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FUNCTION ALGEBRAS OF BESOV AND TRIEBEL-LIZORKIN-TYPE

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Abstract. We prove that in the homogeneous Besov-type space the set of bounded functions constitutes a unital quasi-Banach algebra for the pointwise product. The same result holds for the homogeneous Triebel-Lizorkin-type space.

Keywords: Littlewood-Paley decomposition; Besov-type space; Triebel-Lizorkin-type space

MSC 2020: 46E35, 42B25

1. Introduction and the main result

For the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$, which are defined such that $\|f\|_{\dot{B}^s_{p,q}} = \|f\|_{\dot{F}^s_{p,q}} = 0$ if and only if f is a polynomial on $\mathbb{R}^n$, the subspaces of bounded functions, denoted by $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$, respectively, have played an important role for the composition operators on inhomogeneous Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively, see, e.g., [3], [4], [7], [9]. In these references they have been characterized, in particular:

Proposition 1.1. *For $s > 0$, $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ (or $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$) is a unital quasi-Banach algebra for the pointwise product.*

In this context, we want to extend this proposition to homogeneous Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin-type spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We then introduce the spaces

$$\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = (L_\infty \cap \dot{B}^{s,\tau}_{p,q})(\mathbb{R}^n)$$

and similarly $\mathcal{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ in the $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$-case, see Subsection 2.3 below. We denote by $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ either $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ or $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and by $\mathcal{E}^{s,\tau}_{p,q}(\mathbb{R}^n)$ either $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$

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or $\mathcal{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, if no confusion can arise. We use the abbreviations B-case and F-case to indicate in what space we work. Then we prove the following main result:

Theorem 1.1. *Let $\tau \geqslant 0$, $0 < p, q \leqslant \infty$ ($p < \infty$ in F-case) and $s > (n/p - n)_+$. Then $\mathcal{E}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is a unital quasi-Banach algebra for the pointwise product. Moreover, the inequality*

$$\|fg\|_{\mathcal{E}^{s,\tau}_{p,q}} \leqslant c(\|f\|_\infty \|g\|_{\mathcal{E}^{s,\tau}_{p,q}} + \|g\|_\infty \|f\|_{\mathcal{E}^{s,\tau}_{p,q}})$$

holds for all f, g in $\mathcal{E}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

For the proof we need some preparation, in particular an estimate of the Nikol'skij-type in $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, then we give a result in this direction, see Theorem 3.2 below. On the other hand, another result in this paper is the link between $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and their inhomogeneous counterparts $A^{s,\tau}_{p,q}(\mathbb{R}^n)$, see Theorem 3.1 below ($A^{s,\tau}_{p,q}(\mathbb{R}^n)$ denotes either the Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ or the Triebel-Lizorkin-type space $F^{s,\tau}_{p,q}(\mathbb{R}^n)$).

Also, one can extend the investigations on $\mathcal{E}^{s,\tau}_{p,q}(\mathbb{R}^n)$ using difference operators, it will be presented in future work.

Notation 1.1. All function spaces occurring in this work are defined on the Euclidean space $\mathbb{R}^n$, then we omit it in notations. As usual, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $s \in \mathbb{R}$, then $[s]$ denotes its integer part. If $u := (u_1, \ldots, u_n) \in \mathbb{R}^n$, we put $E(u) := ([u_1], \ldots, [u_n]) \in \mathbb{Z}^n$. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$. The symbol $\hookrightarrow$ indicates a continuous embedding. If $0 < p \leqslant \infty$, we denote by $\|\cdot\|_p$ the quasi-norm in L_p. For $f \in L_1$, the Fourier transform and the inverse are defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}f(x) := (2\pi)^{-n}\hat{f}(-x).$$

We denote by C_{ub} the Banach space of bounded and uniformly continuous functions on $\mathbb{R}^n$ endowed with the supremum norm. We denote by $\mathcal{P}_\infty$ the set of all polynomials in $\mathbb{R}^n$. The symbol $\mathcal{S}_\infty$ is used for the set of functions $\varphi \in \mathcal{S}$ (the Schwartz space) such that $\langle u, \varphi \rangle = 0$ for all $u \in \mathcal{P}_\infty$, its topological dual is denoted by $\mathcal{S}'_\infty$. If $f \in \mathcal{S}'$, then $[f]_\infty$ denotes its equivalence class modulo $\mathcal{P}_\infty$. The mapping which takes any $[f]_\infty$ to the restriction of f to $\mathcal{S}_\infty$ turns out to be an isomorphism from $\mathcal{S}'/\mathcal{P}_\infty$ onto $\mathcal{S}'_\infty$, for this reason, $\mathcal{S}'_\infty$ is the so-called "space of distributions modulo polynomials". For $k \in \mathbb{Z}$ and $\eta \in \mathbb{Z}^n$ we denote by $P_{k,\eta}$ the dyadic cube of all $x \in \mathbb{R}^n$ such that $\eta_l \leqslant 2^k x_l < \eta_l + 1$ ($l = 1, \ldots, n$). The constants $c, c_1, \ldots$ are strictly positive, depend only on the fixed parameters as $n, s, p, \ldots$, their values may change from a line to another.

Finally, in all the paper the parameters s, p, q, τ satisfy: $s \in \mathbb{R}$, $p \in]0, \infty]$ in B-case, $p \in]0, \infty[$ in F-case, $q \in]0, \infty]$ and $\tau \geqslant 0$, unless otherwise stated.

This work is organized as follows. In Section 2, we recall definitions and some properties of both $A^{s,\tau}_{p,q}$, $\dot{A}^{s,\tau}_{p,q}$ and $\mathcal{E}^{s,\tau}_{p,q}$. In Section 3, we state our additional results (see Theorems 3.1–3.2). The last section is devoted to the proofs.

2. Preliminaries

2.1. The Littlewood-Paley setting. To introduce the Littlewood-Paley setting, we choose, once and for all, a standard cut-off function ϱ. More precisely, we assume that ϱ is a radial C^∞ function satisfying $0 \leqslant \varrho \leqslant 1$, $\varrho(\xi) = 1$ for $|\xi| \leqslant 1$, $\varrho(\xi) = 0$ for $|\xi| \geqslant \frac{3}{2}$. We put $\gamma(\xi) := \varrho(\xi) - \varrho(2\xi)$. Then γ is supported by the compact annulus $\frac{1}{2} \leqslant |\xi| \leqslant \frac{3}{2}$ and the following identities hold:

$$\sum_{j\in\mathbb{Z}} \gamma(2^j\xi) = 1, \quad \xi \neq 0, \quad \varrho(2^{-k}\xi) + \sum_{j>k} \gamma(2^{-j}\xi) = 1, \quad k \in \mathbb{Z}.$$

Let us define the convolution operators (S_j) and (Q_j) by $\widehat{S_j f}(\xi) := \varrho(2^{-j}\xi)\hat{f}(\xi)$ and $\widehat{Q_j f}(\xi) := \gamma(2^{-j}\xi)\hat{f}(\xi)$. It is clear that S_j and Q_j are defined on $\mathcal{S}'$. The operators Q_j are also defined on $\mathcal{S}'_\infty$ since $Q_j f = 0$ if and only if f is a polynomial on $\mathbb{R}^n$, then we make use of the following convention:

$$\text{if } f \in \mathcal{S}'_\infty \text{ we set } Q_j f := Q_j g \text{ for all } g \text{ such that } [g]_\infty = f.$$

The operators S_j and Q_j take values in the space of analytical functions of exponential type, see the Paley-Wiener theorem. The families (S_j) and (Q_j) constitute bounded subsets of the normed space $\mathcal{L}(L_p)$ for any $1 \leqslant p \leqslant \infty$ due to Young inequality. Also, we have the following lemma, a classical consequence of Taylor's formula, see, e.g., [8], Proposition 2.5:

Lemma 2.1.
(i) *If $f \in \mathcal{S}$, then $\|Q_j f\|_p = O(2^{-jN})$ as $j \to \infty$ for all $N \in \mathbb{N}_0$.*
(ii) *If $f \in \mathcal{S}_\infty$, then $\|Q_j f\|_p = O(2^{jN})$ and $\|S_j f\|_p = O(2^{jN})$ as $j \to -\infty$ for all $N \in \mathbb{N}_0$.*

The Littlewood-Paley decompositions of a tempered distribution are described in the following statement, which is an immediate consequence of Lemma 2.1.

Proposition 2.1.
(i) *For every $f \in \mathcal{S}_\infty$ (or $f \in \mathcal{S}'_\infty$), it holds that $f = \sum_{j\in\mathbb{Z}} Q_j f$ in $\mathcal{S}_\infty$ (or $\mathcal{S}'_\infty$, respectively).*
(ii) *For every $f \in \mathcal{S}$ (or $f \in \mathcal{S}'$) and every $k \in \mathbb{Z}$, it holds that $f = S_k f + \sum_{j>k} Q_j f$ in $\mathcal{S}$ (or $\mathcal{S}'$, respectively).*

2.2. The Besov and Triebel-Lizorkin spaces. We first define the classical Besov and Triebel-Lizorkin spaces and their homogeneous counterparts.

Definition 2.1. (i) The Besov space $B^s_{p,q}$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{B^s_{p,q}} := \|S_0 f\|_p + \Big(\sum_{j\geqslant 1} (2^{js}\|Q_j f\|_p)^q\Big)^{1/q} < \infty.$$

(ii) The Triebel-Lizorkin space $F^s_{p,q}$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{F^s_{p,q}} := \|S_0 f\|_p + \Big\|\Big(\sum_{j\geqslant 1} (2^{js}|Q_j f|)^q\Big)^{1/q}\Big\|_p < \infty.$$

(iii) The homogeneous Besov space $\dot{B}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{j\in\mathbb{Z}} (2^{js}\|Q_j f\|_p)^q\Big)^{1/q} < \infty.$$

(iv) The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}^s_{p,q}} := \Big\|\Big(\sum_{j\in\mathbb{Z}} (2^{js}|Q_j f|)^q\Big)^{1/q}\Big\|_p < \infty.$$

With usual modifications made when, in the B-case, $p = \infty$ or $q = \infty$ and, in the F-case, $q = \infty$.

Then we define the Besov-type and Triebel-Lizorkin-type spaces and their homogeneous counterparts.

Definition 2.2. (i) The Besov-type space $B^{s,\tau}_{p,q}$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{B^{s,\tau}_{p,q}} := \sup_{k\in\mathbb{Z}} \sup_{\eta\in\mathbb{Z}^n} 2^{kn\tau} \Big(\sum_{j\geqslant k_+} (2^{js}\|Q_j f\|_{L_p(P_{k,\eta})})^q\Big)^{1/q} < \infty \quad (\text{here } Q_0 := S_0).$$

(ii) The Triebel-Lizorkin-type space $F^{s,\tau}_{p,q}$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{F^{s,\tau}_{p,q}} := \sup_{k\in\mathbb{Z}} \sup_{\eta\in\mathbb{Z}^n} 2^{kn\tau} \Big\|\Big(\sum_{j\geqslant k_+} (2^{js}|Q_j f|)^q\Big)^{1/q}\Big\|_{L_p(P_{k,\eta})} < \infty \quad (\text{here } Q_0 := S_0).$$

(iii) The homogeneous Besov-type space $\dot{B}^{s,\tau}_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}} := \sup_{k\in\mathbb{Z}} \sup_{\eta\in\mathbb{Z}^n} 2^{kn\tau} \Big(\sum_{j\geqslant k} (2^{js}\|Q_j f\|_{L_p(P_{k,\eta})})^q\Big)^{1/q} < \infty.$$

(iv) The homogeneous Triebel-Lizorkin-type space $\dot{F}^{s,\tau}_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}} := \sup_{k\in\mathbb{Z}} \sup_{\eta\in\mathbb{Z}^n} 2^{kn\tau} \Big\| \Big(\sum_{j\geqslant k} (2^{js}|Q_j f|)^q \Big)^{1/q} \Big\|_{L_p(P_{k,\eta})} < \infty.$$

With usual modifications made when, in the B-case, $p = \infty$ or $q = \infty$ and, in the F-case, $q = \infty$.

All these spaces are independent of the choice of the function ϱ, see, e.g., [2], [10]–[13] and [15]. Now, in what follows, we will denote for simplicity by $A^s_{p,q}$ either $B^s_{p,q}$ or $F^s_{p,q}$ and by $\dot{A}^s_{p,q}$ either $\dot{B}^s_{p,q}$ or $\dot{F}^s_{p,q}$. We note that $A^{s,0}_{p,q} = A^s_{p,q}$ and $\dot{A}^{s,0}_{p,q} = \dot{A}^s_{p,q}$. We also note that

$$\mathcal{S} \hookrightarrow A^s_{p,q},\ A^{s,\tau}_{p,q} \hookrightarrow \mathcal{S}' \quad \text{and} \quad \mathcal{S}_\infty \hookrightarrow \dot{A}^s_{p,q},\ \dot{A}^{s,\tau}_{p,q} \hookrightarrow \mathcal{S}'_\infty. \tag{2.1}$$

As further embeddings, we have the following two statements.

Proposition 2.2. *The following properties hold:*

(i) $B^{s,\tau}_{p,\min(p,q)} \hookrightarrow F^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,\max(p,q)}$ *and* $A^{s,\tau}_{p,q_1} \hookrightarrow A^{s,\tau}_{p,q_2}$ $(q_1 \leqslant q_2)$.

(ii) $A^{s,\tau}_{p,q} \hookrightarrow B^{s+n\tau-n/p}_{\infty,\infty} = F^{s+n\tau-n/p}_{\infty,\infty}$.

(iii) $A^{s,\tau}_{p,q} \hookrightarrow C_{\rm ub}$ *if* $s + n\tau - n/p > 0$.

P r o o f. For (i)–(ii) we refer to [15], Propositions 2.1, 2.6. In the B-case, (iii) follows by the well-known property of Hölder spaces $B^t_{\infty,\infty}$ for $t > 0$, see, e.g., [11], Theorem 2.2.4/1; in the F-case, we have $F^{s,\tau}_{p,q} \hookrightarrow B^{s,\tau}_{p,\max(p,q)} \hookrightarrow C_{\rm ub}$. □

Proposition 2.3. *The following properties hold:*

(i) $\dot{B}^{s,\tau}_{p,\min(p,q)} \hookrightarrow \dot{F}^{s,\tau}_{p,q} \hookrightarrow \dot{B}^{s,\tau}_{p,\max(p,q)}$ *and* $\dot{A}^{s,\tau}_{p,q_1} \hookrightarrow \dot{A}^{s,\tau}_{p,q_2}$ $(q_1 \leqslant q_2)$.

(ii) *Let* $s_1 > s_2$, $0 < p_1 < p_2 \leqslant \infty$ ($p_2 < \infty$ *in* F-*case*) *and* $0 < r \leqslant \infty$. *If* $s_1 - n/p_1 = s_2 - n/p_2$, *then* $\dot{B}^{s_1,\tau}_{p_1,q} \hookrightarrow \dot{B}^{s_2,\tau}_{p_2,q}$ *and* $\dot{F}^{s_1,\tau}_{p_1,q} \hookrightarrow \dot{F}^{s_2,\tau}_{p_2,r}$.

P r o o f. For (ii) we refer to [14], Proposition 3.3. □

In order to connect with the modified Lebesgue-type space L^τ_p of functions f such that

$$\|f\|_{L^\tau_p} := \sup_{l(P)\geqslant 1} |P|^{-\tau} \|f\|_{L_p(P)} < \infty$$

where the supremum is taken over all dyadic cubes P with side length $l(P) \geqslant 1$, we recall that $L^\tau_p \hookrightarrow \mathcal{S}'$ if $1 \leqslant p \leqslant \infty$ and $L^0_p = L_p$. Thus, we give a relation between $A^{s,\tau}_{p,q}$ and this space and refer to [15], Proposition 2.7.

Proposition 2.4. *If* $s > (n/p - n)_+$, *then it holds* $A^{s,\tau}_{p,q} \hookrightarrow L^\tau_p$.

2.3. Definition of the algebra space.

Definition 2.3. The space $\mathcal{E}^{s,\tau}_{p,q}$ is the set of $f \in L_\infty$ such that $[f]_\infty \in \dot{A}^{s,\tau}_{p,q}$ equipped with the quasi-norm $\|f\|_{\mathcal{E}^{s,\tau}_{p,q}} := \|f\|_\infty + \|[f]_\infty\|_{\dot{A}^{s,\tau}_{p,q}}$.

The following statement is an immediate consequence of the second chain of embeddings given in (2.1).

Proposition 2.5. *The space $\mathcal{E}^{s,\tau}_{p,q}$ is quasi-Banach and the continuous embeddings $\mathcal{S}_\infty \hookrightarrow \mathcal{E}^{s,\tau}_{p,q} \hookrightarrow \mathcal{S}'$ hold.*

3. Further results

The following two theorems are of self-contained interest, since the first one gives us a tool for passing from $A^{s,\tau}_{p,q}$ to $\dot{A}^{s,\tau}_{p,q}$ which is not completely referenced in the literature, see, e.g., [5], Lemma 2.7, Remark 2.8, the second one is Nikol'skij-type inequalities for $\dot{A}^{s,\tau}_{p,q}$.

Theorem 3.1.
(i) *Assume that either $s > (n/p - n)_+$ or $s > n/p - n\tau$ and $\tau \geqslant 1/p$. If $f \in A^{s,\tau}_{p,q}$, then it holds $[f]_\infty \in \dot{A}^{s,\tau}_{p,q}$.*
(ii) *If $s > (n/p-n)_+$, then $A^{s,\tau}_{p,q} = \{f \in L^\tau_p\colon\ [f]_\infty \in \dot{A}^{s,\tau}_{p,q}\}$. Moreover, the expression $\|f\|_{L^\tau_p} + \|[f]_\infty\|_{\dot{A}^{s,\tau}_{p,q}}$ is an equivalent quasi-norm in $A^{s,\tau}_{p,q}$.*

Theorem 3.2. *Let $s > (n/p - n)_+$ and $b > 0$. Let (u_j) be a sequence in $\mathcal{S}'$ such that*

▷ *$\widehat{u_j}$ is supported by the ball $|\xi| \leqslant b2^j$,*

▷ *$A := \sup\limits_{k\in\mathbb{Z}} \sup\limits_{\eta\in\mathbb{Z}^n} 2^{kn\tau}\Big(\sum\limits_{j\geqslant k}(2^{js}\|u_j\|_{L_p(P_{k,\eta})})^q\Big)^{1/q} < \infty$ in the B-case,*

▷ *$A := \sup\limits_{k\in\mathbb{Z}} \sup\limits_{\eta\in\mathbb{Z}^n} 2^{kn\tau}\Big\|\Big(\sum\limits_{j\geqslant k}(2^{js}|u_j|)^q\Big)^{1/q}\Big\|_{L_p(P_{k,\eta})} < \infty$ in the F-case.*

Then the series $\sum\limits_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'_\infty$ and satisfies $\Big\|\sum\limits_{j\in\mathbb{Z}} u_j\Big\|_{\dot{A}^{s,\tau}_{p,q}} \leqslant cA$, where the constant c depends only on n, s, τ, p, q and b.

4. Proofs

P r o o f of Theorem 1.1. We first assume that Theorem 3.2 is indeed valid; a straightforward consequence of this theorem is the following assertion.

Lemma 4.1. *Let $s > (n/p - n)_+$ and $m \in \mathbb{Z}$. For all $f \in L_\infty$ and all $g \in \dot{A}^{s,\tau}_{p,q}$ we put*

$$\pi_m(f,g) := \sum_{j\in\mathbb{Z}} (S_{j-m}f)(Q_j g).$$

Then π_m is a continuous bilinear application from $L_\infty \times \dot{A}^{s,\tau}_{p,q}$ to $\dot{A}^{s,\tau}_{p,q}$.

We turn to the proof of Theorem 1.1 which is similar to that in Theorem 3.26 of [7], we only check some properties needed here. Recall that in this reference the case $p, q \geqslant 1$ is treated which can be extended without difficulty to any $p, q > 0$. Let us take f, g in $\mathcal{E}^{s,\tau}_{p,q}$. By the Abel transform

$$\sum_{k=-j}^{j} (S_k f)(Q_k g) + \sum_{k=-j}^{j-1} (S_k g)(Q_{k+1} f) = (S_j f)(S_j g) - (S_{-j} f)(S_{-j-1} g)$$

holds for all $j > 0$. By Lemma 2.1, $|S_k f(x)| \leqslant c\|f\|_\infty$ and $|S_k g(x)| \leqslant c\|g\|_\infty$ (for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$), and we have (cf. [7], page 253)

$$\lim_{j\to\infty} (S_{-j} f)(S_{-j-1} g) = 0 \text{ in } \mathcal{S}'_\infty \quad \text{and} \quad \lim_{j\to\infty} (S_j f)(S_j g) = fg \text{ in } \mathcal{S}'.$$

Then we get $\pi_0(f, [g]_\infty) + \pi_1(g, [f]_\infty) = [fg]_\infty$ in $\mathcal{S}'_\infty$. Now, by applying Lemma 4.1, we obtain the desired result. □

P r o o f of Theorem 3.1. *Step 1*: Proof of (i) in the B-case. Let us consider $f \in B^{s,\tau}_{p,q}$ and set $U_{k,\eta} := 2^{kn\tau}\Big(\sum_{j\geqslant k} (2^{js}\|Q_j f\|_{L_p(P_{k,\eta})})^q\Big)^{1/q}$ where $k \in \mathbb{Z}$ and $\eta \in \mathbb{Z}^n$. Clearly we have

$$\begin{aligned} U_{k,\eta} &\leqslant 2^{|k|n\tau}\Big(\sum_{j\geqslant |k|} (2^{js}\|Q_j f\|_{L_p(P_{|k|,\eta})})^q\Big)^{1/q} \\ &\quad + 2^{-|k|n\tau}\Big(\sum_{j\geqslant -|k|} (2^{js}\|Q_j f\|_{L_p(P_{-|k|,\eta})})^q\Big)^{1/q} \\ &\leqslant \sup_{l>0} 2^{ln\tau}\Big(\sum_{j\geqslant l}\dots\Big)^{1/q} + \sup_{l\leqslant 0} 2^{ln\tau}\Big(\sum_{j\geqslant l}\dots\Big)^{1/q}, \end{aligned}$$

then we can write

$$\|[f]_\infty\|_{\dot{B}^{s,\tau}_{p,q}} \leqslant \sup_{k>0}\sup_{\eta\in\mathbb{Z}^n} 2^{kn\tau}\Big(\sum_{j\geqslant k}(2^{js}\|Q_jf\|_{L_p(P_{k,\eta})})^q\Big)^{1/q} + \sup_{k\leqslant 0}\sup_{\eta\in\mathbb{Z}^n} 2^{kn\tau}\Big(\sum_{j\geqslant k}(2^{js}\|Q_jf\|_{L_p(P_{k,\eta})})^q\Big)^{1/q}. \tag{4.1}$$

It is clear that the second term is bounded by $\|f\|_{B^{s,\tau}_{p,q}}$. For the third one, since $k\leqslant 0$, we split $\sum_{j\geqslant k}$ into $\sum_{j=k}^{0}+\sum_{j>0}$ and use the equality $\sum_{j>0}\ldots=\sum_{j>k_+}\ldots$, then

$$2^{kn\tau}\Big(\sum_{j>0}(2^{js}\|Q_jf\|_{L_p(P_{k,\eta})})^q\Big)^{1/q} \leqslant \|f\|_{B^{s,\tau}_{p,q}}. \tag{4.2}$$

We reduced the estimation to the term

$$\sup_{k\leqslant 0}\sup_{\eta\in\mathbb{Z}^n} 2^{kn\tau}\Big(\sum_{j=k}^{0}(2^{js}\|Q_jf\|_{L_p(P_{k,\eta})})^q\Big)^{1/q}. \tag{4.3}$$

Substep 1.1: The case $s>(n/p-n)_+$. If $x\in P_{k,\eta}$ and $y\in\mathbb{R}^n$, we have $x-y\in \bigcup_{r=1}^{2^n} P_{k,\eta-E(2^ky)+w_r}$ where $w_r\in\mathbb{Z}^n$ independent of y, an easy proof. Then

$$\|f(\cdot-y)\|_{L_p(P_{k,\eta})} \leqslant c\sum_{r=1}^{2^n}\|f\|_{L_p(P_{k,\eta-E(2^ky)+w_r})} \tag{4.4}$$

for all $y\in\mathbb{R}^n$; this follows from the statement

$$\text{if } a,b\geqslant 0 \text{ then } a^p+b^p\leqslant \max(1,2^{1-p})(a+b)^p. \tag{4.5}$$

▷ If $p\geqslant 1$, then by using (4.4), we get

$$\|Q_jf\|_{L_p(P_{k,\eta})} \leqslant 2^{jn}\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\gamma(2^jy)|\|f(\cdot-y)\|_{L_p(P_{k,\eta})}\,\mathrm{d}y \leqslant c_1 2^{-kn\tau}\|\mathcal{F}^{-1}\gamma\|_1\|f\|_{L_p^\tau} \leqslant c_2 2^{-kn\tau}\|f\|_{L_p^\tau} \quad \forall\, k\leqslant 0. \tag{4.6}$$

As $B^{s,\tau}_{p,q}\hookrightarrow L^\tau_p$ since $s>0$ (see Proposition 2.4), we deduce that the expression given in (4.3) is bounded by $c\|f\|_{B^{s,\tau}_{p,q}}$.

▷ If $0<p<1$, we apply the following lemma.

Lemma 4.2. *We put $d:=\min(1,p)$. Then there exists a constant $c>0$ such that the inequality $\|Q_jf\|_{A^{s,\tau}_{p,q}}\leqslant c\max(1,2^{jn(1-1/d)})\|f\|_{A^{s,\tau}_{p,q}}$ holds for all $f\in A^{s,\tau}_{p,q}$ and all $j\in\mathbb{Z}$.*

We now turn to the estimate (4.3) in which the factor after the text "$\sup_{k\leqslant 0}\sup_{\eta\in\mathbb{Z}^n}$" is bounded by $\Big(\sum_{j\leqslant 0}(2^{js}\|Q_jf\|_{L_p^\tau})^q\Big)^{1/q}$ since $2^{kn\tau}\|Q_jf\|_{L_p(P_{k,\eta})}\leqslant\|Q_jf\|_{L_p^\tau}$. By this lemma we have $Q_jf\in B^{s,\tau}_{p,q}$, then the continuous embedding $B^{s,\tau}_{p,q}\hookrightarrow L_p^\tau$ yields the bound $c_1\|f\|_{B^{s,\tau}_{p,q}}\Big(\sum_{j\leqslant 0}2^{j(s+n-n/p)q}\Big)^{1/q}$ which is itself majorized by $c_2\|f\|_{B^{s,\tau}_{p,q}}$ since

$$s+n-\frac{n}{p}>0. \tag{4.7}$$

Substep 1.2: The case $s>n/p-n\tau$ and $\tau\geqslant 1/p$. By applying Proposition 2.2 (iii), it suffices in the above argument to estimate (4.3) using

$$\begin{aligned}\|Q_jf\|_{L_p(P_{k,\eta})}&\leqslant\|\mathcal{F}^{-1}\gamma\|_1\|f\|_\infty\Big(\int_{P_{k,\eta}}\mathrm{d}x\Big)^{1/p}\\&\leqslant c_1 2^{-nk/p}\|f\|_\infty\leqslant c_2 2^{-nk/p}\|f\|_{B^{s,\tau}_{p,q}}\quad\forall\,k\leqslant 0.\end{aligned} \tag{4.8}$$

Then, we obtain the bound $c\|f\|_{B^{s,\tau}_{p,q}}$. Indeed, it suffices to observe that

$$2^{knq(\tau-1/p)}\sum_{j=k}^{0}2^{jsq}\leqslant\sum_{j\leqslant 0}2^{j(s+n\tau-n/p)q}<\infty.$$

Step 2: Proof of (i) in the F-case. Let $f\in F^{s,\tau}_{p,q}$. We proceed exactly as in (4.1), (4.2) and (4.3) we arrive at

$$\sup_{k\leqslant 0}\sup_{\eta\in\mathbb{Z}^n}2^{kn\tau}\bigg\|\Big(\sum_{j=k}^{0}(2^{js}|Q_jf|)^q\Big)^{1/q}\bigg\|_{L_p(P_{k,\eta})}. \tag{4.9}$$

Substep 2.1: The case $s>(n/p-n)_+$. We consider the following two cases: if $p\geqslant q$, we use the Minkowski inequality, if $p<q$, we use the elementary estimate

$$\sum_j a_j\leqslant\Big(\sum_j a_j^d\Big)^{1/d},\quad a_j\geqslant 0,\ 0<d\leqslant 1$$

with $d:=p/q$ and $a_j:=(2^{js}|Q_jf|)^q$ in the sum $\sum_{j=k}^{0}$. Then in all cases, we obtain

$$\bigg\|\Big(\sum_{j=k}^{0}(2^{js}|Q_jf|)^q\Big)^{1/q}\bigg\|_{L_p(P_{k,\eta})}\leqslant\Big(\sum_{j=k}^{0}(2^{js}\|Q_jf\|_{L_p(P_{k,\eta})})^r\Big)^{1/r} \tag{4.10}$$

where $r:=\min(p,q)$.

▷ If $p \geqslant 1$, the assumption $s > 0$ and (4.6) yield that (4.10) is bounded by $c\|f\|_{L_p^\tau}$. Then, we finish by using $F_{p,q}^{s,\tau} \hookrightarrow L_p^\tau$.

▷ If $0 < p < 1$, combining the estimate (4.10) with the expression (4.9) we obtain the bound $\Big(\sum\limits_{j\leqslant 0}(2^{js}\|Q_j f\|_{L_p^\tau})^r\Big)^{1/r}$. Then as above, we continue by using $F_{p,q}^{s,\tau} \hookrightarrow L_p^\tau$ (i.e., $\|Q_j f\|_{L_p^\tau} \leqslant c\|Q_j f\|_{F_{p,q}^{s,\tau}}$) and Lemma 4.2 because of the assumption (4.7).

Substep 2.2: The case $s > n/p - n\tau$ and $\tau \geqslant 1/p$. This case can be treated as in Substep 1.2 using (4.8) with $F_{p,q}^{s,\tau}$ instead of $B_{p,q}^{s,\tau}$, and (4.10).

Step 3: Proof of (ii). By (i) and Proposition 2.4 we have the embedding in one direction. To prove the converse, let $f \in L_p^\tau$ be such that $[f]_\infty \in \dot{A}_{p,q}^{s,\tau}$. We put for the sake of brevity,

$$B_k := 2^{kn\tau}\Big(\sum_{j\geqslant k_+}(2^{js}\|Q_j f\|_{L_p(P_{k,\eta})})^q\Big)^{1/q} \quad \text{in the } B\text{-case},$$

$$F_k := 2^{kn\tau}\Big\|\Big(\sum_{j\geqslant k_+}(2^{js}|Q_j f|)^q\Big)^{1/q}\Big\|_{L_p(P_{k,\eta})} \quad \text{in the } F\text{-case}.$$

If $k \geqslant 1$ we have $B_k \leqslant \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}$ and $F_k \leqslant \|[f]_\infty\|_{\dot{F}_{p,q}^{s,\tau}}$. Thus, we assume that $k \leqslant 0$ and we have:

$$\begin{aligned}
B_k &= 2^{kn\tau}\Big(\|S_0 f\|_{L_p(P_{k,\eta})}^q + \sum_{j\geqslant 1}\cdots\Big)^{1/q} \\
&\leqslant c2^{kn\tau}\Big(\|S_0 f\|_{L_p(P_{k,\eta})} + \Big(\sum_{j\geqslant 1}\cdots\Big)^{1/q}\Big) \\
&\leqslant c(2^{kn\tau}\|S_0 f\|_{L_p(P_{k,\eta})} + \|[f]_\infty\|_{\dot{B}_{p,q}^{s,\tau}}),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
F_k &= 2^{kn\tau}\Big\|\Big(|S_0 f|^q + \sum_{j\geqslant 1}\cdots\Big)^{1/q}\Big\|_{L_p(P_{k,\eta})} \\
&\leqslant c(2^{kn\tau}\|S_0 f\|_{L_p(P_{k,\eta})} + \|[f]_\infty\|_{\dot{F}_{p,q}^{s,\tau}});
\end{aligned} \tag{4.12}$$

where we used $\sum\limits_{j\geqslant 1}\ldots \leqslant \sum\limits_{j\geqslant k}\ldots$ since $k \leqslant 0$.

▷ The case $p \geqslant 1$: We proceed as in (4.6) by changing Q_j by S_0, we obtain

$$\|S_0 f\|_{L_p(P_{k,\eta})} \leqslant c2^{-kn\tau}\|f\|_{L_p^\tau} \quad \forall\, k \leqslant 0. \tag{4.13}$$

▷ The case $p < 1$: As $S_0 f = f - \sum\limits_{j\geqslant 1} Q_j f$, cf. Proposition 2.1, we get

$$\|S_0 f\|_{L_p(P_{k,\eta})} \leqslant 2^{1/p-1}\Big(\|f\|_{L_p(P_{k,\eta})} + \Big\|\sum_{j\geqslant 1} Q_j f\Big\|_{L_p(P_{k,\eta})}\Big). \tag{4.14}$$

Clearly $\|f\|_{L_p(P_{k,\eta})} \leqslant c2^{-kn\tau}\|f\|_{L_p^\tau}$ for all $k \leqslant 0$. However, for the last factor in (4.14) we first see the B-case. We have

$$\Big\|\sum_{j\geqslant 1} Q_j f\Big\|_{L_p(P_{k,\eta})} \leqslant \Big\|\Big(\sum_{j\geqslant 1}|Q_j f|^p\Big)^{1/p}\Big\|_{L_p(P_{k,\eta})} = \Big(\sum_{j\geqslant 1}\|Q_j f\|^p_{L_p(P_{k,\eta})}\Big)^{1/p},$$

but $2^{js}\|Q_j f\|_{L_p(P_{k,\eta})} \leqslant 2^{-kn\tau}\|[f]_\infty\|_{\dot B^{s,\tau}_{p,q}}$ for all $j \geqslant 1$ since $k \leqslant 0$. Hence, we obtain the bound

$$2^{-kn\tau}\|[f]_\infty\|_{\dot B^{s,\tau}_{p,q}}\Big(\sum_{j\geqslant 1}2^{-jsp}\Big)^{1/p} \leqslant c2^{-kn\tau}\|[f]_\infty\|_{\dot B^{s,\tau}_{p,q}}.$$

In the same way we prove the F-case, i.e., as $2^{js}|Q_j f| \leqslant \Big(\sum_{l\geqslant k}2^{lsq}|Q_l f|^q\Big)^{1/q}$ for all $j \geqslant 1$ since $k \leqslant 0$, we have

$$\Big\|\sum_{j\geqslant 1} Q_j f\Big\|_{L_p(P_{k,\eta})} \leqslant 2^{-kn\tau}\|[f]_\infty\|_{\dot F^{s,\tau}_{p,q}}\Big(\sum_{j\geqslant 1}2^{-js}\Big) \leqslant c2^{-kn\tau}\|[f]_\infty\|_{\dot F^{s,\tau}_{p,q}}.$$

Consequently, from (4.13)–(4.14) we get

$$\|S_0 f\|_{L_p(P_{k,\eta})} \leqslant c2^{-kn\tau}(\|f\|_{L_p^\tau} + \|[f]_\infty\|_{\dot A^{s,\tau}_{p,q}}) \quad \forall\, k \leqslant 0. \tag{4.15}$$

Now, inserting (4.15) into (4.11) and (4.12) in the B- and F-case, respectively, we obtain the bound $c(\|f\|_{L_p^\tau} + \|[f]_\infty\|_{\dot A^{s,\tau}_{p,q}})$. Finally, taking into account the case when $k \geqslant 1$ the result follows. □

P r o o f of Lemma 4.2. We first put $\gamma_j := \gamma(2^{-j}\cdot)$ and $w := \min(1,p,q)$ for brevity. Clearly $Q_l Q_j f \equiv 0$ if $|j-l| \geqslant 2$. Thus, the Fourier transform of the function $y \mapsto \mathcal{F}^{-1}(\gamma_j)(y)Q_l f(x-y)$ has its support in the ball $|\xi| \leqslant (\frac{3}{2}2^j + \frac{3}{2}2^l) \leqslant \frac{9}{2}2^j$ (since $|j-l| \leqslant 1$). Then, by the Bernstein inequality (see, e.g., [12], Remark 1.3.2/1) we get

$$|Q_l Q_j f(x)| \leqslant c2^{jn(1/w-1)}\Big[\int_{\mathbb{R}^n}|\mathcal{F}^{-1}(\gamma_j)(y)Q_l f(x-y)|^w\,\mathrm{d}y\Big]^{1/w}. \tag{4.16}$$

▷ We consider the B-case. By the Minkowski inequality w.r.t. $L_p(P_{k,\eta}; L_w(\mathbb{R}^n))$, it holds

$$\|Q_l Q_j f\|_{L_p(P_{k,\eta})} \leqslant c2^{jn(1/w-1)}\Big[\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\gamma_j(y)|^w\|Q_l f(\cdot-y)\|^w_{L_p(P_{k,\eta})}\,\mathrm{d}y\Big]^{1/w}. \tag{4.17}$$

We put $k_1{:=}\max(k,1)$ and, again by the Minkowski inequality w.r.t. $l_q(\mathbb{Z};L_w(\mathbb{R}^n))$, we obtain

(4.18)
$$\Big(\sum_{l\geqslant k_1}(2^{ls}\|Q_lQ_jf\|_{L_p(P_{k,\eta})})^q\Big)^{1/q} \leqslant c2^{jn(1/w-1)}\bigg[\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\gamma_j(y)|^w\Big(\sum_{l\geqslant k_1}(2^{ls}\|Q_lf(\cdot-y)\|_{L_p(P_{k,\eta})})^q\Big)^{w/q}\,\mathrm{d}y\bigg]^{1/w}.$$

In the right-hand side of (4.18) we use (4.4) with Q_lf instead of f and as

$$2^{jn(1/w-1)}\Big(\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\gamma_j(y)|^w\,\mathrm{d}y\Big)^{1/w}=\|\mathcal{F}^{-1}\gamma\|_w, \tag{4.19}$$

then, by taking conveniently supremum, we obtain

$$\Big(\sum_{l\geqslant k_1}(2^{ls}\|Q_lQ_jf\|_{L_p(P_{k,\eta})})^q\Big)^{1/q}\leqslant c2^{-kn\tau}\|f\|_{B^{s,\tau}_{p,q}}. \tag{4.20}$$

We now treat the term $\|S_0Q_jf\|_{L_p(P_{k,\eta})}$; here $k\leqslant 0$. We put $d:=\min(1,p)$. We have $S_0Q_jf\equiv 0$ if $j\geqslant 2$, and the Fourier transform of the function $y\mapsto\mathcal{F}^{-1}(\gamma_j)(y)S_0f(x-y)$ is supported by the ball $|\xi|\leqslant 3$. Hence, as above in (4.16)–(4.17) we obtain

$$\|S_0Q_jf\|_{L_p(P_{k,\eta})}\leqslant c_1\bigg[\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\gamma_j(y)|^d\|S_0f(\cdot-y)\|^d_{L_p(P_{k,\eta})}\,\mathrm{d}y\bigg]^{1/d},$$

where $c_1:=3^{n(1/d-1)}c$. We again use (4.4) with S_0f instead of f, and take into account that the estimate $\|S_0f\|_{L_p(P_{k,\mu})}\leqslant c2^{-kn\tau}\|f\|_{B^{s,\tau}_{p,q}}$ holds true for all $\mu\in\mathbb{Z}^n$ and all $k\leqslant 0$, then using the equality (4.19) with d instead of w, we obtain

$$\|S_0Q_jf\|_{L_p(P_{k,\eta})}\leqslant c2^{jn(1-1/d)}2^{-kn\tau}\|f\|_{B^{s,\tau}_{p,q}}\quad\forall\, j\leqslant 1,\ \forall\, k\leqslant 0. \tag{4.21}$$

Since $\sum_{l\geqslant k_+}(2^{ls}\|Q_lQ_jf\|_{L_p(P_{k,\eta})})^q$ is bounded by

$$\|S_0Q_jf\|^q_{L_p(P_{k,\eta})}+\sum_{l\geqslant 1}(2^{ls}\|Q_lQ_jf\|_{L_p(P_{k,\eta})})^q\quad\text{if } k\leqslant 0$$

and

$$\sum_{l\geqslant k}(2^{ls}\|Q_lQ_jf\|_{L_p(P_{k,\eta})})^q\quad\text{if } k\geqslant 1,$$

then by dividing each term of the resulting inequality by $2^{-kn\tau}$, using (4.20) and (4.21) and taking supremum, the result follows.

▷ We briefly show the F-case using the same notations. So, we proceed as in (4.16) and (4.18) by applying twice the Minkowski inequality, and we obtain

$$\Big\|\Big(\sum_{l\geqslant k_1}(2^{ls}|Q_lQ_jf|)^q\Big)^{1/q}\Big\|_{L_p(P_{k,\eta})}$$
$$\leqslant c2^{jn(1/w-1)}\Big[\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\gamma_j(y)|^w\Big\|\Big(\sum_{l\geqslant k_1}(2^{ls}|Q_lf(\cdot-y)|)^q\Big)^{1/q}\Big\|^w_{L_p(P_{k,\eta})}\,\mathrm{d}y\Big]^{1/w}.$$

Then, we use (4.4) with $\Big(\sum\limits_{l\geqslant k_1}(2^{ls}|Q_lf(\cdot-y)|)^q\Big)^{1/q}$ instead of $f(\cdot-y)$ and we continue as in B-case. The study of S_0Q_jf on $L_p(P_{k,\eta})$ if $k\leqslant 0$ is exactly similar to the above, which gives us an estimate similar to (4.21) with $B^{s,\tau}_{p,q}$ replaced by $F^{s,\tau}_{p,q}$. The result follows.

□

Proof of Theorem 3.2. *Step 1*: Convergence in $\mathcal{S}'_\infty$. We begin by proving the estimate

$$\sup_{x\in P_{j,w}}|u_j(2x/b)|\leqslant c2^{j(n/p-s-n\tau)}A,\quad\forall\, j\in\mathbb{Z},\ \forall\, w\in\mathbb{Z}^n. \tag{4.22}$$

To do this, we use the following lemma, which is proved in [6], page 782, (2.11), see also [15], proof of Proposition 2.6.

Lemma 4.3. *There exists a constant* $c>0$ *such that the inequality*

$$\sup_{x\in P_{j,w}}|g(x)|\leqslant c2^{jn/p}\sup_{\eta\in\mathbb{Z}^n}\|g\|_{L_p(P_{j,\eta})}\quad\forall\, j\in\mathbb{Z},\ \forall\, w\in\mathbb{Z}^n$$

holds for all $g\in\mathcal{S}'$ *with* $\operatorname{supp}\widehat{g}\subset\{\xi\colon\ |\xi|\leqslant 2^{j+1}\}$.

Applying this lemma with $g(x):=u_j(2x/b)$ and using an easy proof

$$x\in P_{j,\eta}\Rightarrow 2x/b\in\bigcup_{r=0}^{([2/b]+1)^n}P_{j,E(2\eta/b)+w_r},\quad w_r\in\mathbb{Z}^n,$$

we get

$$\|u_j(2(\cdot)/b)\|_{L_p(P_{j,\eta})}\leqslant\Big((2/b)^{-n}\sum_{r=0}^{([2/b]+1)^n}\|u_j\|^p_{L_p(P_{j,E(2\eta/b)+w_r})}\Big)^{1/p}\leqslant c\sup_{\nu\in\mathbb{Z}^n}\|u_j\|_{L_p(P_{j,\nu})}.$$

Hence, it holds

$$\sup_{x\in P_{j,w}}|u_j(2x/b)|\leqslant c2^{jn/p}\sup_{\nu\in\mathbb{Z}^n}\|u_j\|_{L_p(P_{j,\nu})}\quad\forall\, j\in\mathbb{Z},\ \forall\, w\in\mathbb{Z}^n. \tag{4.23}$$

On the other hand, for all $j, l \in \mathbb{Z}$ such that $j \geqslant l$, it holds that

$$(4.24)\qquad \|u_j\|_{L_p(P_{l,\nu})} \leqslant 2^{-js-ln\tau} \times 2^{ln\tau} \Big(\sum_{k \geqslant l} 2^{ksq} \|u_k\|^q_{L_p(P_{l,\nu})}\Big)^{1/q} \leqslant 2^{-js-ln\tau} A$$

in the B-case, and

$$(4.25)\qquad \|u_j\|_{L_p(P_{l,\nu})} \leqslant 2^{-js-ln\tau} \times 2^{ln\tau} \Big\| \Big(\sum_{k \geqslant l} 2^{ksq} |u_k|^q\Big)^{1/q} \Big\|_{L_p(P_{l,\nu})} \leqslant 2^{-js-ln\tau} A$$

in the F-case. Next in (4.23), inserting (4.24) in B-case ((4.25) in F-case) with $j = l$, we obtain the desired estimate, i.e., (4.22).

We now turn to the convergence. Let $f \in \mathcal{S}_\infty$. Due to $\operatorname{supp} \widehat{u_j}$, there exists an integer r, depending only on b, such that $S_{j+r}(u_j) = u_j$, hence $\langle u_j, f\rangle = \langle u_j, S_{j+r} f\rangle$ for all $j \in \mathbb{Z}$. We continue by writing $\sum_{j\in\mathbb{Z}} |\langle u_j, f\rangle| = I_1 + I_2$ where $I_1 := \sum_{j<0} \ldots$ and $I_2 := \sum_{j\geqslant 0} \ldots$

▷ *Estimate of* I_1. By changing variables and by writing $\int_{\mathbb{R}^n} = \sum_{w\in\mathbb{Z}^n} \int_{P_{j,w}}$, we get

$$I_1 \leqslant \Big(\frac{2}{b}\Big)^n \sum_{j<0} \sum_{w\in\mathbb{Z}^n} \int_{P_{j,w}} \Big|u_j\Big(\frac{2x}{b}\Big)\Big| \Big|S_{j+r} f\Big(\frac{2x}{b}\Big)\Big| \,\mathrm{d}x.$$

By Lemma 2.1, we choose an integer N satisfying $\|S_{j+r} f\|_1 = O(2^{jN})$ for $j < 0$ and $N > s + n\tau - n/p$, thus by (4.22) it holds

$$\begin{aligned} I_1 &\leqslant c_1 A \sum_{j<0} 2^{j(n/p-s-n\tau)} \sum_{w\in\mathbb{Z}^n} \int_{P_{j,w}} \Big|S_{j+r} f\Big(\frac{2x}{b}\Big)\Big| \,\mathrm{d}x \\ &\leqslant c_2 A \sum_{j<0} 2^{j(n/p-s-n\tau)} \int_{\mathbb{R}^n} |S_{j+r} f(x)| \,\mathrm{d}x \leqslant c_3 A \sum_{j<0} 2^{j(N+n/p-s-n\tau)} \leqslant c_4 A. \end{aligned}$$

▷ *Estimate of* I_2. We first apply (4.24)–(4.25) with $l = 0$ (i.e., $j \geqslant 0$) to obtain

$$\|u_j\|_{L_p(P_{0,w})} \leqslant 2^{-js} A.$$

On the other hand, let us introduce a parameter d (at our further disposal) such that $0 < d < p$ if $0 < p \leqslant 1$ and $d := 1$ if $p > 1$, thus by the Bernstein inequality and

Hölder inequality with exponent $1/u := 1/d - 1/p$, we obtain
(4.26)

$$\begin{aligned} I_2 &\leqslant c_1 \sum_{j\geqslant 0} 2^{jn(1/d-1)} \left(\int_{\mathbb{R}^n} |u_j(x) S_{j+r} f(x)|^d \,\mathrm{d}x \right)^{1/d} \\ &\leqslant c_2 \sum_{j\geqslant 0} 2^{jn(1/d-1)} \left[\sum_{w\in\mathbb{Z}^n} \left(\int_{P_{0,w}} |u_j(x)|^p \,\mathrm{d}x \right)^{d/p} \left(\int_{P_{0,w}} |S_{j+r} f(x)|^u \,\mathrm{d}x \right)^{d/u} \right]^{1/d} \\ &\leqslant c_2 A \sum_{j\geqslant 0} 2^{j(-s-n+n/d)} \left(\sum_{w\in\mathbb{Z}^n} \left(\int_{P_{0,w}} |S_{j+r} f(x)|^u \,\mathrm{d}x \right)^{d/u} \right)^{1/d}. \end{aligned}$$

We observe that if $x \in P_{0,w}$, then $1 + |w| \leqslant c(1 + |x|)$ with $c := c(n) > 0$; by using the inequality $1 + |x| \leqslant (1 + |y|)(1 + |x - y|)$ and choosing d so that $u \geqslant 1$, i.e.,

$$\frac{p}{p+1} < d < p \leqslant 1 \quad \text{and} \quad d = 1 \text{ if } p > 1, \tag{4.27}$$

we can apply the Minkowski inequality w.r.t. $L_u(P_{0,w}; L_1(\mathbb{R}^n))$ and obtain

$$\begin{aligned} \left(\int_{P_{0,w}} |S_{j+r} f(x)|^u \,\mathrm{d}x \right)^{1/u} &\leqslant c_1 (1 + |w|)^{-N} \left(\int_{P_{0,w}} (1 + |x|)^{Nu} |S_{j+r} f(x)|^u \,\mathrm{d}x \right)^{1/u} \\ &\leqslant c_1 2^{(j+r)n} (1 + |w|)^{-N} \left[\int_{\mathbb{R}^n} (1 + |y|)^N |\mathcal{F}^{-1} \varrho(2^{j+r} y)| \right. \\ &\qquad \left. \times \left(\int_{\mathbb{R}^n} (1 + |x - y|)^{Nu} |f(x - y)|^u \,\mathrm{d}x \right)^{1/u} \right] \mathrm{d}y \\ &\leqslant c_2 2^{(j+r)n} (1 + |w|)^{-N} \int_{\mathbb{R}^n} (1 + |y|)^N |\mathcal{F}^{-1} \varrho(2^{j+r} y)| \,\mathrm{d}y, \end{aligned}$$

where we used $f \in \mathcal{S}_\infty$; the number $N \in \mathbb{N}$ is at our disposal. Trivially we have (recall that $j \geqslant 0$)

$$2^{(j+r)n} \int_{\mathbb{R}^n} (1 + |y|)^N |\mathcal{F}^{-1} \varrho(2^{j+r} y)| \,\mathrm{d}y \leqslant \int_{\mathbb{R}^n} (1 + 2^{-r}|z|)^N |\mathcal{F}^{-1} \varrho(z)| \,\mathrm{d}z \leqslant c,$$

recall that $r := r(b)$. Hence,

$$\left(\int_{P_{0,w}} |S_{j+r} f(x)|^u \,\mathrm{d}x \right)^{1/u} \leqslant c(1 + |w|)^{-N}.$$

Since the assumption $s > (n/p - n)_+$, choose d such that

$$s + n - \frac{n}{d} > 0, \tag{4.28}$$

choose also N such that $Nd \geqslant n+1$ (i.e., $\sum_{w\in\mathbb{Z}^n}(1+|w|)^{-Nd} < \infty$), then from (4.26) we get

$$I_2 \leqslant c_1 A \sum_{j\geqslant 0} 2^{-j(s+n-n/d)} \leqslant c_2 A.$$

Thus, from (4.27)–(4.28) we must find a number d such that

$$d = 1 \quad \text{if} \quad p > 1 \quad \text{and} \quad \frac{1}{p} < \frac{1}{d} < \min\Big(1+\frac{s}{n}, 1+\frac{1}{p}\Big) \quad \text{if } 0 < p \leqslant 1.$$

Step 2: Proof of the estimate $\Big\| \sum_{j\in\mathbb{Z}} u_j \Big\|_{\dot{A}^{s,\tau}_{p,q}} \leqslant cA$. We put $u := \sum_{j\in\mathbb{Z}} u_j \in \mathcal{S}'_\infty$ due to the preceding step. Owing to supp $\widehat{u_j}$, there exists an integer m, depending only on b, such that $Q_k(u_j) \equiv 0$ if $j \leqslant k+m$ (m is the nearest integer to $-\log_2(2b)$). We continue by separating the case $p \geqslant 1$ from that of $p < 1$.

Substep 2.1: Case $p \geqslant 1$. We treat the F-case since the B-case is similar to that given in [1], pages 358–360. We introduce $q_1 \geqslant \max(p,q)$. For $j > k+m$ we have

$$|Q_k u_j(x)| \leqslant 2^{-js}\Big(\sum_{l>k+m} 2^{lsq_1}|Q_k u_l(x)|^{q_1}\Big)^{1/q_1},$$

then it holds

$$\text{(4.29)} \qquad \begin{aligned} |Q_k u(x)| &\leqslant \Big(\sum_{l>k+m} 2^{lsq_1}|Q_k u_l(x)|^{q_1}\Big)^{1/q_1} \sum_{j>k+m} 2^{-js} \\ &\leqslant c2^{-ks}\Big(\sum_{j>k+m} 2^{jsq_1}|Q_k u_j(x)|^{q_1}\Big)^{1/q_1} \end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$. Using the Hölder inequality with $1/p' := 1 - 1/p$, we obtain

$$\text{(4.30)} \qquad \begin{aligned} |Q_k u_j(x)| &\leqslant \sum_{w\in\mathbb{Z}^n} \int_{P_{l,w}} |u_j(y)||2^{kn}\mathcal{F}^{-1}\gamma(2^k(x-y))|\,\mathrm{d}y \\ &\leqslant \sum_{w\in\mathbb{Z}^n} \|u_j\|_{L_p(P_{l,w})}\Big(\int_{P_{l,w}} |2^{kn}\mathcal{F}^{-1}\gamma(2^k(x-y))|^{p'}\,\mathrm{d}y\Big)^{1/p'}. \end{aligned}$$

We now take $x \in P_{l,\eta}$, then if $y \in P_{l,w}$ we obtain

$$1+|w-\eta| \leqslant 2\sqrt{2n}(1+2^l|x-y|) \leqslant 2\sqrt{2n}(1+2^k|x-y|) \quad \text{with} \quad k \geqslant l,$$

which implies, for all $N \in \mathbb{N}$,

$$\text{(4.31)} \quad \Big(\int_{P_{l,w}} |2^{kn}\mathcal{F}^{-1}\gamma(2^k(x-y))|^{p'}\,\mathrm{d}y\Big)^{1/p'} \leqslant c2^{kn-ln/p'}(1+|w-\eta|)^{-N}, \quad k \geqslant l.$$

We choose $N := n+1$. Consequently (4.30) becomes

$$|Q_k u_j(x)| \leqslant c 2^{kn-ln/p'} \sum_{w\in\mathbb{Z}^n} \|u_j\|_{L_p(P_{l,w})}(1+|w-\eta|)^{-(n+1)} \tag{4.32}$$

for all $j\in\mathbb{Z}$, all $(k,l)\in\mathbb{Z}^2$ such that $k\geqslant l$, all $\eta\in\mathbb{Z}^n$ and all $x\in P_{l,\eta}$.

On the other hand, as the two annuli $\frac{1}{2}2^j \leqslant |\xi| \leqslant \frac{3}{2}2^j$ and $\frac{1}{2}2^k \leqslant |\xi| \leqslant \frac{3}{2}2^k$ are disjoint if $|j-k|\geqslant 2$, thus $\sum_{k\in\mathbb{Z}} \widehat{Q_k u}(\xi)$ is locally finite for all $\xi\in\mathbb{R}^n\setminus\{0\}$. Consequently, the sum $\sum_{k\in\mathbb{Z}} Q_k u(x)$ contains at most three non-vanishing terms; then, $k\in\Lambda \subsetneqq \mathbb{Z}$ with $\operatorname{Card}\Lambda = 3$ (Card means cardinality), where the set Λ is constituted by *consecutive elements*, say $\Lambda := \{J, J+1, J+2\}$. Hence, from (4.29) we obtain

$$\sum_{k\geqslant l} 2^{ksq}|Q_k u(x)|^q \leqslant c \sum_{k\geqslant l, k\in\Lambda} \Big(\sum_{j>k+m} 2^{jsq_1}|Q_k u_j(x)|^{q_1}\Big)^{q/q_1} \qquad \forall x\in\mathbb{R}^n. \tag{4.33}$$

Choosing $l := J$ and inserting (4.32) into (4.33) with $x\in P_{J,\eta}$, then, using the estimate $0\leqslant 2^{kn-Jn/p'} \leqslant 2^{2n+Jn/p}$, $k\in\Lambda$ and the Minkowski inequality w.r.t. $l_{q_1}(\mathbb{Z}; l_1(\mathbb{Z}^n))$ (recall that $q_1\geqslant p\geqslant 1$), the right-hand side of (4.33) is bounded by

(4.34)

$$\begin{aligned}
&c_1 2^{Jqn/p} \sum_{k=J}^{J+2} \Big[\sum_{j>k+m} 2^{jsq_1}\Big(\sum_{w\in\mathbb{Z}^n} \|u_j\|_{L_p(P_{J,w})}(1+|w-\eta|)^{-(n+1)}\Big)^{q_1}\Big]^{q/q_1} \\
&\quad\leqslant c_1 2^{Jqn/p} \sum_{k=J}^{J+2} \Big[\sum_{w\in\mathbb{Z}^n}\Big(\sum_{j>k+m} (2^{js}\|u_j\|_{L_p(P_{J,w})})^{q_1}\Big)^{1/q_1}(1+|w-\eta|)^{-(n+1)}\Big]^q \\
&\quad\leqslant c_2 2^{Jqn/p} \sup_{\nu\in\mathbb{Z}^n}\Big(\sum_{j\geqslant J+m} 2^{jsq_1}\|u_j\|^{q_1}_{L_p(P_{J,\nu})}\Big)^{q/q_1}\Big(\sum_{w\in\mathbb{Z}^n}(1+|w-\eta|)^{-(n+1)}\Big)^q \\
&\quad\leqslant c_3 2^{Jqn/p} \sup_{\nu\in\mathbb{Z}^n}\Big(\sum_{j\geqslant J+m} 2^{jsq_1}\|u_j\|^{q_1}_{L_p(P_{J,\nu})}\Big)^{q/q_1}
\end{aligned}$$

where c_1, c_2, c_3 are independent of x, η and J. On the other hand, by the elementary inequality

$$[2^m\nu_l] \leqslant 2^{J+m}x_l < [2^m\nu_l] + [2^m] + 2, \quad x\in P_{J,\nu},\ l=1,\dots,n$$

we obtain

$$P_{J,\nu} \subset \bigcup_{r=0}^{([2^m]+1)^n} P_{J+m, E(2^m\nu)+w_r} \tag{4.35}$$

where $w_r \in \mathbb{Z}^n$. Then, from the last term in (4.34) we find that

$$\begin{aligned}
\sum_{j \geqslant J+m} 2^{jsq_1} \|u_j\|^{q_1}_{L_p(P_{J,\nu})} &\leqslant \sum_{j \geqslant J+m} 2^{jsq_1} \left(\sum_{r=0}^{([2^m]+1)^n} \|u_j\|_{L_p(P_{J+m,E(2^m\nu)+w_r})} \right)^{q_1} \\
&\leqslant c_1 \sum_{r=0}^{([2^m]+1)^n} \sum_{j \geqslant J+m} 2^{jsq_1} \|u_j\|^{q_1}_{L_p(P_{J+m,E(2^m\nu)+w_r})} \\
&\leqslant c_1 \sum_{r=0}^{([2^m]+1)^n} \left\| \left(\sum_{j \geqslant J+m} 2^{jsq_1} |u_j|^{q_1} \right)^{1/q_1} \right\|^{q_1}_{L_p(P_{J+m,E(2^m\nu)+w_r})},
\end{aligned}$$

where we used the Minkowski inequality w.r.t. $l_{q_1}(\mathbb{Z}; L_p(P_{J+m,E(2^m\nu)+w_r}))$. By the embedding $l_q(\mathbb{Z}) \hookrightarrow l_{q_1}(\mathbb{Z})$, the last inequality is bounded by

$$c_2 \sum_{r=0}^{([2^m]+1)^n} \left\| \left(\sum_{j \geqslant J+m} 2^{jsq} |u_j|^q \right)^{1/q} \right\|^{q_1}_{L_p(P_{J+m,E(2^m\nu)+w_r})} \leqslant c_3 (2^{-Jn\tau} A)^{q_1}.$$

Hence, from (4.33) and (4.34) we get

$$\left(\sum_{k \geqslant J} 2^{ksq} |Q_k u(x)|^q \right)^{1/q} \leqslant c 2^{Jn/p - Jn\tau} A \quad \forall\, x \in P_{J,\eta}. \tag{4.36}$$

Now, by dividing both sides of this inequality by $2^{-Jn\tau}$ and by calculating the $L_p(P_{J,\eta})$ norm; as J is arbitrary one obtains the desired result.

Substep 2.2: Case $p < 1$. We follow Step 1/Estimate of I_2, also use the notations of the preceding substep. We observe that the support of the Fourier transform of the function $y \mapsto 2^{kn} u_j(y) \mathcal{F}^{-1}\gamma(2^k(x-y))$ is the ball $|\xi| \leqslant (b + \frac{3}{2} 2^{-m}) 2^j$ (recall that $j > k + m$, cf. the third sentence in Step 2). We introduce two parameters d and u such that $0 < d < p$ and $1/u := 1/d - 1/p$, then by the Bernstein and Hölder inequalities we get

$$\begin{aligned}
|Q_k u_j(x)| &\leqslant c 2^{jn(1/d-1)} \left[\int_{\mathbb{R}^n} |2^{kn} u_j(y) \mathcal{F}^{-1}\gamma(2^k(x-y))|^d \,\mathrm{d}y \right]^{1/d} \\
&\leqslant c 2^{jn(1/d-1)} \left[\sum_{w \in \mathbb{Z}^n} \left(\int_{P_{l,w}} |u_j(y)|^p \,\mathrm{d}y \right)^{d/p} \right. \\
&\qquad \left. \times \left(\int_{P_{l,w}} |2^{kn} \mathcal{F}^{-1}\gamma(2^k(x-y))|^u \,\mathrm{d}y \right)^{d/u} \right]^{1/d}
\end{aligned} \tag{4.37}$$

for all $l \in \mathbb{Z}$. We proceed as in (4.31) by taking $x \in P_{l,\eta}$ and $N \in \mathbb{N}$, then

$$\left(\int_{P_{l,w}} |2^{kn} \mathcal{F}^{-1}\gamma(2^k(x-y))|^u \,\mathrm{d}y \right)^{1/u} \leqslant c 2^{n(k - l/u)} (1 + |w - \eta|)^{-N}, \quad k \geqslant l,$$

the constant c is independent of k, l and x. Inserting this inequality into (4.37) and using the Minkowski inequality w.r.t. $l_{1/d}(\mathbb{Z}; l_1(\mathbb{Z}^n))$, we obtain

(4.38)

$$\begin{aligned} |Q_k u(x)| &\leqslant \sum_{j>k+m} |Q_k u_j(x)| \quad \text{(recall that } k \geqslant l) \\ &\leqslant c 2^{n(k-l/u)} \sum_{j>l+m} 2^{jn(1/d-1)} \Big[\sum_{w\in\mathbb{Z}^n} \|u_j\|^d_{L_p(P_{l,w})} (1+|w-\eta|)^{-Nd} \Big]^{1/d} \\ &\leqslant c 2^{n(k-l/u)} \Bigg(\sum_{w\in\mathbb{Z}^n} \Big[\sum_{j>l+m} 2^{jn(1/d-1)} \|u_j\|_{L_p(P_{l,w})} (1+|w-\eta|)^{-N} \Big]^d \Bigg)^{1/d} \end{aligned}$$

for all $x \in P_{l,\eta}$. Next, using the inclusion (4.35) applied to $P_{l,w}$, and choosing d such that $s+n-n/d>0$, which is possible since $s > n/p - n$, and the fact that

$$\|u_j\|_{L_p(P_{l,w})} \leqslant \Bigg(\sum_{r=0}^{([2^m]+1)^n} \|u_j\|^p_{L_p(P_{l+m,E(2^m w)+w_r})} \Bigg)^{1/p} \leqslant c \sum_{r=0}^{([2^m]+1)^n} \|u_j\|_{L_p(P_{l+m,E(2^m w)+w_r})}$$

where $c := 2^{([2^m]+1)^n(1/p-1)}$ (this follows from (4.5)), it holds that

$$\begin{aligned} \sum_{j>l+m} 2^{jn(1/d-1)} \|u_j\|_{L_p(P_{l,w})} &\leqslant c_1 \sum_{j>l+m} 2^{jn(1/d-1)} \Bigg(\sum_{r=0}^{([2^m]+1)^n} \|u_j\|_{L_p(P_{l+m,E(2^m w)+w_r})} \Bigg) \\ &\leqslant c_2 2^{-ln\tau} A \sum_{j>l+m} 2^{j(n/d-n-s)} \leqslant c_3 2^{l(n/d-n-s-n\tau)} A, \end{aligned}$$

where we used (4.24) in the B-case and (4.25) in the F-case, with $l+m$ instead of l (i.e., $j \geqslant l+m$). Inserting this estimate into (4.38) and choosing N such that $Nd \geqslant n+1$ (i.e., $\sum_{w\in\mathbb{Z}^n} (1+|w-\eta|)^{-Nd} < \infty$), one has

$$|Q_k u(x)| \leqslant c 2^{kn+l(n/p-n-s-n\tau)} A, \quad k \geqslant l, \ x \in P_{l,\eta}. \tag{4.39}$$

Now since we treat the B-case, we have

$$\Bigg(\sum_{k\geqslant l} 2^{ksq} \|Q_k u(x)\|^q_{L_p(P_{l,\eta})} \Bigg)^{1/q} \leqslant c 2^{-l(n+s+n\tau)} A \Bigg(\sum_{k\geqslant l, k\in\Lambda} 2^{k(n+s)q} \Bigg)^{1/q}.$$

Choosing $l := J$ (the integer J is defined by $k \in \Lambda = \{J, J+1, J+2\}$, cf. Substep 2.1), it follows

$$\Bigg(\sum_{k\geqslant J} 2^{ksq} \|Q_k u(x)\|^q_{L_p(P_{J,\eta})} \Bigg)^{1/q} \leqslant c 2^{-Jn\tau} A.$$

Now, by dividing both sides of this inequality by $2^{-Jn\tau}$ and by the arbitrariness of J one obtains the desired result. In the F-case, from (4.39) we choose again $l := J$ and proceed as in (4.36). The proof is complete. □

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