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TENSOR PRODUCTS OF HIGHER ALMOST SPLIT SEQUENCES IN SUBCATEGORIES

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Abstract. We introduce the algebras satisfying the $(\mathcal{B}, n)$ condition. If Λ, Γ are algebras satisfying the $(\mathcal{B}, n)$, $(\mathcal{E}, m)$ condition, respectively, we give a construction of $(m+n)$-almost split sequences in some subcategories $(\mathcal{B} \otimes \mathcal{E})^{(i_0, j_0)}$ of $\operatorname{mod}(\Lambda \otimes \Gamma)$ by tensor products and mapping cones. Moreover, we prove that the tensor product algebra $\Lambda \otimes \Gamma$ satisfies the $((\mathcal{B} \otimes \mathcal{E})^{(i_0, j_0)}, n + m)$ condition for some integers i_0, j_0; this construction unifies and extends the work of A. Pasquali (2017), (2019).

Keywords: n-representation finite algebra; higher almost split sequence; tensor product; mapping cone

MSC 2020: 16G70, 16D90, 16G10

1. Introduction

Almost split sequences (also called Auslander-Reiten sequences) are the main ingredient in Auslander-Reiten theory which play a central role in the representation theory of artin algebras due to Auslander, Reiten and Smalø, see [1], [2], [3], [4]. Recently, as a generalization of the classical Auslander-Reiten theory, Iyama and his co-authors introduced higher Auslander-Reiten theory (see [6], [8], [9], [16]) from the viewpoint of higher homological algebra, which is extensively used in modern representation theory, see [12], [13], [15]. In higher Auslander-Reiten theory, almost split sequences were generalized to n-almost split sequences (see Definition 2.2),

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and Auslander-Reiten translations were generalized to n-Auslander-Reiten translations τ_n, τ_n^-.

As a generalization of representation finite hereditary algebras, n-representation finite algebras and n-complete algebras are introduced and studied in [10], [14], [15], which admit n-almost split sequences in n-cluster tilting subcategories of module categories. Pasquali showed that every n-almost split sequence of an n-representation finite algebra or n-complete algebra is isomorphic to the mapping cone of a suitable chain map of complexes.

Tensor products over field create new algebras in higher representation theory, particularly in the setting of perfect field, see [10], [11], [19]. Let Λ, Γ be, respectively, n-, m-representation finite algebras over perfect field k. Under the condition of l-homogeneous, Herschend and Iyama showed in [10] that their tensor product $\Lambda \otimes_k \Gamma$ is an $(n+m)$-representation finite algebra. In this case, there must exist $(n+m)$-almost split sequences over $\Lambda \otimes_k \Gamma$; Pasquali showed how to describe the structure of such sequences, see [18], Theorem 1.1. Later, using the same method, he also constructed the $(n+m)$-almost split sequences under the tensor product of higher complete algebras, see [19].

However, the tensor product $\Lambda \otimes_k \Gamma$ is not in general $(n+m)$-representation finite. In fact, the condition (l-homogeneity) is a necessary and sufficient condition, see [18], page 648, Remark 2. It is not easy to find the existence of $(n+m)$-almost split sequences when $\Lambda \otimes_k \Gamma$ is not an $(n+m)$-representation finite algebra.

The motivation of this article is to continue the study of the existence of $(n+m)$-almost split sequences but in the setting of the tensor product of more general algebras. We prove that the tensor product of higher almost split sequences is also a higher almost split sequence; this generalizes the result of [18], [19] to a unified framework of the algebras satisfying certain condition.

In this paper, we investigate the algebras satisfying the $(\mathcal{B}, n)$ condition (see Definition 3.1), including n-representation finite algebras and n-complete algebras, and give the construction of n-almost split sequences in subcategory $\mathcal{B}$. Let Λ, Γ be algebras satisfying the $(\mathcal{B}, n)$, $(\mathcal{E}, m)$ conditions, respectively. We will construct new $(m+n)$-almost split sequences in full subcategories $(\mathcal{B} \otimes \mathcal{E})^{(i_0, j_0)}$ of $\operatorname{mod}(\Lambda \otimes_k \Gamma)$ for some $i_0, j_0 \geqslant 0$.

Theorem 1.1. *Let k be a perfect field and Λ, Γ algebras satisfying the $(\mathcal{B}, n)$, $(\mathcal{E}, m)$ condition, respectively. Let $\varphi \in \operatorname{Mor}_r(\mathcal{C}_r^n(\mathcal{B}))$, $\psi \in \operatorname{Mor}_r(\mathcal{C}_r^m(\mathcal{E}))$ such that $\operatorname{Cone}(\varphi)$ is an n-almost split sequence in $\mathcal{B}$ starting in slice i_0, ending in slice i_0+1 for some $i_0 \geqslant 0$, and $\operatorname{Cone}(\psi)$ is an m-almost split sequence in $\mathcal{E}$ starting in slice j_0, ending in slice j_0+1 for some $j_0 \geqslant 0$. Then $\operatorname{Cone}(\varphi \otimes^T \psi)$ is an $(n+m)$-almost split sequence in $(\mathcal{B} \otimes \mathcal{E})^{(i_0, j_0)}$.*

When higher almost split sequences start in common slice, then Theorem 1.1 reduces to [18], Theorem 1.1 for homogeneous higher representation finite algebras, [19], Theorem 4.11 for higher complete algebras. Furthermore, as a corollary, we obtain that the tensor product $\Lambda\otimes_k\Gamma$ satisfies the $((\mathcal{B}\otimes\mathcal{E})^{(i_0,j_0)}, n+m)$ condition.

The paper is organized as follows. In Section 2, we will review some basic definitions and facts needed. In Section 3, we introduce the algebras satisfying the $(\mathcal{B}, n)$ condition and show that every n-almost split sequence in $\mathcal{B}$ is isomorphic to mapping cone of morphism of complexes. In Section 4, we give the proof of the main Theorem 1.1. In Section 5, by Theorem 1.1, we give an example to construct a 2-almost split sequence via almost split sequences starting in a different slice for nonhomogeneous 1-representation finite algebras.

Conventions. Throughout this paper, we denote by k a field, by Λ a finite dimensional algebra over k, and by $\operatorname{mod}\Lambda$ the category of the finitely generated left Λ-modules. All subcategories considered are supposed to be full subcategories, closed under isomorphisms and direct summands. For a class $\mathcal{X}$ of objects in $\operatorname{mod}\Lambda$, we denote by $\operatorname{add}\mathcal{X}$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of direct summands of finite direct sums of objects in $\mathcal{X}$, by $\operatorname{ind}\mathcal{X}$ the class of indecomposable objects of $\mathcal{X}$.

The *Jacobson radical* [1], Appendix A.3 of category $\operatorname{mod}\Lambda$ is the two-sided ideal $\operatorname{rad}_\Lambda$ in $\operatorname{mod}\Lambda$ defined by the formula

$$\operatorname{rad}_\Lambda(X,Y)=\{h\in\operatorname{Hom}_\Lambda(X,Y)\colon\ \mathrm{id}_X-g\circ h \text{ is invertible for every } g\in\operatorname{Hom}_\Lambda(Y,X)\}$$

for all objects $X,Y\in\operatorname{mod}\Lambda$. If objects X,Y in $\operatorname{mod}\Lambda$ with $X\ncong Y$ are indecomposable, then $\operatorname{rad}_\Lambda(X,Y)=\operatorname{Hom}_\Lambda(X,Y)$. For $X,Y\in\operatorname{mod}\Lambda$, we denote by $S_\Lambda(X,Y)$ the quotient vector space $S_\Lambda=\operatorname{Hom}_\Lambda(X,Y)/\operatorname{rad}_\Lambda(X,Y)$ and we will simplify the notation by writing $\operatorname{rad}(X,Y)$ instead of $\operatorname{rad}_\Lambda(X,Y)$.

2. Preliminaries

Let $\mathcal{B}$ be a full subcategory of $\operatorname{mod}\Lambda$, we denote by $\mathcal{C}(\mathcal{B})$ the category of complexes of $\mathcal{B}$, by $\mathcal{K}(\mathcal{B})$ the homotopy category of $\mathcal{C}(\mathcal{B})$, and by $\mathcal{C}_r(\mathcal{B})$ the full subcategory of $\mathcal{C}(\mathcal{B})$ whose objects are chain complexes where differentials are radical morphisms. Let n be a nonnegative integer, we denote by $\mathcal{C}^n(\mathcal{B})$ the full subcategory of $\mathcal{C}(\mathcal{B})$ whose objects are defined by $\{(A_\bullet, d_\bullet^A)\in\mathcal{C}(\mathcal{B})\colon\ A_i=0 \text{ for all } i<0 \text{ or } i>n\}$. Let $f\colon A_\bullet\to B_\bullet$ be a morphism of chain complexes. The *mapping cone* $\operatorname{Cone}(f)$ of f is defined as the complex $\operatorname{Cone}(f)=(\operatorname{Cone}(f)_i, d_i^{\operatorname{Cone}(f)})$ with

$$\operatorname{Cone}(f)_i=A[-1]_i\oplus B_i \text{ and the differentials } d_i^{\operatorname{Cone}(f)}=\begin{bmatrix} d_i^{A[-1]} & 0\\ f[-1]_i & d_i^B\end{bmatrix}$$

for every $i \in \mathbb{Z}$, where $[-]$ is the shift functor of $\mathcal{C}(\mathcal{B})$, that is, for every $m \in \mathbb{Z}$ and $A_\bullet = (A_i, d_i^A) \in \mathcal{C}(\mathcal{B})$, $A_\bullet[m] = (A[m]_i, d_i^{A[m]})$ with $A[m]_i = A_{i+m}$ and the differentials $d_i^{A[m]} = (-1)^m d_{i+m}^A$ for every $i \in \mathbb{Z}$, the shift $f[m]$ of f is the morphism of complexes $f[m] = (f[m]_i)_{i\in\mathbb{Z}}\colon A_\bullet[m] \to B_\bullet[m]$ with components $f[m]_i = f_{i+m}$.

Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}(\mathcal{B})$. We denote by $\mathrm{Mor}(\mathcal{A})$ the *morphism category* whose objects are all morphisms of complexes in $\mathcal{A}$. Let $(A_\bullet^0 \xrightarrow{f} B_\bullet^0)$, $(A_\bullet^1 \xrightarrow{g} B_\bullet^1) \in \mathrm{Mor}(\mathcal{A})$. The morphism $f \to g$ is a pair (φ, ψ) of morphisms in $\mathcal{A}$ with $\varphi\colon A_\bullet^0 \to A_\bullet^1$ and $\psi\colon B_\bullet^0 \to B_\bullet^1$ such that $\psi f = g\varphi$. The composition of morphisms $(\varphi, \psi) \in \mathrm{Hom}_{\mathrm{Mor}(\mathcal{A})}(f, g)$ and $(\varphi', \psi') \in \mathrm{Hom}_{\mathrm{Mor}(\mathcal{A})}(g, z)$ is the morphism $(\varphi'\varphi, \psi'\psi) \in \mathrm{Hom}_{\mathrm{Mor}(\mathcal{A})}(f, z)$. Let $\mathrm{Mor}_r(\mathcal{A})$ denote the full subcategory of $\mathrm{Mor}(\mathcal{A})$ whose objects are given by $\{f\colon A_\bullet \to B_\bullet \in \mathrm{Mor}(\mathcal{A})\colon f_i \in \mathrm{rad}(A_i, B_i)$ for every $i \in \mathbb{Z}\}$; the objects are called the *radical morphisms*.

The following lemma is used repeatedly in the sequel.

Lemma 2.1 ([7]). *For any $f \in \mathrm{Mor}(\mathcal{C}(\mathcal{B}))$, $\mathrm{Cone}(f)$ is an acyclic complex of $\mathcal{C}(\mathrm{mod}\,\Lambda)$ if and only if f is a quasi-isomorphism.*

We give the following general definition of n-almost split sequence defined in [13], [15].

Definition 2.2. Let n be a positive integer and $\mathcal{B}$ a full subcategory of $\mathrm{mod}\,\Lambda$.

(1) The complex $\ldots \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0$ with terms in $\mathcal{B}$ is called a *sink sequence* in $\mathcal{B}$ of C_0 if $f_i \in \mathrm{rad}(C_i, C_{i-1})$ for every integer $i \geqslant 0$, and the following sequence is exact on $\mathcal{B}$:

$$\cdots \xrightarrow{f_3\circ -} \mathrm{Hom}_\Lambda(-, C_2) \xrightarrow{f_2\circ -} \mathrm{Hom}_\Lambda(-, C_1) \xrightarrow{f_1\circ -} \mathrm{rad}_\Lambda(-, C_0) \longrightarrow 0.$$

(2) The complex $C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} C_{n-2} \xrightarrow{f_{n-2}} \ldots$ with terms in $\mathcal{B}$ is called a *source sequence* in $\mathcal{B}$ of C_n if $f_i \in \mathrm{rad}(C_i, C_{i-1})$ for every integer $i \leqslant n$, and the following sequence is exact on $\mathcal{B}$:

$$\cdots \xrightarrow{-\circ f_{n-2}} \mathrm{Hom}_\Lambda(C_{n-2}, -) \xrightarrow{-\circ f_{n-1}} \mathrm{Hom}_\Lambda(C_{n-1}, -) \xrightarrow{-\circ f_n} \mathrm{rad}_\Lambda(C_n, -) \longrightarrow 0.$$

(3) The exact sequence $C_\bullet\colon 0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$ with terms in $\mathcal{B}$ is called *n-almost split sequence* in $\mathcal{B}$ if $C_\bullet$ is a sink sequence in $\mathcal{B}$ of C_0 and a source sequence in $\mathcal{B}$ of C_{n+1} simultaneously.

It is well known that the rightmost (or leftmost) objects of sink (or source) sequences are nonprojective (or noninjective) in $\mathcal{B}$ and indecomposable, see [13]. We say that $\mathcal{B}$ has n-almost split sequences for subcategories $\mathcal{B}_I$, $\mathcal{B}_P$ of $\mathcal{B}$ if for each indecomposable object $C \in \mathcal{B}_P$, there is an n-almost split sequence $0 \to C_{n+1} \to C_n \to \ldots \to C_1 \to C \to 0$; and for each indecomposable object $X \in \mathcal{B}_I$, there is an n-almost split sequence $0 \to X \to X_n \to \ldots \to X_1 \to X_0 \to 0$.

Remark 2.3. If $\mathcal{B}$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$, the definition of n-almost split sequences in $\mathcal{B}$ is simpler, see [14], Proposition 2.10. This simple definition was used in [18].

Fixed Λ-module $X \in \mathcal{B}$, Pasquali defined two functors F_X, G_X^n from $\mathcal{C}_r(\mathcal{B})$ to $\mathcal{C}(\operatorname{mod} k)$ as subfunctors of $\operatorname{Hom}_\Lambda(X,-)$ and $\operatorname{Hom}_\Lambda(-,X)$, respectively, [18], that is, for any

$$C_\bullet\colon \ \ldots \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} \ldots \xrightarrow{f_1} C_0 \xrightarrow{f_0} \ldots$$

in $\mathcal{C}_r(\mathcal{B})$,

$$F_X(C_\bullet)\colon \ \ldots \xrightarrow{f_{i+1}\circ -} \operatorname{Hom}_\Lambda(X, C_i) \xrightarrow{f_i\circ -} \cdots \xrightarrow{f_1\circ -} \operatorname{rad}_\Lambda(X, C_0) \xrightarrow{f_0\circ -} \cdots$$

is the complex given by replacing $\operatorname{Hom}_\Lambda(X, C_0)$ in complex $\operatorname{Hom}_\Lambda(X, C_\bullet)$ with $\operatorname{rad}(X, C_0)$, and the differentials of $F_X(C_\bullet)$ equal to the differentials of $\operatorname{Hom}_\Lambda(X, C_\bullet)$. Similarly,

$$G_X^n(C_\bullet)\colon \ \ldots \xrightarrow{-\circ f_i} \operatorname{Hom}_\Lambda(C_i, X) \xrightarrow{-\circ f_{i+1}} \cdots \xrightarrow{-\circ f_{n+1}} \operatorname{rad}_\Lambda(C_{n+1}, X) \xrightarrow{-\circ f_{n+2}} \cdots$$

The functors F_X, G_X^n can be used to characterize n-almost split sequences.

Lemma 2.4 ([19], Lemma 4.5). *Let $C_\bullet\colon \ \ldots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \ldots \xrightarrow{f_1} C_0 \xrightarrow{f_0} \ldots$ be a complex in $\mathcal{C}_r(\mathcal{B})$. Then:*

(1) *If $C_i = 0$ for $i < 0$, then $C_\bullet$ is a sink sequence of C_0 in $\mathcal{B}$ if and only if $F_X(C_\bullet)$ is acyclic for every $X \in \mathcal{B}$.*
(2) *If $C_i = 0$ for $i > n+1$, then $C_\bullet$ is a source sequence of C_{n+1} in $\mathcal{B}$ if and only if $G_X^n(C_\bullet)$ is acyclic for every $X \in \mathcal{B}$.*
(3) *If $C_\bullet \in \mathcal{C}^{n+1}(\mathcal{B})$ is acyclic in $\operatorname{mod}\Lambda$, then $C_\bullet$ is an n-almost split sequence in $\mathcal{B}$ if and only if $F_X(C_\bullet)$ and $G_X^n(C_\bullet)$ are acyclic for every $X \in \mathcal{B}$.*

Remark 2.5. If moreover $\mathcal{B}$ is a Krull-Schmidt subcategory, by the additivity of F_X and G_X^n, we can replace "every $X \in \mathcal{B}$" by "every $X \in \operatorname{ind}\mathcal{B}$" in the above lemma.

3. Construction of n-almost split sequence

In this section, we show that every n-almost split sequence is isomorphic to a mapping cone in subcategory $\mathcal{B}$ for algebras satisfying the $(\mathcal{B}, n)$ condition, and give a characterization of n-almost split sequences. This unifies and extends the results of Pasquali for n-representation finite algebras (see [18]) and n-complete algebras, see [19].

3.1. Algebras satisfying the $(\mathcal{B}, n)$ **conditions.** Let us start by giving the definition of algebras satisfying the $(\mathcal{B}, n)$ condition.

Definition 3.1. Let $\mathcal{B}$ be a subcategory of $\operatorname{mod}\Lambda$. We say that Λ satisfies the $(\mathcal{B}, n)$ condition if there are full subcategories $\mathcal{B}_i$ $(i \geqslant 0)$ of $\mathcal{B}$ satisfying the following conditions:

(1) $\mathcal{B} = \bigoplus_{i\geqslant 0} \mathcal{B}_i$, that is, for every $X \in \mathcal{B}$, there exist finite many nonzero objects $X^i \in \mathcal{B}_i$ such that $X = \bigoplus_{i\geqslant 0} X^i$.

(2) If $i > j$, then $\operatorname{Hom}_\Lambda(X, Y) = 0$ for all objects $X \in \mathcal{B}_i$, $Y \in \mathcal{B}_j$.

(3) $\mathcal{B}$ has n-almost split sequences for some subcategories $\mathcal{B}_I$, $\mathcal{B}_P$ of $\mathcal{B}$.

We now present some classes of algebras satisfying the $(\mathcal{B}, n)$ condition. The following notations τ_n, τ_n^- are n-Auslander-Reiten translations, see [13], [15].

Example 3.2. Let Λ be an n-representation finite algebra (see [16]) with n-cluster tilting module M, we have $\mathcal{M} := \operatorname{add} M = \bigoplus_{i\geqslant 0} \mathcal{M}_i$ with $\mathcal{M}_i = \operatorname{add} \tau_n^{-i}\Lambda$ being an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$. Then Λ satisfies the $(\mathcal{M}, n)$ condition and $\mathcal{M}$ has n-almost split sequences for subcategories $\mathcal{M}_I = \operatorname{add} M \setminus \operatorname{add}(D\Lambda)$, $\mathcal{M}_P = \operatorname{add} M \setminus (\operatorname{add}\Lambda)$ of $\mathcal{M}$, see Theorem 3.3.1 of [13] or Proposition 2.1 of [18].

Example 3.3. Let Λ be an n-complete algebra (see [15]), there exists a positive minimum integer l such that $\tau_n^l(D\Lambda) = 0$. We define the subcategory $\mathcal{M} = \bigoplus_{i=0}^{l-1} \mathcal{M}_i$ of $\operatorname{mod}\Lambda$ with $\mathcal{M}_i = \operatorname{add} \tau_n^{l-1-i} D\Lambda$, then Λ satisfies the $(\mathcal{M}, n)$ condition and $\mathcal{M}$ has n-almost split sequences for the following subcategories of $\mathcal{M}$, see Proposition 4.2 and Theorem 4.4 of [19]:

$$\mathcal{M}_I = \{X \in \mathcal{M}\colon\ X \text{ has no nonzero summands in } \operatorname{add} D\Lambda\},$$
$$\mathcal{M}_P = \{X \in \mathcal{M}\colon\ X \text{ has no nonzero summand } N \text{ such that } \tau_n N = 0\}.$$

Example 3.4. Let Λ be an n-representation infinite algebra, see [11], Definition 2.7. We can define the subcategory $\mathscr{P} = \mathscr{P}_\Lambda = \operatorname{add}\{\nu_n^{-i}(\Lambda)\colon\ i \geqslant 0\} = \operatorname{add}\{\tau_n^{-i}(\Lambda)\colon\ i \geqslant 0\}$ of $\operatorname{mod}\Lambda$, see [11], Definition 4.7. We get functors $\nu_n^{-i} = \tau_n^{-i}$

for $i \geqslant 0$ on $\mathscr{P}$ by [11], Proposition 4.21. Thus, $\mathscr{P} = \bigoplus_{i\geqslant 0} \mathscr{P}_i$ equipped with $\mathscr{P}_i = \operatorname{add} \tau_n^{-i}\Lambda$. Then Λ satisfies the $(\mathscr{P}, n)$ condition and $\mathscr{P}$ has n-almost split sequences for subcategories $\mathscr{P}_I = \mathscr{P}$ and

$$\mathscr{P}_P = \{X \in \mathscr{P}\colon\ X \text{ has no nonzero summands in } \operatorname{add}\Lambda\}$$

of $\mathscr{P}$, see Proposition 2.3 (b) and Theorem 4.25 of [11]. Note that $\mathscr{P}$ is not an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ in general.

Remark 3.5. By [15], Proposition 1.13 (b), an n-representation finite algebra is a special n-complete algebra. For n-representation infinite algebra Λ, the functor ν_n is an auto-equivalence of the bounded derived category of $\operatorname{mod}\Lambda$ (see [11]), so $\nu_n^i(D\Lambda) \neq 0$ for $i \geqslant 0$. By [11], Proposition 4.21, we have $\tau_n^i(D\Lambda) = \nu_n^i(D\Lambda) \neq 0$ for $i \geqslant 0$. Hence, n-complete algebras and n-representation infinite algebras are two disjoint classes of algebras.

Assume that Λ satisfies the $(\mathcal{B}, n)$ condition and that $0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$ is an n-almost split sequence in $\mathcal{B}$. Because $\mathcal{B} = \bigoplus_{i\geqslant 0} \mathcal{B}_i$ and $\dim C_m < \infty$, there exist nonnegative integer l such that

$$C_m = \bigoplus_{i=0}^{l} C_m^i \quad \text{with } C_m^i \in \mathcal{B}_i$$

for $0 \leqslant m \leqslant n+1$.

Definition 3.6. Let Λ be an algebra satisfying the $(\mathcal{B}, n)$ condition with $\mathcal{B} = \bigoplus_{i\geqslant 0} \mathcal{B}_i$, $C_\bullet\colon\ 0 \to C_{n+1} \to C_n \to \ldots \to C_1 \to C_0 \to 0$ be an n-almost split sequence in $\mathcal{B}$ and i_0, j_0 two nonnegative integers. We say that $C_\bullet$ starts in slice i_0 if $C_{n+1} \in \mathcal{B}_{i_0}$ and ends in slice j_0 if $C_0 \in \mathcal{B}_{j_0}$.

Example 3.7. Let Λ be an n-representation finite algebra and $C_\bullet\colon\ 0 \to C_{n+1} \to C_n \to \ldots \to C_1 \to C_0 \to 0$ be an n-almost split sequence in $\mathcal{M}$. By [13], Theorem 3.3.1, we obtain $C_0 = \tau_n^- C_{n+1}$ and $C_{n+1} = \tau_n C_0$. Therefore, if $C_\bullet$ starts in slice i_0, then $C_\bullet$ ends in slice $i_0 + 1$. Similar results hold for n-complete algebras (see Example 3.3) and n-representation infinite algebras (see Example 3.4).

Proposition 3.8. *Let Λ be an algebra satisfying the $(\mathcal{B}, n)$ condition with $\mathcal{B} = \bigoplus_{i\geqslant 0} \mathcal{B}_i$, $C_\bullet\colon\ 0 \to C_{n+1} \to C_n \to \ldots \to C_0 \to 0$ be an n-almost split sequence in $\mathcal{B}$ with $C_m = \bigoplus_{i=0}^{l} C_m^i$ for some $C_m^i \in \mathcal{B}_i$, $0 \leqslant m \leqslant n+1$, $i \geqslant 0$. Then we have*

(1) *if $C_\bullet$ ends in slice j_0, then $C_m^i = 0$ for $i > j_0$;*
(2) *if $C_\bullet$ starts in slice i_0, then $C_m^i = 0$ for $i < i_0$;*
(3) *if $C_\bullet$ starts in slice i_0 and ends in slice j_0, then $i_0 \leqslant j_0$.*

P r o o f. (1) Suppose that $C_q^j \neq 0$ for some $j > j_0$ and $n+1 \geqslant q > 0$. Let

$$b = \max\{i\colon\ i > j_0,\ C_m^i \neq 0 \text{ and } n+1 \geqslant m > 0\},$$
$$a = \min\{m\colon\ C_m^b \neq 0,\ n+1 \geqslant m > 0\}.$$

Observe that by the definition of $(\mathcal{B}, n)$ condition, $\operatorname{Hom}_\Lambda(C_m^i, C_{m'}^{i'}) = 0$, $i > i'$ for $n+1 \geqslant m,\ m' \geqslant 0$. Consider $C_m = \bigoplus_{i=0}^{l} C_m^i$ for $0 \leqslant m \leqslant a-1$, using the minimality of a and maximality of b, we get $C_m^i = 0$ for $b \leqslant i \leqslant l$, hence

$$\operatorname{Hom}_\Lambda(C_a^b, C_m) = \operatorname{Hom}_\Lambda\left(C_a^b, \bigoplus_{i=0}^{l} C_m^i\right) \cong \bigoplus_{i=0}^{b-1} \operatorname{Hom}_\Lambda(C_a^b, C_m^i) = 0.$$

For $C_a = \bigoplus_{i=0}^{l} C_a^i$, by the maximality of b, we have $C_a^i = 0$ for $b+1 \leqslant i \leqslant l$, so

$$\operatorname{Hom}_\Lambda(C_a^b, C_a) = \operatorname{Hom}_\Lambda\left(C_a^b, C_a^b \oplus \left(\bigoplus_{i=0}^{b-1} C_a^i\right)\right) = \operatorname{Hom}_\Lambda(C_a^b, C_a^b).$$

Because $C_\bullet$ is an n-almost split sequence in $\mathcal{B}$ and $C_a^b \in \mathcal{B}$,

$$F_{C_a^b}(C_\bullet)\colon\ 0 \longrightarrow \operatorname{Hom}_\Lambda(C_a^b, C_{n+1}) \xrightarrow{f_{n+1}\circ -} \cdots$$
$$\longrightarrow \operatorname{Hom}_\Lambda(C_a^b, C_{a+1}) \xrightarrow{f_{a+1}\circ -} \operatorname{Hom}_\Lambda(C_a^b, C_a^b) \longrightarrow 0$$

is an exact sequence of k-vector spaces. The relation $\mathrm{id}_{C_a^b} \in \operatorname{rad}_\Lambda(C_a^b, C_a^b)$ follows from the fact that $f_{a+1} \in \operatorname{rad}_\Lambda(C_{a+1}, C_a)$, this contradicts the fact that $C_a^b = 0$. Thus, $C_m^i = 0$ for $i > j_0$.

(2) The proof is similar to (1).

(3) Suppose that $i_0 > j_0$. Since $C_{n+1} \in \mathcal{B}_{i_0}$ and $C_0 \in \mathcal{B}_{j_0}$, for $0 < m < n+1$, we have $C_m^i = 0$ for $i_0 \leqslant i \leqslant l$ by (1), and $C_m^i = 0$ for $0 \leqslant i \leqslant i_0 - 1$ by (2). Hence,

$$C_m = \bigoplus_{i=0}^{l} C_m^i = \left(\bigoplus_{i=0}^{i_0-1} C_m^i\right) \oplus \left(\bigoplus_{i=i_0}^{l} C_m^i\right) = 0,$$

this contradicts the fact that $C_m \neq 0$. □

As a consequence of the above proposition, we have the following corollary.

Corollary 3.9. *Let* Λ *be an algebra satisfying the* $(\mathcal{B}, n)$ *condition with* $\mathcal{B} = \bigoplus_{i\geqslant 0} \mathcal{B}_i$, $C_\bullet\colon 0 \to C_{n+1} \to C_n \to \ldots \to C_0 \to 0$ *be an* n*-almost split sequence in* $\mathcal{B}$. *Then:*

(1) *If* $C_\bullet$ *starts in slice* i_0 *and ends in slice* j_0 *with* $j_0 \geqslant i_0 \geqslant 0$, *then there exist* $C_m^i \in \mathcal{B}_i$ *such that* $C_m = \bigoplus_{i=i_0}^{j_0} C_m^i$ *for every* $0 \leqslant m \leqslant n+1$.

(2) *If* $C_\bullet$ *starts in slice* i_0 *and ends in slice* i_0 *with* $i_0 \geqslant 0$, *then* $C_\bullet \in \mathcal{C}_r(\mathcal{B}_{i_0})$.

Proof. (1) Under the assumptions, we have $C_m^i = 0$ for $j_0 + 1 \leqslant i \leqslant l$ by Proposition 3.8 (1), and $C_m^i = 0$ for $0 \leqslant i \leqslant i_0 - 1$ by Proposition 3.8 (2), the statement follows.

(2) easily follows from (1). □

3.2. Mapping cone of n**-almost split sequences.** In this subsection, we investigate the relationship between mapping cones and n-almost split sequences.

Theorem 3.10. *Let* Λ *be an algebra satisfying the* $(\mathcal{B}, n)$ *condition with* $\mathcal{B} = \bigoplus_{i\geqslant 0} \mathcal{B}_i$, $C_\bullet\colon 0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \ldots \xrightarrow{f_1} C_0 \longrightarrow 0$ *be an* n*-almost split sequence in* $\mathcal{B}$ *starting in slice* i_0 *and ending in slice* j_0 *with* $0 \leqslant i_0 < j_0$. *Then for each* $i_0 \leqslant b < j_0$, *there exist complexes* $A_\bullet^0 \in \mathcal{C}_r^n\Big(\bigoplus_{i=i_0}^{b} \mathcal{B}_i\Big)$, $A_\bullet^1 \in \mathcal{C}_r^n\Big(\bigoplus_{i=b+1}^{j_0} \mathcal{B}_i\Big)$ *and radical morphism* $\varphi\colon A_\bullet^0 \to A_\bullet^1$ *such that* $C_\bullet = \mathrm{Cone}(\varphi)$ *in* $\mathcal{C}(\mathcal{B})$.

Proof. For each $i_0 \leqslant b < j_0$, by Corollary 3.9 (1), $C_m = \bigoplus_{i=i_0}^{j_0} C_m^i$ for every $0 \leqslant m \leqslant n+1$, then C_m can be written as

$$C_m = A_{m-1}^0 \oplus A_m^1,$$

where $A_{m-1}^0 = \bigoplus_{i=i_0}^{b} C_m^i \in \bigoplus_{i=i_0}^{b} \mathcal{B}_i$, $A_m^1 = \bigoplus_{i=b+1}^{j_0} C_m^i \in \bigoplus_{i=b+1}^{j_0} \mathcal{B}_i$, and f_m can be written as

$$f_m = \begin{bmatrix} \alpha_m^b & \beta_m \\ \gamma_m & \alpha_m^{b+1} \end{bmatrix}\colon C_m \to C_{m-1},$$

where $\alpha_m^b\colon A_{m-1}^0 \to A_{m-2}^0$, $\beta_m\colon A_m^1 \to A_{m-2}^0$, $\gamma_m\colon A_{m-1}^0 \to A_{m-1}^1$, $\alpha_m^{b+1}\colon A_m^1 \to A_{m-1}^1$. Note that $\beta_m = 0$ since $i_0 < j_0$.

We may write $A^0_\bullet = (A^0_m,\ d^{A^0}_m)$, $A^1_\bullet = (A^1_m,\ d^{A^1}_m)$ and $\varphi = (\varphi_m)$, where $d^{A^0}_m = -\alpha^b_{m+1}$, $d^{A^1}_m = \alpha^{b+1}_m$, and $\varphi_m = \gamma_{m+1}\colon A^0_m \to A^1_m$ for every m. We note that $f_{m-1}f_m = 0$ and $f_m \in \operatorname{rad}(C_m, C_{m-1})$ for every m, it is easily seen that $\varphi\colon A^0_\bullet \to A^1_\bullet$ is a radical morphism and $A^0_\bullet \in \mathcal{C}^n_r\Big(\bigoplus\limits_{i=i_0}^{b} \mathcal{B}_i\Big)$, $A^1_\bullet \in \mathcal{C}^n_r\Big(\bigoplus\limits_{i=b+1}^{j_0} \mathcal{B}_i\Big)$. We have

$$\operatorname{Cone}(\varphi)_m = A^0_\bullet[-1]_m \oplus A^1_m = A^0_{m-1} \oplus A^1_m = C_m$$

and

$$d^{\operatorname{Cone}(\varphi)}_m = \begin{bmatrix} -d^{A^0}_{m-1} & 0 \\ \varphi_{m-1} & d^{A^1}_m \end{bmatrix} = \begin{bmatrix} \alpha^b_m & 0 \\ \gamma_m & \alpha^{b+1}_m \end{bmatrix} = f_m$$

for every m, therefore $C_\bullet = \operatorname{Cone}(\varphi)$ in $\mathcal{C}(\mathcal{B})$. □

Theorem 3.10 shows that under the $(\mathcal{B}, n)$ condition, every n-almost split sequence is isomorphic to a mapping cone of certain complex morphism. This result generalizes Theorem 2.4 of [18] and Theorem 4.7 of [19].

The following corollary follows directly from Theorem 3.10, Examples 3.4 and 3.7.

Corollary 3.11. *Let Λ be an n-representation infinite algebra with $\mathscr{P}_\Lambda = \bigoplus\limits_{i\geqslant 0} \operatorname{add}\tau_n^{-i}\Lambda$, $C_\bullet$ be an n-almost split sequence in $\mathscr{P}_\Lambda$ which starts in slice i_0 for some $i_0 \geqslant 0$. Then there exist complexes $A^0_\bullet \in \mathcal{C}^n_r(\operatorname{add}\tau_n^{-i_0}\Lambda)$, $A^1_\bullet \in \mathcal{C}^n_r(\operatorname{add}\tau_n^{-(i_0+1)}\Lambda)$ and radical morphism $\varphi\colon A^0_\bullet \to A^1_\bullet$ such that $C_\bullet = \operatorname{Cone}(\varphi)$ in $\mathcal{C}(\mathscr{P}_\Lambda)$.*

Remark 3.12. In Theorem 3.10, assume that $\mathcal{B} = \bigoplus\limits_{i\geqslant 0} \mathcal{B}_i$ and n-almost split sequence $C_\bullet$ in $\mathcal{B}$ starts in slice i_0 and ends in slice j_0. In order to investigate the tensor products of n-almost split sequences in Section 4, we can decompose $\mathcal{B}$ as $\mathcal{B} = \mathcal{E} = \bigoplus\limits_{j\geqslant 0} \mathcal{E}_j$ equipped with

$$\mathcal{E}_j = \begin{cases} \mathcal{B}_j, & 0 \leqslant j < i_0, \\ \bigoplus\limits_{i=i_0}^{b} \mathcal{B}_i, & j = i_0, \\ \bigoplus\limits_{i=b+1}^{j_0} \mathcal{B}_i, & j = i_0 + 1, \\ \mathcal{B}_{j+j_0-i_0-1}, & j > i_0 + 1, \end{cases}$$

then $C_\bullet$ in $\mathcal{C}(\mathcal{E})$ starts in slice i_0 and ends in slice $i_0 + 1$, and Λ satisfies the $(\mathcal{E}, n)$ condition.

The following proposition shows that under the $(\mathcal{B}, n)$ condition and isomorphism, there is a uniqueness of φ of mapping cone $\operatorname{Cone}(\varphi)$.

Proposition 3.13. *Let Λ be an algebra satisfying the $(\mathcal{B},n)$ condition with $\mathcal{B}=\bigoplus_{i\geqslant 0}\mathcal{B}_i$. For any complexes $A^0_\bullet, B^0_\bullet \in \mathcal{C}\Big(\bigoplus_{i=i_0}^{b}\mathcal{B}_i\Big)$, $A^1_\bullet, B^1_\bullet \in \mathcal{C}\Big(\bigoplus_{i=b+1}^{j_0}\mathcal{B}_i\Big)$ and morphisms $\varphi\colon A^0_\bullet\to A^1_\bullet$, $\psi\colon B^0_\bullet\to B^1_\bullet$, the following statements are equivalent:*

(1) $\mathrm{Cone}(\varphi)\cong\mathrm{Cone}(\psi)$ *in* $\mathcal{C}(\mathcal{B})$.

(2) *There are isomorphisms $f\colon A^0_\bullet\to B^0_\bullet$, $g\colon A^1_\bullet\to B^1_\bullet$ in $\mathcal{C}(\mathcal{B})$ such that the following square is commutative in the homotopy category $\mathcal{K}(\mathcal{B})$:*

$$\begin{array}{ccc} A^0_\bullet & \xrightarrow{f} & B^0_\bullet \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ A^1_\bullet & \xrightarrow{g} & B^1_\bullet \end{array} \tag{3.1}$$

Proof. Let $\varepsilon=(\varepsilon_m)\colon \mathrm{Cone}(\varphi)\to\mathrm{Cone}(\psi)$ be a morphism of complexes with

$$\varepsilon_m=\begin{bmatrix}\alpha_m & r_m\\ q_m & \beta_m\end{bmatrix}\colon A^0_{m-1}\oplus A^1_m\to B^0_{m-1}\oplus B^1_m,$$

then $r_m=0$ for $m\in\mathbb{Z}$, because $A^1_m\in\bigoplus_{i=b+1}^{j_0}\mathcal{B}_i$, $B^0_{m-1}\in\bigoplus_{i=i_0}^{b}\mathcal{B}_i$ and $(\mathcal{B},n)$ condition (2). Because (ε_m) is a chain map, we have $\varepsilon_{m-1}d_m^{\mathrm{Cone}(\varphi)}=d_m^{\mathrm{Cone}(\psi)}\varepsilon_m$ for $m\in\mathbb{Z}$, that is

$$\begin{bmatrix}\alpha_{m-1} & 0\\ q_{m-1} & \beta_{m-1}\end{bmatrix}\begin{bmatrix}-d^{A^0}_{m-1} & 0\\ \varphi_{m-1} & d^{A^1}_m\end{bmatrix}=\begin{bmatrix}-d^{B^0}_{m-1} & 0\\ \psi_{m-1} & d^{B^1}_m\end{bmatrix}\begin{bmatrix}\alpha_m & 0\\ q_m & \beta_m\end{bmatrix} \tag{3.2}$$

$$\Leftrightarrow\begin{cases}\alpha_{m-1}d^{A^0}_{m-1}=d^{B^0}_{m-1}\alpha_m & \forall\, m\in\mathbb{Z}\\ \beta_{m-1}d^{A^1}_m=d^{B^1}_m\beta_m & \forall\, m\in\mathbb{Z}\\ -q_{m-1}d^{A^0}_{m-1}+\beta_{m-1}\varphi_{m-1}=\psi_{m-1}\alpha_m+d^{B^1}_m q_m & \forall\, m\in\mathbb{Z}\end{cases}$$

$$\Leftrightarrow\begin{cases}\alpha=(\alpha_m)\colon A^0[-1]_\bullet\to B^0[-1]_\bullet \text{ is a chain map.}\\ \beta=(\beta_m)\colon A^1_\bullet\to B^1_\bullet \text{ is a chain map.}\\ \beta_{m-1}\varphi_{m-1}=\psi_{m-1}\alpha_m+d^{B^1}_m q_m+q_{m-1}d^{A^0}_{m-1} \quad \forall\, m\in\mathbb{Z}.\end{cases}$$

(1) $\Rightarrow$ (2) Assume that $(\varepsilon_m)\colon\mathrm{Cone}(\varphi)\to\mathrm{Cone}(\psi)$ is an isomorphism in $\mathcal{C}(\mathcal{B})$, then there is an isomorphism of complexes $(\eta_m)\colon\mathrm{Cone}(\psi)\to\mathrm{Cone}(\varphi)$ with $\eta_m=\begin{bmatrix}x_m & 0\\ p_m & y_m\end{bmatrix}\colon B^0_{m-1}\oplus B^1_m\to A^0_{m-1}\oplus A^1_m$, such that $(\varepsilon_m)(\eta_m)=\mathrm{id}_{\mathrm{Cone}(\psi)}$, $(\eta_m)(\varepsilon_m)=\mathrm{id}_{\mathrm{Cone}(\varphi)}$, we have $\alpha_m x_m=\mathrm{id}_{B^0_{m-1}}$ and $x_m\alpha_m=\mathrm{id}_{A^0_{m-1}}$, $\beta_m y_m=\mathrm{id}_{B^1_m}$ and $y_m\beta_m=\mathrm{id}_{A^1_m}$ for $m\in\mathbb{Z}$. Therefore, $\alpha_m\colon A^0_{m-1}\to B^0_{m-1}$ and $\beta_m\colon A^1_m\to B^1_m$ are isomorphisms, then by (3.2), $\alpha[1]\colon A^0_\bullet\to B^0_\bullet$ and $\beta\colon A^1_\bullet\to B^1_\bullet$ are isomorphisms of complexes in $\mathcal{C}(\mathcal{B})$.

We write $f = \alpha[1]$, $g = \beta$, then we have chain maps $\psi f,\ g\varphi\colon\ A^0_\bullet \to B^1_\bullet$. The fact that $g_m\varphi_m = \psi_m f_m + d^{B^1}_{m+1} q_{m+1} + q_m d^{A^0}_m$ for every m implies that the diagram (3.1) commutes in the homotopy category $\mathcal{K}(\mathcal{B})$.

(2) ⇒ (1). Assume that the isomorphisms of complexes $f\colon\ A^0_\bullet \to B^0_\bullet$, $g\colon\ A^1_\bullet \to B^1_\bullet$ in $\mathcal{C}(\mathcal{B})$ induced a commutative diagram (3.1) in $\mathcal{K}(\mathcal{B})$. Then there exist morphisms $q_m\colon\ A^0_{m-1} \to B^1_m$ such that

$$g_m\varphi_m - \psi_m f_m = d^{B^1}_{m+1} q_{m+1} + q_m d^{A^0}_m \quad \text{for every } m. \tag{3.3}$$

Note that f and g are isomorphisms, thus for every m, f_m and g_m are invertible and

$$f_{m-1} d^{A^0}_m = d^{B^0}_m f_m, \quad g_{m-1} d^{A^1}_m = d^{B^1}_m g_m. \tag{3.4}$$

We define for every m

$$\varepsilon_m = \begin{bmatrix} f_{m-1} & 0 \\ q_m & g_m \end{bmatrix}\colon\ A^0_{m-1} \oplus A^1_m \to B^0_{m-1} \oplus B^1_m,$$
$$\eta_m = \begin{bmatrix} f^{-1}_{m-1} & 0 \\ -g^{-1}_m q_m f^{-1}_{m-1} & g^{-1}_m \end{bmatrix}\colon\ B^0_{m-1} \oplus B^1_m \to A^0_{m-1} \oplus A^1_m,$$

then ε_m, η_m are mutually-inverse morphisms, $(\varepsilon_m)\colon\ \mathrm{Cone}(\varphi) \to \mathrm{Cone}(\psi)$ and $(\eta_m)\colon\ \mathrm{Cone}(\psi) \to \mathrm{Cone}(\varphi)$ are morphisms in $\mathcal{C}(\mathcal{B})$ by (3.3) and (3.4), thus we have $\mathrm{Cone}(\varphi) \cong \mathrm{Cone}(\psi)$ in $\mathcal{C}(\mathcal{B})$. □

Since n-almost split sequences are unique up to isomorphism of complexes [13], Proposition 3.1.1, as a consequence of Proposition 3.13, the morphisms of complexes whose mapping cone corresponds to an n-almost split sequence in $\mathcal{B}$ is unique up to homology.

3.3. Characterization of n-almost split sequences. We see that every n-almost split sequence is isomorphic to the mapping cone $\mathrm{Cone}(\varphi)$ for a morphism of complexes φ (see Theorem 3.10) under $(\mathcal{B}, n)$ condition. In this subsection, we will relate the properties of n-almost split sequences to the mapping cone.

Fix object X in $\mathcal{B}$ and $n \in \mathbb{Z}$, we consider two functors $\widetilde{F}_X$ and $\widetilde{G}^n_X$ from $\mathrm{Mor}_r(\mathcal{C}_r(\mathcal{B}))$ to $\mathrm{Mor}(\mathcal{C}(\mathrm{mod}\, k))$ in [18], by putting

$$\widetilde{F}_X(\varphi) = \varphi \circ -\colon\ \mathrm{Hom}(X, A_\bullet) \to F_X(B_\bullet),$$
$$\widetilde{G}^n_X(\varphi) = - \circ (\varphi[-1])\colon\ \mathrm{Hom}(B_\bullet[-1], X) \to G^n_X(A_\bullet[-1])$$

for $\varphi\colon\ A_\bullet \to B_\bullet$. We consider the mapping cone functor $\mathrm{Cone}\colon\ \mathrm{Mor}_r(\mathcal{C}_r(\mathcal{B})) \to \mathcal{C}(\mathcal{B})$, and for category of k-vector space $\mathrm{mod}\, k$, we still denote by Cone the mapping cone functor $\mathrm{Cone}\colon\ \mathrm{Mor}(\mathcal{C}(\mathrm{mod}\, k)) \to \mathcal{C}(\mathrm{mod}\, k)$.

Lemma 3.14. *The diagram*

$$\begin{array}{ccccc}
\mathrm{Mor}(\mathcal{C}(\mathrm{mod}\,k)) & \xleftarrow{\widetilde{G}_X^n} & \mathrm{Mor}_r(\mathcal{C}_r^n(\mathcal{B})) & \xrightarrow{\widetilde{F}_X} & \mathrm{Mor}(\mathcal{C}^n(\mathrm{mod}\,k)) \\
\Big\downarrow{\scriptstyle \mathrm{Cone}} & & \Big\downarrow{\scriptstyle \mathrm{Cone}} & & \Big\downarrow{\scriptstyle \mathrm{Cone}} \\
\mathcal{C}(\mathrm{mod}\,k) & \xleftarrow{\quad G_X^n \quad} & \mathcal{C}_r(\mathcal{B}) & \xrightarrow{\quad F_X \quad} & \mathcal{C}(\mathrm{mod}\,k)
\end{array}$$

commutes for every object $X \in \mathcal{B}$ and nonnegative integer n.

P r o o f. The right-hand square commutes by [18], Lemma 2.6. We only need to prove that the left-hand square is commutative. For any object $\varphi\colon A_\bullet \to B_\bullet$ in $\mathrm{Mor}_r(\mathcal{C}_r^n(\mathcal{B}))$, note that $G_X^n(A_\bullet[-1])_i = 0$ for $i > -1$ or $i < -(n+1)$. The complex $G_X^n(A_\bullet[-1])$ with

$$\begin{aligned}
[\mathrm{Cone}(\widetilde{G}_X^n(\varphi))]_i &= [\mathrm{Hom}(B_\bullet[-1], X)]_{i-1} \oplus [G_X^n(A_\bullet[-1])]_i \\
&= \begin{cases} \mathrm{Hom}(B_{-i}, X) \oplus \mathrm{Hom}(A_{-i-1}, X), & i \neq -(n+1), \\ \mathrm{Hom}(B_{n+1}, X) \oplus \mathrm{rad}(A_n, X) = \mathrm{rad}(A_n, X), & i = -(n+1) \end{cases}
\end{aligned}$$

and the differential $d_i^{\mathrm{Cone}(\widetilde{G}_X^n(\varphi))}\colon [\mathrm{Cone}(\widetilde{G}_X^n(\varphi))]_i \to [\mathrm{Cone}(\widetilde{G}_X^n(\varphi))]_{i-1}$ is given by

$$d_i^{\mathrm{Cone}(\widetilde{G}_X^n(\varphi))} = \begin{bmatrix} -\circ d_{-i+1}^B & 0 \\ -\circ \varphi_{-i} & -(-\circ d_{-i}^A) \end{bmatrix}.$$

On the other hand, consider the complex

$$G_X^n(\mathrm{Cone}(\varphi)) = ([G_X^n(\mathrm{Cone}(\varphi))]_i, d_i^{G_X^n(\mathrm{Cone}(\varphi))}),$$

where

$$\begin{aligned}
&[G_X^n(\mathrm{Cone}(\varphi))]_i \\
&\quad = \begin{cases} \mathrm{Hom}(B_{-i} \oplus A_{-i-1}, X) = \mathrm{Hom}(B_{-i}, X) \oplus \mathrm{Hom}(A_{-i-1}, X), & i \neq -(n+1), \\ \mathrm{rad}(A_n \oplus B_{n+1}, X) = \mathrm{rad}(A_n, X), & i = -(n+1) \end{cases}
\end{aligned}$$

and the differential $d_i^{G_X^n(\mathrm{Cone}(\varphi))}\colon [G_X^n(\mathrm{Cone}(\varphi))]_i \to [G_X^n(\mathrm{Cone}(\varphi))]_{i-1}$ is given by

$$d_i^{G_X^n(\mathrm{Cone}(\varphi))} = -\circ d_{-i+1}^{\mathrm{Cone}(\varphi)} = \begin{bmatrix} -\circ d_{-i+1}^B & 0 \\ -\circ \varphi_{-i} & -(-\circ d_{-i}^A) \end{bmatrix}.$$

□

Using Lemma 3.14, we have a characterization of n-almost split sequences by mapping cone.

Proposition 3.15. *Let n be a positive integer. For any object $\varphi\colon A^0_\bullet \to A^1_\bullet$ in $\mathrm{Mor}_r(\mathcal{C}^n_r(\mathcal{B}))$, the following two conditions are equivalent:*

(1) *The mapping cone* $\mathrm{Cone}(\varphi)$ *is an n-almost split sequence in $\mathcal{B}$.*

(2) $\mathrm{Cone}(\varphi) \in \mathcal{C}^{n+1}_r(\mathcal{B})$ *is acyclic, and $\widetilde{F}_X(\varphi)$, $\widetilde{G}^n_X(\varphi)$ are quasi-isomorphisms for every object $X \in \mathcal{B}$.*

Proof. (1) $\Rightarrow$ (2). Under the assumptions, it is clear that $\mathrm{Cone}(\varphi) \in \mathcal{C}^{n+1}_r(\mathcal{B})$ and is acyclic. By Lemma 2.4 (3), $F_X(\mathrm{Cone}(\varphi))$ and $G^n_X(\mathrm{Cone}(\varphi))$ are acyclic for every object $X \in \mathcal{B}$, and by Lemma 3.14, we have

$$\mathrm{Cone}(\widetilde{F}_X(\varphi)) = F_X(\mathrm{Cone}(\varphi)), \quad \mathrm{Cone}(\widetilde{G}^n_X(\varphi)) = G^n_X(\mathrm{Cone}(\varphi)),$$

which yields that $\widetilde{F}_X(\varphi)$, $\widetilde{G}^n_X(\varphi)$ are quasi-isomorphisms for every $X \in \mathcal{B}$ by Lemma 2.1.

(2) $\Rightarrow$ (1). $\widetilde{F}_X(\varphi)$, $\widetilde{G}^n_X(\varphi)$ being quasi-isomorphisms for every $X \in \mathcal{B}$ implies that $\mathrm{Cone}(\widetilde{F}_X(\varphi)) = F_X(\mathrm{Cone}(\varphi))$ and $\mathrm{Cone}(\widetilde{G}^n_X(\varphi)) = G^n_X(\mathrm{Cone}(\varphi))$ are acyclic by Lemmas 2.1 and 3.14. By Lemma 2.4 (3), we know that $\mathrm{Cone}(\varphi)$ is an n-almost split sequence in $\mathcal{B}$. $\square$

The following lemma shows that if $\widetilde{F}_X(\varphi)$ and $\widetilde{G}^n_X(\varphi)$ are quasi-isomorphisms, $\widetilde{F}_Z(\varphi)$, $\widetilde{G}^n_Z(\varphi)$ are also quasi-isomorphisms for any direct summand Z of X and $\varphi \in \mathrm{Mor}_r(\mathcal{C}^n_r(\mathcal{B}))$.

Lemma 3.16. *Let n be a nonnegative integer. For any $X = X_1 \oplus X_2 \in \mathcal{B}$ and $\varphi \in \mathrm{Mor}_r(\mathcal{C}^n_r(\mathcal{B}))$, we have:*

(1) *$\widetilde{F}_X(\varphi)$ is a quasi-isomorphism if and only if $\widetilde{F}_{X_1}(\varphi)$, $\widetilde{F}_{X_2}(\varphi)$ are quasi-isomorphisms.*

(2) *$\widetilde{G}^n_X(\varphi)$ is a quasi-isomorphism if and only if $\widetilde{G}^n_{X_1}(\varphi)$, $\widetilde{G}^n_{X_2}(\varphi)$ are quasi-isomorphisms.*

Proof. We only prove (1); the proof of (2) is similar. Observe that $\widetilde{F}_X(\varphi)$ is a quasi-isomorphism if and only if $\mathrm{Cone}(\widetilde{F}_X(\varphi)) = F_X(\mathrm{Cone}(\varphi)) \cong F_{X_1}(\mathrm{Cone}(\varphi)) \oplus F_{X_2}(\mathrm{Cone}(\varphi))$ is acyclic by Lemma 2.1 and Lemma 3.14, and if and only if $F_{X_1}(\mathrm{Cone}(\varphi)) = \mathrm{Cone}(\widetilde{F}_{X_1}(\varphi))$ and $F_{X_2}(\mathrm{Cone}(\varphi)) = \mathrm{Cone}(\widetilde{F}_{X_2}(\varphi))$ are acyclic, the conclusion follows at once from Lemma 2.1. $\square$

4. Tensor products of mapping cone

Let Λ, Γ be finite dimensional algebras over field k. The aim of this section is to study the tensor products of complexes over field k and provide proof of the main Theorem 1.1.

4.1. Tensor products of complexes. We use the symbol $\otimes$ for tensor product of modules and $\otimes^T$ for tensor product complexes over field k. The tensor product bifunctor $-\otimes^T -\colon \mathcal{C}(\operatorname{mod}\Lambda)\times\mathcal{C}(\operatorname{mod}\Gamma)\to\mathcal{C}(\operatorname{mod}(\Lambda\otimes\Gamma))$ follows [5], IV 4.4 and IV 4.5. For complexes $A_\bullet\in\mathcal{C}(\operatorname{mod}\Lambda)$ and $B_\bullet\in\mathcal{C}(\operatorname{mod}\Gamma)$, $A_\bullet\otimes^T B_\bullet$ is the complex with $(A_\bullet\otimes^T B_\bullet)_p=\bigoplus\limits_{j\in\mathbb{Z}} A_j\otimes B_{p-j}$ and the differential given by

$$d_p^{A_\bullet\otimes^T B_\bullet}(v\otimes w)=d_j^A(v)\otimes w+(-1)^j v\otimes d_{p-j}^B(w)\quad \forall\, v\otimes w\in A_j\otimes^T B_{p-j}.$$

For chain maps $\varphi\colon A_\bullet^0\to A_\bullet^1$ in $\mathcal{C}(\operatorname{mod}\Lambda)$ and $\psi\colon B_\bullet^0\to B_\bullet^1$ in $\mathcal{C}(\operatorname{mod}\Gamma)$, $\varphi\otimes^T\psi\colon A_\bullet^0\otimes^T B_\bullet^0\to A_\bullet^1\otimes^T B_\bullet^1$ is the chain map of complexes with

$$(\varphi\otimes^T\psi)_p=\bigoplus_{j\in\mathbb{Z}}(\varphi_j\otimes\psi_{p-j})\colon \bigoplus_{j\in\mathbb{Z}}A_j^0\otimes B_{p-j}^0\to\bigoplus_{j\in\mathbb{Z}}A_j^1\otimes B_{p-j}^1\quad\text{for } p\in\mathbb{Z}.$$

The following lemma tells us that the shift functors are related to functor G_X^n.

Corollary 4.1. *Let Λ, Γ be finite dimensional algebras over field k. For any $A_\bullet\in\mathcal{C}(\operatorname{mod}\Lambda)$, $B_\bullet\in\mathcal{C}(\operatorname{mod}\Gamma)$ and $M\in\operatorname{mod}\Lambda$, $N\in\operatorname{mod}\Gamma$, we have*

$$G_{M\otimes N}^{n+m}(A_\bullet[i]\otimes^T B_\bullet[j])[-m]=G_{M\otimes N}^{n}((A_\bullet\otimes^T B_\bullet)[m+i+j])$$

for every $i,j,m,n\in\mathbb{Z}$.

Proof. We have

$$\begin{aligned}G_{M\otimes N}^{n+m}(A_\bullet[i]\otimes^T B_\bullet[j])[-m]&=G_{M\otimes N}^{n}((A_\bullet[i]\otimes^T B_\bullet[j])[m])\\&=G_{M\otimes N}^{n}(((A_\bullet\otimes^T B_\bullet)[i+j])[m])\\&=G_{M\otimes N}^{n}((A_\bullet\otimes^T B_\bullet)[m+i+j]).\end{aligned}$$

□

We need the following two results.

Lemma 4.2 ([18], Lemma 3.1)**.** *Let Λ, Γ be finite dimensional algebras over field k and $\mathcal{B}$, $\mathcal{E}$ be full subcategories of $\operatorname{mod}\Lambda$, $\operatorname{mod}\Gamma$, respectively. Then, for any quasi-isomorphisms $\varphi\in\operatorname{Mor}(\mathcal{C}(\mathcal{B}))$ and $\psi\in\operatorname{Mor}(\mathcal{C}(\mathcal{E}))$, $\varphi\otimes^T\psi$ is a quasi-isomorphism.*

Proposition 4.3 ([18], Proposition 3.4)**.** *Let Λ, Γ be finite dimensional algebras over field k. For each $M, N \in \operatorname{mod}\Lambda$ and $M', N' \in \operatorname{mod}\Gamma$, the canonical map $\operatorname{Hom}_\Lambda(M,N) \otimes_k \operatorname{Hom}_\Gamma(M',N') \to \operatorname{Hom}_{\Lambda\otimes_k\Gamma}(M \otimes_k M', N \otimes_k N')$ given by $f \otimes g \to f \otimes g$ is an isomorphism of k-vector spaces.*

When k is perfect, tensor product preserves some nice properties, such as finite global dimension and semi-simple algebras, so it is natural to consider the tensor products of n-representation finite, n-complete and n-representation infinite algebras over perfect field, see [10], [11], [18], [19]. Now we assume k to be a perfect field. The following result can be easily shown by modifying Lemma 3.6 in [18].

Lemma 4.4. *Let k be a perfect field and Λ, Γ finite dimensional algebras over k, let $M, N \in \operatorname{mod}\Lambda$ and $M', N' \in \operatorname{mod}\Gamma$. Then we have*

$$(4.1)\qquad 0 \longrightarrow \operatorname{rad}(M,N) \otimes \operatorname{rad}(M',N') \xrightarrow{\left[\begin{smallmatrix}\alpha\\ -\alpha\end{smallmatrix}\right]} \begin{matrix}\operatorname{rad}(M,N) \otimes \operatorname{Hom}(M',N')\\ \oplus \\ \operatorname{Hom}(M,N) \otimes \operatorname{rad}(M',N')\end{matrix} \xrightarrow{[\alpha,\alpha]} \operatorname{rad}(M \otimes M', N \otimes N') \longrightarrow 0,$$

where $\alpha\colon f \otimes g \to f \otimes g$.

Let $\mathcal{B}$, $\mathcal{E}$ be full subcategories of $\operatorname{mod}\Lambda$, $\operatorname{mod}\Gamma$, respectively. For any complexes $A_\bullet \in \mathcal{C}_r^n(\mathcal{B})$ and $B_\bullet \in \mathcal{C}_r^m(\mathcal{E})$, and modules $M \in \mathcal{B}$ and $N \in \mathcal{E}$, we define the morphism of complexes

$$\iota\colon \operatorname{Hom}(M, A_\bullet) \otimes^T \operatorname{Hom}(N, B_\bullet) \to \operatorname{Hom}(M \otimes N, A_\bullet \otimes^T B_\bullet)$$

equipped with

$$\iota_p = \bigoplus_{i+j=p} \iota_{ij}\colon \bigoplus_{i+j=p} \operatorname{Hom}(M, A_\bullet)_i \otimes \operatorname{Hom}(N, B_\bullet)_j \to \bigoplus_{i+j=p} \operatorname{Hom}(M \otimes N, A_i \otimes B_j)$$

for every $p \in \mathbb{Z}$, where $\iota_{ij}\colon \operatorname{Hom}(M, A_\bullet)_i \otimes \operatorname{Hom}(N, B_\bullet)_j \to \operatorname{Hom}(M \otimes N, A_i \otimes B_j)$ is given by the canonical map $f \otimes g \to f \otimes g$. Then ι_{ij} is a canonical isomorphism, see Proposition 4.3.

By Lemma 4.4, when ι is restricted to $F_M(A_\bullet) \otimes^T F_N(B_\bullet)$, then it induces an injective morphism

$$\iota'\colon F_M(A_\bullet) \otimes^T F_N(B_\bullet) \to F_{M\otimes N}(A_\bullet \otimes^T B_\bullet).$$

Similarly, we obtain two morphisms of complexes

$$u\colon \operatorname{Hom}(A_\bullet[-1], M) \otimes^T \operatorname{Hom}(B_\bullet[-1], N) \to \operatorname{Hom}(A_\bullet[-1] \otimes^T B_\bullet[-1], M \otimes N),$$
$$u'\colon G^n_M(A_\bullet[-1]) \otimes^T G^m_N(B_\bullet[-1]) \to G^{n+m+1}_{M\otimes N}(A_\bullet[-1] \otimes^T B_\bullet[-1]).$$

Proposition 4.5. *Let k be a perfect field. Under the above notations, the morphisms ι, u are isomorphisms of complexes and ι', u' are injective morphisms of complexes.*

Proof. It follows easily from Proposition 4.3. □

We next consider the case when ι', u' are quasi-isomorphisms.

Proposition 4.6. *Let k be a perfect field and Λ, Γ finite dimensional algebras over k. Let $\mathcal{B}$, $\mathcal{E}$ be full subcategories of $\operatorname{mod}\Lambda$, $\operatorname{mod}\Gamma$, respectively. Let $\varphi \in \operatorname{Mor}(\mathcal{C}^n_r(\mathcal{B}))$, $\psi \in \operatorname{Mor}(\mathcal{C}^m_r(\mathcal{E}))$, where $\varphi\colon A^0_\bullet \to A^1_\bullet$, $\psi\colon B^0_\bullet \to B^1_\bullet$ such that $\operatorname{Cone}(\varphi)$ is an n-almost split sequence in $\mathcal{B}$ and $\operatorname{Cone}(\psi)$ is an m-almost split sequence in $\mathcal{E}$. For any $M \in \mathcal{B}$, $N \in \mathcal{E}$, we have:*

(1) *If $\iota'\colon F_M(A^1_\bullet) \otimes^T F_N(B^1_\bullet) \to F_{M\otimes N}(A^1_\bullet \otimes^T B^1_\bullet)$ is a quasi-isomorphism, then $\widetilde{F}_{M\otimes N}(\varphi \otimes^T \psi)$ is also a quasi-isomorphism.*

(2) *If $u'\colon G^n_M(A^0_\bullet[-1]) \otimes^T G^m_N(B^0_\bullet[-1]) \to G^{n+m+1}_{M\otimes N}(A^0_\bullet[-1] \otimes^T B^0_\bullet[-1])$ is a quasi-isomorphism, then $\widetilde{G}^{n+m}_{M\otimes N}(\varphi \otimes^T \psi)$ is also a quasi-isomorphism.*

Proof. We only prove (1); the proof of (2) is similar. The following diagram is commutative:

(4.2)
$$\begin{array}{ccc} \operatorname{Hom}(M, A^0_\bullet) \otimes^T \operatorname{Hom}(N, B^0_\bullet) & \xrightarrow[\simeq]{\iota} & \operatorname{Hom}(M \otimes N, A^0_\bullet \otimes^T B^0_\bullet) \\ {\scriptstyle \widetilde{F}_M(\varphi)\otimes^T \widetilde{F}_N(\psi)}\downarrow \simeq & & \downarrow {\scriptstyle \widetilde{F}_{M\otimes N}(\varphi\otimes^T\psi)} \\ F_M(A^1_\bullet) \otimes^T F_N(B^1_\bullet) & \xrightarrow{\iota'} & F_{M\otimes N}(A^1_\bullet \otimes^T B^1_\bullet), \end{array}$$

where $\simeq$ is the quasi-isomorphism. The maps $\widetilde{F}_M(\varphi)$ and $\widetilde{F}_N(\psi)$ are quasi-isomorphisms by Proposition 3.15, and then $\widetilde{F}_M(\varphi) \otimes^T \widetilde{F}_N(\varphi)$ is also a quasi-isomorphism by Lemma 4.2. Assume that $\iota'\colon F_M(A^1_\bullet) \otimes^T F_N(B^1_\bullet) \to F_{M\otimes N}(A^1_\bullet \otimes^T B^1_\bullet)$ is a quasi-isomorphism, then $\iota'(\widetilde{F}_M(\varphi) \otimes^T \widetilde{F}_N(\varphi))$ is a quasi-isomorphism. This implies that $(\widetilde{F}_{M\otimes N}(\varphi \otimes^T \psi))\iota$ is a quasi-isomorphism by the commutative diagram (4.2), and then $\widetilde{F}_{M\otimes N}(\varphi \otimes^T \psi)$ is a quasi-isomorphism because ι is a quasi-isomorphism. □

The following proposition is needed in the proof of the main theorem.

Proposition 4.7. *Under the above notations, we have:*

(1) *The cokernel of* ι' *is the complex* $\operatorname{Coker}\iota' \cong D^1_\bullet \oplus D^2_\bullet$, *where*

$$\begin{aligned} D^1_\bullet &= F_M(A_\bullet) \otimes_k S(N, B_0), & D^2_\bullet &= S(M, A_0) \otimes_k F_N(B_\bullet),\\ D^1_i &= F_M(A_\bullet)_i \otimes_k S(N, B_0), & d^{D^1}_i &= d^{F_M(A_\bullet)}_i \otimes \mathrm{id}_{S(N,B_0)},\\ D^2_i &= S(M, A_0) \otimes_k F_N(B_\bullet)_i, & d^{D^2}_i &= \mathrm{id}_{S(M,A_0)} \otimes d^{F_N(B_\bullet)}_i. \end{aligned}$$

(2) *The cokernel of* u' *is the complex* $\operatorname{Coker} u' \cong E^1_\bullet \oplus E^2_\bullet$, *where*

$$\begin{aligned} E^1_\bullet &= G^n_M(A_\bullet[-1]) \otimes_k S(B_m, N), & E^2_\bullet &= S(A_n, M) \otimes_k G^m_N(B_\bullet[-1])[n+1],\\ E^1_i &= G^n_M(A_\bullet[-1])_{i+m+1} \otimes_k S(B_m, N), & d^{E^1}_i &= d^{G^n_M(A_\bullet[-1])}_{i+m+1} \otimes \mathrm{id}_{S(B_m,N)},\\ E^2_i &= S(A_n, M) \otimes_k G^m_N(B_\bullet[-1])_{i+n+1}, & d^{E^2}_i &= \mathrm{id}_{S(A_n,M)} \otimes (-1)^{n+1} d^{G^m_N(B_\bullet[-1])}_{i+n+1}. \end{aligned}$$

P r o o f. The proof of (1) is similar to the proof in [18], Section 3.3, pages 660–662. (2) follows from the duality of (1). □

Proposition 4.8. *Let* k *be a perfect field and* Λ, Γ *finite dimensional algebras over* k. *Let* $\varphi \in \operatorname{Mor}_r(\mathcal{C}^n_r(\operatorname{mod}\Lambda))$ *and* $\psi \in \operatorname{Mor}_r(\mathcal{C}^m_r(\operatorname{mod}\Gamma))$ *such that* $\operatorname{Cone}(\varphi)$ *and* $\operatorname{Cone}(\psi)$ *are acyclic complexes. Then* $\operatorname{Cone}(\varphi \otimes^T \psi) \in \mathcal{C}^{n+m+1}_r(\operatorname{mod}(\Lambda \otimes \Gamma))$ *is acyclic.*

P r o o f. Let $\varphi\colon A^0_\bullet \to A^1_\bullet$, $\psi\colon B^0_\bullet \to B^1_\bullet$ such that $\operatorname{Cone}(\varphi)$ and $\operatorname{Cone}(\psi)$ are acyclic, then φ and ψ are quasi-isomorphisms by Lemma 2.1, this implies that $\operatorname{Cone}(\varphi \otimes^T \psi)$ is an acyclic complex by Lemma 4.2 and Lemma 2.1. Observing that $\varphi \otimes^T \psi \in \operatorname{Mor}(\mathcal{C}^{n+m}(\operatorname{mod}(\Lambda \otimes \Gamma)))$, by Lemma 4.4, we know that $d^{A^0_\bullet \otimes B^0_\bullet}_i \in \operatorname{rad}((A^0_\bullet \otimes^T B^0_\bullet)_i, (A^0_\bullet \otimes^T B^0_\bullet)_{i-1})$, $d^{A^1_\bullet \otimes B^1_\bullet}_i \in \operatorname{rad}((A^1_\bullet \otimes^T B^1_\bullet)_i, (A^1_\bullet \otimes^T B^1_\bullet)_{i-1})$, and $(\varphi \otimes^T \psi)_i \in \operatorname{rad}((A^0_\bullet \otimes^T B^0_\bullet)_i, (A^1_\bullet \otimes^T B^1_\bullet)_i)$ for every $i \in \mathbb{Z}$. Hence, $\operatorname{Cone}(\varphi \otimes^T \psi) \in \mathcal{C}^{n+m+1}_r(\operatorname{mod}(\Lambda \otimes \Gamma))$. □

Let Λ be an algebra satisfying the $(\mathcal{B}, n)$ condition with $\mathcal{B} = \bigoplus_{i=0}^{b_1} \mathcal{B}_i$, and Γ an algebra satisfying the $(\mathcal{E}, m)$ condition with $\mathcal{E} = \bigoplus_{i=0}^{b_2} \mathcal{E}_i$, here $b_1, b_2 \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. Fix integers $0 \leqslant i_0 < b_1$, $0 \leqslant j_0 < b_2$. Let $l_1 = \max\{0, i_0 - j_0\}$ and $l_2 = \min\{b_1, b_2 + i_0 - j_0\}$. We construct the subcategory $(\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)} = \bigoplus_{i=l_1}^{l_2} \operatorname{add}\mathcal{H}_i \subseteq \operatorname{mod}(\Lambda \otimes_k \Gamma)$, where for $l_1 \leqslant i \leqslant l_2$,

$$\mathcal{H}_i = \{M \otimes_k N\colon\ M \in \operatorname{ind}\mathcal{B}_i,\ N \in \operatorname{ind}\mathcal{E}_{i-i_0+j_0}\}.$$

4.2. Proof of Theorem 1.1. We may suppose the morphisms $\varphi\colon A^0_\bullet \to A^1_\bullet$ and $\psi\colon B^0_\bullet \to B^1_\bullet$. Since $\operatorname{Cone}(\varphi)$ is starting in slice i_0 and ending in slice i_0+1, by Theorem 3.10, we have $A^0_\bullet \in \mathcal{C}^n_r(\mathcal{B}_{i_0})$ and $A^1_\bullet \in \mathcal{C}^n_r(\mathcal{B}_{i_0+1})$, similarly, $B^0_\bullet \in \mathcal{C}^m_r(\mathcal{E}_{j_0})$ and $B^1_\bullet \in \mathcal{C}^m_r(\mathcal{E}_{j_0+1})$. Then $A^0_\bullet \otimes^T B^0_\bullet \in \mathcal{C}^{n+m}_r(\operatorname{add}\mathcal{H}_{i_0})$ and $A^1_\bullet \otimes^T B^1_\bullet \in \mathcal{C}^{n+m}_r(\operatorname{add}\mathcal{H}_{i_0+1})$, and hence

$$\operatorname{Cone}(\varphi \otimes^T \psi) \in \mathcal{C}\left(\bigoplus_{i=i_0}^{i_0+1} \operatorname{add}\mathcal{H}_i\right) \subseteq \mathcal{C}((\mathcal{B}\otimes\mathcal{E})^{(i_0,j_0)}).$$

By Proposition 4.8, $\operatorname{Cone}(\varphi \otimes^T \psi) \in \mathcal{C}^{n+m+1}_r((\mathcal{B}\otimes\mathcal{E})^{(i_0,j_0)})$ is acyclic.

Notice that every $X \in \operatorname{ind}(\mathcal{B}\otimes\mathcal{E})^{(i_0,j_0)}$ is a direct summand of an object $Y' \in \bigcup_{i=l_1}^{l_2} \mathcal{H}_i$. If $\widetilde{F}_Y(\varphi \otimes^T \psi)$ and $\widetilde{G}^{n+m}_Y(\varphi \otimes^T \psi)$ are quasi-isomorphisms for every object $Y \in \bigcup_{i=l_1}^{l_2} \mathcal{H}_i$, by Lemma 3.16, $\widetilde{F}_X(\varphi\otimes^T\psi)$ and $\widetilde{G}^{n+m}_X(\varphi\otimes^T\psi)$ are quasi-isomorphisms for every $X \in \operatorname{ind}(\mathcal{B}\otimes\mathcal{E})^{(i_0,j_0)}$, this implies that $\operatorname{Cone}(\varphi \otimes^T \psi)$ is an $(n+m)$-almost split sequence in $(\mathcal{B}\otimes\mathcal{E})^{(i_0,j_0)}$ by Proposition 3.15.

Fix an object $M \otimes_k N \in \bigcup_{i=l_1}^{l_2} \mathcal{H}_i$, there exists i' with $l_1 \leqslant i' \leqslant l_2$ such that $M \in \operatorname{ind}\mathcal{B}_{i'}$ and $N \in \operatorname{ind}\mathcal{E}_{i'-i_0+j_0}$. Now, it suffices to show that $\widetilde{F}_{M\otimes N}(\varphi \otimes^T \psi)$ and $\widetilde{G}^{n+m}_{M\otimes N}(\varphi \otimes^T \psi)$ are quasi-isomorphisms. We only prove the second quasi-isomorphism; the proof of the first is similar. Consider the morphism

$$u'\colon G^n_M(A^0_\bullet[-1]) \otimes^T G^m_N(B^0_\bullet[-1]) \to G^{n+m+1}_{M\otimes N}(A^0_\bullet[-1] \otimes^T B^0_\bullet[-1]).$$

By Proposition 4.7 (2), we have

$$\operatorname{Coker} u' \cong (G^n_M(A^0_\bullet[-1]) \otimes_k S(B^0_m, N)) \oplus (S(A^0_n, M) \otimes_k G^m_N(B^0_\bullet[-1])[n+1]);$$

here $G^n_M(A^0_\bullet[-1]) \otimes_k S(B^0_m, N)$ is acyclic. Indeed, if $S(B^0_m, N) = 0$, the statement is trivial. If $S(B^0_m, N) \neq 0$, then N is a direct summand of B^0_m, it follows from $B^0_m \in \mathcal{E}_{j_0}$ that $N \in \mathcal{E}_{j_0}$ and $M \in \mathcal{B}_{i_0}$. By $A^1_\bullet \in \mathcal{C}^n_r(\mathcal{B}_{i_0+1})$ and Definition 3.1 (2), we have $\operatorname{Hom}(A^1_\bullet[-1], M) = 0$. Since $\operatorname{Cone}(\varphi)$ is an n-almost split sequence in $\mathcal{B}$, by Proposition 3.15, we deduce that $\widetilde{G}^n_M(\varphi)$ is a quasi-isomorphism, and then $G^n_M(A^0_\bullet[-1]) \cong \operatorname{Cone}(\widetilde{G}^n_M(\varphi))$ is acyclic by Lemma 2.1. Applying the exact tensor functor $-\otimes_k S(B^0_m, N)$ over k, we get an acyclic complex $G^n_M(A^0_\bullet[-1]) \otimes_k S(B^0_m, N)$. By symmetry, we obtain that $S(A^0_n, M) \otimes_k G^m_N(B^0_\bullet[-1])[n+1]$ is acyclic. Therefore, $\operatorname{Coker} u'$ is acyclic, this implies that the injective $u'\colon G^n_M(A^0_\bullet[-1]) \otimes^T G^m_N(B^0_\bullet[-1]) \to G^{n+m+1}_{M\otimes N}(A^0_\bullet[-1] \otimes^T B^0_\bullet[-1])$ is a quasi-isomorphism. Consequently, by Proposition 4.6 (2), the morphism $\widetilde{G}^{n+m}_{M\otimes N}(\varphi \otimes^T \psi)$ is a quasi-isomorphism. □

Remark 4.9. Let Λ, Γ be respectively n-, m-representation finite l-homogeneous algebras over perfect field k, M_Λ and M_Γ are, respectively, n-, m-cluster tilting modules of Λ, Γ. Herschend and Iyama showed in [10] that $\Lambda \otimes_k \Gamma$ is an $(n+m)$-representation finite algebra admitting an $(n+m)$-cluster tilting module $M_{\Lambda\otimes_k\Gamma} = \bigoplus_{i=0}^{l-1}(\tau_n^{-i}\Lambda\otimes_k\tau_m^{-i}\Gamma)$. Then we obtain $\operatorname{add} M_{\Lambda\otimes_k\Gamma} = \bigoplus_{i=0}^{l-1}\operatorname{add}(\tau_n^{-i}\Lambda\otimes_k\tau_m^{-i}\Gamma) = (\operatorname{add} M_\Lambda, \operatorname{add} M_\Gamma)^{(i_0,i_0)}$ for any i_0 with $0 \leqslant i_0 \leqslant l-1$. Hence, the main Theorem 1.1 generalizes [18], Theorem 1.1. Similar constructions hold for higher complete algebras [19]. Theorem 1.1 states that we do not have to care about l-homogeneity and whether the respectively n-, m-almost split sequences start in common slice.

For n-representation infinite algebras [11], by Theorem 1.1, we have the following corollary.

Corollary 4.10. *Let k be a perfect field. Let Λ, Γ be, respectively, n-, m-representation infinite algebras. Let $\varphi \in \operatorname{Mor}_r(\mathcal{C}_r^n(\mathscr{P}_\Lambda))$, $\psi \in \operatorname{Mor}_r(\mathcal{C}_r^m(\mathscr{P}_\Gamma))$ such that $\operatorname{Cone}(\varphi)$ and $\operatorname{Cone}(\psi)$ are respectively n-, m-almost split sequences in $\mathscr{P}_\Lambda$ and $\mathscr{P}_\Gamma$ starting in common slice $i_0 \geqslant 0$. Then $\operatorname{Cone}(\varphi \otimes^T \psi)$ is an $(n+m)$-almost split sequence in $\mathscr{P}_{\Lambda\otimes\Gamma}$.*

If $M_1, M_2 \in \operatorname{mod}\Lambda$ and $N_1, N_2 \in \operatorname{mod}\Gamma$, by Lemma 4.10 of [19], there is a functorial isomorphism

$$\operatorname{Ext}^i_{\Lambda\otimes\Gamma}(M_1\otimes N_1, M_2\otimes N_2) \cong \bigoplus_{p+q=i}\operatorname{Ext}^p_\Lambda(M_1, M_2)\otimes\operatorname{Ext}^q_\Gamma(N_1, N_2).$$

This shows that $M_1 \otimes N_1$ is projective if and only if M_1, N_1 are projective; $M_2 \otimes N_2$ is injective if and only if M_2, N_2 are injective. If moreover k is an algebraically closed field, then $M \in \operatorname{ind}(\operatorname{mod}\Lambda)$ and $N \in \operatorname{ind}(\operatorname{mod}\Gamma)$ if and only if $M \otimes N \in \operatorname{ind}(\operatorname{mod}\Lambda\otimes\Gamma)$. Indeed, $M \otimes N$ being indecomposable implies that $\operatorname{End}_\Lambda(M)\otimes\operatorname{End}_\Gamma(N) = \operatorname{End}_{\Lambda\otimes\Gamma}(M\otimes N)$ is a local algebra, it is follows from [17], Theorem 3 that $\operatorname{End}_\Lambda(M)$ and $\operatorname{End}_\Gamma(N)$ are local. Hence, M and N are indecomposable. On the contrary, assume $M \in \operatorname{ind}(\operatorname{mod}\Lambda), N \in \operatorname{ind}(\operatorname{mod}\Gamma)$, then $(\operatorname{End}_\Lambda(M)/\operatorname{rad}(\operatorname{End}_\Lambda(M)))\otimes(\operatorname{End}_\Gamma(N)/\operatorname{rad}(\operatorname{End}_\Gamma(N))) \cong k\otimes k \cong k$ is a local algebra. By [17], Theorem 4 again, $\operatorname{End}_\Lambda(M)\otimes\operatorname{End}_\Gamma(N) \cong \operatorname{End}_{\Lambda\otimes\Gamma}(M\otimes N)$ is a local algebra, hence $M\otimes N \in \operatorname{ind}(\operatorname{mod}\Lambda\otimes\Gamma)$.

We have the following result for tensor product of algebras.

Corollary 4.11. *Let k be a perfect field. Let Λ be an algebra satisfying the $(\mathcal{B}, n)$ condition with $\mathcal{B} = \bigoplus_{i=0}^{b_1}\mathcal{B}_i$, and Γ an algebra satisfying the $(\mathcal{E}, m)$ condition with $\mathcal{E} = \bigoplus_{i=0}^{b_2}\mathcal{E}_i$. Assume that any n-almost split sequences in $\mathcal{B}$ starts in slice i, ends in* *slice $i+1$ for some $i \geqslant 0$, and any m-almost split sequences in $\mathcal{E}$ starts in slice j, ends in slice $j+1$ for some $j \geqslant 0$. Fix integer $0 \leqslant i_0 < b_1$, $0 \leqslant j_0 < b_2$.*

(1) *For any indecomposable objects $M \in \mathcal{B}_I \cap \mathcal{B}_s$ and $N \in \mathcal{E}_I \cap \mathcal{E}_{s-i_0+j_0}$ for an integer $s \geqslant 0$, there exists an $(n+m)$-almost split sequence in $(\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)}$ starting at $M \otimes N$.*

(2) *For any indecomposable objects $M \in \mathcal{B}_P \cap \mathcal{B}_s$ and $N \in \mathcal{E}_P \cap \mathcal{E}_{s-i_0+j_0}$ for an integer $s \geqslant 0$, there exists an $(n+m)$-almost split sequence in $(\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)}$ ending at $M \otimes N$.*

(3) *If field k is algebraically closed, then $(\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)}$ has $(n+m)$-almost split sequences for the following subcategories of $(\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)}$:*

$$(\mathcal{B} \otimes \mathcal{E})_I^{(i_0,j_0)} = \operatorname{add}\{M \otimes N \in (\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)} \colon M \in \mathcal{B}_I \cap \mathcal{B}_s \text{ and } N \in \mathcal{E}_I \cap \mathcal{E}_{s-i_0+j_0} \text{ for some } s\},$$
$$(\mathcal{B} \otimes \mathcal{E})_P^{(i_0,j_0)} = \operatorname{add}\{M \otimes N \in (\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)} \colon M \in \mathcal{B}_P \cap \mathcal{B}_s \text{ and } N \in \mathcal{E}_P \cap \mathcal{E}_{s-i_0+j_0} \text{ for some } s\}.$$

(4) *If field k is algebraically closed, then $\Lambda \otimes \Gamma$ satisfies the $((\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)}, n+m)$ condition.*

Proof. (1) Suppose indecomposable objects $M \in \mathcal{B}_I \cap \mathcal{B}_s$ and $N \in \mathcal{E}_I \cap \mathcal{E}_{s-i_0+j_0}$ for an integer $s \geqslant 0$, then there exists n-almost split sequence $C^1_\bullet$ in $\mathcal{B}$ starting at object M and m-almost split sequence $C^2_\bullet$ in $\mathcal{E}$ starting at object N. It is clear from Theorem 3.10 that there exists $\varphi \in \operatorname{Mor}_r(\mathcal{C}^n_r(\mathcal{B}))$ and $\psi \in \operatorname{Mor}_r(\mathcal{C}^m_r(\mathcal{E}))$ such $C^1_\bullet = \operatorname{Cone}(\varphi)$ and $C^2_\bullet = \operatorname{Cone}(\psi)$. By Theorem 1.1, $\operatorname{Cone}(\varphi \otimes^T \psi)$ is an $(n+m)$-almost split sequence in $(\mathcal{B} \otimes \mathcal{E})^{(s,s-i_0+j_0)} = (\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)}$ starting at module $M \otimes N$.

(2) is dual to (1), and (3) easily follows from (1) and (2).

(4) $\Lambda \otimes \Gamma$ is a finite dimension algebra since $\dim_k(\Lambda \otimes_k \Gamma) = \dim_k(\Lambda) \times \dim_k(\Gamma)$ is finite. By (3) and the construction of $(\mathcal{B} \otimes \mathcal{E})^{(i_0,j_0)}$, it is enough to show that $\operatorname{Hom}_{\Lambda\otimes\Gamma}(X,Y) = 0$ for all $l_2 \geqslant s > t \geqslant l_1$ and every $X \in \mathcal{H}_s$, $Y \in \mathcal{H}_t$.

For $s > t$, suppose $M \otimes N \in \mathcal{H}_s$ and $M' \otimes N' \in \mathcal{H}_t$, then $M \in \mathcal{B}_s$, $M' \in \mathcal{E}_t$ and $N \in \mathcal{B}_{s-i_0+j_0}$, $N' \in \mathcal{E}_{t-i_0+j_0}$. It follows from Proposition 4.3 and Definition 3.1 (2) that

$$\operatorname{Hom}_{\Lambda\otimes\Gamma}(M \otimes N, M' \otimes N') \cong \operatorname{Hom}_\Lambda(M, M') \otimes \operatorname{Hom}_\Gamma(N, N') = 0.$$

Therefore, $\operatorname{Hom}_{\Lambda\otimes\Gamma}(M \otimes N, M' \otimes N') = 0$, the proof is complete. □

The tensor product algebra $\Lambda \otimes \Gamma$ in Corollary 4.11 behaves well, particularly, under certain conditions, tensor product of algebras preserves n-representation finiteness [10], n-completeness [19] and n-representation infiniteness [11].

5. Example

In this section, we give an example for nonhomogeneous, 1-representation finite algebra, and by Theorem 1.1, construct a 2-almost split sequence via (1-)almost split sequences starting in different slice.

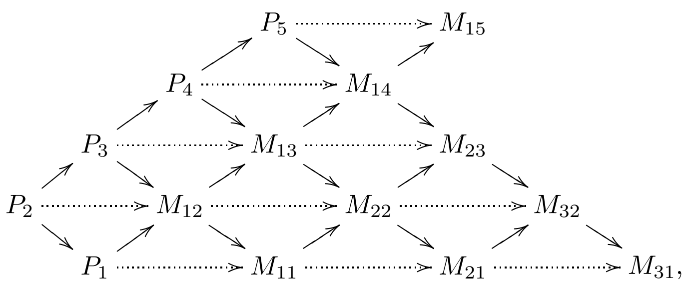
Let $\Lambda = kQ$ be the path algebra defined by the quiver

$$Q\colon \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet} \longleftarrow \overset{4}{\bullet} \longleftarrow \overset{5}{\bullet}$$

Λ is a nonhomogeneous, 1-representation finite algebra, see [10]. We consider the algebra $\Gamma = \Lambda \otimes_k \Lambda$, which is not 2-representation finite since Λ is nonhomogeneous. For $1 \leqslant i \leqslant 5$, let P_i and I_i be the indecomposable projective, injective Λ-modules corresponding vertex i, respectively. The 15 nonisomorphic indecomposable modules in $\operatorname{mod}\Lambda$ given by the following dimension vectors:

$$\begin{array}{lll}
P_1 : (11000), & M_{11} : (00100), & M_{21} : (00010), \\
P_2 : (01000), & M_{12} : (11100), & M_{22} : (00110), \\
P_3 : (01100), & M_{13} : (11110), & M_{23} = I_3 : (00111), \\
P_4 : (01110), & M_{14} = I_2 : (11111), & M_{31} = I_5 : (00001), \\
P_5 : (01111), & M_{15} = I_1 : (10000), & M_{32} = I_4 : (00011).
\end{array}$$

The Auslander-Reiten quiver of Λ is given as



where the dashed arrows show the Auslander-Reiten translation τ_1^-.

Notice that Λ satisfies the $(\mathcal{B}, 1)$ condition with $\mathcal{B} = \operatorname{mod}\Lambda = \bigoplus_{i=0}^{3} \mathcal{B}_i$ where

$$\begin{array}{ll}
\mathcal{B}_0 = \operatorname{add}\{P_i\colon\ 1 \leqslant i \leqslant 5\}, & \mathcal{B}_1 = \operatorname{add}\{M_{1i}\colon\ 1 \leqslant i \leqslant 5\}, \\
\mathcal{B}_2 = \operatorname{add}\{M_{2i}\colon\ 1 \leqslant i \leqslant 3\}, & \mathcal{B}_3 = \operatorname{add}\{M_{31}, M_{32}\}.
\end{array}$$

There are the following two (1-)almost split sequences in $\mathcal{B}$:

$$
\begin{aligned}
A_\bullet\colon\ 0 \to P_3 \xrightarrow{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} P_4 \oplus M_{12} \xrightarrow{[c,d]} M_{13} \to 0,\\
B_\bullet\colon\ 0 \to M_{13} \xrightarrow{\left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]} M_{14} \oplus M_{22} \xrightarrow{[g,h]} M_{23} \to 0.
\end{aligned}
$$

We see that $A_\bullet$ starts in slice 0, ends in slice 1 and $B_\bullet$ starts in slice 1, ends in slice 2. By Theorem 3.10, $A_\bullet = \mathrm{Cone}(\varphi)$ and $B_\bullet = \mathrm{Cone}(\psi)$, where

$$
\varphi\colon\ \begin{array}{ccccccccc}
\cdots \to & 0 & \to & P_3 & \xrightarrow{-a} & P_4 & \to & 0 & \to \cdots\\
 & \downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow & \\
\cdots \to & 0 & \to & M_{12} & \xrightarrow{d} & M_{13} & \to & 0 & \to \cdots
\end{array}
\qquad
\psi\colon\ \begin{array}{ccccccccc}
\cdots \to & 0 & \to & M_{13} & \xrightarrow{-e} & M_{14} & \to & 0 & \to \cdots\\
 & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow & \\
\cdots \to & 0 & \to & M_{22} & \xrightarrow{h} & M_{23} & \to & 0 & \to \cdots
\end{array}
$$

Then tensor product $\varphi \otimes^T \psi$ is the following morphism:

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & P_3 \otimes M_{13} & \xrightarrow{\left[\begin{smallmatrix} -(1\otimes e) \\ -(a\otimes 1) \end{smallmatrix}\right]} & (P_3 \otimes M_{14}) \oplus (P_4 \otimes M_{13}) & \\
 & \downarrow & & \downarrow{\scriptstyle b\otimes f} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} b\otimes g & 0 \\ 0 & c\otimes f \end{smallmatrix}\right]} & \\
\cdots \longrightarrow & 0 & \longrightarrow & M_{12} \otimes M_{22} & \xrightarrow{\left[\begin{smallmatrix} 1\otimes h \\ d\otimes 1 \end{smallmatrix}\right]} & (M_{12} \otimes M_{23}) \oplus (M_{13} \otimes M_{22}) &
\end{array}
$$

$$
\begin{array}{ccccc}
\xrightarrow{[-(a\otimes 1),-(1\otimes e)]} & P_4 \otimes M_{14} & \longrightarrow & 0 & \longrightarrow \cdots\\
 & \downarrow{\scriptstyle c\otimes g} & & \downarrow & \\
\xrightarrow{[d\otimes 1,1\otimes h]} & M_{13} \otimes M_{23} & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}
$$

We construct the subcategory $(\mathcal{B} \otimes \mathcal{B})^{(0,1)} = \bigoplus_{i=0}^{2} \operatorname{add} \mathcal{H}_i$, where $\mathcal{H}_0 = \{P_i \otimes M_{1j}\colon 1 \leqslant i,\ j \leqslant 5\}$, $\mathcal{H}_1 = \{M_{1i} \otimes M_{2j}\colon 1 \leqslant i \leqslant 5,\ 1 \leqslant j \leqslant 3\}$ and $\mathcal{H}_2 = \{M_{2i} \otimes M_{3j}\colon k1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant 2\}$. By Theorem 1.1, we obtain that the mapping cone $\mathrm{Cone}(\varphi \otimes^T \psi)$

$$
0 \longrightarrow P_3 \otimes M_{13} \xrightarrow{\left[\begin{smallmatrix} 1\otimes e \\ a\otimes 1 \\ b\otimes f \end{smallmatrix}\right]} \begin{matrix} P_3 \otimes M_{14} \\ \oplus \\ P_4 \otimes M_{13} \\ \oplus \\ M_{12} \otimes M_{22} \end{matrix} \xrightarrow{\left[\begin{smallmatrix} a\otimes 1 & 1\otimes e & 0 \\ b\otimes g & 0 & 1\otimes h \\ 0 & c\otimes f & d\otimes 1 \end{smallmatrix}\right]} \begin{matrix} P_4 \otimes M_{14} \\ \oplus \\ M_{12} \otimes M_{23} \\ \oplus \\ M_{13} \otimes M_{22} \end{matrix}
$$

$$
\xrightarrow{[c\otimes g,d\otimes 1,1\otimes h]} M_{13} \otimes M_{23} \longrightarrow 0
$$

is a 2-almost split sequence in $(\mathcal{B} \otimes \mathcal{B})^{(0,1)}$ starting in slice 0, ending in slice 1. The algebra Γ satisfies the $((\mathcal{B} \otimes \mathcal{B})^{(0,1)}, 2)$ condition by Corollary 4.11.

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