Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 73 (2023), No. 4, 1229–1254

Persistent URL: http://dml.cz/dmlcz/151957

Czechoslovak Mathematical Journal, 73 (148) (2023), 1229–1254

PERIODIC LINEAR GROUPS FACTORIZED BY MUTUALLY PERMUTABLE SUBGROUPS

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Received October 21, 2022. Published online November 2, 2023.

Abstract. The aim is to investigate the behaviour of (homomorphic images of) periodic linear groups which are factorized by mutually permutable subgroups. Mutually permutable subgroups have been extensively investigated in the finite case by several authors, among which, for our purposes, we only cite J. C. Beidleman and H. Heineken (2005). In a previous paper of ours (see M. Ferrara, M. Trombetti (2022)) we have been able to generalize the first main result of J. C. Beidleman, H. Heineken (2005) to periodic linear groups (showing that the commutator subgroups and the intersection of mutually permutable subgroups are subnormal subgroups of the whole group), and, in this paper, we completely generalize all other main results of J. C. Beidleman, H. Heineken (2005) to (homomorphic images of) periodic linear groups.

Keywords: mutually permutable subgroup; periodic linear group

MSC 2020: 20D40, 20F19, 20H20

1. Introduction

Let G be a group. We say that the subgroups A and B of G are *mutually permutable* if $AY = YA$ and $XB = BX$ for all subgroups X of A and Y of B. Of course, any two normal subgroups are mutually permutable, while the example in [1] (see page 454) shows that there are non-supersoluble groups which are factorized by two mutually permutable proper (supersoluble) subgroups but not by two proper normal subgroups. Groups which are products of two mutually permutable subgroups have been recently investigated by several authors, and we refer to the monograph (see [4])

The authors are supported by GNSAGA (INdAM) and are members of the non-profit association "Advances in Group Theory and Applications" (`www.advgrouptheory.com`). Open Access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

DOI: 10.21136/CMJ.2023.0485-22

for questions and results concerning this subject, see also [6], [12], [9] and their reference lists. In particular, Beidleman and Heineken in [6] proved that if $G = AB$ is a finite group which is factorized by two mutually permutable subgroups, then $A \cap B$, A' and B' are subnormal subgroups of G. This result has been partially extended in [9] to Černikov groups, while in the recent paper [12] we have been able to prove that not only the result of Beidleman and Heineken holds for Černikov groups to its full extent, but that this is the case for every periodic linear group.

The aim of this paper is to continue the investigation of periodic linear groups that are factorized by mutually permutable subgroups, generalizing all relevant results of [6] and providing a new insight concerning infinite groups. The layout of the paper is the following one. In Section 2 we study the behaviour of chief factors of periodic linear groups with respect to the mutually permutable subgroups into which the group is factorized; we summarize the results as follows:

Theorem 1.1. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B, and let N be a minimal normal subgroup of G. Then:*

- ▷ *either $N \leqslant A$ or $N \cap A = \{1\}$;*
- ▷ *if N is non-abelian, then $N = (N \cap A)(N \cap B)$;*
- ▷ *if $N \leqslant A$ and $N \cap B = \{1\}$, then either $[N, A] = \{1\}$ or $[N, B] = \{1\}$ (and the latter certainly holds if N is not cyclic);*
- ▷ *if $N \cap A = N \cap B = \{1\}$, then N is cyclic of prime order and either $[N, A] = \{1\}$ or $[N, B] = \{1\}$.*

In Section 3 we employ the results obtained in Section 2 to investigate how requirements on mutually permutable subgroups factorizing a group influence the structure of the whole group. In particular, we consider the following classes of groups: soluble groups, hyperabelian groups, supersoluble groups, hypercyclic groups, paranilpotent groups, and their “localizations” to a prime p, see Section 3 for the definitions.

Our notation is standard and can be found for instance in [16] and [20]. If $\mathfrak{X}$ is any group theoretical property, we generally say that a normal section N/M of a group G is $\mathfrak{X}$*-ly embedded* in G if “N/M satisfies the requirements of $\mathfrak{X}$ with respect to G/M”. For instance, if we say that N/M is *hypercentrally embedded* in G, we mean that N/M lies in the hypercentre of G/M; if we say that N/M is *hypercyclically embedded* in G, we mean that N/M has an ascendant G/M-invariant series with cyclic factors; and so on ...

2. Minimal normal subgroups

The aim of this section is to study the behaviour of minimal normal subgroups of (homomorphic images of) periodic linear groups which are factorized by mutually permutable subgroups. These results are employed in the proof of Theorem 3.19, which is in turn used in Corollary 3.23 to give a bound for the p-length of the whole group in terms of that of the factors. Before getting into the matter we need to recall some known facts about linear groups. If G is any group, we denote by $S = S(G)$ the *soluble radical* of G, i.e., the product of all its normal soluble subgroups. It is well known that if G is linear, then S is a soluble subgroup of G which is closed in the Zariski topology and contains all soluble ascendant subgroups of G (see for instance Lemma 2.11 of [11]); moreover, if G is periodic linear, then G/S contains a normal subgroup B/S which is the direct product of finitely many simple non-abelian groups, and the index $|G : B|$ is finite, see 5.1.5 and Theorem 5.1.6 of [18], or [11], Lemma 2.15. We denote by $u(G)$ the *unipotent radical* of a linear group G, i.e., its largest unipotent normal subgroup. If G is periodic linear of characteristic p, then $u(G) = \{1\}$ if $p = 0$, and $u(G) = O_p(G)$ if $p > 0$. It has been proved in [15] that if G is a periodic group with trivial unipotent radical, then all its homomorphic images are linear groups; observe that if G is linear, then also $G/u(G)$ is linear, see for instance Lemma 2.13 of [11].

Now, we start by generalizing Theorem B of [12] to homomorphic images of periodic linear groups.

Theorem 2.1. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. Then $A \cap B$, A' and B' are subnormal subgroups of G.*

Proof. Let N be a normal subgroup of a periodic linear group H such that $G = H/N$. With a slight abuse of notation, we may assume that $H/N = (A/N) \cdot (B/N)$ is factorized by the two mutually permutable subgroups A/N and B/N. Then it is clear that $H = AB$ is factorized by the two mutually permutable subgroups A and B, so Theorem B of [12] yields that $A \cap B$, A' and B' are subnormal subgroups of H. Then $(A/N)' = A'N/N$ and $(B/N)' = B'N/N$ are subnormal subgroups of G. Moreover, since $N \leqslant A \cap B$, it follows that also $(A/N) \cap (B/N) = (A \cap B)/N$ is a subnormal subgroup of G. The statement is proved. □

The following lemmas are certainly well known and the first one of them should be seen in comparison with Corollary 9.21 of [20], but we did not find them explicitly stated anywhere and so we state (and prove) them here.

Lemma 2.2. *Let G be a homomorphic image of a periodic linear group. Then every locally nilpotent subgroup is hypercentral.*

Proof. Let N be a normal subgroup of a periodic linear group such that $G = H/N$. Let X/N be a locally nilpotent subgroup of H/N. Then X/N is the direct product of its primary components. Let $P/N = O_p(X/N)$ for a prime p. By Theorem 9.20 of [20] there exists a p-subgroup L of H such that $LN = P$. Of course, L is a locally nilpotent linear group, so it is hypercentral. The arbitrariness of p yields that X/N is hypercentral. □

If G is a group, we denote by $\varrho^*_{\mathfrak{N}}(G)$ the nilpotent residual of G. It has been proved in [11], Corollary 3.9, that if G is a homomorphic image of a periodic linear group, then there exists a positive integer c for which $\varrho^*_{\mathfrak{N}}(G) = \gamma_c(G)$.

Lemma 2.3. *Let G be a homomorphic image of a periodic linear group. If $N \leqslant \zeta_\omega(G)$, then $\gamma_c(G) \leqslant C_G(N)$ for some c. In particular, $C_G(N)$ is subnormal in G.*

Proof. As we remarked above, $\gamma_c(G) = \gamma_{c+1}(G)$ for a positive integer c. Moreover, it is well known that $[\gamma_m(G), \zeta_m(G)] = \{1\}$ for any positive integer m, so $[\gamma_c(G), \zeta_\omega(G)] = \{1\}$. This completes the proof of the statement. □

Lemma 2.4. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. Then $(A \cap B)^G/(A \cap B)_G$ is hypercentral.*

Proof. Let $X = A \cap B$, so X is a permutable subgroup of both A and B, see for instance Lemma 3.1 of [12]. Theorem 7 of [7] yields that both X/X_A and X/X_B are hypercentral, so also $X/X_A \cap X_B$ is hypercentral. On the other hand,

$$X_A = (A \cap B)_A = A \cap B_A = A \cap B_G \quad \text{and} \quad X_B = (A \cap B)_B = A_B \cap B = A_G \cap B,$$

so $X_A \cap X_B = A_G \cap B_G = X_G$ and hence, X/X_G is hypercentral. By Theorem 2.1, we also have that X is a subnormal subgroup of G, so X^G/X_G is locally nilpotent, and even hypercentral by Lemma 2.2. □

It is proved in the recent paper in preparation[1] that if X is a Zariski closed permutable subgroup of the linear group G, then X^G/X_G is nilpotently embedded in G, so if we require in the above lemma that $A \cap B$ is Zariski closed, then (using similar arguments but replacing Theorem 7 of [7] by the result we just quoted) we see that $(A \cap B)^G/(A \cap B)_G$ is nilpotent. Of course, $A \cap B$ is Zariski closed in particular in the following cases:
(1) both A and B are Zariski closed;
(2) $A \cap B$ is finite.

[1] F. de Giovanni, M. Trombetti, B. A. F. Wehrfritz: Permutability in linear groups

Recall that two subgroups A and B of a group G are said to be *totally permutable* if $HK = KH$ whenever $H \leqslant A$ and $K \leqslant B$. Of course, totally permutable subgroups are mutually permutable. It has been proved in [3] that the product of two finite totally permutable supersoluble subgroups is supersoluble, so any locally finite group which is factorized by locally supersoluble subgroups is locally supersoluble.

Now, we are ready to study the behaviour of chief factors of periodic linear groups with respect to mutually permutable subgroups factorizing the whole group.

Theorem 2.5. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If N is a non-abelian minimal normal subgroup of G, then either $N \leqslant A$ or $N \cap A = \{1\}$. Moreover, $N = (N \cap A)(N \cap B)$.*

P r o o f. Lemma 2.1 of [12] yields that $X = (N \cap A)(N \cap B)$ is a normal subgroup of G, so either $X = N$ or $X = \{1\}$. Moreover, if $N \cap A = N \cap B = \{1\}$, then $\langle A', B' \rangle \leqslant C_G(N)$ (recall that A' and B' are subnormal in G by Theorem 2.1), so $G/C_G(N)$ is soluble (actually metabelian by Ito's theorem) and hence $N \leqslant C_G(N)$, a contradiction. Thus, the latter part of the theorem follows from the former one.

Assume $\{1\} \neq N \cap A \neq N$, so $X = N$. Theorem 2.1 shows that $A \cap B$ is subnormal in G, so if $A \cap B \cap N$ were not trivial, then $A \cap B \cap N$ would be a locally nilpotent proper subnormal subgroup of N, see Lemma 2.4. The minimality of N gives that N is locally nilpotent and so hypercentral (by Lemma 2.2) and even abelian, a contradiction. Thus, $A \cap B \cap N = \{1\}$. Now, Lemmas 2.1 and 3.5 of [12] yield that N is factorized by the two totally permutable subgroups $N \cap A$ and $N \cap B$.

Now, $R_A = \varrho^*_{\mathfrak{N}}(N \cap A)$ centralizes $N \cap B$ by Theorem 1 of [5], so R_A is a normal subgroup of N; moreover, $(N \cap A)/R_A$ is nilpotent, see [11], Theorem 3.8. Similarly, $R_B = \varrho^*_{\mathfrak{N}}(N \cap B)$ is normal in N, and $(N \cap B)/R_B$ is nilpotent. Then $N/R_A R_B$ is a product of two nilpotent subgroups which are totally permutable, so it is locally soluble by Corollary 2.4 of [12] and hence even soluble by Corollary 9.21 of [20]. The minimality of N yields that $N = R_A R_B = R_A \times R_B = (N \cap A) \times (N \cap B)$.

Let C be any abelian subgroup of $N \cap B$. Then

$$AC \cap N = (A \cap N) \times C \quad \text{and} \quad C = \zeta_1(AC \cap N),$$

so C is a normal subgroup of AC, and A induces in $N \cap B$ power automorphisms by conjugation, so $N \cap B$ is normal in G and hence either $N \cap B = \{1\}$ or $N \cap B = N$. If $N \cap B = \{1\}$, then $N \cap A = N$, a contradiction, and, on the other hand, if $N \leqslant B$, then $\{1\} = A \cap B \cap N = A \cap N$, again a contradiction. The statement is proved. □

Corollary 2.6. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If N is the largest perfect normal subgroup of A, then N is normal in G.*

P r o o f. Let $L = N^G$. Since L is generated by conjugates of N, it follows that $L' = L$. Assume by contradiction that $A' \cap L$ is properly contained in L. Since A' is subnormal in G by Theorem 2.1, the structure of periodic linear groups implies the existence of a (non-abelian) chief factor L/M of G such that $(A' \cap L)^L \leqslant M$. Now, NM/M is non-abelian, so Theorem 2.5 gives $L \leqslant AM$ and hence even $L \leqslant A'M$ since $L/L \cap A'M$ is abelian. On the other hand,

$$L = A'M \cap L = (A' \cap L)M = M$$

and we obtain a contradiction. Thus, $L \leqslant A'$ and so $L = N$, as we wanted. □

Theorem 2.7. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If N is a minimal normal subgroup of G, then either $N \leqslant A$ or $N \cap A = \{1\}$.*

P r o o f. By Theorem 2.5, we may assume N is abelian of prime exponent p, and $\{1\} \neq N \cap A \neq N$, so (as in the proof of Theorem 2.5) $N = (N \cap A)(N \cap B)$ is factorized by the two mutually permutable subgroups $N \cap A$ and $N \cap B$. Of course, if $N \cap B = \{1\}$, then $N = N \cap A$, so $N \cap B \neq \{1\}$.

Put $X = A \cap B$, so $N \nleqslant X$ and in particular, $N \cap X_G = \{1\}$. Lemma 2.4 yields that X^G/X_G is hypercentral. Thus, the minimality of N shows that either $NX_G \cap X^G = X_G$ or $NX_G/X_G \leqslant \zeta_1(X^G/X_G)$. In any case $[N, X] \leqslant N \cap X_G = \{1\}$, so $X \leqslant C = C_G(N)$.

Since $X \leqslant C$, it follows from Lemma 4 of [9] that G/C is factorized by the two totally permutable subgroups AC/C and BC/C. Theorem 1 of [5] yields that A^*C/C centralizes B/C, where $A^* = \varrho^*_{\mathfrak{N}}(A)$; in particular, A^*C is normal in G. Now, $N \cap A$ is normalized by A and N (which is abelian), and

$$A/(N \cap A) \ltimes N/(N \cap A) = A/(N \cap A) \ltimes (N \cap B)(N \cap A)/(N \cap A),$$

so

$$\langle b\rangle(N \cap A)/(N \cap A) = (A\langle b\rangle/(N \cap A)) \cap (N/(N \cap A))$$

is normalized by A for any $b \in N \cap B$. Therefore, A acts as a group of power automorphisms on $N/N \cap A$; in particular, $[N, A'A^{p-1}] \leqslant N \cap A$. Symmetrically, $[N, B'B^{p-1}] \leqslant N \cap B$.

It follows from $N \cap A < N$ that any normal subgroup of G contained in $[N, A'A^{p-1}]$ must be trivial; in particular, $[A^*C, N] = [A^*, N]$ is trivial and so $A^* \leqslant C$, which means that AC/C is nilpotent. In order to obtain the same conclusion for the normal subgroups of G which are contained in $[N, B'B^{p-1}]$, we must show that $N \cap B < N$.

Suppose $N \leqslant B$. As we know, X is permutable in B (see Lemma 3.1 of [12]), so $X/X_B \leqslant \zeta_\omega(B/X_B)$ by Theorem 7 of [7]. Since $X_B = A_G \cap B$ and $N \leqslant B$, we have that $N \cap X = N \cap A$ is contained in $\zeta_\omega(B)$. In particular, if Q is the subgroup of B generated by all its p'-elements, then $Q \leqslant C_B(N \cap A)$. Since $(N \cap A)^G = (N \cap A)^B = N$, it follows that $Q \leqslant C_B(N)$, so $B/C_B(N)$ is a p-group. Now, BC/C is locally nilpotent, while AC/C is nilpotent, so G/C is locally supersoluble, see [3].

Let W/C be the subgroup generated by all q-elements of AC/C, where $q > p$. Then W/C is normalized by BC/C since G/C is locally supersoluble, and so W is a normal subgroup of G which is contained in AC/C. Now, $[W, N] \leqslant [A^{p-1}, N]$ and so $W \leqslant C$. Therefore, $\pi(G/C) \subseteq \{1, \ldots, p\}$.

Let V/C be the largest p-subgroup of G/C. Then there is a p-subgroup P of V such that $V = PC$, see Theorem 9.20 of [20]. But then PN is hypercentral and so $N \leqslant \zeta_1(PN)$ by minimality of N (note that PN is a normal subgroup of G), which means $P \leqslant C$. Thus, actually, $\pi(G/C) \subseteq \{1, \ldots, p-1\}$, and in particular, $B \leqslant C$. Then $N \cap A$ is a normal subgroup of G and this contradicts the minimality of N.

We have thus proved that $N \cap B$ is properly contained in N, and so, as we remarked above, every normal subgroup of G which is contained in $[N, B'B^{p-1}]$ must be trivial; in particular, the nilpotent residual of B is contained in C, which means that BC/C is nilpotent. It follows that G/C is locally supersoluble, see [3].

Let $K/C = O_\pi(G/C)$, where $\pi = \{2, \ldots, p\}$, and assume $K < G$. Theorem 3 of [5] yields that either AK/K or BK/K contains a nontrivial normal subgroup of G/K. Without loss of generality we may assume that L/K is a nontrivial normal subgroup of G/K which is contained in AK/K. Of course, we may assume $\pi(L/K) \subseteq \pi'$. Then there is a π'-subgroup P/C of AC/C such that $PK = L$. Since G/C is locally supersoluble, we have that $L/C = P/C \times K/C$ and so P/C is normal in G/C. Moreover, $P \leqslant A^{p-1}C$, so $[P, N]$ is a G-invariant subgroup of $[A^{p-1}, N]$, and hence $P \leqslant C$. It follows that $K = G$ and consequently $\pi(G/C) \subseteq \pi$.

Finally, observe that $N \cap A$ is not normal in G, so there is an element $x \in B$ such that $(N \cap A)^x \neq N \cap A$. Since $\pi(G/C) \subseteq \pi$, we can choose x of prime power order q^n for a prime $q < p$. Clearly,

$$(N \cap A)^x \leqslant N \cap (\langle x \rangle A) = (N \cap (\langle x \rangle A))^x.$$

Let Y be a Sylow p-subgroup of A. Then Y is a Sylow p-subgroup of $\langle x \rangle A$, and so

$$N \cap A = N \cap Y = N \cap (\langle x \rangle A).$$

Then $(N \cap A)^x \leqslant (N \cap A)$ and so even (since x has finite order) $(N \cap A)^x = (N \cap A)$, the final contradiction. □

Theorem 2.8. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. Let N be a minimal normal subgroup of G which is contained in A and such that $N \cap B = \{1\}$. Then either $[N, A] = \{1\}$ or $[N, B] = \{1\}$. Moreover, if N is non-cyclic, then $[N, B] = \{1\}$.*

P r o o f. Let K be any subgroup of N. Then $KB = BK$ and so $N \cap KB = K(N \cap B) = K$ is normalized by KB; in particular, K is normalized by B. Thus, B induces power automorphisms by conjugation on N.

We immediately note that if A centralizes N, then every cyclic subgroup of N is normal in G and so N must be cyclic. It follows that if N is not cyclic, then A does not centralize N, so the second part of the statement follows from the first part.

Assume first that N is non-abelian. Then N is the direct product of locally finite simple groups, see the introduction of this section. Since $[N, B] \leqslant \zeta_1(N) = \{1\}$ (see [17], Theorem 1.5.2), the statement is proved in this case.

Assume now N is an elementary abelian p-group for a prime number p and $A \nleqslant C = C_G(N)$. Then $B/C_B(N)$ is a nontrivial cyclic group of order dividing $p - 1$, and we may therefore choose $b \in B$ such that $B = \langle b \rangle C_B(N)$. Suppose that AC/C is a p-group. Then G/C is locally supersoluble (being the product of two totally permutable locally nilpotent subgroups) and so AC/C is a normal p-group. Let P be any p-subgroup of A such that $AC = PC$, see [20], Theorem 9.20. It follows that NP is hypercentral, so $N \cap \zeta_1(AC)$ is not trivial and hence $N \leqslant \zeta_1(AC)$, which means that $A \leqslant C$, a contradiction. Therefore, AC/C admits a nontrivial element of prime order $q \neq p$. Let y be a q-element in $A \setminus C_A(N)$ such that $y^q \in C_A(N)$.

Let $x \in (NB \cap \langle y \rangle B) \setminus B$ and take $b_1, b_2 \in B$, $y_1 \in \langle y \rangle$, $g \in N$ with $x = y_1 b_1 = g b_2$. Put $S = \langle x, B \rangle = \langle y_1 \rangle B = \langle g \rangle B$. Clearly,

$$|\langle g \rangle B : B| = |\langle g \rangle| \quad \text{and} \quad |\langle y_1 \rangle B : B| = |\langle y_1 \rangle : B \cap \langle y_1 \rangle|,$$

so $|S : B|$ is at the same time a p-number and a q-number, which means $x \in S = B$. In particular, $NB \cap \langle y \rangle B = B$ and so $N \cap \langle y \rangle B = \{1\}$.

Since $N \cap B = \{1\}$, we have $N \nleqslant X = A \cap B$ and in particular, $N \cap X_G = \{1\}$. Lemma 2.4 yields that X^G/X_G is hypercentral. Thus, the minimality of N shows that either $NX_G \cap X^G = X_G$ or $NX_G/X_G \leqslant \zeta_1(X^G/X_G)$. In any case, $[N, X] \leqslant N \cap X_G = \{1\}$, so $X \leqslant C$, and consequently y does not belong to B.

Now, Lemma 2.1 of [12] yields that $W = \langle y \rangle B$ is factorized by the mutually permutable subgroups $\langle y \rangle (A \cap B)$ and B. The same lemma applied to $C_W(N)$ shows that $V = \langle y^q \rangle C_B(N)$ is a normal subgroup of W. Since V contains $A \cap B$, it follows that W/V is factorized by the totally permutable subgroups $\langle y \rangle V/V$ and BV/V, see for instance Lemma 4 of [9]. Moreover, $(\langle y \rangle C_B(N)) \cap (\langle y^q \rangle B) = V$. Thus, W/V is locally supersoluble (see [3]) and $\langle y\rangle V/V$ is normalized by all r-subgroups of BV/V, where r is a prime with $r < q$. Similarly, every r-subgroup of BV/V with $r > q$ is normalized by y. Moreover, if Q/V is the Sylow q-subgroup of BV/V, then Q/V has index q in $\langle y\rangle Q/V$, so Q/V is normalized by y.

Direct computation shows that $[b, y]$ centralizes N for any $b \in B$. Thus, $\langle y\rangle V/V$ is actually centralized by all r-subgroups of BV/V when r is a prime such that $r < q$; and every r-subgroup of BV/V with $r \geqslant q$ is centralized by y. It follows that W/V is abelian, so $[B, y] \leqslant V$.

Choose $u \in N$ such that $[y, u] \neq 1$, and $b \in B$. Replacing y by y^u (recall that $N \leqslant A$) we see that $[b, y^u]$ lies in V (note that $(y^u)^q = y^q$). Clearly,

$$[b, y^u] = [b, y[y, u]] = [b, [y, u]][b, y],$$

and $\langle [b, y^u], [b, y]\rangle \leqslant V$. Therefore, $[b, [y, u]]$ lies in $V \cap N = \{1\}$. On the other hand, B acts as a power automorphism on N and $[y, u]$ is not trivial, so actually $[b, N] = \{1\}$ and hence $[B, N] = \{1\}$. □

Theorem 2.9. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. Let N be a minimal normal subgroup of G such that $N \cap A = N \cap B = \{1\}$. Then $|N| = p$ for a prime p, and either $[N, A] = \{1\}$ or $[N, B] = \{1\}$.*

Proof. Let $\widehat{G}$ be a periodic linear group of characteristic q and degree n, and $\widehat{H}$ a normal subgroup of $\widehat{G}$ such that $G = \widehat{G}/\widehat{H}$. By Theorem 2.5, N is abelian and consequently an elementary abelian p-group for a prime p.

Suppose first N is finite and the statement is false. In this case we may find a finite subgroup F of G which is factorized by the mutually permutable subgroups $F \cap A$ and $F \cap B$, and is such that N is a minimal normal subgroup of F, see for instance Lemma 2.1 of [12]. Then Lemma 2 of [6] shows that $|N| = p$, so there must be elements $a \in A$ and $b \in B$ such that $[a, N] \neq \{1\} \neq [b, N]$. But then we can choose F containing a and b, obtaining a contradiction again by Lemma 2 of [6]. Thus, the statement is true whenever N is finite.

The structure of periodic linear groups we have outlined above shows that N is finite whenever $q = 0$ or q is a prime distinct from p, so in the following we may assume that $q = p$ is a prime and N is infinite.

Let F be a finite subgroup of G such that $F = (F \cap A)(F \cap B)$ is factorized by the two mutually permutable subgroups $F \cap A$ and $F \cap B$, see for instance Lemma 2.1 of [12]. Let N_1 be a minimal normal subgroup of F contained in $N \cap F$. It follows from Lemma 2 of [6] that $|N_1| = p$. Repeating this argument we find an F-invariant series

$$\{1\} = N_0 \leqslant N_1 \leqslant N_2 \leqslant \ldots \leqslant N_t = N \cap F$$

such that N_{i+1}/N_i is cyclic of order p. Let $C = \bigcap_i C_F(N_{i+1}/N_i)$, so F/C has exponent $p-1$. Now, $C/C_F(N)$ is nilpotent (stabilizing a finite series) and hence is a p-group. A combination of 2.6 and Theorem 9.20 of [20] yields that $C/C_F(N)$ has exponent at most p^e, where e is such that $p^{e-1} < n \leqslant p^e$. Therefore, $(F^{p-1})^{p^e} \leqslant C_F(N)$ and the arbitrariness of F gives that $G^{p-1}C_G(N)/C_G(N)$ is a p-group (of exponent at most p^e).

Let P be a p-subgroup of G such that $G^{p-1}C_G(N) = PC_G(N)$. Then NP is a p-group, so hypercentral (see Lemma 2.2) and hence $N \cap \zeta_1(NP) \neq \{1\}$. Thus,

$$N \cap \zeta_1(G^{p-1}) \neq \{1\}$$

and so $N \leqslant \zeta_1(G^{p-1})$, which means $G^{p-1} \leqslant C_G(N)$.

Now, G/G^{p-1} is a locally finite group satisfying the minimal condition on abelian subgroups, because otherwise if Q/G^{p-1} is a q-subgroup of G/G^{p-1}, then Q_1 is Černikov (see [20], 2.6 and Theorem 9.20 of [20]) and so even finite. A well known theorem of Šunkov shows that G/G^{p-1} is Černikov and consequently finite. Therefore, $G/C_G(N)$ is finite and so also N is finite, the final contradiction. □

3. Primary localizations

In this section we study groups which are factorized by mutually permutable subgroups satisfying certain "primary localizations" of some well known concepts.

Recall that if p is any prime, the *upper p-series* of G is the normal series $\{\sigma^p_\alpha(G)\}_\alpha$ defined recursively as follows: $\sigma^p_0(G) = \{1\}$; if λ is a successor ordinal, write $\lambda = \mu + n$ for a limit ordinal μ and a positive integer n, and put $\sigma^p_\lambda(G)/\sigma^p_{\lambda-1}(G) = O_\pi(G/\sigma^p_{\lambda-1}(G))$, where $\pi = \{q \in \mathbb{P}\colon q \neq p\}$ if n is odd, and $\pi = \{p\}$ otherwise. We denote by $\sigma^p(G)$ the last term of the upper p-series of G, and we say that G is *p-hyperabelian* if $G = \sigma^p(G)$. The group G is called *p-soluble* if it coincides with a finite term of the upper p-series; in this case, the number of nontrivial factors of $\{\sigma^p_\alpha(G)\}_\alpha$ that are p-groups is called the *p-length* of G, see [14]. Furthermore, G is *p-supersoluble* (*p-hypercyclic*) if it is p-soluble (p-hyperabelian) and the nontrivial p-factors of its upper p-series admit a finite G-invariant series (an ascending G-invariant series) with cyclic factors, see [6]. Finally, recall that G is called *p-nilpotent* if $G = O_{p'p}(G) = \sigma^p_2(G)$; of course, $O_{p'p}(G)$ is the largest p-nilpotent normal subgroup of a locally finite group G.

A couple of terminology remarks:

- ▷ We could have defined *p-nilpotency* by requiring that the factor group $G/O_{p'}(G)$ were a nilpotent p-group, *p-hypercentrality* by requiring that $G/O_{p'}(G)$ were a hypercentral p-group, and *p-(local nilpotency)* by requiring that $G/O_{p'}(G)$ were a locally finite p-group. With respect to these definitions, p-nilpotent (p-hypercentral) means the group is p-soluble and the p-factors of its upper p-series are nilpotently (hypercentrally) embedded in the group; this is proved using arguments which are very similar to those employed in the proof of Lemma 3.4. Although this would have been more consistent with the previous definitions and would have certainly give some other piece of information (e.g., in the statement of Lemma 3.4 we could have derived p-hypercentrality in general, and p-nilpotency for p-supersoluble groups), we have decided not to complicate this more than necessarily since our main results do not deal directly with these concepts, and the definition of p-nilpotency we gave is essentially established also for infinite groups. For similar reasons we did not introduce the concept of *p-(locally supersoluble)* group.

▷ As the reader certainly noted, the definition of p-soluble (p-hyperabelian) does not require that the p-factors are soluble (hyperabelianly embedded in G), differently from the other local definitions. This is because our main results deal with homomorphic images of linear groups, and, for such groups, the properties of being hyperabelian/locally soluble/soluble are equivalent.

Before studying groups which are factorized by mutually permutable subgroups, we need some lemmas, which at the same time illustrate the relation between the "p-local" and "non p-local" concepts in homomorphic images of periodic linear groups, and, more generally, in locally finite groups.

Lemma 3.1. *Let G be a locally finite group and let p be a prime.*

(1) *If G is p-hyperabelian, then G is locally p-soluble.*
(2) *If G is p-hypercyclic, then G is locally p-supersoluble.*
(3) *If H is an ascendant subgroup of G such that $H = \sigma^p_\alpha(H)$ for an ordinal number α, then $H \leqslant \sigma^p_\alpha(G)$. In particular, the product of any collection of ascendant subgroups of p-length at most n has p-length at most n.*
(4) *G is locally p-nilpotent if and only if G is p-nilpotent.*

Proof. The proof of this result is standard and we omit it. □

It is easy to see that the wreath product

$$\ldots \wr C_3 \wr C_2 \wr \ldots \wr C_2 \wr C_3 \wr C_2$$

is a locally finite group which is locally soluble but is not 2-hyperabelian: in fact it does not contain any nontrivial normal locally nilpotent subgroup. On the other hand, it is known that a locally soluble homomorphic image of a periodic linear group is soluble (see [20], Corollary 9.21) and our next result extends this fact to local p-solubility.

Lemma 3.2. *Let p be a prime and let G be a homomorphic image of a periodic linear group. If G is locally p-soluble, then G is p-hyperabelian.*

Proof. Suppose first that G is a non-abelian locally finite simple linear group. If G does not contain elements of order p, then G is p-hyperabelian, and we are done. If g is a nontrivial element of G having order a power of p, it follows that g is contained in a finite simple subgroup H of G having order strictly larger than p, see for instance Corollary 9.32 of [20]. On the other hand, H is p-soluble, so it must be a p-group and its order must be p, a contradiction.

Assume now that G is a homomorphic image of a periodic linear group. The structure of a periodic linear group (we outlined at the beginning of Section 2) and the previous paragraph show that G is soluble-by-finite and hence p-hyperabelian. □

It is clear that soluble (or even hyperabelian) locally finite groups are p-hyperabelian for any prime p. On the other hand, the consideration of any locally finite p-group which is not hyperabelian shows that the converse of the statement does not hold in general. The situation is much better within the universe of periodic linear groups.

Lemma 3.3. *Let G be a homomorphic image of a periodic linear group which is p-hyperabelian for any prime p. Then G is soluble.*

Proof. It is easily seen that every finite subgroup of G is soluble, so G is locally soluble and hence soluble, see Corollary 9.21 of [20]. □

Lemma 3.4. *Let p be a prime and let G be a locally finite p-hypercyclic group. Then G' is p-nilpotent.*

Proof. It is clear that G' centralizes every chief factor of G whose order is p. Let $N = O_{p'}(G')$ and assume $N < G'$. Then there are G-invariant subgroups $M \leqslant L$ of G' such that M/N is a nontrivial p-group which is hypercyclically embedded in G, while L/M is a p'-group; in particular, M/N lies in the hypercentre of G'/N and so in the hypercentre of L/N. Let F/M be any finite subgroup of L/M. The main theorem of [8] shows that there is a finite normal subgroup U/N of F/N such that F/U is hypercentral. Now, there exists n such that $P/N = O_p(U/N) \leqslant \zeta_n(U/N)$ and U/P is a p'-group, so $\gamma_{n+1}(U)N/N$ is a finite p'-group by Baer's theorem, see Corollary 2 to Theorem 4.21 of [16]. If we put $V/N = O_{p'}(U/N)$, then $(U/N)/(V/N)$ is a p-group, and hence is contained in the hypercentre of $(F/N)/(V/N)$. Therefore, F/V is hypercentral and the p'-elements of F/N form a subgroup. It follows that the set of all p'-elements of L/N is a subgroup, so L/N is a p-group and hence $L = M$. Thus, G'/N must be a p-group and the statement is proved. □

Corollary 3.5. *Let p be a prime and let G be a locally finite p-hypercyclic group. Then G is p-soluble of p-length at most 1.*

Clearly, any supersoluble (hypercyclic) locally finite group is p-supersoluble (p-hypercyclic) for any prime p. On the other hand, the consideration of any periodic group of rank one with infinitely many finite nontrivial Sylow subgroups is an example of a group which is not supersoluble but it is p-supersoluble for any prime p.

Lemma 3.6. *Let G be a locally finite group. If G is p-hypercyclic for any prime p, then G is hypercyclic.*

Proof. It follows from Lemma 3.4 that G' is p-nilpotent for any prime p. Since G' is periodic, it embeds in the direct product $\mathrm{Dr}_p(G'/O_{p'}(G'))$, so G admits an ascending normal series with primary factors. But any of the factors of this series is hypercyclically embedded in G, and hence G is hypercyclic. □

Lemma 3.7. *Let p be a prime and let G be a homomorphic image of a periodic linear group. If N is any normal p-nilpotent subgroup of G, then N centralizes every p-chief factor of G.*

Proof. Let L/M be a p-chief factor of G. Of course we may assume $M = \{1\}$, so L is a minimal normal subgroup of G. Let $K = O_{p'}(N)$, then $K \cap L = \{1\}$, so $K \leqslant C_G(L)$ and we may also assume $K = \{1\}$. Now, N is a p-group, so LN is a hypercentral normal subgroup of G by Lemma 2.2. It follows that $L \leqslant \zeta_1(LN)$ and hence $N \leqslant C_G(L)$. The statement is proved. □

Lemma 3.8. *Let p be a prime and let G be a locally finite group. If X is any subgroup of G centralizing every p-chief factor of G, then X is p-nilpotent.*

Proof. Let F be any finite subgroup of X and let $\{G_i\}_{i\in I}$ be any chief series of G. Then $\{F \cap G_i\}_{i\in I}$ is a finite series of F (once we have removed all duplicates) in which the p-factors are central. An argument similar to the one employed in Lemma 3.4 shows that F is p-nilpotent, so its p'-elements form a subgroup. The arbitrariness of F in X yields that the p'-elements of X form a subgroup and consequently that X is p-nilpotent. □

A combination of Lemmas 3.7 and 3.8 gives the following result, which is analogous to the finite case.

Corollary 3.9. *Let p be a prime and let G be a homomorphic image of a periodic linear group. Then $O_{p'p}(G)$ coincides with the intersection of the centralizers of the p-chief factors of G.*

The consideration of any locally finite p-group which is not hypercyclic shows that there exist locally p-supersoluble groups which are not p-hypercyclic. On the other hand, it is known that a linear group which is locally supersoluble is also hypercyclic (see for instance Theorem 11.21 of [20]), and our next result generalizes this fact to homomorphic images of periodic linear groups.

Theorem 3.10. *Let p be a prime and let G be a homomorphic image of a periodic linear group. If G is locally p-supersoluble, then G is p-hypercyclic.*

P r o o f. Let $\widehat{G}$ be a periodic linear group of characteristic q such that $G = \widehat{G}/\widehat{N}$ for a normal subgroup $\widehat{N}$ of $\widehat{G}$.

Of course, we may assume $O_{p'}(G) = \{1\}$. Let $P = O_p(G)$. Lemma 3.4 shows that G' is locally p-nilpotent, so Lemma 3.1 yields that G' is p-nilpotent, and hence that G' is a p-group; in particular, $G' \leqslant P$ and G/P is a p'-group. In order to complete the proof it is enough to prove that P contains a nontrivial G-invariant cyclic subgroup.

By Lemma 2.2, P is hypercentral. Let Z be the socle of $\zeta_1(P)$ and choose any finite subgroup F of G such that $F \cap Z \neq \{1\}$. Since G is locally p-supersoluble, we may find an F-invariant series

$$\{1\} = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_s = Z \cap F$$

with cyclic factors of order p. Let

$$C = \bigcap_{0 \leqslant i \leqslant s-1} C_F(A_{i+1}/A_i).$$

Then $C/C_F(Z \cap F)$ is nilpotent (stabilizing a finite series) and so a p-group. On the other hand, every p-element of G centralizes Z, so actually $C = C_F(Z \cap F)$. Of course, $F^{p-1} \leqslant C$, so the arbitrariness of F in G yields that $G^{p-1} \leqslant C_G(Z)$.

If $q = p$, then $G/C_G(Z)$ is a locally a finite group satisfying the minimal condition on abelian subgroups (see [20], 2.6 and Theorem 9.20), so it is Černikov by a well known theorem of Šunkov (see [16], Part I, page 98) and hence even finite, being of finite exponent. Let E be a finite subgroup of G such that $G = EC_G(Z)$ and $E \cap Z \neq \{1\}$. Then E normalizes a nontrivial cyclic subgroup W of $E \cap Z$, which is thus the required nontrivial G-invariant cyclic subgroup.

If $q \neq p$, then P is Černikov (see [20], 2.6 and Theorem 9.20) and Z is finite. Let $\mathcal{X}$ be the set of all nontrivial cyclic subgroups of Z, and let $\mathcal{F}$ be the set of all finite subgroups of G. For any $X \in \mathcal{X}$, let $\mathcal{F}_X = \{F \in \mathcal{F}\colon\ F \leqslant N_G(X)\}$. Since the set $\mathcal{X}$ is finite, there is $Y \in \mathcal{X}$ such that $\mathcal{F}_Y$ is a local system of G. Thus, X is normal in G and we are done. □

Combining Theorem 3.10 and Lemma 3.6 we have the following result.

Corollary 3.11. *Let G be a homomorphic image of a periodic linear group. If G is locally supersoluble, then G is hypercyclic.*

It is impossible to generalize the last two results to periodic homomorphic images of non-periodic linear groups. In fact, every free group of countable rank is linear but there exist countable locally finite p-groups which are not hypercyclic.

It is time to study groups which are factorized by mutually permutable subgroups which satisfy some "p-local" conditions. Clearly, Sym(3), the symmetric group of degree 3, is factorized by two totally permutable subgroups which are 3-nilpotent, but it is not 3-nilpotent. On the other hand, in Corollary 3.20 we show that the product of two mutually permutable 2-nilpotent subgroups is 2-nilpotent. Moreover, observe that if p is a prime, a group (even finite) which is factorized by two mutually permutable p-hypercyclic groups need not be p-hypercyclic. In fact, if this were the case, then a finite group G which is factorized by two mutually permutable supersoluble subgroups would be q-supersoluble for any prime q, and so even supersoluble by Lemma 3.6; but this is not true as shown by an example in [1], page 454.

Lemma 3.12. *Let p be a prime and let $G = AB$ be a locally finite group which is factorized by two mutually permutable subgroups A and B. If A and B are locally p-soluble, then G is locally p-soluble.*

Proof. By Lemma 2.1 of [12], the set of all finite subgroups F which are factorized by the two subgroups $F \cap A$ and $F \cap B$ is a local system of G, so G is locally p-soluble by Corollary 2 of [6]. □

Corollary 3.13. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are p-hyperabelian (p-soluble), then G is p-hyperabelian (p-soluble).*

Proof. By Lemma 3.12, G is locally p-soluble, so G is p-hyperabelian by Lemma 3.2. If both subgroups A and B are p-soluble, it follows from Theorem 2.1 and Lemma 3.1 that A' and B' lie in a finite term $N = \sigma_n^p(G)$ of the upper p-series of G. Furthermore, G/N is the product of two abelian groups, so it is metabelian by Ito's theorem, and hence $G = \sigma_{n+4}^p(G)$ is p-soluble. □

Corollary 3.14. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are soluble, then G is soluble.*

Proof. This follows from Corollary 3.13 and Lemma 3.3. □

Our main aim is now to give a bound on the p-length of a group which is factorized by two mutually permutable subgroups: in order to do this we start by discussing products of mutually permutable p-hypercyclic subgroups.

Lemma 3.15. *Let p be a prime and let $G = AB$ be a p-hypercyclic locally finite group which is factorized by two mutually permutable subgroups A and B. If A and B are p-supersoluble, then G is p-supersoluble.*

Proof. By Lemma 3.4, G' is p-nilpotent, so we need to prove that $O_p(G/O_{p'}(G))$ is finite. Of course, it is possible to assume $O_{p'}(G) = \{1\}$, and we put $N = O_p(G)$; clearly, N is hypercyclically embedded in G.

Let F be a finite subgroup of G such that $F = (F \cap A)(F \cap B)$, and let p^n be the product of the orders of the p-factors of the upper p-series's of A and B. Since $|F| = |F \cap A||F \cap B|/|F \cap A \cap B|$, it follows that $|N \cap F|$ divides p^n. The arbitrariness of F yields that N is finite and the statement is proved. □

Lemma 3.16. *Let p be a prime and let $G = AB$ be a locally finite group which is factorized by two mutually permutable subgroups A and B. If A and B are locally p-supersoluble, then G is locally p-supersoluble provided that at least one among the subgroups A, B, G' is p-nilpotent.*

Proof. This follows from a combination of Corollary 5 of [6] and the local argument we employed in Theorem 2.9. □

As a consequence of Lemmas 3.16 and 3.6, we have the following result.

Lemma 3.17. *Let $G = AB$ be a locally finite group which is factorized by two mutually permutable locally supersoluble subgroups A and B. If G' is locally nilpotent, then G is locally supersoluble.*

Theorem 3.18. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are p-hypercyclic (p-supersoluble), then G is p-hypercyclic (p-supersoluble) provided that at least one among the subgroups A, B, G' is p-nilpotent.*

P r o o f. By Lemma 3.1, A and B are locally p-supersoluble subgroups, so Lemma 3.16 yields that G is locally p-supersoluble. Thus, G is p-hypercyclic by Theorem 3.10. In the case that A and B are p-supersoluble, the statement follows from Lemma 3.15. □

Although the product $G = AB$ of two mutually permutable p-hypercyclic locally finite subgroups A and B need not be p-hypercyclic, the following result shows that we can at least bound the p-length of G.

Theorem 3.19. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are p-hypercyclic, then $G/O_{p'p}(G)$ is metabelian of exponent dividing $(p-1)^3$.*

P r o o f. Let L/M be a p-chief factor of G, and put $C = C_G(L/M)$. In order to prove that G/C is abelian of exponent dividing $(p-1)^3$, we may assume $M = \{1\}$, so L is a minimal normal subgroup of G and it is a p-group.

If L is cyclic, it has order p, so G/C is abelian of order dividing $p-1$, and we are done. Thus, we may assume that L is not cyclic. In particular, we cannot have $L \leqslant A$ and $B \leqslant C$, or, similarly, $L \leqslant B$ and $A \leqslant C$. In fact, if for instance $L \leqslant A$ and $B \leqslant C$, then since $G = AB$, the subgroup L is a minimal normal subgroup of A, and so is cyclic since A is p-hypercyclic.

Now, Theorem 2.7 shows that $\{L \cap A, L \cap B\} \subseteq \{L, \{1\}\}$, while Theorem 2.9 shows that we cannot have $A \cap L = B \cap L = \{1\}$. If $L \leqslant A$ and $L \cap B = \{1\}$, then Theorem 2.8 shows that $B \leqslant C$, and we have already dealt with this case; the case $L \leqslant B$ and $L \cap A = \{1\}$ is similar. Assume $L \leqslant A \cap B$. The argument employed in the last part of the proof of Theorem 2.9 yields that $A^{p-1}C_A(L)/C_A(L)$ and $B^{p-1}C_B(L)/C_B(L)$ are p-groups, so a local application of Theorem 2 of [6] gives that G/C admits a normal Sylow p-subgroup P/C. But then P centralizes L (see Lemma 2.2 of this paper and Theorem 9.20 of [20]) and so $P = C$. Moreover, A' and B' are p-nilpotent subnormal subgroups of G (see Theorem 2.1 and Lemma 3.4), so they are contained in C by Lemma 3.7. Since G/C is factorized by the two mutually permutable abelian subgroups AC/C and BC/C, both of the exponent dividing $p-1$, it follows that G/C is metabelian (by Ito's theorem) and has the exponent dividing $(p-1)^3$: in fact, $G/(AC \cap BC)$ is factorized by two totally permutable abelian subgroups of exponent $p-1$ and so has the exponent dividing $(p-1)^2$.

In order to complete the proof it is enough to apply Corollary 3.9. □

It should be noted that in the proof of Theorem 4 (i) of [6] (which is analogous to our Theorem 3.19), the case $L \leqslant A$ and $L \cap B = \{1\}$ seems to be missing. It is not possible to replace "metabelian" by "abelian" in the above statement: if this were possible, Theorem 3.18 would show that any group which is the product of two mutually permutable finite supersoluble subgroups is supersoluble, but we have already observed that this is not true.

Corollary 3.20. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are 2-nilpotent, then G is 2-nilpotent.*

P r o o f. This follows at once from Lemma 2.2 and Theorem 3.19. □

Corollary 3.21. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are 2-hypercyclic, then G is 2-hypercyclic.*

P r o o f. This follows at once from Theorem 3.19 and Lemma 2.2. □

Corollary 3.22. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B.*

(1) *If A and B are of p-length n, then $G/\sigma^p_{2n+1}(G)$ is hypercyclic and metabelian.*
(2) *If $A/\sigma^p_{2n+1}(A)$ and $B/\sigma^p_{2n+1}(B)$ are p-hypercyclic, then $G = \sigma^p_{2n+3}(G)$.*

P r o o f. The implication (1) is just a combination of Theorem 2.1, Lemma 3.1, Ito's theorem, Lemma 3.17, and Corollary 3.11.

In order to prove the implication (2), note that A/A' is hypercyclic (being abelian) and $A'/\sigma^p_{2n+1}(A')$ is p-hypercyclically embedded in $A/\sigma^p_{2n+1}(A')$. Thus, $A/\sigma^p_{2n+1}(A')$ is p-hypercyclic and symmetrically $B/\sigma^p_{2n+1}(B')$ is p-hypercyclic. Now, it follows from Theorem 2.1 and Lemma 3.1 that

$$\sigma^p_{2n+1}(G) \geqslant \langle \sigma^p_{2n+1}(A'), \sigma^p_{2n+1}(B') \rangle,$$

so $G/\sigma^p_{2n+1}(G)$ is factorized by two mutually permutable p-hypercyclic subgroups, and hence it is of p-length 1 by Theorem 3.19. Therefore, $G = \sigma^p_{2n+3}(G)$ and the statement is proved. □

Corollary 3.23. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are p-soluble of p-length n, then G is p-soluble of p-length $n + 1$.*

In the final part of the paper, we deal with some generalizations of further results of Beidleman and Heineken (see [6]).

Lemma 3.24. *Let p be a prime and let $G = AB$ be a locally finite group which is factorized by two mutually permutable subgroups A and B. If A and B are locally p-supersoluble, then $G/(A \cap B)_G$ is locally p-supersoluble.*

Proof. We may assume $(A \cap B)_G = \{1\}$. Let $\mathcal{F}$ be the set of all finite subgroups of G such that $F = (F \cap A)(F \cap B)$ is factorized by the two mutually permutable subgroups $F \cap A$ and $F \cap B$. As we have already mentioned several times, $\mathcal{F}$ is a local system of G.

Let $E \in \mathcal{F}$. It follows from Theorem 6 of [6] that $E/(E \cap A \cap B)_E$ is p-supersoluble. Thus, the set $\mathcal{P}(E)$ made by all E-invariant subgroups L of $E \cap A \cap B$ such that E/L is p-supersoluble is nonempty. Now, a standard inverse limit argument shows that G admits a normal subgroup N such that $N \leqslant A \cap B$ and G/N is locally p-supersoluble. The statement is proved. □

Theorem 3.25. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are p-hypercyclic (p-supersoluble), then $G/(A \cap B)_G$ is p-hypercyclic (p-supersoluble).*

Proof. By Lemma 3.24, $G/(A \cap B)_G$ is locally p-supersoluble, so even p-hypercyclic by Theorem 3.10. Finally, if A and B are p-supersoluble, it follows from Lemma 3.15 that $G/(A \cap B)_G$ is p-supersoluble. □

Recall that a group is a *T-group* if normality is a transitive relation.

Corollary 3.26. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are p-hypercyclic (p-supersoluble) and B is a T-group, then G is p-hypercyclic (p-supersoluble).*

Proof. Theorem 3.25 yields that G/N is p-hypercyclic, where $N = (A \cap B)_G$. Since B is a T-group, the ascending A-invariant series $\{N \cap \sigma^p_\alpha(A)\}_\alpha$ of N is even G-invariant, and its p-factors are hypercyclically embedded in G. Hence, G is p-hypercyclic. Finally, if A and B are p-supersoluble, it follows from Lemma 3.15 that $G/(A \cap B)_G$ is p-supersoluble. □

Recall that a group G is *paranilpotent* if there exists a finite series

$$\{1\} = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_t = G$$

of normal subgroups of G such that for $i = 0, \ldots, t-1$, the factor group G_{i+1}/G_i is abelian and each of its subgroups is normal in G/G_i (the smallest possible length of a paranilpotent series is called the *paraheight* of G). These groups were usually referred to in literature as *parasoluble groups* (see [19], where these groups were introduced), but here, we prefer to follow [10] and speak of paranilpotent groups. Clearly, paranilpotent groups are hypercyclic and so locally supersoluble. Now, if p is a prime, we introduce the concept of p-paranilpotency as follows: we say that a locally finite group G is *p-paranilpotent* if it is p-soluble and the p-factors of its upper p-series are paranilpotenly embedded in G. Of course, a p-nilpotent group is p-hypercyclic. On the other hand, the standard wreath product $C_{3^\infty} \wr C_3$ is 3-hypercyclic but is not 3-paranilpotent, and, actually, since any p-paranilpotent group is nilpotent (see for instance Lemma 2.3 of [13]), any non-nilpotent Černikov p-group is p-hypercyclic but not p-paranilpotent.

There exist locally finite groups which are p-paranilpotent for any prime p but are not paranilpotent. To see this, it is enough to consider the direct product $G = \mathrm{Dr}_{p \in \mathbb{P}} G_p$, where for any prime p, G_p is a finite paranilpotent p-group of paraheight at least p. The situation is much better for homomorphic images of periodic linear groups.

Lemma 3.27. *Let G be a homomorphic image of a periodic linear group. If G is p-paranilpotent for any prime p, then G is paranilpotent.*

P r o o f. Let $\widehat{G}$ be a periodic linear group of characteristic q such that $G = \widehat{G}/\widehat{N}$ for a normal subgroup $\widehat{N}$ of $\widehat{G}$.

By Lemma 3.3, the group G is soluble, so the structure of periodic linear groups yields that G admits a normal subgroup M such that G/M is finite and $M/O_q(G)$ is an abelian q'-group of finite rank. Since G is q-paranilpotent, it follows that $O_q(G)$ is paranilpotently embedded in G, so we may assume $O_q(G) = \{1\}$. Similarly, we may assume $O_\pi(M) = \{1\}$, where $\pi = \pi(G/M)$. By Lemma 3.6, G is hypercyclic, and hence Lemma 11.7 of [20] yields that M is paranilpotently embedded in G. Therefore, G is paranilpotent and the statement is proved. □

The behaviour of p-paranilpotent groups is different from that of p-supersoluble groups. It turns out for instance that there is no analogue of Lemma 3.15. To see this let $G = H \ltimes (A \times B)$, where $H = \langle x \rangle \ltimes (\langle y \rangle \times \langle z \rangle) \simeq \mathrm{Dih}(8)$, $A \simeq B \simeq C_{2^\infty}$, z acts as the inversion on $A \times B$, y acts as the inversion on A and centralizes B,

x switches A and B. It is easy to see that G is hypercentral but not paranilpotent, and it is factorized by the two normal paranilpotent subgroups $U = \langle x, z, A, B\rangle$ and $V = \langle y, z, A, B\rangle$; of course, G is linear, being Černikov. This example is a 2-group and the intersection of U and V is infinite: it could not be otherwise, as shown by the following results.

Lemma 3.28. *Let G be a homomorphic image of a periodic linear group. If G is a 2-group and it is factorized by two mutually permutable paranilpotent subgroups A and B, then G is paranilpotent provided that one of the following conditions holds:*
(1) *$(A \cap B)_G$ has finite exponent,*
(2) *A and B are totally permutable.*

Proof. Let $\widehat{G}$ be a periodic linear group of characteristic q such that $G = \widehat{G}/\widehat{N}$ for a normal subgroup $\widehat{N}$ of $\widehat{G}$.

If $q = 2$, then G is nilpotent by 2.6 and Theorem 9.20 of [20]. Thus, we may assume $q \neq 2$, so by the same results, G is Černikov. Let A^*, B^* and G^* be the finite residuals of A, B and G, respectively. Then A^*B^* has finite index in $G = AB$ and A^*B^* is divisible abelian, so $G^* = A^*B^*$.

Suppose now that $(A \cap B)_G$ has finite exponent, so it is finite, and let D be the finite residual of $X = A \cap B$. Then X is a permutable subgroup of both A and B (see for instance Lemma 3.1 of [12]), so, if y is any element of $A \cup B$, we have that $Y = X\langle y\rangle = \langle y\rangle X$; on the other hand, D has a finite index in X, and hence D has a finite index in Y, which means that D is the finite residual of Y (being divisible) and, in particular, D is normalized by y. Thus, D is normal in $G = AB$, so $D = \{1\}$ (being finite and divisible) and consequently, X is finite. Let E and F be finite subgroups of G containing X and such that $A = EA^*$ and $B = FB^*$. Now, if P is any divisible subgroup of A^*, it follows from Lemma 3.5 of [12] that $PF = FP$, so P is normalized by F (and consequently by B) since P is the finite residual of FP. Similarly, A normalizes every divisible subgroup of B^*. Therefore, A^* and B^* are paranilpotently embedded in G, which means that G^* is paranilpotently embedded in G. Since G/G^* is a finite 2-group, it follows that G is paranilpotent.

Finally, if A and B are totally permutable subgroups, then the argument we employed in the previous paragraph shows that every divisible subgroup of A^* and B^* is normal in G, so actually G^* is paranilpotently embedded in G and (since G/G^* is a finite 2-group) G is paranilpotent. □

Lemma 3.29. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by the two mutually permutable subgroups A and B. If A and B are 2-paranilpotent and $(A \cap B)_G$ has finite exponent, then G is 2-paranilpotent.*

P r o o f. Let $\widehat{G}$ be a periodic linear group of characteristic q such that $G = \widehat{G}/\widehat{N}$ for a normal subgroup $\widehat{N}$ of $\widehat{G}$. If $q = 2$, then G is nilpotent by 2.6 and Theorem 9.20 of [20]. Thus, we may assume $q \neq 2$.

It follows from Theorem 3.19 that $G = O_{2'2}(G)$, so if we put $L = O_{2'}(G)$, then G/L is a 2-group. By Theorem 9.20 of [20], it is possible to find 2-subgroups U and V of A and B, respectively, such that

$$U \ltimes (L \cap A) = A \quad \text{and} \quad V \ltimes (L \cap B) = B.$$

By Corollary 9.14 of [20] the Sylow 2-subgroups of G are conjugate, so we may find an element $x \in G$ such that $\langle U, V^x \rangle$ is a 2-group. Write $x = ba$ for some $a \in A$ and $b \in B$. Then $B^x = B^a$, so A and B^x are still mutually permutable subgroups and $A \cap B^x = A \cap B^a = (A \cap B)^a$ has a finite exponent. Thus, we may replace B by B^x, assuming consequently that U and V are contained in the same Sylow 2-subgroup W of G.

Now, Lemma 9.13 of [20] yields that $G = WL$. Moreover,

$$A \cap B = (W \cap A \cap B) \ltimes (L \cap A \cap B).$$

Let $H = (A \cap B)_G$ and $K = (A \cap B)^G$. By Lemma 2.4, K/H is locally nilpotent, so $[W \cap A \cap B, L] \leqslant H$, and hence every W-invariant subgroup of $W \cap A \cap B$ is contained in H. It follows that W satisfies the hypotheses of the statement and therefore the result follows from Lemma 3.28. □

Lemma 3.30. *Let p be an odd prime number and let G be a homomorphic image of a periodic linear group. If G is p-hypercyclic and it is factorized by two mutually permutable p-paranilpotent subgroups A and B, then G is p-paranilpotent.*

P r o o f. Let $\widehat{G}$ be a periodic linear group of characteristic q such that $G = \widehat{G}/\widehat{N}$ for a normal subgroup $\widehat{N}$ of $\widehat{G}$.

It is certainly possible to assume $O_{p'}(G) = \{1\}$. If $M = O_p(G)$, then G/M is metabelian of exponent dividing $(p-1)^3$ by Theorem 3.19, so G is soluble.

The structure of periodic linear groups shows that G/M is finite regardless of the fact that $p = q$ or $p \neq q$. Now, if $p = q$, then M is nilpotent and so an application of Lemma 11.7 of [20] yields that M is paranilpotently embedded in G, thus completing the proof.

Assume $p \neq q$, so G is Černikov. Let A^* and B^* be the finite residuals of A and B, respectively. Then the finite residual G^* of G is abelian and contains A^* and B^*. On the other hand, $|A : A^*|$ and $|B : B^*|$ are finite, so $|G : A^*B^*|$ is finite (see for instance Lemma 1.2.5 of [2]) and hence $A^*B^* = G^*$. Let P

be any infinite locally cyclic subgroup of B^*. Then $AP = PA$ and A^*P is normal in AP, being its finite residual. It easily follows that $AB^* = AG^*$ is paranilpotent. Similarly, BG^* is paranilpotent and we may consequently assume $G^* \leqslant A \cap B$.

Let $U = O_p(A)$ and $V = O_p(B)$. By Lemma 2.3 of [13], U and V are nilpotent, so $G^* \leqslant \zeta_1(U) \cap \zeta_1(V)$. It follows that $G^* \leqslant \zeta_1(X)$, where $X = \langle U^G, V^G \rangle$. On the other hand, G/X is a finite group which is factorized by two mutually permutable p'-subgroups, so actually $X = M$ and $G^* \leqslant \zeta_1(M)$, which means that M is nilpotent. Now, an application of Lemma 11.7 of [20] yields that M is paranilpotently embedded in G, thus completing the proof also in this case. □

Theorem 3.31. *Let p be a prime and let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are p-paranilpotent, then $G/(A \cap B)_G$ is p-paranilpotent.*

Proof. This is a combination of Lemmas 3.29, 3.30 and Theorem 3.25. □

As a consequence of the above theorem and Lemma 3.27, we have the next result.

Corollary 3.32. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. If A and B are paranilpotent, then $G/(A \cap B)_G$ is paranilpotent.*

In the final result of the paper we describe what a counterexample to Lemma 3.30 for $p = 2$ should look like: it turns out that it resembles the example discussed before Lemma 3.28.

Theorem 3.33. *Let $G = AB$ be a homomorphic image of a periodic linear group which is factorized by two mutually permutable subgroups A and B. Assume further that G is not 2-paranilpotent and $O_{p'}(G) = \{1\}$. Then G is a Černikov 2-group and there are normal subgroups $N \leqslant D$ of G such that:*

(1) *N and G/D are finite, while D/N is divisible;*
(2) *AD/D and BD/D are elementary abelian normal subgroups of G/D, which intersect nontrivially;*
(3) *if X/D is a cyclic subgroup of $(AD \cap BD)/D$, then X/N acts as the inversion on its nilpotent residual, which coincides with $(X/N)'$ and is not trivial;*
(4) *$D/N = C_{G/N}(D/N)$.*

Furthermore, if the finite residual of G has the smallest possible rank, then we may even require that $(AD \cap BD)/D = (G/D)'$ is cyclic of order 2.

P r o o f. Let $\widehat{G}$ be a periodic linear group of characteristic q such that $G = \widehat{G}/\widehat{N}$ for a normal subgroup $\widehat{N}$ of $\widehat{G}$.

By Theorem 3.19, G is a 2-group. Moreover, as usual, we may assume $q \neq 2$, so G is Černikov, see [20], 2.6 and Theorem 9.20. Let A^* and B^* be the finite residuals of A and B, respectively. Since G is Černikov, its finite residual G^* is abelian and contains A^* and B^*. On the other hand, $|A : A^*|$ and $|B : B^*|$ are finite, so $|G : A^*B^*|$ is finite (see for instance Lemma 1.2.5 of [2]) and hence $A^*B^* = G^*$.

Let $C = C_G(G^*)$ and choose a finite subgroup F of C such that $C = FG^*$. It is clear that F is contained in the set of all elements of order at most $|F|$ of C, so we may factor it out and assume consequently that $C = G^*$, see also Theorem 3.14 of [16].

Let P be any infinite locally cyclic subgroup of B^*. Then $AP = PA$ and A^*P is normal in AP, being its finite residual. It easily follows that $H = AB^* = AG^*$ is paranilpotent. Now, H^2 is nilpotent because periodic automorphisms of infinite locally cyclic 2-groups have order at most 2, so $G^* \leqslant \zeta_1(H^2)$ (see Theorem 3.14 of [16]) and hence $G^* = \zeta_1(H^2)$; in particular, AG^*/G^* is an elementary abelian 2-group. Similarly, BG^*/G^* is an elementary abelian 2-group. Furthermore, it is possible to assume that $G^* \leqslant A \cap B$.

If Y/G^* is a cyclic subgroup of B/G^*, then $AY = YA$, so A has the index at most 2 in AY and hence A/G^* is normalized by Y/G^*. Thus, A is normal in G; similarly B is normal in G.

If $A \cap B \leqslant G^*$, then G' is abelian, so G is paranilpotent by Lemma 2.5 of [10], a contradiction. It follows that $A \cap B > G^*$. Let X/G^* be a cyclic subgroup of $(A \cap B)/G^*$.

Now, we need to discuss the nilpotent residual R of X. Let P, Q be subgroups of G^* such that $P \simeq Q/P \simeq C_{2^\infty}$, $[P, X] = \{1\}$, $[Q, X]P = Q$ (which means that X acts as the inversion on Q/P). Since P is divisible, we may find a subgroup U such that $Q = P \times U$. Choose $x \in X \setminus C_X(Q/P)$, write

$$U = \langle u_0, u_1, \ldots, u_n, \ldots :\ u_0^2 = 1,\ u_{n+1}^2 = u_n,\ n \in \mathbb{N}_0 \rangle$$

and put, for any i, $u_i^x = v_i u_i^{-1}$ for some $v_i \in P$; it is clear that $v_{j+1}^2 = v_j$ for any j. Define

$$V = \langle v_{i+1}^{-1} u_i \colon\ i \in \mathbb{N}_0 \rangle \simeq C_{2^\infty}$$

and note that

$$(v_{i+1}^{-1} u_i)^x = v_{i+1}^{-1} v_i u_i^{-1} = v_{i+1} v_i^{-1} v_i u_i^{-1} = (v_{i+1}^{-1} u_i)^{-1},$$

so X acts as the inversion on V. It follows that R admits a finite X-invariant series whose (nontrivial) factors are groups of type 2^∞ on which X acts as the inversion. Since $[G^*, X] \leqslant R$, we have $[G^*, X] = R$.

Suppose P, Q are divisible subgroups of G^* such that x acts as the inversion on P and on Q/P. As in the previous paragraph we may write $Q = P \times U$ for a subgroup U. Let $k \in U$. Then $k^x = k^{-1}w$ for some $w \in P$, and hence

$$k = k^{x^2} = (k^{-1}w)^x = kw^{-2},$$

so $w^2 = 1$. On the other hand, if $w \neq 1$ and $k_1 \in U$ is such that $k_1^2 = k$, then $k_1^x = k_1^{-1}v$, where $v^2 = w \neq 1$, a contradiction. Thus, x acts as the inversion on R. Note also that $R \neq \{1\}$ since G^* coincides with its own centralizer.

Finally, suppose G^* has the smallest possible rank. Let E_A and E_B be finite subgroups such that $A = E_A G^*$ and $B = E_B G^*$. If $R < G^*$, then G/R is paranilpotent by the minimality assumption; moreover, $L = \langle E_A, E_B, R\rangle$ is factorized by the mutually permutable subgroups $L \cap A$ and $L \cap B$ (see Lemma 2.1 of [12]), so again the minimality assumption yields that R is paranilpotently embedded in G. Thus, G^* is paranilpotently embedded in G and we obtain a contradiction. It follows that $R = G^*$.

If $y \in A \cap B \setminus X$, then y acts as the inversion on G^*, but then xy centralizes G^*, a contradiction. Thus, $A \cap B/G^*$ has order 2 and the statement is proved. $\square$

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