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A TWISTED CLASS NUMBER FORMULA AND GROSS'S SPECIAL UNITS OVER AN IMAGINARY QUADRATIC FIELD

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Abstract. Let F/k be a finite abelian extension of number fields with k imaginary quadratic. Let O_F be the ring of integers of F and $n \geqslant 2$ a rational integer. We construct a submodule in the higher odd-degree algebraic K-groups of O_F using corresponding Gross's special elements. We show that this submodule is of finite index and prove that this index can be computed using the higher "twisted" class number of F, which is the cardinal of the finite algebraic K-group $K_{2n-2}(O_F)$.

Keywords: algebraic K-theory; Dedekind zeta function; Artin L-function; Beilinson regulator; generalized index; Lichtenbaum conjecture

MSC 2020: 11R70, 19F27

1. Introduction

Let F be a number field. The Quillen higher algebraic K-groups of the ring of integers O_F of the field F have the following property:

Theorem 1.1 (Borel). *For integers $n \geqslant 2$, the groups $K_{2n-2}(O_F)$ are finite and the groups $K_{2n-1}(F) = K_{2n-1}(O_F)$ are of finite type over $\mathbb{Z}$,*

$$\operatorname{rank}_{\mathbb{Z}}(K_{2n-1}(F)) := d_n = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd,} \\ r_2 & \text{if } n \text{ is even,} \end{cases}$$

where r_1 and r_2 denote, respectively, the number of real and complex places of F.

Let O_F^* be the group of units of O_F. Let $\zeta_F(s)$ denote the Dedekind zeta function associated with the number field F and defined for $(\operatorname{Re}(s) > 1)$ by the following

Euler product over all prime ideals of O_F:

$$\zeta_F(s) = \prod_{\mathfrak{p} \subseteq O_F} \frac{1}{1 - \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-s}},$$

where $\mathrm{N}_{F/\mathbb{Q}}$ is the norm map relative to the extension $F/\mathbb{Q}$. The function $\zeta_F(s)$ has an analytic continuation to the complex plane $\mathbb{C}$ as a meromorphic function with a simple pole at $s = 1$. Moreover, it is known that the order of vanishing of the function $\zeta_F(s)$ at $s = 0$ is exactly the $\mathbb{Z}$-rank of the group of units O_F^* equal by Dirichlet's unit theorem to $r_1 + r_2 - 1$.

At strictly negative integers $1 - n$ ($n \geqslant 2$), the zeta function $\zeta_F(s)$ has the order of vanishing equal to d_n, the $\mathbb{Z}$-rank of the group $K_{2n-1}(F)$: This shows that the group $K_{2n-1}(F)$ plays at negative integers the role the unit group O_F^* plays at zero. This is the first observation of a far-reaching analogy between higher algebraic K-theory and the classical arithmetic objects associated with F: In fact, for $n \geqslant 2$, the odd degree K-groups $K_{2n-1}(F)$ are the analogues of the unit group O_F^* and the even degree finite K-groups $K_{2n-2}(O_F)$ are the analogues of the class group Cl_F of F. To go even further with this analogy, Siegel and Klingen showed in [12], [20] that when F is totally real, $\zeta_F(s)$ is a nonzero rational number at negative odd integers. Borel in [1] later built on this result and showed that the special value $\zeta_F^*(1-n) := \lim_{s \to 1-n} (s+n-1)^{-d_n} \zeta_F(s)$ is always the product of a rational number q_n by the Borel regulator $R_n^B(F) \in \mathbb{R}^\times$. The Borel regulator is exactly the Beilinson regulator described below in Subsection 2.1 multiplied by a fixed power of 2 (by [3], this fixed power is exactly 2^{d_n}). This motivated Lichtenbaum to conjecture the exact expression of the rational number q_n in terms of higher algebraic K-groups: This is the famous Lichtenbaum conjecture (see Subsection 4 for its statement). Another remarkable example is the work of Kurihara (see [14]) to prove Vandiver's conjecture using K-theory. Kurihara showed that the conjecture is equivalent to the vanishing of the groups $K_n(\mathbb{Z})$ whenever n is a multiple of 4. Later, Soulé in [25] expanded Kurihara's result using similar methods, see for example the article of [9] for more details on this work and the proof of Vandiver's conjecture using K-theory.

The next step in this analogy framework concerns "special" units which are related to the special values of L-functions and which play an important role in number theory. These are for example cyclotomic units for abelian extensions of $\mathbb{Q}$ and more generally Stark units, see [27]. Gross formulated in the 1970s a conjecture which asserts that the leading term at any strictly negative integer of the Artin L-function should be equal, within an undetermined algebraic factor, to the image by the Beilinson regulator of a certain element in odd degree higher algebraic K-group (Gross's conjecture was published more recently in [10]). Gross's conjecture is the natural analogue in higher algebraic K-theory of Stark's conjecture.

In the absolute abelian case, Gross's conjectural units are given by Beilinson's elements (see [16]) and p-adically by the Beilinson-Deligne-Soulé elements, see e.g., [24]. An application of Beilinson's units is the following equality which holds (see [13], proof of Theorem 6.4) whenever F is an abelian number field and p is an odd rational prime ($n \geqslant 2$):

$$\left|\frac{K_{2n-1}(F)_{/\mathrm{tors}}}{B_n(F)}\right| \sim_p |K_{2n-2}(O_F)|, \tag{1.1}$$

where $K_{2n-1}(F)_{/\mathrm{tors}}$ is the torsion-free quotient of $K_{2n-1}(F)$ and $B_n(F)$ is a submodule of $K_{2n-1}(F)_{/\mathrm{tors}}$ constructed from Beilinson's units ($\sim_p$ refers to equality of p-primary parts). This equation requires the Quillen-Lichtenbaum conjecture (i.e., the local higher algebraic K-groups coincide with cohomological K-groups for all odd p), which is now a theorem thanks to work of Rost, Weibel and Voevodsky, see [29]. There are also refinements of this result, see e.g., [5], [6] among others. Equation (1.1) is a natural analogue of the following classical equality of cardinals proved by Sinnott, see [21], Section 4:

$$|O_F^*/C_F| = |\mathrm{Cl}_{F^+}| \cdot c_F, \tag{1.2}$$

where C_F is the subgroup of circular units (see [21], Section 4), F^+ is the maximal real subfield of F and c_F is a rational number whose definition does not involve $|\mathrm{Cl}_{F^+}|$.

In the nonabsolute abelian case, we can mention partial results obtained by Snaith, see [22], and Nickel, see [18]. Nickel shows that the Equivariant Tamagawa Number conjecture, or ETNc for short, (see e.g., [8] for an excellent survey on the ETNc) implies the existence of certain elements constructed using leading terms of Artin L-functions and higher algebraic K-theory which verify a refinement of the Coates-Sinnott conjecture.

When the field F is "almost" abelian (i.e., F is a finite abelian extension of an imaginary quadratic field), the ETNc provides the exact definition of the special units mentioned before, see Subsection 2.2 below for more details. The aim of the following work is to generalize equation (1.1) to the case of abelian extensions of an imaginary quadratic field (see Theorem 2.2 below) using these special units. For this purpose we combine techniques of calculation of generalized indices (see Subsection 3) and the Lichtenbaum conjecture.

1.1. Notations. We adopt the following notations throughout this paper:

- ▷ F/k is a finite abelian extension of number fields with Galois group $G := \mathrm{Gal}(F/k)$. From Subsection 2.2 onward we suppose that k is imaginary quadratic.
- ▷ If K is any number field, we let O_K denote its ring of integers and we fix an algebraic closure $\overline{K}$ of K. We set $G_K := \mathrm{Gal}(\overline{K}/K)$.

- ▷ For any nonzero integer r we write $\mathbb{Q}/\mathbb{Z}(r)$ for the group $\mathbb{Q}/\mathbb{Z}$ regarded as a G_F-module by setting $g.x := \chi_{\text{cyc}}(g)^r x$ for all $g \in G_F$ and $x \in \mathbb{Q}/\mathbb{Z}$, where χ_{cyc} is the cyclotomic character, see e.g., [17], Definition (7.3.6).
- ▷ If A is a finite group we let $|A|$ denote the cardinal of A (i.e., the number of elements of A).

2. Special "twisted" units in an odd-degree algeberic K-theory over an imaginary quadratic number field

2.1. The Beilinson regulator map. Let $n \geqslant 2$ be a rational integer. The Beilinson regulator defined over the K-group of the field of complex numbers (see e.g., [19] for more details) is a map

$$\mathrm{reg}_n\colon\ K_{2n-1}(\mathbb{C}) \to H^1_D(\mathrm{Spec}(\mathbb{C}), (2\pi\mathrm{i})^n\mathbb{R}) \cong (2\pi\mathrm{i})^{n-1}\mathbb{R},$$

where $H^1_D()$ is the first group of Deligne's cohomology and $\mathrm{i} := \sqrt{-1}$.

We assign a number field F with the map

$$K_{2n-1}(F) \to \prod_{\sigma\colon F\to\mathbb{C}} K_{2n-1}(\mathbb{C}),$$

where the product is taken over all embeddings $\sigma \in \mathrm{Hom}(F,\mathbb{C})$ of F in the field of complex numbers. We obtain a map

$$\mathrm{reg}_n(F)\colon\ K_{2n-1}(F) \to (X_F \otimes (2\pi\mathrm{i})^{n-1}\mathbb{R})^+,$$

where $X_F := \mathbb{Z}[\mathrm{Hom}(F,\mathbb{C})]$ (the free abelian group over the set $\mathrm{Hom}(F,\mathbb{C})$) and $(X_F \otimes (2\pi\mathrm{i})^{n-1}\mathbb{R})^+$ is the "plus part" of $(X_F \otimes (2\pi\mathrm{i})^{n-1}\mathbb{R})$, i.e., the submodule that is invariant under the action of complex conjugation.

Note that for all $n \geqslant 2$ we have $K_{2n-1}(O_F) = K_{2n-1}(F)$, see e.g., [23]. Thus, the map $\mathrm{reg}_n(F)$ can be equivalently defined over $K_{2n-1}(O_F)$.

The image of $\mathrm{reg}_n(F)$ is a complete lattice of the $\mathbb{R}$-vector-space $(X_F\otimes(2\pi\mathrm{i})^{n-1}\mathbb{R})^+$. The covolume $\mathrm{R}_n(F)$ of this lattice is also called the *Beilinson regulator*. Borel in [1] proved that the kernel of the Borel regulator is equal to $K_{2n-1}(F)_{\text{tors}}$, where $K_{2n-1}(F)_{\text{tors}}$ is the $\mathbb{Z}$-torsion subgroup of $K_{2n-1}(F)$. Since the Beilinson and the Borel regulators agree up to a fixed power of 2 (see [3]), we also have

$$\ker(\mathrm{reg}_n(F)) = K_{2n-1}(F)_{\text{tors}}.$$

Hence, if we write $K_{2n-1}(F)_{/\text{tors}}$ for the torsion-free quotient of $K_{2n-1}(F)$, then $\mathrm{reg}_n(F)$ induces the injective map

$$\mathrm{reg}_n(F)\colon\ K_{2n-1}(F)_{/\text{tors}} \hookrightarrow (X_F \otimes (2\pi\mathrm{i})^{n-1}\mathbb{R})^+,$$

which becomes an isomorphism if we tensor the left-hand side with $\mathbb{R}$.

2.2. Special "twisted" units. Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$. Let $\mathrm{Ram}(F/k)$ denote the set of finite places of k which ramify in F/k and S_∞ the set of Archimedean places of k. If $\chi \in \widehat{G}$, the Artin L-function attached to χ is defined for $\{s \in \mathbb{C}\colon \mathrm{Re}(s) > 1\}$ by

$$L(s,\chi) = \prod_{\mathfrak{p} \notin \mathrm{Ram}(F/k) \cup S_\infty} (1 - \chi(\sigma_{\mathfrak{p}}) \mathrm{N}\mathfrak{p}^{-s})^{-1},$$

where $\sigma_{\mathfrak{p}} \in G$ is the Frobenius of the (unramified) prime $\mathfrak{p}$. This function can be analytically continued to a meromorphic function on $\mathbb{C}$.

We can view $L(s,\chi)$ as the L-function of a Hecke character (one dimensional Artin representation) of the absolute Galois group G_k. The functional equation relating $L(s,\chi)$ and $L(1-s,\chi^{-1})$ shows that the order of vanishing $r_\chi(1-n)$ of $L(s,\chi)$ at $1-n$, $n > 1$, is given for any nontrivial character χ by

$$r_\chi(1-n) = r_2(k) + (\text{the number of real places of } k, \text{ where } \chi \text{ has sign } (-1)^{1-n}),$$

where the integer $r_2(k)$ denotes the number of complex places of k. Let $L'(s,\chi)$ denote the first derivative of the Artin L-function $L(s,\chi)$.

In the rest of the paper we assume that k is an imaginary quadratic number field and that $n \geqslant 2$.

Since k is imaginary quadratic, we get for all $\chi \in \widehat{G}$

$$L(1-n,\chi) = 0 \quad \text{and} \quad L'(1-n,\chi) \neq 0. \tag{2.1}$$

This is also shown in [4], (3.4), where a formula for the first derivative of the L-function $L(s,\chi)$ is also provided at negative integers.

Let $\chi \in \widehat{G}$. Let $\mathbb{Z}[\chi]$ be the ring generated over $\mathbb{Z}$ by the values of the character χ. Again since k is imaginary quadratic and taking into account the Galois action on $(X_F \otimes (2\pi \mathrm{i})^{n-1}\mathbb{R})^+$, we can see that

$$(X_F \otimes (2\pi \mathrm{i})^{n-1}\mathbb{R})^+ \cong \mathbb{R}[G] \quad (\text{as } \mathbb{R}[G] \text{ modules}).$$

Indeed, since k is imaginary quadratic, for each $\sigma \in \mathrm{Hom}(F,\mathbb{C})$ there is a unique $\widetilde{\sigma} \in G$ such that either $\sigma = \widetilde{\sigma}$ or $\sigma = \tau \circ \widetilde{\sigma}$, where τ stands for complex conjugation. If we add to this the fact that $(X_F \otimes (2\pi \mathrm{i})^{n-1}\mathbb{R})^+$ is generated (as an $\mathbb{R}$-module) by elements of the form $\sigma \otimes (2\pi\mathrm{i})^{n-1} \pm \tau\sigma \otimes (2\pi\mathrm{i})^{n-1}$ ($\sigma \in \mathrm{Hom}(F,\mathbb{C})$ and "$\pm$" depending on the parity of n), we can form the isomorphism above by sending each such element to $\widetilde{\sigma}^{-1}$ ($\widetilde{\sigma}^{-1}$ and not $\widetilde{\sigma}$ since the G-action on X_F is contragredient) and extending it by scalars over $\mathbb{R}$.

Thus, by extension of scalars, we can rewrite the Beilinson regulator as

$$\mathrm{reg}_n(F)\colon\ K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}[\chi] \hookrightarrow \mathbb{R}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[\chi].$$

Conjecture 2.1. *Suppose that $n \geqslant 2$. There exists a unique element $\varepsilon(\chi) \in K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}[\chi]$ (a special "twisted" unit) such that*

$$\mathrm{reg}_n(F)(\varepsilon(\chi)) = w_n(F^{\ker(\chi)})L'(1-n, \chi^{-1})|G|e_\chi,$$

where:
(1) $w_n(F^{\ker(\chi)}) := |H^0(\mathrm{Gal}(\overline{\mathbb{Q}}/F^{\ker(\chi)}), \mathbb{Q}/\mathbb{Z}(n))|$,
(2) $e_\chi := |G|^{-1} \sum_{\sigma \in G} \chi^{-1}(\sigma)\sigma$ *is the idempotent associated with* χ.

Note that the group $H^0(\mathrm{Gal}(\overline{\mathbb{Q}}/F^{\ker(\chi)}), \mathbb{Q}/\mathbb{Z}(n))$ is a "twisted" analogue of the group of roots of unity in $F^{\ker(\chi)}$, and is known to be finite for all $n \neq 0$, see e.g., [17], Proposition (7.3.10) (i). Further, we also know that for twists $n \geqslant 2$, the group $H^0(\mathrm{Gal}(\overline{\mathbb{Q}}/F^{\ker(\chi)}), \mathbb{Q}/\mathbb{Z}(n))$ agrees with the finite torsion group $K_{2n-1}(F^{\ker(\chi)})_{\mathrm{tors}}$ modulo a power of 2, see [13], Lemma 2.2 (1).

The strongest evidence in support of Conjecture 2.1 is the following theorem:

Theorem 2.1. *Conjecture* 2.1 *holds locally (i.e., its p-primary part holds) for all rational primes p which are split in k and such that $p \nmid 6$.*

Proof. The existence of $\varepsilon(\chi)$, and consequently the fact that

$$w_n(F^{\ker(\chi)})L'(1-n, \chi^{-1})|G|e_\chi \in \mathbb{R}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[\chi],$$

is a consequence of the Equivariant Tamagawa Number conjecture. This is shown in detail in [7].

By [11], the local Equivariant Tamagawa Number conjecture holds in the case of a finite extension of an imaginary quadratic number field for all primes p which are split in k and such that $p \nmid 6$. □

In the rest of this work we assume that Conjecture 2.1 holds (i.e., holds locally for all primes).

The element $\varepsilon(\chi)$ is unique since $\mathrm{reg}_n(F)$ is injective over $K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}[\chi]$.

Remark 2.1. We call the element $\varepsilon(\chi)$ a "twisted unit" since the group $K_{2n-1}(F) = K_{2n-1}(O_F)$ is considered in the literature to be a "Tate-twisted" analogue of the group of units of the ring of integers O_F of F, see for example the introduction of [13].

The element $\varepsilon(\chi)$ is said to be special, since it maps through the Beilinson regulator to the "special" value of the L-function $L(s, \chi)$ at $1-n$.

2.3. The main result. Let $\mathcal{E}$ denote the field extension of $\mathbb{Q}$ generated by all values of characters $\chi \in \widehat{G}$ and let $\mathcal{O}$ be its ring of integers. Let $\mathfrak{d} := \mathrm{Rank}_{\mathbb{Z}}(\mathcal{O})$.

We define $\mathcal{Y}_F$ as the $\mathcal{O}[G]$-module generated by the set $\{\varepsilon(\chi), \chi \in \widehat{G}\}$ and written additively. It is clear that $\mathcal{Y}_F$ is a submodule of $K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}$.

Our main result is the following theorem:

Theorem 2.2. *Suppose that the Lichtenbaum conjecture (see Section 4 below) holds for F. Then the quotient $(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O})/\mathcal{Y}_F$ is finite and we have*

$$\left|\frac{K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}}{\mathcal{Y}_F}\right| \overset{2}{=} \left(\frac{\left(\prod_{\chi \in \widehat{G}} w_n(F^{\ker(\chi)})\right)|G|^{|G|}}{|K_{2n-1}(O_F)_{\mathrm{tors}}|}|K_{2n-2}(O_F)|\right)^{\mathfrak{d}},$$

where "$\overset{2}{=}$" denotes equality up to a power of 2.

3. The notion of generalized index

For any two $\mathbb{Z}$-modules A and B such that $B \subseteq A$ and A/B is finite, we write $[A : B]$ to refer to the usual index of B in A, equal to the cardinal of the finite group A/B.

Let O be an integral domain which is also a free finite-extension of $\mathbb{Z}$, i.e., $O \cong \mathbb{Z}^k$ for some $k \in \mathbb{Z}_{\geqslant 1}$ (e.g., the ring of integers of a number field) and E a field containing O and flat as an O-module. Let $\mathcal{V}$ denote a finite dimensional E-vector space. An O-lattice M of $\mathcal{V}$ of rank d_M is a free O-submodule of $\mathcal{V}$ which verifies

$$d_M := \mathrm{rk}_O M = \dim_E EM,$$

where $EM := E \otimes_O M$. Let $M \neq 0$ (respectively N) be such O-lattices of rank d_M (respectively d_N) of a given E-vector space $\mathcal{V}$. We recall that the generalized index $(M : N)_O$ over O of N in M is defined as the O-module

$$(M : N)_O := \langle \det(u),\ u \in \mathrm{End}_E(\mathcal{V}') \text{ and } u(M) \subseteq N \rangle,$$

where $\mathcal{V}'$ is the E-vector subspace of $\mathcal{V}$ generated by M and N. If

$$d_M = \mathrm{rk}_O M = \dim_E \mathcal{V},$$

we say that M is a complete O-lattice in the E-vector space $\mathcal{V} = EM$. Let M and N be such complete lattices of the vector space $\mathcal{V} = EM = EN$. Since we suppose $\mathrm{rk}_O M = \mathrm{rk}_O N = \dim_E \mathcal{V}$, we have

$$(M : N)_O = \det_E(u) O$$

for any given endomorphism u of $\mathcal{V}$ such that $u(M) = N$.

For instance, if $d := d_M = d_N$, and $(e_i)_{1\leqslant i\leqslant d}$ (respectively $(\varepsilon_i)_{1\leqslant i\leqslant d}$) is an O-basis of M (respectively of N), then one has

$$(M : N)_O = \det(\mathrm{Mat}_{(e_i)_{1\leqslant i\leqslant d}}(\varepsilon_1, \ldots, \varepsilon_d))O,$$

where $\mathrm{Mat}_{(e_i)_{1\leqslant i\leqslant d}}(\varepsilon_1, \ldots, \varepsilon_d)$ is the matrix of the vectors $(\varepsilon_i)_{1\leqslant i\leqslant d}$ written in the basis $(e_i)_{1\leqslant i\leqslant d}$ of $\mathcal{V}$.

In the following, we are only interested in the case, where both M and N are complete lattices of $\mathcal{V}$. The following properties can be inferred directly from its definition:

Let M, N and L be complete lattices of the E-vector space $\mathcal{V}$.

(P1) The generalized index is multiplicative (multiplication of submodules of the O-algebra E):

$$(M : N)_O = (M : L)_O \cdot (L : N)_O.$$

(P2) We have $(M : N)_O \cdot (N : M)_O = O$ and we write

$$(M : N)_O^{-1} := (N : M)_O.$$

(P3) If φ is any automorphism of V, then $(M : N)_O = (\varphi(M) : \varphi(N))_O$. In particular, if $\psi\colon A \hookrightarrow B$ is an injective homomorphism of O-modules such that $EA = EB = \mathcal{V}$ and $M, N \subseteq A$, then $(M : N)_O = (\psi(M) : \psi(N))_O$.

(P4) If M and N are complete $\mathbb{Z}$-lattices of an E-vector space $\mathcal{V}$, then $M \otimes_{\mathbb{Z}} O$ and $N \otimes_{\mathbb{Z}} O$ are complete O-lattices of the E-vector space $\mathcal{V}$ and

$$(M \otimes_{\mathbb{Z}} O : N \otimes_{\mathbb{Z}} O)_O = (M : N)_{\mathbb{Z}} \cdot O.$$

If additionally $N \subseteq M$, then

$$(M \otimes_{\mathbb{Z}} O : N \otimes_{\mathbb{Z}} O)_O = [M : N]O.$$

(P5) If M and N are complete O-lattices of an E-vector space $\mathcal{V}$, then M and N are complete $\mathbb{Z}$-lattices of the E-vector space $\mathcal{V} \otimes_{\mathbb{Z}} O \cong \mathcal{V}^{\mathrm{rank}_{\mathbb{Z}} O}$ (since M and N are O-generated by bases of $\mathcal{V}$ and O is free of finite rank over $\mathbb{Z}$) and

$$(M : N)_O^{\mathrm{rank}_{\mathbb{Z}} O} = (M : N)_{\mathbb{Z}} O.$$

Indeed, $(M : N)_O = \det(\mathcal{M})O$, where $\mathcal{M} := \mathrm{Mat}_{(e_1,\ldots,e_d)}(\varepsilon_1, \ldots, \varepsilon_d)$ and $(e_i)_{1\leqslant i\leqslant d}$ (respectively $(\varepsilon_i)_{1\leqslant i\leqslant d}$) is an O-basis of M (respectively of N). Let

$(x_1,\dots,x_m)$ be a $\mathbb{Z}$-basis of O, then $(x_ie_j)_{1\leqslant j\leqslant d}^{1\leqslant i\leqslant m}$ (respectively $(x_i\varepsilon_j)_{1\leqslant j\leqslant d}^{1\leqslant i\leqslant m}$) is a $\mathbb{Z}$-basis of M (respectively of N) and

$$(M:N)_{\mathbb{Z}} = \det\Big(\mathrm{Mat}_{(x_ie_j)_{1\leqslant j\leqslant d}^{1\leqslant i\leqslant m}}((x_i\varepsilon_j)_{1\leqslant j\leqslant d}^{1\leqslant i\leqslant m})\Big)\mathbb{Z}.$$

If we order the vectors $(x_ie_j)_{i,j}$ (respectively $(x_i\varepsilon_j)_{i,j}$) as $x_1e_1, x_1e_2,\dots,x_ie_j, x_ie_{j+1},\dots,x_me_d$ (respectively $x_1\varepsilon_1, x_1\varepsilon_2,\dots,x_i\varepsilon_j, x_i\varepsilon_{j+1},\dots,x_m\varepsilon_d$), then we get

$$\mathrm{Mat}_{(x_ie_j)_{1\leqslant j\leqslant d}^{1\leqslant i\leqslant m}}((x_i\varepsilon_j)_{1\leqslant j\leqslant d}^{1\leqslant i\leqslant m}) = \begin{bmatrix} \mathcal{M} & 0 & \dots & 0 \\ 0 & \mathcal{M} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathcal{M} \end{bmatrix}$$

with $\mathcal{M}$ appearing m times in the diagonal of the block matrix above.

3.1. Case of a Dedekind domain. When O is a Dedekind domain, we can extend the definition of generalized index to finitely generated torsion-free O-modules. For this we follow the definition provided in [28].

Let M and N be such O-modules (finitely generated and torsion-free). Again suppose that E is a field containing the Dedekind domain O and that there is a nonzero E-vector space $\mathcal{V}$ such that

$$EM = EN = \mathcal{V}.$$

The generalized index $(M:N)'_O$ is defined exactly as above. It is the O-module generated as follows:

$$(M:N)'_O := \langle \det(u), u \in \mathrm{End}_E(\mathcal{V}) \text{ and } u(M) \subseteq N\rangle.$$

Since M and N are finitely generated and torsion-free and O is a Dedekind domain, they are projective O-modules of finite rank (the rank of a finitely generated projective module is well defined over integral domains or more generally, over any ring with connected spectrum (see e.g., [2], Chapter II, Section 5, 3)). Further, since E is flat over O (every torsion-free module over a Dedekind domain is flat), the equality $EM = EN = \mathcal{V}$ shows that M and N have the same O-rank equal to the E-dimension of $\mathcal{V}$.

Let $d := \dim_E \mathcal{V}$. Since $EM = EN = \mathcal{V}$, by the structure theorem of finitely generated modules over a Dedekind domain (see e.g., [2], Chapter VII, Sections 4, 10, Proposition 24) there exists a fractional ideal $\mathfrak{m}$ (respectively $\mathfrak{n}$) and an E-basis $(e_i)_{1\leqslant i\leqslant d}$ (respectively $(\varepsilon_i)_{1\leqslant i\leqslant d}$) of $\mathcal{V}$ such that

$$M = \mathfrak{m}e_1 \oplus \bigoplus_{i=2}^{d} e_iO \quad \text{respectively} \quad N = \mathfrak{n}\varepsilon_1 \oplus \bigoplus_{i=2}^{d} \varepsilon_iO.$$

Then (by [28], Proposition 2.1)

$$(M:N)'_O = \mathfrak{n}\mathfrak{m}^{-1}\det(\mathrm{Mat}_{(e_i)_{1\leqslant i\leqslant d}}(\varepsilon_1,\dots,\varepsilon_d)).$$

The generalized index of finitely generated projective modules over a Dedekind domain verifies the invertibility and multiplicativity properties listed above (by [28], Proposition 2.4 and Corollary 2.2). It is also straightforward to check that it verifies Property (P3) listed above and to observe that the index $(M:N)'_O$ coincides with $(M:N)_O$ when M and N are free O-modules. Further, if there exists an endomorphism u of $\mathcal{V}$ such that $u(M)=N$, then $(M:N)'_O$ is generated over O by $\det_E(u)$ (by [28], Proposition 2.2):

$$(M:N)'_O = \det_E(u)O.$$

Remark 3.1. The index $(M:N)'_O$ is (obviously) slightly easier to compute when M and N are free O-modules (i.e., $(M:N)'_O = (M:N)_O$). This is why we opted to note $(M:N)_O$ and $(M:N)'_O$ differently to keep track of the nature of modules in question: If we write $(M:N)_O$, the reader should automatically understand that M and N are free. If we write $(M:N)'_O$, it means that at least one of the two O-modules is not necessarily free (or is not proven to be so).

4. Proof of Theorem 2.2

Proposition 4.1. *The $\mathcal{O}[G]$-modules $K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}$ and $\mathcal{Y}_F$ are both complete $\mathcal{O}$-lattices of the $\mathbb{C}$-vector space $K_{2n-1}(F)\otimes_{\mathbb{Z}}\mathbb{C}$.*

Proof. Since k is imaginary quadratic, the group $K_{2n-1}(F)_{/\mathrm{tors}}$ is free over $\mathbb{Z}$ with rank $\mathrm{rank}_{\mathbb{Z}}(K_{2n-1}(F)_{/\mathrm{tors}}) = r_2(F) = |G|$, where $r_2(F)$ is the number of complex places of F, see the introduction. Thus, $\dim_{\mathbb{C}}(K_{2n-1}(F)\otimes_{\mathbb{Z}}\mathbb{C}) = |G|$. Moreover, $\mathcal{O}$ is $\mathbb{Z}$-flat (since it is free), hence, the $\mathcal{O}$-module $K_{2n-1}(F)_{/\mathrm{tors}}\otimes_{\mathbb{Z}}\mathcal{O}$ is free and

$$\mathrm{rank}_{\mathcal{O}}(K_{2n-1}(F)_{/\mathrm{tors}}\otimes_{\mathbb{Z}}\mathcal{O}) = \mathrm{rank}_{\mathbb{Z}}(K_{2n-1}(F)_{/\mathrm{tors}}) = \dim_{\mathbb{C}}(K_{2n-1}(F)\otimes_{\mathbb{Z}}\mathbb{C}) = |G|.$$

On the other hand, consider the isomorphism

$$\mathrm{reg}_n(F)\otimes \mathrm{Id}_{\mathbb{C}} : K_{2n-1}(F)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{\sim} \mathbb{C}[G].$$

Since $(\mathrm{reg}_n(F)\otimes\mathrm{Id}_{\mathbb{C}})(e_\chi\varepsilon(\chi)) = (\mathrm{reg}_n(F)\otimes\mathrm{Id}_{\mathbb{C}})(\varepsilon(\chi))$, we get $e_\chi\varepsilon(\chi) = \varepsilon(\chi)$. Hence, (as a submodule of $K_{2n-1}(F)\otimes_{\mathbb{Z}}\mathbb{C}$), we have

$$\begin{aligned}\mathcal{Y}_F &:= \sum_{\chi\in\widehat{G}} \varepsilon(\chi)\mathcal{O}[G] = \sum_{\chi\in\widehat{G}} (e_\chi\varepsilon(\chi))\mathcal{O}[G] \\ &= \bigoplus_{\chi\in\widehat{G}} (e_\chi\varepsilon(\chi))\mathcal{O} \quad (\text{since } e_\chi\mathcal{O}[G] = e_\chi\mathcal{O} \text{ and } \varepsilon(\chi)\neq 0).\end{aligned}$$

Consequently, the $\mathcal{O}$-module $\mathcal{Y}_F$ is free of rank $|G|$ and $\mathcal{Y}_F\otimes_{\mathcal{O}}\mathbb{C} = K_{2n-1}(F)\otimes_{\mathbb{Z}}\mathbb{C}$. □

Corollary 4.1. *The quotient* $(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O})/\mathcal{Y}_F$ *is finite and*

$$\left|\frac{K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}}{\mathcal{Y}_F}\right|\mathcal{O} = ((K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O} : \mathcal{Y}_F)_{\mathcal{O}})^{\mathfrak{d}}.$$

Proof. By Proposition 4.1 the index $(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O} : \mathcal{Y}_F)_{\mathcal{O}}$ is well defined. Now use Property (P5) of Section 3. □

Proposition 4.2. *We have*

$$(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O} : \mathcal{Y}_F)_{\mathcal{O}} = \prod_{\chi \in \widehat{G}} (e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}) : e_\chi \mathcal{Y}_F)'_{\mathcal{O}}.$$

Proof. Let f be an endomorphism of the $\mathbb{C}$-vector space $V := K_{2n-1}(F) \otimes_{\mathbb{Z}} \mathbb{C}$ such that $f(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}) = \mathcal{Y}_F$. By definition we have

$$(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O} : \mathcal{Y}_F)_{\mathcal{O}} = \det_{\mathbb{C}}(f)\mathcal{O}.$$

Since V is a $\mathbb{C}[G]$-module, we can define for each $\chi \in \widehat{G}$ the induced maps

$$f_\chi \colon\ e_\chi V \to e_\chi V, \quad e_\chi x \mapsto e_\chi f(x),$$

and we get $\det_{\mathbb{C}}(f) = \prod_{\chi \in \widehat{G}} \det_{\mathbb{C}}(f_\chi)$. Again, by definition, we also have

$$\det_{\mathbb{C}}(f_\chi)\mathcal{O} = (e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}) : e_\chi \mathcal{Y}_F)'_{\mathcal{O}},$$

which gives

$$(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O} : \mathcal{Y}_F)_{\mathcal{O}} = \prod_{\chi \in \widehat{G}} (e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}) : e_\chi \mathcal{Y}_F)'_{\mathcal{O}}.$$

It remains to show that the indices $(e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}) : e_\chi \mathcal{Y}_F)'_{\mathcal{O}}$ are well defined, which is equivalent to showing that the $\mathcal{O}$-modules $e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O})$ and $e_\chi \mathcal{Y}_F$ are both finitely generated (torsion-free) submodules of the $\mathbb{C}$-vector space $V_\chi = e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{C})$ and that

$$\mathrm{rank}_{\mathcal{O}}(e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O})) = \mathrm{rank}_{\mathcal{O}}(e_\chi \mathcal{Y}_F) = \dim_{\mathbb{C}} V_\chi.$$

If A is an $\mathcal{O}$-module generated over $\mathcal{O}$ by $\{a_1, \ldots, a_i\}$ for an integer i, then $e_\chi A$ is generated over $\mathcal{O}$ by $\{e_\chi a_1, \ldots, e_\chi a_i\}$. This shows that $e_\chi \mathcal{Y}_F$ and $e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O})$ are finitely generated $\mathcal{O}$-modules.

Since $e_\chi \mathcal{Y}_F \subseteq e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}) \subseteq |G|^{-1}(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O})$, both $e_\chi \mathcal{Y}_F$ and $e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O})$ are torsion-free (as $\mathcal{O}$-modules) and they can be viewed as $\mathcal{O}$-submodules of V_χ. These observations also show that they are projective of finite rank. On the one hand,

$$\mathrm{rank}_\mathcal{O}(e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O})) = \dim_\mathbb{C} V_\chi$$

(since $e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}) \otimes_\mathcal{O} \mathbb{C} = V_\chi$). On the other hand, the regulator $\mathrm{reg}_n(F)$ induces the following isomorphism:

$$V_\chi = e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathbb{C}) \simeq e_\chi \mathbb{C}[G] \simeq \mathbb{C}.$$

Hence, $\mathrm{rank}_\mathcal{O}(e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O})) = 1$. Moreover, by the proof of Proposition 4.1, we have $e_\chi \mathcal{Y}_F = \varepsilon(\chi)\mathcal{O} \neq 0$. Thus, $e_\chi \mathcal{Y}_F$ is a nonzero finitely generated and projective submodule of a rank-one module and hence it also has rank one over $\mathcal{O}$. □

Let $\zeta_F(s)$ denote the Dedekind zeta function associated with the field F and $\zeta_F^*(1-n)$ its special value at $1-n$ (i.e., the coefficient of the first nonzero term in the Taylor expansion of $\zeta_F(s)$ at $1-n$). Then:

Proposition 4.3. *We have*

$$\begin{aligned}(K_{2n-1}(F)_{/\mathrm{tors}} &\otimes_\mathbb{Z} \mathcal{O} : \mathcal{Y}_F)_\mathcal{O} \\ &= |G|^{|G|} \zeta_F^*(1-n) \prod_{\chi \in \widehat{G}} (w_n(F^{\ker(\chi)})(e_\chi \mathrm{reg}_n(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}) : e_\chi \mathcal{O})'_\mathcal{O}).\end{aligned}$$

P r o o f. Let $\mathrm{reg}_{n,\chi}$ be the following map induced from reg_n:

$$\mathrm{reg}_{n,\chi}\colon\ e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}) \hookrightarrow e_\chi(\mathbb{R}[G] \otimes \mathcal{O}), \quad e_\chi x \to e_\chi \mathrm{reg}_n(x).$$

The injectivity of $\mathrm{reg}_{n,\chi}$ is a direct consequence of the injectivity of reg_n and the fact that reg_n is G-linear (meaning $\mathrm{reg}_n(e_\chi a) = e_\chi \mathrm{reg}_n(a)$ for any $a \in K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}$). By the previous points and Property (P3) of Section 3 we get

$$\begin{aligned}(e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} &\otimes_\mathbb{Z} \mathcal{O}) : e_\chi \mathcal{Y}_F)'_\mathcal{O} \\ &= (\mathrm{reg}_{n,\chi}(e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O})) : \mathrm{reg}_{n,\chi}(e_\chi \varepsilon(\chi)\mathcal{O}))'_\mathcal{O} \\ &= (e_\chi \mathrm{reg}_n(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}) : w_n(F^{\ker(\chi)}) L'(1-n, \chi^{-1}) |G| e_\chi \mathcal{O})'_\mathcal{O},\end{aligned}$$

which gives by multiplicativity of generalized indices (see Property (P1) in Section 3):

$$\begin{aligned}(e_\chi(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}) : e_\chi \mathcal{Y}_F)'_\mathcal{O} &= (e_\chi \mathrm{reg}_n(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_\mathbb{Z} \mathcal{O}) : e_\chi \mathcal{O})'_\mathcal{O} \\ &\quad \times (e_\chi \mathcal{O} : w_n(F^{\ker(\chi)}) L'(1-n, \chi^{-1}) |G| e_\chi \mathcal{O})_\mathcal{O}.\end{aligned}$$

The proof ends by using Proposition 4.2 and noticing that

$$(e_\chi\mathcal{O} : w_n(F^{\ker(\chi)})L'(1-n,\chi^{-1})|G|e_\chi\mathcal{O})_{\mathcal{O}} = |G|w_n(F^{\ker(\chi)})L'(1-n,\chi^{-1})\mathcal{O}.$$

□

Recall that $\mathrm{R}_n(F)$ ($\neq 0$) refers to the covolume of the regulator map $\mathrm{reg}_n(F)$:

$$\mathrm{reg}_n(F) : K_{2n-1}(F) \to \mathbb{R}[G].$$

Proposition 4.4. *We have*

$$\prod_{\chi\in\widehat{G}} (e_\chi \mathrm{reg}_n(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O}) : e_\chi\mathcal{O})'_{\mathcal{O}} = \mathrm{R}_n(F)^{-1}\mathcal{O}.$$

P r o o f. By the definition of the regulator $\mathrm{R}_n(F)$ we have

$$\mathrm{R}_n(F) = (\mathbb{Z}[G] : \mathrm{reg}_n(F)(K_{2n-1}(F)))_{\mathbb{Z}}.$$

Now we can apply Property (P4) of Section 3 to get

$$\mathrm{R}_n(F)\mathcal{O} = (\mathcal{O}[G] : \mathrm{reg}_n(F)(K_{2n-1}(F) \otimes_{\mathbb{Z}} \mathcal{O}))_{\mathcal{O}}.$$

Let f be an endomorphism of the $\mathbb{C}$-vector space $K_{2n-1}(F) \otimes_{\mathbb{Z}} \mathbb{C}$ such that

$$(\mathcal{O}[G] : \mathrm{reg}_n(F)(K_{2n-1}(F) \otimes_{\mathbb{Z}} \mathcal{O}))_{\mathcal{O}} = \det_{\mathbb{C}}(f)\mathcal{O}.$$

Again, as in the proof of Proposition 4.2, we can write

$$\det_{\mathbb{C}}(f) = \prod_{\chi\in\widehat{G}} \det_{\mathbb{C}}(f_\chi),$$

where for each element $\chi \in \widehat{G}$, f_χ denotes the induced endomorphism defined over $e_\chi(K_{2n-1}(F) \otimes_{\mathbb{Z}} \mathbb{C})$. By definition,

$$\det_{\mathbb{C}}(f_\chi)\mathcal{O} = (e_\chi\mathcal{O} : e_\chi(\mathrm{reg}_n(F)(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O})))'_{\mathcal{O}}.$$

□

Corollary 4.2. *The following equality holds:*

$$(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O} : \mathcal{Y}_F)_{\mathcal{O}} = \left(\prod_{\chi\in\widehat{G}} w_n(F^{\ker(\chi)})\right)|G|^{|G|}\frac{\zeta_F^*(1-n)}{\mathrm{R}_n(F)}\mathcal{O}.$$

P r o o f. Combine Propositions 4.3 and 4.4. □

We now consider the Lichtenbaum conjecture for a number field K:

The Lichtenbaum conjecture. *For a number field K and for all $n > 1$ we have*

$$\zeta_K^*(1-n) \overset{2}{=} \pm \frac{|K_{2n-2}(O_K)|}{|K_{2n-1}(O_K)_{\mathrm{tors}}|} \mathrm{R}_n(K).$$

The Lichtenbaum conjecture holds if K is an abelian number field (see [15]) or if K is a real number field, see [30]. In the case of a finite abelian extension of an imaginary quadratic field, one of the most promising results is given in [26]. It shows that the Lichtenbaum conjecture should hold in this case modulo a given finite set of primes.

It is also worth noting that the original statement of the Lichtenbaum conjecture involves the Borel regulator $R_n^B(K)$ instead of the Beilinson regulator $\mathrm{R}_n(K)$. By [3], we know that the two regulators agree modulo a fixed power of 2 (the Borel regulator map is twice the Beilinson map $\mathrm{reg}_n(K)$). Thus, it is also correct to state the conjecture using $\mathrm{R}_n(K)$.

Using Corollary 4.2 and assuming the Lichtenbaum conjecture yields the following corollary.

Corollary 4.3. *We have*

$$(K_{2n-1}(F)_{/\mathrm{tors}} \otimes_{\mathbb{Z}} \mathcal{O} : \mathcal{Y}_F)_{\mathcal{O}} \overset{2}{=} \frac{(\prod_{\chi \in \widehat{G}} w_n(F^{\ker(\chi)}))|G|^{|G|}}{|K_{2n-1}(O_F)_{\mathrm{tors}}|} |K_{2n-2}(O_F)|\mathcal{O}.$$

Theorem 2.2 ensues from Corollaries 4.3 and 4.1.

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