} - \frac{(m+1)(4m+1)}{m-1}\Phi_m \begin{bmatrix} 0,0 \\ 0,1 \end{bmatrix} = \nabla_m \times \begin{cases} \frac{-2(2m+1)(3m+2)}{(m-1)(4m+3)}, & m \equiv_3 0 \text{ and } m \neq 0, \\ \frac{-1}{2m}, & m \equiv_3 1 \text{ and } m \neq 1, \\ \frac{(4m+1)(3m^2+4m-1)}{4m(m+1)(m-1)}, & m \equiv_3 2. \end{cases} \tag{24}$$

3.4. The fourth contiguous relation.

Proposition 7. *Let $m \in \mathbb{N}_0$. Then*

$$\Phi_m \begin{bmatrix} a,b \\ c,d \end{bmatrix} = \frac{c(a+d-m)}{(a+c+m)(d-2m)}\Phi_m \begin{bmatrix} a,b \\ c-1,d+1 \end{bmatrix} - \frac{(c-d+2m)(a+m)}{(a+c+m)(d-2m)}\Phi_m \begin{bmatrix} a-1,b \\ c,d+1 \end{bmatrix}.$$

In particular, when $a=b=c=1$ and $d=0$, we have the equation

$$\Phi_m\begin{bmatrix}1,1\\1,0\end{bmatrix}=\frac{m-1}{2m(m+2)}\Phi_m\begin{bmatrix}1,1\\0,1\end{bmatrix}+\frac{(1+2m)(1+m)}{2m(m+2)}\Phi_m\begin{bmatrix}0,1\\1,1\end{bmatrix}.$$

From this, we derive the summation formula

$$\begin{aligned}(25)\qquad \Phi_m\begin{bmatrix}0,1\\1,1\end{bmatrix}&=\frac{2m(m+2)}{(1+2m)(1+m)}\Phi_m\begin{bmatrix}1,1\\1,0\end{bmatrix}-\frac{m-1}{(1+2m)(1+m)}\Phi_m\begin{bmatrix}1,1\\0,1\end{bmatrix}\\&=\nabla_m\times\begin{cases}\dfrac{2(2+3m)}{(m+1)(4m+3)}, & m\equiv_3 0 \text{ and } m\neq 0,\\[2mm] \dfrac{-(3m+1)}{2m(m+1)}, & m\equiv_3 1,\\[2mm] \dfrac{4m+1}{4m(m+1)^2}, & m\equiv_3 2.\end{cases}\end{aligned}$$

3.5. The fifth contiguous relation.

Proposition 8. *Let $m\in\mathbb{N}_0$. Then*

$$\Phi_m\begin{bmatrix}a,b\\c,d\end{bmatrix}=\Phi_m\begin{bmatrix}a,b\\c,d-1\end{bmatrix}-\frac{2m(a+m)(2b+1)}{(1+c)(2m-d)(1+2m-d)}\Phi_{m-1}\begin{bmatrix}a,b+1\\c+1,d-1\end{bmatrix}.$$

In particular, when $a=d=1$ and $b=c=0$, we deduce the summation formula

$$\begin{aligned}(26)\qquad \Phi_m\begin{bmatrix}1,0\\0,1\end{bmatrix}&=\Phi_m\begin{bmatrix}1,0\\0,0\end{bmatrix}-\frac{m+1}{2m-1}\Phi_{m-1}\begin{bmatrix}1,1\\1,0\end{bmatrix}\\&=\nabla_m\times\begin{cases}\dfrac{-(2m+4)}{(m-1)(4m+3)}, & m\equiv_3 0 \text{ and } m\neq 0,\\[2mm] \dfrac{-1}{2m}, & m\equiv_3 1 \text{ and } m\neq 1,\\[2mm] \dfrac{7m^2-1}{4m(m+1)(m-1)}, & m\equiv_3 2.\end{cases}\end{aligned}$$

3.6. The sixth contiguous relation.

Proposition 9. *Let $m\in\mathbb{N}_0$. Then*

$$\Phi_m\begin{bmatrix}a,b\\c,d\end{bmatrix}=\frac{1-d+2m}{1-d+m}\Phi_m\begin{bmatrix}a,b\\c,d-1\end{bmatrix}-\frac{m}{1-d+m}\Phi_{m-1}\begin{bmatrix}a+1,b\\c,d-2\end{bmatrix}.$$

In particular, when $a = b = c = 0$ and $d = 2$, we deduce the summation formula

$$\Phi_m \begin{bmatrix} 0,0 \\ 0,2 \end{bmatrix} = \frac{2m-1}{m-1}\Phi_m \begin{bmatrix} 0,0 \\ 0,1 \end{bmatrix} - \frac{m}{m-1}\Phi_{m-1} \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix}$$
$$= \nabla_m \times \begin{cases} \dfrac{2(m-2)(2m-1)}{(m-1)^2(4m+3)}, & m \equiv_3 0 \text{ and } m \neq 0, \\ \dfrac{-(m+2)(2m-1)(5m-1)}{2m(m-1)(m+2)(4m-1)}, & m \equiv_3 1 \text{ and } m \neq 1, \\ \dfrac{(2m-1)^2}{4m(m-1)^2}, & m \equiv_3 2. \end{cases} \tag{27}$$

3.7. The seventh contiguous relation.

Proposition 10. *Let $m \in \mathbb{N}_0$. Then*

$$\Phi_m \begin{bmatrix} a,b \\ c,d \end{bmatrix} = \frac{a+m}{a+c+m}\Phi_m \begin{bmatrix} a-1,b \\ c,d \end{bmatrix} + \frac{c}{a+c+m}\Phi_m \begin{bmatrix} a,b \\ c-1,d \end{bmatrix}.$$

In particular, when $a = c = 1$ and $b = d = 0$, we deduce the summation formula

$$\Phi_m \begin{bmatrix} 1,0 \\ 1,0 \end{bmatrix} = \frac{1}{m+2}\Phi_m \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} + \frac{m+1}{m+2}\Phi_m \begin{bmatrix} 0,0 \\ 1,0 \end{bmatrix}$$
$$= \nabla_m \times \begin{cases} \dfrac{2}{4m+3}, & m \equiv_3 0 \text{ and } m \neq 0, \\ \dfrac{m-1}{2m(m+2)}, & m \equiv_3 1, \\ \dfrac{4m+1}{8m(m+1)}, & m \equiv_3 2. \end{cases} \tag{28}$$

3.8. The eighth contiguous relation.

Proposition 11. *Let $m \in \mathbb{N}_0$. Then*

$$\Phi_m \begin{bmatrix} a,b \\ c,d \end{bmatrix} = \frac{1-d+2m}{1-a-d+m}\Phi_m \begin{bmatrix} a,b \\ c,d-1 \end{bmatrix} - \frac{a+m}{1-a-d+m}\Phi_m \begin{bmatrix} a-1,b \\ c,d \end{bmatrix}.$$

In particular, when $a = 1$ and $b = c = d = 0$, we have the equation

$$\Phi_m \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} = \frac{1+2m}{m}\Phi_m \begin{bmatrix} 1,0 \\ 0,-1 \end{bmatrix} - \frac{1+m}{m}\Phi_m \begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}.$$

From this we deduce the summation formula

$$\begin{aligned}\Phi_m\begin{bmatrix}1,0\\0,-1\end{bmatrix} &= \frac{1+m}{1+2m}\Phi_m\begin{bmatrix}0,0\\0,0\end{bmatrix}+\frac{m}{1+2m}\Phi_m\begin{bmatrix}1,0\\0,0\end{bmatrix}\\ &= \nabla_m\times\begin{cases}\dfrac{1}{1+2m}, & m\equiv_3 0,\\[2mm] \dfrac{-1}{2(1+2m)}, & m\equiv_3 1,\\[2mm] \dfrac{4m+1}{4(m+1)(1+2m)}, & m\equiv_3 2.\end{cases}\end{aligned}\tag{29}$$

3.9. The ninth contiguous relation.

Proposition 12. *Let* $m\in\mathbb{N}_0$. *Then*

$$\Phi_m\begin{bmatrix}a,b\\c,d\end{bmatrix}=\frac{1+2b}{1+2b-2c}\Phi_m\begin{bmatrix}a,b+1\\c,d\end{bmatrix}-\frac{2c}{1+2b-2c}\Phi_m\begin{bmatrix}a,b\\c-1,d\end{bmatrix}.$$

Three summation formulae can further be established as follows:

▷ $a=b=d=0$ and $c=1$:

$$\Phi_m\begin{bmatrix}0,1\\1,0\end{bmatrix}=2\Phi_m\begin{bmatrix}0,0\\0,0\end{bmatrix}-\Phi_m\begin{bmatrix}0,0\\1,0\end{bmatrix}=\nabla_m\times\begin{cases}\dfrac{1}{m+1}, & m\equiv_3 0,\\[2mm] \dfrac{-1}{2(m+1)}, & m\equiv_3 1,\\[2mm] \dfrac{-(4m+1)}{8(m+1)^2}, & m\equiv_3 2.\end{cases}\tag{30}$$

▷ $a=d=0$ and $b=-c=-1$:

$$\begin{aligned}\Phi_m\begin{bmatrix}0,-1\\1,0\end{bmatrix} &= \frac{1}{3}\Phi_m\begin{bmatrix}0,0\\1,0\end{bmatrix}+\frac{2}{3}\Phi_m\begin{bmatrix}0,-1\\0,0\end{bmatrix}\\ &= \nabla_m\times\begin{cases}\dfrac{1+8(m+1)^2}{3(m+1)(4m+3)}, & m\equiv_3 0,\\[2mm] \dfrac{4m+5}{6(m+1)}, & m\equiv_3 1,\\[2mm] \dfrac{16m^2+36m+17}{24(m+1)^2}, & m\equiv_3 2.\end{cases}\end{aligned}\tag{31}$$

▷ $a=b=c=d=0$ in Proposition 5:

$$\Phi_m\begin{bmatrix}0,0\\0,0\end{bmatrix}=\frac{2m+2}{4m+3}\Phi_m\begin{bmatrix}0,1\\1,0\end{bmatrix}+\frac{2m+1}{4m+3}\Phi_m\begin{bmatrix}0,0\\1,-1\end{bmatrix},$$

we can further evaluate the series

$$\Phi_m\begin{bmatrix}0,0\\1,-1\end{bmatrix}=\frac{4m+3}{2m+1}\Phi_m\begin{bmatrix}0,0\\0,0\end{bmatrix}-\frac{2m+2}{2m+1}\Phi_m\begin{bmatrix}0,1\\1,0\end{bmatrix}$$
$$=\nabla_m\times\begin{cases}\dfrac{1}{2m+1}, & m\equiv_3 0,\\ \dfrac{1}{2m+1}, & m\equiv_3 1,\\ \dfrac{4m+1}{4(m+1)(2m+1)}, & m\equiv_3 2.\end{cases}\tag{32}$$

4. Reversal series of argument $\frac{1}{4}$

For $a,c\in\mathbb{N}_0$ and $b,d\in\mathbb{Z}$, by inverting the summation order, we can reformulate

$$\Phi_m\begin{bmatrix}a,b\\c,d\end{bmatrix}={}_3F_2\left[\begin{matrix}-m,-m-a,b+\frac{1}{2}\\1+c,d-2m\end{matrix}\,\middle|\,4\right]$$
$$=(-1)^m\begin{bmatrix}1+a,\frac{1}{2}+b,1-d\\1+c,\frac{1-d}{2},1-\frac{d}{2}\end{bmatrix}_m{}_3F_2\left[\begin{matrix}-m,-m-c,1-d+m\\1+a,\frac{1}{2}-b-m\end{matrix}\,\middle|\,\frac{1}{4}\right].$$

Letting $\Psi_m\left[\begin{smallmatrix}a,b\\c,d\end{smallmatrix}\right]$ stand for the series

$$\Psi_m\begin{bmatrix}a,b\\c,d\end{bmatrix}={}_3F_2\left[\begin{matrix}-m,-m-a,1-b+m\\1+c,\frac{1}{2}-d-m\end{matrix}\,\middle|\,\frac{1}{4}\right],\tag{33}$$

we have established the reciprocal relation

$$\Psi_m\begin{bmatrix}a,b\\c,d\end{bmatrix}=(-1)^m\Phi_m\begin{bmatrix}c,d\\a,b\end{bmatrix}\times\begin{bmatrix}1+a,\frac{1-b}{2},1-\frac{b}{2}\\1+c,1-b,\frac{1}{2}+d\end{bmatrix}_m.\tag{34}$$

In particular $a=b=c=d=0$, we get a remarkable reciprocity

$$\Psi_m\begin{bmatrix}0,0\\0,0\end{bmatrix}=(-1)^m\Phi_m\begin{bmatrix}0,0\\0,0\end{bmatrix}=(-1)^m\Delta_m\chi(m\equiv_3 0).\tag{35}$$

We record six further summation formulae for those $\Psi_m\left[\begin{smallmatrix}a,b\\c,d\end{smallmatrix}\right]$ with all the parameters a, b, c, d being zero except for one parameter equal to ±1.

Corollary 13. *Let $m \in \mathbb{N}_0$. Then*

(a)
$$\Psi_m \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} = (-1)^m \nabla_m \times \begin{cases} \dfrac{2m+3}{4m+3}, & m \equiv_3 0, \\ \dfrac{1}{2}, & m \equiv_3 1, \\ \dfrac{4m+1}{8(m+1)}, & m \equiv_3 2. \end{cases}$$

(b)
$$\Psi_m \begin{bmatrix} 0,0 \\ 1,0 \end{bmatrix} = (-1)^m \nabla_m \times \begin{cases} \dfrac{1}{(m+1)(3+4m)}, & m \equiv_3 0 \text{ and } m \neq 0, \\ \dfrac{-1}{2m(m+1)}, & m \equiv_3 1, \\ \dfrac{4m+1}{4m(m+1)^2}, & m \equiv_3 2. \end{cases}$$

(c)
$$\Psi_m \begin{bmatrix} 0,1 \\ 0,0 \end{bmatrix} = (-1)^m \nabla_m \times \begin{cases} \dfrac{2}{4m+3}, & m \equiv_3 0 \text{ and } m \neq 0, \\ \dfrac{-1}{4m}, & m \equiv_3 1, \\ \dfrac{1}{8m}, & m \equiv_3 2. \end{cases}$$

(d)
$$\Psi_m \begin{bmatrix} 0,-1 \\ 0,0 \end{bmatrix} = (-1)^m \nabla_m \times \begin{cases} \dfrac{1}{m+1}, & m \equiv_3 0, \\ \dfrac{1}{2m+2}, & m \equiv_3 1, \\ 0, & m \equiv_3 2. \end{cases}$$

(e)
$$\Psi_m \begin{bmatrix} 0,0 \\ 0,1 \end{bmatrix} = (-1)^m \nabla_m \times \begin{cases} \dfrac{1}{2m+1}, & m \equiv_3 0, \\ \dfrac{-1}{2m+1}, & m \equiv_3 1, \\ 0, & m \equiv_3 2. \end{cases}$$

(f)
$$\Psi_m \begin{bmatrix} 0,0 \\ 0,-1 \end{bmatrix} = (-1)^m (1-2m) \nabla_m.$$

5. Identities of binomial sums

In recent papers [8], [9], [13], [15], several alternating binomial sums containing Catalan numbers were evaluated in closed forms. In particular, the conjectured identity (1) was proposed as an open problem in [10]. This section will illustrate that the hypergeometric series treated in the last section can be utilized to confirm (1) and further to evaluate explicitly more binomial sums of similar nature.

The examples exhibited in this section show that for this class of binomial sums, the closed expressions obtained via the hypergeometric series approach are more elegant than those produced in [3] by a computer algebra system.

5.1. The first one is the following formula anticipated in "Introduction", which can be shown without difficulty by invoking $\Psi_n \begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}$:

$$\begin{aligned} \mathrm{W}_1(n) &= \sum_{k=0}^{n} (-1)^k \binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{2k} \\ &= 2(-1)^n \chi(n \equiv_3 0) \binom{\lfloor \frac{2n-1}{3} \rfloor + 1}{\lfloor \frac{n}{3} \rfloor} \binom{2\left(\lfloor \frac{2n-1}{3} \rfloor\right) + 1}{\lfloor \frac{2n-1}{3} \rfloor}. \end{aligned} \tag{36}$$

5.2. Another example comes from a problem posed recently in [10], where the authors demanded to show the following binomial identity (here we have corrected a typo by exchanging the odd case and the even case), which is confirmed by combining $\Psi_n \begin{bmatrix} 0,0 \\ 1,0 \end{bmatrix}$ with $\Phi_n \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix}$:

$$\begin{aligned} \mathrm{W}_2(n) &= \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{2k} \\ &= \binom{\lfloor \frac{2n-1}{3} \rfloor + 1}{\lfloor \frac{n}{3} \rfloor} \binom{2\left(\lfloor \frac{2n-1}{3} \rfloor\right) + 1}{\lfloor \frac{2n-1}{3} \rfloor} \\ &\quad \times \frac{(-1)^{\lfloor n/3 \rfloor}}{n(n+1)} \begin{cases} 2 + 4\left\lfloor \frac{n+1}{6} \right\rfloor & \text{if } n \text{ is odd;} \\ 4 + 4\left\lfloor \frac{n-1}{6} \right\rfloor & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{37}$$

5.3. The next sum was examined recently by Campbell [3], Theorem 2, which follows easily from the difference $\mathrm{W}_3(n) = \mathrm{W}_1(n) - \mathrm{W}_2(n)$:

$$\begin{aligned} \mathrm{W}_3(n) &= \sum_{k=0}^{n} \frac{k(-1)^k}{k+1} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{2k} \\ &= \binom{\lfloor \frac{2n-1}{3} \rfloor + 1}{\lfloor \frac{n}{3} \rfloor} \binom{2\left(\lfloor \frac{2n-1}{3} \rfloor\right) + 1}{\lfloor \frac{2n-1}{3} \rfloor} \\ &\quad \times \frac{2(-1)^n}{3n(n+1)} \begin{cases} n(2+3n), & n \equiv_3 0; \\ 2+n, & n \equiv_3 1; \\ -4-n, & n \equiv_3 2. \end{cases} \end{aligned} \tag{38}$$

5.4. By utilizing $\Psi_n\left[\begin{smallmatrix}0,0\\0,0\end{smallmatrix}\right]$ and $\Psi_n\left[\begin{smallmatrix}0,0\\1,0\end{smallmatrix}\right]$, we can deduce the binomial identity below:

$$\begin{aligned}\text{(39)}\qquad \mathrm{W}_4(n) &= \sum_{k=0}^{n}(-1)^k\binom{2k+1}{k}\binom{2n-2k}{n-k}\binom{n+k}{2k}\\ &= (-1)^n\binom{2n}{n}\nabla_n\times\begin{cases}\dfrac{6n+5}{(n+1)(4n+3)}, & n\equiv_3 0 \text{ and } n\neq 0;\\ \dfrac{1}{2n(n+1)}, & n\equiv_3 1;\\ \dfrac{-(4n+1)}{4n(n+1)^2}, & n\equiv_3 2.\end{cases}\end{aligned}$$

5.5. By applying $\Psi_n\left[\begin{smallmatrix}1,0\\0,1\end{smallmatrix}\right]$ and $\Phi_n\left[\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right]$, we can evaluate the next binomial sum:

$$\begin{aligned}\text{(40)}\qquad \mathrm{W}_5(n) &= \sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2n-2k+1}{n-k}\binom{n+k}{2k}\\ &= (-1)^n\binom{2n}{n}\nabla_n\times\begin{cases}\dfrac{1}{n+1}, & n\equiv_3 0;\\ \dfrac{-1}{2(n+1)}, & n\equiv_3 1;\\ \dfrac{-(4n+1)}{8(n+1)^2}, & n\equiv_3 2.\end{cases}\end{aligned}$$

5.6. By making use of $\Psi_n\left[\begin{smallmatrix}1,-1\\0,0\end{smallmatrix}\right]$ and $\Phi_n\left[\begin{smallmatrix}0,0\\1,-1\end{smallmatrix}\right]$, we can show the identity below:

$$\begin{aligned}\text{(41)}\qquad \mathrm{W}_6(n) &= \sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k+1}{2k}\\ &= (-1)^n\binom{2n}{n}\nabla_n\times\begin{cases}1, & n\equiv_3 0;\\ 1, & n\equiv_3 1;\\ \dfrac{4n+1}{4n+4}, & n\equiv_3 2.\end{cases}\end{aligned}$$

5.7. The following binomial identity is derived by appealing to $\Psi_n\left[\begin{smallmatrix}1,0\\1,0\end{smallmatrix}\right]$ and $\Phi_n\left[\begin{smallmatrix}1,0\\1,0\end{smallmatrix}\right]$:

$$\text{(42)}\qquad \mathrm{W}_7(n) = \sum_{k=0}^{n}\frac{(-1)^k}{(k+1)(n-k+1)}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{2k}$$

$$= (-1)^n \binom{2n}{n} \nabla_n \times \begin{cases} \dfrac{2}{(n+1)(4n+3)}, n \equiv_3 0 \text{ and } n \neq 0; \\ \dfrac{n-1}{2n(n+1)(n+2)}, n \equiv_3 1; \\ \dfrac{4n+1}{8n(n+1)^2}, n \equiv_3 2. \end{cases}$$

5.8. By combining $\Phi_n \left[\begin{smallmatrix} 0,1 \\ 1,0 \end{smallmatrix}\right]$ with $\Phi_n \left[\begin{smallmatrix} 1,1 \\ 1,0 \end{smallmatrix}\right]$, we find the following binomial identity:

$$\text{W}_8(n) = \sum_{k=0}^{n} (-1)^k \binom{2k+1}{k} \binom{2n-2k+1}{n-k} \binom{n+k}{2k} \tag{43}$$

$$= (-1)^n \binom{2n}{n} \nabla_n \times \begin{cases} \dfrac{2}{n+1}, & n \equiv_3 0 \text{ and } n \neq 0; \\ \dfrac{(1-n)(2n+3)}{2n(n+1)(n+2)}, & n \equiv_3 1; \\ \dfrac{-(2n+3)(4n+1)}{8n(n+1)^2}, & n \equiv_3 2. \end{cases}$$

5.9. Finally, the following binomial identity is established by employing the two contiguous values $\Psi_n \left[\begin{smallmatrix} 0,-1 \\ 1,0 \end{smallmatrix}\right]$ and $\Phi_n \left[\begin{smallmatrix} 1,0 \\ 0,-1 \end{smallmatrix}\right]$:

$$\text{W}_9(n) = \sum_{k=0}^{n} (-1)^k \binom{2k+1}{k} \binom{2n-2k}{n-k} \binom{n+k+1}{2k+1} \tag{44}$$

$$= (-1)^n \binom{2n}{n} \nabla_n \times \begin{cases} \dfrac{1}{n+1}, & n \equiv_3 0; \\ \dfrac{-1}{2(n+1)}, & n \equiv_3 1; \\ \dfrac{4n+1}{4(n+1)^2}, & n \equiv_3 2. \end{cases}$$

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