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GLOBAL SOLVABILITY IN THE PARABOLIC-ELLIPTIC CHEMOTAXIS SYSTEM WITH SINGULAR SENSITIVITY AND LOGISTIC SOURCE

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Abstract. We study the chemotaxis system with singular sensitivity and logistic-type source: $u_t = \Delta u - \chi\nabla\cdot(u\nabla v/v) + ru - \mu u^k$, $0 = \Delta v - v + u$ under the non-flux boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $\chi, r, \mu > 0$, $k > 1$ and $n \geqslant 1$. It is shown with $k \in (1, 2)$ that the system possesses a global generalized solution for $n \geqslant 2$ which is bounded when $\chi > 0$ is suitably small related to $r > 0$ and the initial datum is properly small, and a global bounded classical solution for $n = 1$.

Keywords: chemotaxis; singular sensitivity; global solvability

MSC 2020: 92C17, 35K55, 35B45

1. Introduction

Chemotaxis is a spontaneous cross-diffusion phenomenon by which organisms direct their movements in regard to a stimulating chemical. In 1970, Keller and Segel proposed a model to represent the chemotaxis phenomena, i.e., the oriented or partially oriented movement of cells with respect to a chemical signal produced by cells themselves, see [8]:

$$
(1.1)\qquad \begin{cases} u_t = \Delta u - \chi\nabla\cdot\Big(\dfrac{u}{v}\nabla v\Big), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\end{cases}
$$

where $\tau \in \{0, 1\}$, $\chi > 0$. The singular chemotactic sensitive function χ/v is derived by the Weber-Fechner law on the response of the cells u to the stimulating chem-

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ical signal v. With the singularity determined by the sensitive function χ/v, the cellular movements are governed by the taxis flux $\chi\nabla v/v$, which may be unbounded when $v \approx 0$. Different to the classical Keller-Segel model (i.e., replacing the singular sensitive function χ/v by the constant function χ in (1.1)), it seems from the known results that the lower bound on v is important to study the global dynamical behavior of a solution to a chemotaxis system with singular sensitivity. Fortunately, this can be arrived at by a point-in-wise estimate, see [7]

$$\inf_{x\in\Omega} v(x,t) \geqslant \min\left\{\frac{1}{2}\inf_{x\in\Omega} v_0, c_0 \inf_{s\in[0,t]} \int_\Omega u(x,s)\,\mathrm{d}x\right\}, \quad t>0$$

with some $c_0 = c_0(|\Omega|, n) > 0$. Due to the mass conservation of cells u in system (1.1), it is known that the singularity involved in sensitive function χ/v is in fact absent. Generally, chemotactic sensitive coefficient $\chi > 0$ sufficiently small favors the global existence-boundedness of solutions to system (1.1), which can be observed in [1], [3], [4], [5], [11], [14]. It is pointed that for the parabolic-elliptic case of system (1.1) ($\tau = 0$) with radial assumption, Nagai and Senba proved in [12] that there exists a finite time blow-up solution if $\chi > 2n/(n-2)$ with $n \geqslant 3$, and $\int_\Omega u_0|x|^2$ sufficiently small.

Consider the classical chemotaxis system with logistic-type source as follows:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + ru - \mu u^k, & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

where $\chi, r, \mu > 0$, $k > 1$ and $\tau \in \{0,1\}$. Such proliferation-death mechanism involved in the logistic-type source generally benefits the global dynamic of solutions. For parabolic-elliptic case of (1.2) ($\tau = 0$), the system with $k = 2$ possesses a global weak solution if $\mu > 0$ and a global bounded classical solution if $\mu > \chi(n-2)/n$, see [17]. If $k > 2 - n^{-1}$ with $n \geqslant 1$, there exists a global very weak solution, which is globally bounded provided μ is sufficiently large and u_0 sufficiently small, see [20]. For the parabolic-parabolic case of (1.2) ($\tau = 1$), if $k = 2$, $n = 2$ (see [13]), or $n \geqslant 3$ with $\mu > 0$ sufficiently large (see [22]), the problem possesses globally bounded classical solutions. In particular, if $n = 3$ with $k = 2$, it is proved that the model admits a global weak solution which is eventually smooth in [10]. If $k > 2 - n^{-1}$ with $n \geqslant 1$, there exist global very weak solutions (see [18]), which are globally bounded provided r/μ and the initial data are sufficiently small for $n = 3$, see [19]. Recently, the damping exponent k for ensuring the global existence of a solution in a generalized framework has been extended to $k > (2n+4)/(n+4)$ in [24] and $k > 1$ in [25]. For more properties of solutions to (1.2) refer to [16], [23].

Recall the chemotaxis system with singular sensitivity and logistic source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\left(\dfrac{u}{v}\nabla v\right) + ru - \mu u^k, & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{1.3}$$

where $\chi, r, \mu > 0$, $k > 1$ and $\tau \in \{0,1\}$. The difficulty in studying global solvability of solutions comes from the hazardous combination of a singular sensitive chemotactic mechanism and the self-limiting growth mechanism involved in logistic source. Due to the missing mass conservation for u, the singularity contained in the chemotactic term may be presence. For the parabolic-elliptic case of (1.3) ($\tau = 0$), it is proved in [6], [9] that there exists a global bounded classical solution for $n, k \geqslant 2$, if

$$r > \begin{cases} \dfrac{\chi^2}{4}, & 0<\chi\leqslant 2,\\ \chi - 1, & \chi > 2. \end{cases} \tag{1.4}$$

Notice that condition (1.4) is sufficient to ensure that the chemical signal v admits a uniformly positive lower bound. Similar results on the boundedness of a classical solution to (1.3) with $\tau = 1$ have been obtained in [28], [29]. Recently, we obtained the global boundedness of a classical solution when $k > 1$ and χ is suitably small independent of r, i.e., without the procedure in establishing the uniformly lower bound for v, see [27]. In addition, for the parabolic-parabolic case of (1.3) ($\tau = 1$), it is proved that system (1.3) possesses a global weak solution for $k = 2$ in [2], or a global very weak solution for $k > 2 - n^{-1}$ with χ suitably small related to r, k in [29]. In the framework of a generalized solution, the global solvability to (1.3) with $\tau = 1$ and $k > 1$ has been established in [26].

In this paper, we continuously consider the chemotaxis system with singular sensitivity and a logistic-type source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\left(\dfrac{u}{v}\nabla v\right) + ru - \mu u^k, & x\in\Omega,\ t>0,\\ 0 = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.5}$$

where $\chi, \mu > 0$, $r \in \mathbb{R}$ and $k > 1$. Further $\Omega \subset \mathbb{R}^n$ ($n \geqslant 1$) is a smooth bounded domain, $\partial/\partial\nu$ denotes the derivation with respect to the outer normal of $\partial\Omega$. The initial data satisfy

$$u_0(x) \in C^0(\overline{\Omega}), \quad u_0(x) \geqslant 0 \quad \text{and} \quad u_0(x) \not\equiv 0,\ x\in\overline{\Omega}. \tag{1.6}$$

We mainly concentrate on the global solvability of a solution to system (1.5) in the generalized framework for $n \geqslant 2$ and global boundedness of a classical solution for $n = 1$. Inspired by [1], [11] and [25], the definitions of a generalized solution will be introduced by combing a *weak subsolution* and a *weak supersolution* in a suitable framework. Then we establish the global existence of the classical solution to the corresponding regularized problem by means of some necessary compactness procedure of this solution. Thereafter, upon selecting a suitable subsequence, we will obtain a global generalized solution to (1.5) for $k \in (1,2)$ and $n \geqslant 2$ by a standard compactness argument. Finally, global boundedness of the classical solution to (1.5) will be obtained in the one-dimension setting. Now, we state the main result of this paper.

Theorem 1.1. *Let* $n \geqslant 2$ *and* $k \in (1,2)$. *Then system* (1.5) *possesses at least a global generalized solution.*

Theorem 1.2. *Assume that* $n \geqslant 2$ *and* $k \in (1,2)$. *If* $r, \chi > 0$ *fulfilling*

$$r > \begin{cases} \dfrac{\chi^2}{4}, & 0 < \chi \leqslant 2, \\ \chi - 1, & \chi > 2, \end{cases} \tag{1.7}$$

then there exist $\eta, \lambda > 0$ *so small that the generalized solution established in Theorem* 1.1 *is globally bounded provided* $r/\mu < \eta$ *and* $\int_\Omega u_0^p \,\mathrm{d}x < \lambda$ *with* $p \in (\frac{1}{2}n(n+2)/(n+1), n]$.

Theorem 1.3. *For* $n = 1$, *system* (1.5) *with* $k \in (1,2)$ *admits a global bounded classical solution.*

Remark 1.1. Recall from [25] that the classical parabolic-parabolic chemotaxis system (1.2) possesses a global generalized solution if $k \in (1,2)$ in logistic source $ru - \mu u^k$. Theorem 1.1 shows that $k \in (1,2)$ is also permitted for obtaining the global solvability to system (1.5) under the generalized framework, which coincides with the fully parabolic case of (1.5) in [26] and extends it to the parabolic-elliptic case of (1.5).

Remark 1.2. It is shown by [20] that global boundedness of a small-data solution to (1.2) with $\tau = 0$ will be established when $k \in (2 - n^{-1}, 2)$ for $n \geqslant 2$. In fact, this damping exponent k can be extended to $k \in (1,2)$ via a similar discussion as in Theorem 1.1. In addition, under the one dimensional setting, Theorem 1.3 says that the classical solution is globally bounded without any restrictions on the initial datum and the coefficients involved in (1.5).

2. Global generalized solution for $n \geqslant 2$

2.1. Definition of the generalized solution. To study the global solvability of system (1.5) for more general exponent $k > 1$ in the logistic-type source $ru - \mu u^k$, we introduce the generalized solution to (1.5) via the following definitions inspired by [1], [25].

Definition 2.1. A pair (u, v) of nonnegative functions

$$u \in L^k_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)), \quad v \in L^1_{\mathrm{loc}}([0,\infty); W^{1,1}(\Omega))$$

will be called a *weak subsolution* to (1.5) if

$$\int_\Omega u(\cdot,t) \leqslant \int_\Omega u_0 + r\int_0^t \int_\Omega u - \mu \int_0^t \int_\Omega u^k \quad \text{for a.e. } t > 0, \tag{2.1}$$

and if

$$-\int_0^\infty \int_\Omega v\psi_t - \int_\Omega v_0 \psi(\cdot,0) + \int_0^\infty \int_\Omega \nabla v \cdot \nabla \psi + \int_0^\infty \int_\Omega v\psi = \int_0^\infty \int_\Omega u\psi \tag{2.2}$$

holds for all $\psi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

Definition 2.2. Let $\gamma \in (0,1)$. A pair of nonnegative functions $u \in L^\gamma_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$, $v \in L^1_{\mathrm{loc}}([0,\infty); W^{1,1}(\Omega)) \cap L^{\gamma+1}_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty))$ forms a *weak γ-entropy supersolution* to (1.5) if

$$|\nabla u^{\gamma/2}|^2, \quad \frac{u^\gamma}{v^2}|\nabla v|^2 \quad \text{and} \quad \frac{u^{\gamma+1}}{v} \quad \text{belong to } L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)),$$

and if

$$\begin{aligned} -\int_0^\infty \int_\Omega u^\gamma \varphi_t - \int_\Omega u_0^\gamma \varphi(\cdot,0) &\geqslant \frac{4(1-\gamma)(1-\gamma\chi)}{\gamma} \int_0^\infty \int_\Omega |\nabla u^{\gamma/2}|^2 \varphi + \int_0^\infty \int_\Omega u^\gamma \Delta\varphi \\ &+ 4(1-\gamma)\chi \int_0^\infty \int_\Omega \Big|\nabla u^{\gamma/2} - \frac{u^{\gamma/2}}{2v}\nabla v\Big|^2 \varphi \\ &- (1-2\gamma)\chi \int_0^\infty \int_\Omega \frac{u^\gamma}{v} \nabla v \cdot \nabla \varphi + (1-\gamma)\chi \int_0^\infty \int_\Omega \frac{u^{\gamma+1}}{v}\varphi \\ &- (\chi - \gamma\chi - r\gamma) \int_0^\infty \int_\Omega u^\gamma \varphi - \mu\gamma \int_0^\infty \int_\Omega u^{\gamma+k-1}\varphi \end{aligned} \tag{2.3}$$

for each nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ with $\partial\varphi/\partial\nu = 0$ on $\partial\Omega \times (0,\infty)$ along with equality (2.2).

Definition 2.3. A couple of functions (u, v) will be called a *generalized solution* to (1.5) if it is both a weak subsolution and a weak γ-entropy supersolution of (1.5).

2.2. Regularized problems. To deal with the global solvability of generalized solutions to (1.5) for $k \in (1,2)$, we consider the following regularized problems corresponding to (1.5):

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi \nabla \cdot \left(\dfrac{u_\varepsilon}{v_\varepsilon} \nabla v_\varepsilon \right) + r u_\varepsilon - \mu u_\varepsilon^k - \varepsilon u_\varepsilon^2, & x \in \Omega,\ t > 0, \\ 0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{2.4}$$

with $\varepsilon \in (0,1)$. The local classical solution of the regularized problems (2.4) with general $k > 1$ can be obtained by similar arguments as in [6]. That is:

Lemma 2.1. *Assume that* u_0 *satisfies* (1.6). *Let* $k > 1$, $\chi, \mu > 0$ *and* $r \in \mathbb{R}$. *Then for each* $\varepsilon \in (0,1)$ *there exist* $T_{\max,\varepsilon} \in (0, +\infty]$ *and a unique pair* $(u_\varepsilon, v_\varepsilon)$ *of functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ v_\varepsilon \in C^{2,0}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \end{cases}$$

satisfying (2.4) *in the classical sense with* $u_\varepsilon, v_\varepsilon > 0$ *in* $\overline{\Omega} \times (0, T_{\max,\varepsilon})$. *Moreover, either* $T_{\max,\varepsilon} = \infty$, *or*

$$\limsup_{t \to T_{\max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty, \quad \text{or} \quad \liminf_{t \to T_{\max,\varepsilon}} \inf_{x \in \Omega} v_\varepsilon(x,t) = 0.$$

Let $(u_\varepsilon, v_\varepsilon)$ be the local classical solution to system (2.4) for each $\varepsilon \in (0,1)$. Denote $r_+ = \max\{0, r\}$ and $r_- = -\min\{0, r\}$ throughout this paper. Then we have the following a priori estimates.

Lemma 2.2. *With* $k \in (1,2)$, *it holds for all* $\varepsilon \in (0,1)$ *that*

$$\int_\Omega u_\varepsilon \leqslant m^*, \quad t \in (0, T_{\max,\varepsilon}), \tag{2.5}$$

$$\varepsilon \int_0^T \int_\Omega u_\varepsilon^2 + \int_0^T \int_\Omega u_\varepsilon^k \leqslant M_1(1+T), \quad T \in (0, T_{\max,\varepsilon}) \tag{2.6}$$

with $m^* = \max\{\int_\Omega u_0, |\Omega|(r_+/\mu)^{1/(k-1)}\}$ *and* $M_1 = (1 + r_+)m^*/\mu$, *and*

$$\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \leqslant |\Omega|, \quad T \in (0, T_{\max,\varepsilon}). \tag{2.7}$$

Proof. Integrate $(2.4)_1$ over Ω with Hölder's inequality to know that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon \leqslant r_+\int_\Omega u_\varepsilon - \mu\int_\Omega u_\varepsilon^k - \varepsilon\int_\Omega u_\varepsilon^2 \tag{2.8}$$

$$\leqslant r_+\int_\Omega u_\varepsilon - \frac{\mu}{|\Omega|^{k-1}}\Big(\int_\Omega u_\varepsilon\Big)^k, \quad t\in(0,T_{\max,\varepsilon}). \tag{2.9}$$

We get (2.5) by Bernoulli's inequality with (2.9). The estimate (2.6) comes from integrating (2.8) from 0 to T and from (2.5). In addition, multiply $(2.4)_2$ by v_ε^{-1} to get

$$0=\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} - |\Omega| + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}, \quad t\in(0,T_{\max,\varepsilon}),$$

which yields (2.7). □

In order to deal with the global existence of classical solution to (2.4) for each $\varepsilon\in(0,1)$, we give a point-in-wise lower bound of v_ε for all $(x,t)\in\Omega\times(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$.

Lemma 2.3. *Let $k\in(1,2)$. Then there exist some $M_2, M_3>0$ such that*

$$v_\varepsilon(x,t)\geqslant M_3\mathrm{e}^{-M_2(1+t)} \quad \text{for all } \varepsilon\in(0,1) \text{ and } (x,t)\in\Omega\times(0,T_{\max,\varepsilon}). \tag{2.10}$$

Proof. A direct computation with $(2.4)_1$ shows

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \ln u_\varepsilon = \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} - \chi\int_\Omega \frac{\nabla u_\varepsilon\cdot\nabla v_\varepsilon}{u_\varepsilon v_\varepsilon} + r|\Omega| - \mu\int_\Omega u^{k-1} - \varepsilon\int_\Omega u_\varepsilon \tag{2.11}$$

$$\geqslant -\frac{\chi^2}{4}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + (r-\mu)|\Omega| - (\mu+1)\int_\Omega u_\varepsilon, \quad t\in(0,T_{\max,\varepsilon})$$

by Young's inequality for all $\varepsilon\in(0,1)$. This together with (2.5) and (2.7) entails

$$\begin{aligned}\int_\Omega \ln u_\varepsilon(x,t) &\geqslant -\frac{\chi^2}{4}\int_0^t\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \int_\Omega \ln u_0 + (r-\mu)|\Omega|t - (\mu+1)m^*t \\ &\geqslant \int_\Omega \ln u_0 - \frac{\chi^2}{4}|\Omega|(1+t) + (r-\mu)|\Omega|t - (\mu+1)m^*t \\ &\geqslant -C_1(1+t), \quad t\in(0,T_{\max,\varepsilon})\end{aligned} \tag{2.12}$$

with $C_1=\max\{1,|\int_\Omega \ln u_0 - \frac14\chi^2|, |(r-\mu-\frac14\chi^2)|\Omega| - (\mu+1)m^*|\}>0$. By means of the Jensen inequality and the concavity of the logarithmic function, we get from (2.12) that

$$\int_\Omega u_\varepsilon \geqslant |\Omega|\exp\Big(\frac{1}{|\Omega|}\int_\Omega \ln u_\varepsilon\Big) \geqslant |\Omega|\exp\Big(\frac{-C_1}{|\Omega|}(1+t)\Big), \quad t\in(0,T_{\max,\varepsilon}),$$

and hence,

$$v_\varepsilon(x,t) \geqslant \eta \int_\Omega u_\varepsilon \geqslant M_3 \mathrm{e}^{-M_2(1+t)}, \quad t \in (0, T_{\max,\varepsilon})$$

by [7], Lemma 2.1 with $M_2 = C_1/|\Omega|$ and $M_3 = \eta|\Omega|$. The proof is complete. □

The local classical solution $(u_\varepsilon, v_\varepsilon)$ for each $\varepsilon \in (0,1)$ in Lemma 2.1 is in fact global.

Lemma 2.4. *Let $k \in (1,2)$. Then for each $\varepsilon \in (0,1)$ system (2.4) possesses a global classical solution $(u_\varepsilon, v_\varepsilon)$.*

Proof. For each $\varepsilon \in (0,1)$, let $(u_\varepsilon, v_\varepsilon)$ be the local classical solution to the regularized problem (2.4) with $k \in (1,2)$. Based on Lemmas 2.1 and 2.2, similar arguments as that in [9], Theorem 1.2 will establish global existence of the classical solution to (2.4). □

Remark 2.1. The estimates in Lemmas 2.1 and 2.2 are valid with $T_{\max,\varepsilon} = \infty$ for all $\varepsilon \in (0,1)$. For convenience, we omit the new marks on these estimates.

2.3. Construction of limit function. Now, we deal with a spatio-temporal integral estimate on u_ε for all $\varepsilon \in (0,1)$.

Lemma 2.5. *If $k \in (1,2)$ and $\gamma \in (0, \min\{\chi^{-1}, 1\})$, there exists some $M_4 > 0$ such that*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2 + \int_0^T \int_\Omega \Big|\nabla u_\varepsilon^{\gamma/2} - \frac{u_\varepsilon^{\gamma/2} \nabla v_\varepsilon}{2v_\varepsilon}\Big|^2 + \int_0^T \int_\Omega \frac{u_\varepsilon^\gamma |\nabla v_\varepsilon|^2}{v_\varepsilon^2} \leqslant M_4(1+T), \quad T > 0 \tag{2.13}$$

for all $\varepsilon \in (0,1)$. Moreover, there exists some $p > 1$ along with $M_5 > 0$ such that

$$\int_0^T \int_\Omega \Big(\frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon}\Big)^p \leqslant M_5 \mathrm{e}^{M_5 T}, \quad T > 0 \tag{2.14}$$

for all $\varepsilon \in (0,1)$.

Proof. A direct calculation with (2.4) for $\gamma > 0$ shows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_\varepsilon^\gamma &= \gamma \int_\Omega u_\varepsilon^{\gamma-1} \Big[\Delta u_\varepsilon - \chi \nabla \cdot \Big(\frac{u_\varepsilon}{v_\varepsilon} \nabla v_\varepsilon\Big) + r u_\varepsilon - \mu u_\varepsilon^k - \varepsilon u_\varepsilon^2\Big] \\ &= \gamma(1-\gamma) \int_\Omega u_\varepsilon^{\gamma-2} |\nabla u_\varepsilon|^2 - \chi\gamma(1-\gamma) \int_\Omega \frac{u_\varepsilon^{\gamma-1}}{v_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\ &\quad + r\gamma \int_\Omega u_\varepsilon^\gamma - \mu\gamma \int_\Omega u_\varepsilon^{\gamma+k-1} - \varepsilon\gamma \int_\Omega u_\varepsilon^{\gamma+1}, \quad t > 0. \end{aligned} \tag{2.15}$$

In addition, multiply $(2.4)_2$ by $u_\varepsilon^\gamma/v_\varepsilon$ to know

$$
\begin{aligned}
(2.16)\qquad 0 &= \int_\Omega \frac{u_\varepsilon^\gamma}{v_\varepsilon}[\Delta v_\varepsilon - v_\varepsilon + u_\varepsilon] \\
&= \int_\Omega u_\varepsilon^\gamma \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} - \gamma \int_\Omega \frac{u_\varepsilon^{\gamma-1}}{v_\varepsilon}\nabla u_\varepsilon \cdot \nabla v_\varepsilon - \int_\Omega u_\varepsilon^\gamma + \int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon}, \quad t>0.
\end{aligned}
$$

Then we have from (2.15) and (2.16) that

$$
\begin{aligned}
(2.17)\quad \frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^\gamma &= \gamma(1-\gamma)\int_\Omega u_\varepsilon^{\gamma-2}|\nabla u_\varepsilon|^2 - 2\chi\gamma(1-\gamma)\int_\Omega \frac{u_\varepsilon^{\gamma-1}}{v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\quad + \chi(1-\gamma)\int_\Omega u_\varepsilon^\gamma\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + (r\gamma+\chi\gamma-\chi)\int_\Omega u_\varepsilon^\gamma \\
&\quad + \chi(1-\gamma)\int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon} - \mu\gamma\int_\Omega u_\varepsilon^{\gamma+k-1} - \varepsilon\gamma\int_\Omega u_\varepsilon^{\gamma+1} \\
&= \frac{4(1-\gamma)(1-\chi\gamma)}{\gamma}\int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2 + 4\chi(1-\gamma)\int_\Omega \Big|\nabla u_\varepsilon^{\gamma/2} - \frac{u_\varepsilon^{\gamma/2}\nabla v_\varepsilon}{2v_\varepsilon}\Big|^2 \\
&\quad + \chi(1-\gamma)\int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon} + (r\gamma+\chi\gamma-\chi)\int_\Omega u_\varepsilon^\gamma \\
&\quad - \mu\gamma\int_\Omega u_\varepsilon^{\gamma+k-1} - \varepsilon\gamma\int_\Omega u_\varepsilon^{\gamma+1}, \quad t>0.
\end{aligned}
$$

If $\gamma \in (0, \min\{\chi^{-1}, 1\})$, a combination of (2.17) with (2.5) and (2.6) yields

$$
\begin{aligned}
(2.18)\quad & \frac{4(1-\gamma)(1-\chi\gamma)}{\gamma}\int_0^T\int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2 + 4\chi(1-\gamma)\int_0^T\int_\Omega \Big|\nabla u_\varepsilon^{\gamma/2} - \frac{u_\varepsilon^{\gamma/2}\nabla v_\varepsilon}{2v_\varepsilon}\Big|^2 \\
&\quad + \chi(1-\gamma)\int_0^T\int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon} \\
&\leqslant \int_\Omega u_\varepsilon(\cdot,T) + ((r_- + \mu)\gamma + \chi)\int_0^T\int_\Omega u_\varepsilon^k + \varepsilon\gamma\int_0^T\int_\Omega u_\varepsilon^2 \\
&\quad + |\Omega| + ((r_- + \mu + 1)\gamma + \chi)|\Omega| T \\
&\leqslant C_2(1+T), \quad T>0
\end{aligned}
$$

by Young's inequality with some $C_2 > 0$ independent of $\varepsilon \in (0,1)$. Again by (2.16) and Young's inequality, we have

$$
\begin{aligned}
\int_\Omega u_\varepsilon^\gamma\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} &= \gamma\int_\Omega \frac{u_\varepsilon^{\gamma-1}}{v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon + \int_\Omega u_\varepsilon^\gamma - \int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon} \\
&\leqslant \frac{1}{2}\int_\Omega u_\varepsilon^\gamma\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + 2\int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2 + \int_\Omega u_\varepsilon^\gamma, \quad t>0,
\end{aligned}
$$

which integrating from 0 to T entails (2.13) by (2.18) with some $M_4 > 0$.

Selecting $\gamma < \beta \in (0, \min\{1, \chi^{-1}\})$ and replacing γ by β in (2.18), we know by Lemma 2.3 that

$$\begin{aligned}\int_0^T\int_\Omega\Big(\frac{u_\varepsilon^\gamma}{v_\varepsilon}\Big)^{(\beta+1)/(\gamma+1)} &\leqslant \int_0^T\int_\Omega \frac{u_\varepsilon^\beta}{v_\varepsilon}\frac{1}{v_\varepsilon^{(\beta-\gamma)/(\gamma+1)}} \leqslant \Big(\frac{\mathrm{e}^{M_2(1+T)}}{M_3}\Big)^{(\beta-\gamma)/(\gamma+1)}\int_0^T\int_\Omega\frac{u_\varepsilon^\beta}{v_\varepsilon}\\ &\leqslant M_5\mathrm{e}^{M_5(1+T)}, \quad T>0\end{aligned}$$

for all $\varepsilon \in (0,1)$ with $M_5 = \max\{2(\beta-\gamma)M_2/(\gamma+1), C_2M_3^{-(\beta-\gamma)/(\gamma+1)}\}$. This concludes (2.14). $\square$

Next, we deal with the estimate on the time derivative of $(1+u_\varepsilon)^{\gamma/2}$ for all $\varepsilon \in (0,1)$.

Lemma 2.6. *For $k \in (1,2)$ with $\gamma \in (0, \min\{\chi^{-1}, 1\})$, there exists $M_6 > 0$ such that*

$$\int_0^T \Big\|\frac{\mathrm{d}}{\mathrm{dt}}(1+u_\varepsilon)^{\gamma/2}\Big\|_{(W_0^{1,\infty}(\Omega))^*}\,\mathrm{d}s \leqslant M_6(1+T), \quad T>0 \tag{2.19}$$

for all $\varepsilon \in (0,1)$.

Proof. Let $\phi \in W_0^{1,\infty}(\Omega)$. Then we have from $(2.4)_1$ that

(2.20)

$$\begin{aligned}&\int_\Omega \frac{\mathrm{d}}{\mathrm{dt}}(1+u_\varepsilon)^{\gamma/2}\phi\\
&= \frac{\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}\phi\Big[\Delta u_\varepsilon - \chi\nabla\cdot\Big(\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon\Big) + ru_\varepsilon - \mu u_\varepsilon^k - \varepsilon u_\varepsilon^2\Big]\\
&= -\frac{\gamma}{2}\int_\Omega \nabla((1+u_\varepsilon)^{(\gamma-2)/2}\phi)\cdot\Big(\nabla u_\varepsilon - \chi\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon) + \frac{r\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}u_\varepsilon\phi\\
&\quad - \frac{\mu\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}u_\varepsilon^k\phi - \frac{\varepsilon\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}u_\varepsilon^2\phi\\
&= \frac{\gamma(2-\gamma)}{4}\int_\Omega(1+u_\varepsilon)^{(\gamma-4)/2}|\nabla u_\varepsilon|^2\phi - \frac{\gamma\chi(2-\gamma)}{4}\int_\Omega(1+u_\varepsilon)^{(\gamma-4)/2}\frac{u_\varepsilon}{v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\phi\\
&\quad - \frac{\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}\nabla u_\varepsilon\cdot\nabla\phi + \frac{\gamma\chi}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla\phi\\
&\quad + \frac{r\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}u_\varepsilon\phi - \frac{\mu\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}u_\varepsilon^k\phi - \frac{\varepsilon\gamma}{2}\int_\Omega(1+u_\varepsilon)^{(\gamma-2)/2}u_\varepsilon^2\phi\\
&\leqslant C_3\Big(\int_\Omega|\nabla u_\varepsilon^{\gamma/2}|^2\Big)\|\phi\|_{L^\infty(\Omega)} + C_3\Big(\int_\Omega|\nabla u_\varepsilon^{\gamma/2}|^2\Big)^{1/2}\Big(\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Big)^{1/2}\|\phi\|_{L^\infty(\Omega)}\\
&\quad + C_3\Big(\int_\Omega|\nabla u_\varepsilon^{\gamma/2}|^2\Big)^{1/2}\|\nabla\phi\|_{L^2(\Omega)} + C_3\Big(\int_\Omega(1+u_\varepsilon)^\gamma\Big)^{1/2}\Big(\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Big)^{1/2}\|\nabla\phi\|_{L^\infty(\Omega)}\\
&\quad + C_3\Big(\int_\Omega(1+u_\varepsilon)^\gamma + \int_\Omega u_\varepsilon^k + \varepsilon\int_\Omega u_\varepsilon^{\gamma+1}\Big)\|\phi\|_{L^\infty(\Omega)}, \quad t>0\end{aligned}$$

by Hölder's inequality with some $C_3 > 0$. In addition, it is known by Young's inequality with (2.20) and (2.5) that

$$\tag{2.21} \begin{aligned} &\Big|\int_\Omega \frac{\mathrm{d}}{\mathrm{dt}}(1+u_\varepsilon)^{\gamma/2}\phi\Big| \\ &\leqslant C_4\Big(1+\int_\Omega u_\varepsilon^k + \int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2 + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \varepsilon\int_\Omega u_\varepsilon^2\Big)\|\phi\|_{W_0^{1,\infty}(\Omega)}, \quad t>0 \end{aligned}$$

with some $C_4 > 0$. Since $W_0^{n+1,2}(\Omega) \hookrightarrow W_0^{1,\infty}(\Omega)$, integrating (2.21) from 0 to T, we obtain from (2.6), (2.7) and (2.13) that

$$\begin{aligned} &\int_0^T \Big\|\frac{\mathrm{d}}{\mathrm{dt}}(1+u_\varepsilon)^{\gamma/2}\Big\|_{(W_0^{n+1,2}(\Omega))^*}\,\mathrm{d}s \\ &\leqslant \sup_{\substack{\phi\in W_0^{n+1,2}(\Omega)\\ \|\phi\|_{W_0^{n+1,2}(\Omega)}\leqslant 1}} \int_0^T \Big|\int_\Omega \frac{\mathrm{d}}{\mathrm{dt}}(1+u_\varepsilon)^{\gamma/2}\phi\Big|\,\mathrm{d}s \\ &\leqslant C_4\Big(T+\int_0^T\int_\Omega u_\varepsilon^k + \int_0^T\int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2 + \int_0^T\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \varepsilon\int_0^T\int_\Omega u_\varepsilon^2\Big) \\ &\leqslant M_6(1+T), \quad T>0 \end{aligned}$$

with some $M_6 > 0$. The proof is complete. □

We now perform a subsequence extraction procedure to obtain a limit object (u, v), i.e., a generalized solution to problem (1.5).

Lemma 2.7. *If $k \in (1,2)$ and $\gamma \in (0, \min\{1, \chi^{-1}\})$, then for $p \in [1,k)$ there exist $u \in L^1_{\mathrm{loc}}(\Omega \times (0,\infty))$ and $v \in L^1_{\mathrm{loc}}((0,\infty), W^{1,1}(\Omega))$ such that*

$$\tag{2.22} u_\varepsilon^{\gamma/2} \rightharpoonup u^{\gamma/2}, \quad \textit{in } L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)),$$

$$\tag{2.23} u_\varepsilon \to u, \quad \textit{a.e. in } \Omega\times(0,\infty) \textit{ and in } L^p_{\mathrm{loc}}(\Omega\times[0,\infty)),$$

$$\tag{2.24} v_\varepsilon \to v, \quad \textit{a.e. in } \Omega\times(0,\infty) \textit{ and in } L^1_{\mathrm{loc}}([0,\infty); W^{1,1}(\Omega)),$$

$$\tag{2.25} \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon} \to \frac{u^{\gamma+1}}{v}, \quad \textit{in } L^1_{\mathrm{loc}}(\Omega\times[0,\infty)),$$

$$\tag{2.26} \frac{u_\varepsilon^{\gamma/2}\nabla v_\varepsilon}{v_\varepsilon} \rightharpoonup \frac{u^{\gamma/2}\nabla v}{v}, \quad \textit{in } L^2_{\mathrm{loc}}(\Omega\times(0,\infty))$$

for $\varepsilon = \varepsilon_j \searrow 0$.

Proof. Since $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow (W_0^{1,\infty}(\Omega))^*$, we have by the Aubin-Lions lemma with (2.13) and (2.19) that $u_\varepsilon^{\gamma/2} \to u^{\gamma/2}$ in $L^2(\Omega\times(0,T))$ and $u_\varepsilon \to u$ a.e. in $\Omega\times(0,T)$, and (2.22) holds as $\varepsilon = \varepsilon_j \searrow 0$. For $p \in (1,k)$, it is known from (2.6) that $\{u_\varepsilon^p\}_{\varepsilon\in(0,1)} \subset L^{k/p}_{\rm loc}(\overline{\Omega}\times[0,\infty))$. This together with the Vitali convergence theorem and $u_\varepsilon \to u$ a.e. in $\Omega\times(0,T)$ indicates (2.23). The standard elliptic regularity theory and (2.23) ensure that there exists a nonnegative function v defined on $\Omega\times(0,\infty)$ such that (2.24) holds. Again by the Vitali convergence theorem with (2.14), (2.23) and (2.24), we obtain (2.25). Finally, in view of (2.13) with (2.23) and (2.7), we can get estimate (2.26). □

Proof of Theorem 1.1. For $k\in(1,2)$, we will firstly demonstrate that the function (u,v) obtained in Lemma 2.7 is a *weak subsolution* of (1.5). Multiplying $(2.4)_1$ and integrating by parts, we have for all $\varepsilon\in(0,1)$ that

$$\int_\Omega u_\varepsilon(\cdot,t) - \int_\Omega u_0 = r\int_0^t\int_\Omega u_\varepsilon - \mu\int_0^t\int_\Omega u_\varepsilon^k - \varepsilon\int_0^t\int_\Omega u_\varepsilon^{k+1}. \tag{2.27}$$

By (2.23), we know

$$r\int_0^t\int_\Omega u_\varepsilon \to r\int_0^t\int_\Omega u \tag{2.28}$$

as $\varepsilon=\varepsilon_j\searrow 0$. In addition, based on the Tonelli Theorem there exists a null set $N\subset(0,\infty)$ such that $u_\varepsilon(\cdot,t)\to u(\cdot,t)$ a.e. in Ω for all $t\in(0,\infty)\setminus N$ as $\varepsilon=\varepsilon_j\searrow 0$. And hence, by (2.27) with (2.28) and the Fatou Lemma along with the positivity of $\varepsilon\int_0^T\int_\Omega u_\varepsilon^2$, we obtain

$$\begin{aligned}\int_\Omega u(\cdot,t) + \mu\int_0^t\int_\Omega u^k &\leqslant \liminf_{\varepsilon=\varepsilon_j\searrow 0}\Big(\int_\Omega u_\varepsilon(\cdot,t) + \mu\int_0^t\int_\Omega u_\varepsilon^k\Big)\\ &= \lim_{\varepsilon=\varepsilon_j\searrow 0}\Big(\int_\Omega u_0 + r\int_0^t\int_\Omega u_\varepsilon - \varepsilon\int_\Omega u_\varepsilon^2\Big)\\ &\leqslant \int_\Omega u_0 + r\int_0^t\int_\Omega u\end{aligned} \tag{2.29}$$

for all $t\in(0,\infty)\setminus N$. This concludes (2.1). In addition, let $\psi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, and multiply $(2.4)_2$ by ψ and integrate by parts, then we get

$$-\int_0^\infty\int_\Omega v_\varepsilon\psi_t - \int_\Omega v_0\psi(\cdot,0) + \int_0^\infty\int_\Omega \nabla v_\varepsilon\cdot\nabla\psi + \int_0^\infty\int_\Omega v_\varepsilon\psi = \int_0^\infty\int_\Omega u_\varepsilon\psi. \tag{2.30}$$

According to (2.24) and (2.23), we get (2.2) as $\varepsilon=\varepsilon_j\searrow 0$. This together with (2.29) indicates that (u,v) is a global *weak subsolution* of (1.5).

Now, select a nonnegative $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ satisfying $\partial\varphi/\partial\nu = 0$ on $\partial\Omega\times(0,\infty)$. Then multiply $(1.5)_1$ by $\gamma u_\varepsilon^{\gamma-1}\varphi$ to get

$$\begin{aligned}
(2.31)\quad & -\int_0^\infty \int_\Omega u_\varepsilon^\gamma \varphi_t - \int_\Omega u_0^\gamma \varphi(\cdot,0) \\
&= \gamma(1-\gamma)\int_0^\infty \int_\Omega u_\varepsilon^{\gamma-2}|\nabla u_\varepsilon|^2\varphi - \chi\gamma(1-\gamma)\int_0^\infty \int_\Omega \frac{u_\varepsilon^{\gamma-1}}{v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\varphi \\
&\quad + \int_0^\infty \int_\Omega u_\varepsilon^\gamma\Delta\varphi + \chi\gamma\int_0^\infty \int_\Omega \frac{u_\varepsilon^\gamma}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla\varphi + r\gamma\int_0^\infty \int_\Omega u_\varepsilon^\gamma\varphi \\
&\quad - \mu\gamma\int_0^\infty \int_\Omega u_\varepsilon^{\gamma+k-1}\varphi - \varepsilon\gamma\int_0^\infty \int_\Omega u_\varepsilon^{\gamma+1}\varphi.
\end{aligned}$$

In addition, multiplying $(2.4)_2$ by $u^\gamma\varphi/v_\varepsilon$, we know

$$\begin{aligned}
(2.32)\qquad 0 &= \int_0^\infty \int_\Omega \frac{u^\gamma\varphi}{v_\varepsilon}[\Delta v_\varepsilon - v_\varepsilon + u_\varepsilon] \\
&= -\gamma\int_0^\infty \int_\Omega \frac{u_\varepsilon^{\gamma-1}}{v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\varphi + \int_0^\infty \int_\Omega u_\varepsilon^\gamma\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\varphi \\
&\quad - \int_0^\infty \int_\Omega \frac{u_\varepsilon^\gamma}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla\varphi - \int_0^\infty \int_\Omega u_\varepsilon^\gamma\varphi + \int_0^\infty \int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon}\varphi.
\end{aligned}$$

Hence, we have by (2.31) and (2.32) that

(2.33)

$$\begin{aligned}
& -\int_0^\infty \int_\Omega u_\varepsilon^\gamma\varphi_t - \int_\Omega u_0^\gamma\varphi(\cdot,0) \\
&= \frac{4(1-\gamma)(1-\chi\gamma)}{\gamma}\int_0^\infty \int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2\varphi + 4\chi(1-\gamma)\int_0^\infty \int_\Omega \Big|\nabla u_\varepsilon^{\gamma/2} - \frac{u_\varepsilon^{\gamma/2}\nabla v_\varepsilon}{2v_\varepsilon}\Big|^2 \\
&\quad + \int_0^\infty \int_\Omega u_\varepsilon^\gamma\Delta\varphi + \chi\int_0^\infty \int_\Omega u_\varepsilon^\gamma\frac{\nabla v_\varepsilon}{v_\varepsilon}\cdot\nabla\varphi + \chi(1-\gamma)\int_0^\infty \int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon}\varphi \\
&\quad + (r\chi+\gamma\chi-\chi)\int_0^\infty \int_\Omega u_\varepsilon^\gamma\varphi - \mu\gamma\int_0^\infty \int_\Omega u_\varepsilon^{\gamma+k-1}\varphi - \varepsilon\gamma\int_0^\infty \int_\Omega u_\varepsilon^{\gamma+1}\varphi.
\end{aligned}$$

By (2.23), we have

$$(2.34)\qquad -\int_0^\infty \int_\Omega u_\varepsilon^\gamma\varphi_t \to -\int_0^\infty \int_\Omega u^\gamma\varphi_t,$$

$$(2.35)\qquad (r\chi+\gamma\chi-\chi)\int_0^\infty \int_\Omega u_\varepsilon^\gamma\varphi \to (r\chi+\gamma\chi-\chi)\int_0^\infty \int_\Omega u^\gamma\varphi,$$

$$(2.36)\qquad -\mu\gamma\int_0^\infty \int_\Omega u_\varepsilon^{\gamma+k-1}\varphi \to -\mu\gamma\int_0^\infty \int_\Omega u_\varepsilon^{\gamma+k-1}\varphi$$

$$(2.37)\qquad \int_0^\infty \int_\Omega u_\varepsilon^\gamma\Delta\varphi \to \int_0^\infty \int_\Omega u^\gamma\Delta\varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, whereas (2.25) implies that

$$\chi(1-\gamma)\int_0^\infty\int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon}\varphi \to \chi(1-\gamma)\int_0^\infty\int_\Omega \frac{u^{\gamma+1}}{v}\varphi \tag{2.38}$$

as $\varepsilon = \varepsilon_j \searrow 0$. In addition, it follows from (2.23) and (2.26) that

$$\begin{aligned}\chi\int_0^\infty\int_\Omega \frac{u_\varepsilon^\gamma}{v_\varepsilon}\nabla v_\varepsilon\cdot\nabla\varphi &= \chi\int_0^\infty\int_\Omega u_\varepsilon^{\gamma/2}\frac{u_\varepsilon^{\gamma/2}}{v}{}_\varepsilon\nabla v_\varepsilon\cdot\nabla\varphi\\ &\to \chi\int_0^\infty\int_\Omega u^{\gamma/2}\frac{u^{\gamma/2}}{v}\nabla v\cdot\nabla\varphi = \chi\int_0^\infty\int_\Omega \frac{u^\gamma}{v}\nabla v\cdot\nabla\varphi\end{aligned} \tag{2.39}$$

as $\varepsilon = \varepsilon_j \searrow 0$. A simple calculation with (2.6) shows that

(2.40)

$$|-\varepsilon\gamma\int_0^\infty\int_\Omega u_\varepsilon^{\gamma+1}\varphi| \leqslant \varepsilon^{(1-\gamma)/2}\Big(\varepsilon\int_0^\infty\int_\Omega u_\varepsilon^2\Big)^{(1+\gamma)/2}\Big(\int_0^\infty\int_\Omega \varphi^{2/(1-\gamma)}\Big)^{(1-\gamma)/2} \to 0$$

as $\varepsilon = \varepsilon_j \searrow 0$. Hence, we obtain from (2.33) with (2.34)–(2.40) and the Fatou Lemma that

(2.41)

$$\begin{aligned}&\frac{4(1-\gamma)(1-\chi\gamma)}{\gamma}\int_0^\infty\int_\Omega |\nabla u^{\gamma/2}|^2\varphi + 4\chi(1-\gamma)\int_0^\infty\int_\Omega\Big|\nabla u^{\gamma/2} - \frac{u^{\gamma/2}\nabla v}{2v}\Big|^2\\
&\leqslant \liminf_{\varepsilon=\varepsilon_j\searrow 0}\Big[\frac{4(1-\gamma)(1-\chi\gamma)}{\gamma}\int_0^\infty\int_\Omega |\nabla u_\varepsilon^{\gamma/2}|^2\varphi\\
&\qquad + 4\chi(1-\gamma)\int_0^\infty\int_\Omega\Big|\nabla u_\varepsilon^{\gamma/2} - \frac{u_\varepsilon^{\gamma/2}\nabla v_\varepsilon}{2v_\varepsilon}\Big|^2\Big]\\
&= \lim_{\varepsilon=\varepsilon_j\searrow 0}\Big(-\int_0^\infty\int_\Omega u_\varepsilon^\gamma\varphi_t - \int_\Omega u_0^\gamma\varphi(\cdot,0) - \int_0^\infty\int_\Omega u_\varepsilon^\gamma\Delta\varphi - \chi\int_0^\infty\int_\Omega u_\varepsilon^\gamma\frac{\nabla v_\varepsilon}{v_\varepsilon}\cdot\nabla\varphi\\
&\qquad - \chi(1-\gamma)\int_0^\infty\int_\Omega \frac{u_\varepsilon^{\gamma+1}}{v_\varepsilon}\varphi - (r\chi+\gamma\chi-\chi)\int_0^\infty\int_\Omega u_\varepsilon^\gamma\varphi\\
&\qquad + \mu\gamma\int_0^\infty\int_\Omega u_\varepsilon^{\gamma+k-1}\varphi + \varepsilon\gamma\int_0^\infty\int_\Omega u_\varepsilon^{\gamma+1}\varphi\Big)\\
&= -\int_0^\infty\int_\Omega u^\gamma\varphi_t - \int_\Omega u_0^\gamma\varphi(\cdot,0) - \int_0^\infty\int_\Omega u^\gamma\Delta\varphi\\
&\quad - \chi\int_0^\infty\int_\Omega u^\gamma\frac{\nabla v}{v}\cdot\nabla\varphi - \chi(1-\gamma)\int_0^\infty\int_\Omega\frac{u^{\gamma+1}}{v}\varphi\\
&\quad - (r\chi+\gamma\chi-\chi)\int_0^\infty\int_\Omega u^\gamma\varphi + \mu\gamma\int_0^\infty\int_\Omega u^{\gamma+k-1}\varphi\end{aligned}$$

as $\varepsilon = \varepsilon_j \searrow 0$. This together with (2.30) yields that (u, v) is a *weak γ-entropy supersolution* of system (1.5) as well.

Consequently, the proof of Theorem 1.1 is complete. □

3. BOUNDEDNESS OF SMALL-DATA SOLUTION FOR $n \geqslant 2$

To study the dynamic behavior of solutions to (1.5) for $k > 1$, we should pay attention to establishing a positive uniform-in-time lower bound for v. With the aid of the following crucial ODE inequality (see [15]), this will be realized by a uniform-in-time upper estimate on the integral $\int_\Omega u^{-m}\,\mathrm{d}x$ with some $m > 0$, see [6].

Lemma 3.1 ([15], Lemma 3.4). *Let $T > 0$, and suppose that y is a nonnegative absolutely continuous function on $[0,T)$ satisfying*

$$y'(t) + ay(t) \leqslant f(t) \quad \text{for almost every } t \in (0,T)$$

with some $a > 0$ and a nonnegative function $f \in L^1_{\rm loc}([0,T))$ for which there exists $b > 0$ such that

$$\int_t^{t+1} f(s)\,\mathrm{d}x \leqslant b \quad \text{for all } t \in [0,T-1).$$

Then

$$y(t) \leqslant \max\Big\{y(0)+b, \frac{b}{a}+2b\Big\} \quad \text{for all } t \in (0,T).$$

Now, we have:

Lemma 3.2. *Let $k > 1$, $\mu > 0$ and $r, \chi > 0$ satisfy*

$$r > \begin{cases} \dfrac{\chi^2}{4}, & 0 < \chi \leqslant 2, \\ \chi - 1, & \chi > 2. \end{cases} \tag{3.1}$$

Then there exists some $\delta_0 > 0$ such that

$$v_\varepsilon \geqslant \delta_0 \quad \text{for all } \varepsilon \in (0,1) \text{ and } (x,t) \in \Omega \times (0,\infty). \tag{3.2}$$

Proof. It is shown by Lemma 2.1 of [7] that

$$v_\varepsilon(x,t) \geqslant \eta \int_\Omega u_\varepsilon(x,t)\,\mathrm{d}x \quad \text{for all } \varepsilon \in (0,1) \text{ and } (x,t) \in \Omega \times (0,\infty) \tag{3.3}$$

with some $\eta = \eta(\Omega) > 0$. Now, let $c_1 := \|u_0\|_{L^\infty(\Omega)} + 1$, $c_2 := \frac{1}{2}\int_\Omega u_0\,\mathrm{d}x$ and define for each $\varepsilon \in (0,1)$

$$T_\varepsilon := \sup\{T \in [0,\infty)\colon\ \|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant c_1 \text{ and } (I-\Delta)^{-1}u_\varepsilon(\cdot,t) \geqslant c_2\eta,\ t \in [0,T]\}, \tag{3.4}$$

and furthermore

$$T_0 := \inf_{\varepsilon \in (0,1)} T_\varepsilon. \tag{3.5}$$

Then can be readily seen by Lemma 2.4 with (3.4) and (3.5) that $T_\varepsilon > 0$ for $\varepsilon \in (0,1)$ along with $T_0 \geqslant 0$ being well-defined. Based on well-known estimates on the Neumann heat semigroup $\{\mathrm{e}^{t\Delta}\}_{t\geqslant 0}$ in [21], Lemma 1.3 and $v_\varepsilon(\cdot,s) = (I-\Delta)^{-1}u_\varepsilon \geqslant c_2$ for all $t \in [0,T_\varepsilon]$ with $\varepsilon \in (0,1)$, we know for $\varepsilon \in (0,1)$ that

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(\Omega)} &\leqslant \|\mathrm{e}^{t\Delta}u_0\|_{L^\infty(\Omega)} + \chi\int_0^t \Big\|\mathrm{e}^{(t-s)\Delta}\nabla\cdot\Big(\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon\Big)\Big\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\quad + r\int_0^t \|\mathrm{e}^{(t-s)\Delta}u_\varepsilon\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\leqslant \|u_0\|_{L^\infty(\Omega)} + \frac{2K_1\chi c_1^2}{c_2}T_\varepsilon^{1/2} + rK_2c_1T_\varepsilon, \quad t\in[0,T_\varepsilon],\end{aligned} \tag{3.6}$$

where K_1, K_2 are certain positive constants. Let

$$T_{\varepsilon 1} = \frac{1}{rK_2c_1}\bigg(\sqrt{\Big(\frac{K_1\chi c_1^2}{c_2\sqrt{rK_2c_1}}\Big)^2 + \frac{1}{2}} - \frac{K_1\chi c_1^2}{c_2\sqrt{rK_2c_1}}\bigg)^2. \tag{3.7}$$

Then $T_{\varepsilon 1} > 0$ is independent of $\varepsilon \in (0,1)$ and $(2K_1\chi c_1^2/c_2)T_{\varepsilon 1}^{1/2} + rK_2c_1T_{\varepsilon 1} = \frac{1}{2}$. Moreover, we know from (3.6) and (3.7) for all $\varepsilon \in (0,1)$ that

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + \frac{1}{2}, \quad t \in [0, \min\{T_\varepsilon, T_{\varepsilon 1}\}]. \tag{3.8}$$

In addition, we get for $\varepsilon \in (0,1)$ that

$$\begin{aligned}\int_\Omega u_\varepsilon\,\mathrm{d}x &= \int_\Omega \mathrm{e}^{t\Delta}u_0\,\mathrm{d}x + r\int_0^t\int_\Omega \mathrm{e}^{(t-s)\Delta}u\,\mathrm{d}x - \mu\int_0^t\int_\Omega \mathrm{e}^{(t-s)\Delta}u^k\,\mathrm{d}x \\ &\quad - \varepsilon\int_0^t\int_\Omega \mathrm{e}^{(t-s)\Delta}u^2\,\mathrm{d}x \\ &\geqslant \int_\Omega u_0\,\mathrm{d}x - (\mu c_1^k + c_1^2)T_\varepsilon, \quad t\in[0,T_\varepsilon],\end{aligned}$$

which yields

$$(I-\Delta)^{-1}u_\varepsilon \geqslant 2\eta c_2 - \eta(\mu c_1^k + c_1^2)T_\varepsilon, \quad t \in [0,T_\varepsilon]. \tag{3.9}$$

Let

$$T_{\varepsilon 2} = \frac{c_2}{2(\mu c_1^k + c_1^2)}. \tag{3.10}$$

It is easy to see that $T_{\varepsilon 2} > 0$ is independent of $\varepsilon \in (0,1)$ and $\eta(\mu c_1^k + c_1^2)T_{\varepsilon 2} = \frac{1}{2}\eta c_2$. Hence, we obtain from (3.9) and (3.10) that

$$(I-\Delta)^{-1}u_\varepsilon \geqslant \frac{3}{2}\eta c_2, \quad t \in [0, \min\{T_\varepsilon, T_{\varepsilon 2}\}]. \tag{3.11}$$

Therefore, according to the definitions of T_ε and T_0 in (3.4) and (3.5), a combination of (3.7), (3.8), (3.10) and (3.11) ensures that

$$\begin{aligned}(3.12)\quad T_0 = \inf_{\varepsilon\in(0,1)} T_\varepsilon &\geqslant \min\{T_{\varepsilon 1}, T_{\varepsilon 2}\}\\ &= \min\Big\{\frac{1}{rK_2c_1}\Big(\sqrt{\Big(\frac{K_1\chi c_1^2}{c_2\sqrt{rK_2c_1}}\Big)^2+\frac{1}{2}}-\frac{K_1\chi c_1^2}{c_2\sqrt{rK_2c_1}}\Big)^2, \frac{c_2}{2(\mu c_1^k + c_1^2)}\Big\} > 0,\end{aligned}$$

which is independent of $\varepsilon \in (0,1)$, and moreover

$$(3.13)\qquad \|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 1 \quad \forall\, \varepsilon\in(0,1) \text{ and } t\in(0,T_0],$$

$$(3.14)\qquad v_\varepsilon(x,t) \geqslant \frac{\eta}{2}\int_\Omega u_0\,\mathrm{d}x \quad \forall\, \varepsilon\in(0,1) \text{ and } (x,t)\in\Omega\times(0,T_0].$$

Now, we deal with the uniformly lower bound of v_ε for all $\varepsilon\in(0,1)$ and $(x,t)\in\Omega\times(T_0,\infty)$. For $m>0$, it is known from $(2.4)_1$ that

$$\begin{aligned}(3.15)\quad \frac{1}{m}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x =& -\int_\Omega u_\varepsilon^{-m-1}\Big[\Delta u_\varepsilon - \chi\nabla\cdot\Big(\frac{u_\varepsilon}{v_\varepsilon}\nabla v\Big) + ru_\varepsilon - \mu u_\varepsilon^k - \varepsilon u_\varepsilon^2\Big]\,\mathrm{d}x\\ =& -(m+1)\int_\Omega u_\varepsilon^{-m-2}|\nabla u|^2\,\mathrm{d}x + \chi(m+1)\int_\Omega u_\varepsilon^{-m-1}\frac{\nabla u_\varepsilon\cdot\nabla v_\varepsilon}{v_\varepsilon}\,\mathrm{d}x\\ & - r\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x + \mu\int_\Omega u_\varepsilon^{-m-1+k}\,\mathrm{d}x + \varepsilon\int_\Omega u_\varepsilon^{-m+1}, \quad t\in(T_0,\infty).\end{aligned}$$

If $a\in(0,\chi)$, we get from $(2.4)_2$ that

$$\begin{aligned}(3.16)\quad (m+1)a\int_\Omega u_\varepsilon^{-m-1}\frac{\nabla u_\varepsilon\cdot\nabla v_\varepsilon}{v_\varepsilon}\,\mathrm{d}x =& -\frac{(m+1)a}{m}\int_\Omega \nabla u_\varepsilon^{-m}\cdot\frac{\nabla v_\varepsilon}{v_\varepsilon}\,\mathrm{d}x\\ \leqslant& -\frac{(m+1)a}{m}\int_\Omega u_\varepsilon^{-m}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\,\mathrm{d}x\\ & + \frac{(m+1)a}{m}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x,\end{aligned}$$

and by Young's inequality that

$$\begin{aligned}(3.17)\quad (m+1)(\chi-a)&\int_\Omega u_\varepsilon^{-m-1}\frac{\nabla u_\varepsilon\cdot\nabla v_\varepsilon}{v_\varepsilon}\,\mathrm{d}x\\ &\leqslant (m+1)\int_\Omega u_\varepsilon^{-m-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x + \frac{(m+1)(\chi-a)^2}{4}\int_\Omega u_\varepsilon^{-m}\frac{|\nabla v_\varepsilon|^2}{v^2}\,\mathrm{d}x.\end{aligned}$$

Let $m := 4a/(\chi-a)^2$ for $a\in(0,\chi)$. Then $\frac{1}{4}(m+1)(\chi-a)^2 = (m+1)a/m$. Denote

$$f(a) := -4\Big(r - \frac{(m+1)a}{m}\Big) = a^2 - (2\chi-4)a + \chi^2 - 4r.$$

A direct calculation shows that $\Delta = 16(r+1-\chi) > 0$ for $r > \max\{\chi-1, 0\}$, and hence $f(a) < 0$ for $a\in(a_-, a_+)$, here $a_\pm = \chi - 2 \pm 2\sqrt{r+1-\chi}$. By the Viète formula, we know $a_- < 0 < a_+$ if $r > \frac{1}{4}\chi^2$ with $\chi > 0$, and $0 < a_- < \chi < a_+$ if $\chi - 1 < r \leqslant \frac{1}{4}\chi^2$ with $\chi > 2$. Therefore, if $r, \chi > 0$ satisfying (3.1), there exists some $c_0 > 0$ such that $-(r - (m+1)a/m) = \frac{1}{4}f(p) \leqslant -c_0 < 0$ for $a \in (0, \chi) \cap (a_-, a_+)$. Hence, combining (3.15)–(3.17), we have

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x \leqslant -c_0\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x + \mu\int_\Omega u_\varepsilon^{-m-1+k}\,\mathrm{d}x + \int_\Omega u_\varepsilon^{-m+k}\,\mathrm{d}x \tag{3.18}$$

for $t \in (T_0, \infty)$.

If $k - m \leqslant 0$, we have from (3.18) by Young's inequality that

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x \leqslant -\frac{c_0}{2}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x + \Big(1 + \Big(\frac{4}{c_0}\Big)^{(m-k)/k} + \mu\Big(\frac{4\mu}{c_0}\Big)^{(m+1-k)/(k-1)}\Big)|\Omega|. \tag{3.19}$$

Similarly, if $k - 1 - m \leqslant 0 < k - m$, we get

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x \leqslant -\frac{c_0}{2}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x + \int_\Omega u_\varepsilon^{k}\,\mathrm{d}x + \Big(2 + \mu\Big(\frac{2\mu}{c_0}\Big)^{(m+1-k)/(k-1)}\Big)|\Omega|, \tag{3.20}$$

and if $k - 1 - m > 0$,

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x \leqslant -c_0\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x + \int_\Omega u_\varepsilon^{k}\,\mathrm{d}x + (\mu(2\mu)^{(k-1-m)/(m+1)} + 2^{(k-m)/m})|\Omega|. \tag{3.21}$$

Estimates (3.19)–(3.21) yield for $k > 1$, $\mu > 0$ and $r, \chi > 0$ satisfying (3.1) that

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x \leqslant -\frac{c_0}{2}\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x + \int_\Omega u_\varepsilon^{k}\,\mathrm{d}x + C_5, \quad t \in (T_0, \infty) \tag{3.22}$$

with some $C_5 > 0$ independent of $\varepsilon \in (0,1)$. In addition, it is known by (2.8) with (2.5) that

$$\int_t^{t+1}\int_\Omega u_\varepsilon^k\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{(r+1)m_*}{\mu}, \quad t \in (0, \infty).$$

This together with (3.13), (3.22) and Lemma 3.1 implies

$$\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x \leqslant C_6, \quad t \in (T_0, \infty) \tag{3.23}$$

with some $C_6 > 0$ independent of $\varepsilon \in (0,1)$.

Let $\alpha := m/(m+1) \in (0,1)$. Then for $k > 1$ we obtain from (3.3) and (3.23) that

$$v_\varepsilon \geqslant \eta\int_\Omega u_\varepsilon\,\mathrm{d}x \geqslant \eta|\Omega|^{(m+1)/m}\Big(\int_\Omega u_\varepsilon^{-m}\,\mathrm{d}x\Big)^{-1/m} \geqslant \eta C_2^{-1/m}|\Omega|^{(m+1)/m} =: \eta_0$$

by Hölder's inequality for all $(x,t) \in \Omega \times (T_0, \infty)$. This together with (3.14) concludes (3.2) with $\delta_0 = \min\{\frac{1}{2}\eta\int_\Omega u_0\,\mathrm{d}x, \eta_0\}$ independent of $\varepsilon \in (0,1)$. □

Next, we will prove that the global generalized solution to (1.5) is globally bounded. At first, we give a crucial estimate on $\int_\Omega u_\varepsilon^p\,\mathrm{d}x$ for all $\varepsilon \in (0,1)$.

Lemma 3.3. *Let $n \geqslant 2$ and $(u_\varepsilon, v_\varepsilon)$ be the global classical solution of problem (2.4) established in Lemma 2.4. Then for $p \in (\frac{1}{2}n(n+2)/(n+1), n]$ with $q \in (p, (n+2)p/n)$ it holds*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^p\,\mathrm{d}x \leqslant {}& -\int_\Omega u_\varepsilon^p\,\mathrm{d}x + M_7\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{(q-qa)/(p-qa)} + M_7\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{q/p}\\ &+ M_7\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{2q/(p(q-p))} + M_7\int_\Omega u_\varepsilon\,\mathrm{d}x, \quad t>0\end{aligned} \tag{3.24}$$

with $a = \frac{1}{2}n(q-p)/q \in (0,1)$ and some $M_7 > 0$ independent of $\varepsilon \in (0,1)$.

Proof. It follows from $(2.4)_1$ and (3.2) for $1 < p < q$ that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u_\varepsilon^p\,\mathrm{d}x ={}& \int_\Omega u_\varepsilon^{p-1}\Big[\Delta u_\varepsilon - \chi\nabla\cdot\Big(\frac{u_\varepsilon}{v_\varepsilon}\nabla v_\varepsilon\Big) + ru_\varepsilon - \mu u_\varepsilon^k - \varepsilon u_\varepsilon^{k+1}\Big]\,\mathrm{d}x\\ \leqslant{}& -\frac{1}{p}\int_\Omega u_\varepsilon^p\,\mathrm{d}x - (p-1)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ &+ \chi(p-1)\int_\Omega \frac{u_\varepsilon^{p-1}}{v_\varepsilon^2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\,\mathrm{d}x\\ &+ (r+1)\int_\Omega u_\varepsilon^p\,\mathrm{d}x - \mu\int_\Omega u_\varepsilon^{p+k-1}\,\mathrm{d}x\\ \leqslant{}& -\frac{1}{p}\int_\Omega u_\varepsilon^p\,\mathrm{d}x - \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x + \frac{\chi^2(p-1)}{2\delta_0^2}\int_\Omega u_\varepsilon^p|\nabla v_\varepsilon|^2\,\mathrm{d}x\\ &+ (r+1)\int_\Omega u_\varepsilon^p\,\mathrm{d}x - \frac{\mu}{2}\int_\Omega u_\varepsilon^{p+k-1}\,\mathrm{d}x\\ \leqslant{}& -\frac{1}{p}\int_\Omega u_\varepsilon^p\,\mathrm{d}x - \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x + \frac{\chi^2(p-1)}{2\delta_0^2}\int_\Omega u_\varepsilon^q\,\mathrm{d}x\\ &+ \frac{\chi^2(p-1)}{2\delta_0^2}\int_\Omega |\nabla v_\varepsilon|^{2q/(q-p)}\,\mathrm{d}x + C_7\int_\Omega u_\varepsilon\,\mathrm{d}x, \quad t>0\end{aligned} \tag{3.25}$$

by Young's inequality with $C_7 = (r+2)^{(p+k-1)/(k-1)}(2/\mu)^{n/(k-1)}$. If $q < np/(n-2)$, invoke Gagliardo-Nirenberg's inequality to get

$$\begin{aligned}\|u_\varepsilon\|_{L^q(\Omega)} = \|u_\varepsilon^{p/2}\|^{2/p}_{L^{2q/p}(\Omega)} &\leqslant C_{GN}\|u_\varepsilon^{p/2}\|^{2a/p}_{W^{1,2}(\Omega)}\|u_\varepsilon^{p/2}\|^{2(1-a)/p}_{L^2(\Omega)}\\ &\leqslant 2^{pa/2}C_{GN}(\|u_\varepsilon^{(p-2)/2}\nabla u\|^{2a/p}_{L^2(\Omega)}\|u_\varepsilon^{p/2}\|^{2(1-a)/p}_{L^2(\Omega)} + \|u_\varepsilon^{p/2}\|^{2/p}_{L^2(\Omega)}).\end{aligned} \tag{3.26}$$

where $a = n(q-p)/2q \in (0,1)$. Moreover, if $1 < p < q < (n+2)/np$, we know $2qa/p < 1$. This fact together with (3.26) yields

$$\frac{\chi^2(p-1)}{2\delta_0^2}\int_\Omega u_\varepsilon^q \,\mathrm{d}x \leqslant \frac{p-1}{2}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2\,\mathrm{d}x + C_8\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{q(1-a)/(p-qa)} + C_9\Big(\int_\Omega u_\varepsilon^p x\Big)^{q/p} \tag{3.27}$$

by Young's inequality with $C_8 = (p-1)^{p/(qa)}(2^{q+pqa/2}\chi^2/\delta_0^2 C_{GN}^q)^{p/(p-qa)}$ and $C_9 = 2^{q+pqa/2}C_{GN}^q\chi^2(p-1)/\delta_0^2$. Now, let $p \in \frac{1}{2}n(n+2)/(n+1), n]$ with $p < q < ((n+2)/n)p$. Then $2q/(q-p) < np/(n-p)$ and $W^{1,2q/(q-p)}(\Omega)$ is continuously embedded into $W^{2,p}(\Omega)$. By the elliptic regularity theory in [6], Lemma 4.3, we obtain

$$\begin{aligned}\frac{\chi^2(p-1)}{2\delta_0^2}\int_\Omega |\nabla v_\varepsilon|^{2q/(q-p)}\,\mathrm{d}x &= \frac{\chi^2(p-1)}{2\delta_0^2}\|\nabla v_\varepsilon\|_{L^{2q/(q-p)}(\Omega)}^{2q/(q-p)}\\ &\leqslant C_{10}\frac{\chi^2(p-1)}{2\delta_0^2}\|\nabla v_\varepsilon\|_{W^{1,p}(\Omega)}^{2q/(q-p)} \leqslant C_{11}\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{2q/(p(q-p))}, \quad t>0\end{aligned} \tag{3.28}$$

with some $C_{10}, C_{11} > 0$. Combining (3.25) with (3.27) and (3.28), we have

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^p\,\mathrm{d}x \leqslant &-\frac{1}{p}\int_\Omega u_\varepsilon^p\,\mathrm{d}x + C_8\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{(q-qa)/(p-qa)} + C_9\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{q/p}\\ &+ C_{11}\Big(\int_\Omega u_\varepsilon^p\,\mathrm{d}x\Big)^{2q/(p(q-p))} + C_7\int_\Omega u_\varepsilon\,\mathrm{d}x, \quad t>0.\end{aligned} \tag{3.29}$$

This completes conclusion (3.24) with $M_7 = p\max\{C_7, C_8, C_9, C_{11}\}$. □

Now, we establish the following uniform-in-time estimate on $\int_\Omega u_\varepsilon^p\,\mathrm{d}x$ for all $\varepsilon \in (0,1)$ with the initial data u_0 and r/μ suitably small.

Lemma 3.4. *Let $n \geqslant 2$ and $(u_\varepsilon, v_\varepsilon)$ be the global classical solution of problem* (2.4) *established in Lemma* 2.4. *Then for $p \in (\frac{1}{2}n(n+2)/(n+1), n]$ there exist $\eta, \lambda > 0$ such that*

$$\int_\Omega u_\varepsilon^p\,\mathrm{d}x \leqslant M_8, \quad t>0, \tag{3.30}$$

provided $r/\mu < \eta$ and $\int_\Omega u_0^p\,\mathrm{d}x < \lambda$ with some $M_8 > 0$ independent of $\varepsilon \in (0,1)$.

Proof. Let $F_\varepsilon(t) := \int_\Omega u_\varepsilon(x,t)^p\,\mathrm{d}x$, $t>0$. Then we have from (3.29) and (4.3) that

$$\begin{cases} F_\varepsilon'(t) \leqslant -F_\varepsilon(t) + M_7F_\varepsilon(t)^{(q-qa)/(p-qa)} + M_6F_\varepsilon(t)^{q/p}\\ \qquad\quad +M_7F_\varepsilon(t)^{2q/(p(q-p))} + M_6m^*, \quad t>0,\\ F_\varepsilon(0) = \int_\Omega u_0^p\,\mathrm{d}x.\end{cases} \tag{3.31}$$

Since $p \in (\frac{1}{2}n(n+2)/(n+1), n]$ and $p < q < (n+2)p/n$, we know $(q-qa)/(p-qa)$, q/p, $2q/(p(q-a)) > 1$. Denote

$$h(s, m^*) := -s + M_7 s^{(q-qa)/(p-qa)} + M_6 s^{\frac{q}{p}} + M_6 s^{2q/(p(q-p))} + M_6 m^*, \quad s > 0.$$

Then there exists $m_0^* > 0$ such that $h(s, m_0^*)$ has the unique positive root s_0. Furthermore, $M(t) \equiv s_0$ verifies the ODE problem

$$\begin{cases} M'(t) = h(M(t), m_0^*), & t > 0, \\ M(0) = s_0. \end{cases} \tag{3.32}$$

If $m^* < m_0^*$, it follows by a continuous dependence argument that function $h(s, m^*)$, with $h(s, m^*) < h(s, m_0^*)$, has exactly two positive roots $0 < s_1 < s_0 < s_2$. Now let

$$\eta := \Big(\frac{m_0^*}{|\Omega|}\Big)^{k-1} \quad \text{and} \quad \lambda := \min\Big\{s_0, \frac{m_0^{*p}}{|\Omega|^{p-1}}\Big\}.$$

Then for $r/\mu < \eta$ and $\int_\Omega u_0^p \,\mathrm{d}x < \lambda$,

$$\int_\Omega u_0 \,\mathrm{d}x < |\Omega|^{(p-1)/p} \Big(\int_\Omega u_0^p \,\mathrm{d}x\Big)^{1/p} < m_0^*$$

and

$$\int_\Omega u_\varepsilon \,\mathrm{d}x \leqslant \max\Big\{\int_\Omega u_0 \,\mathrm{d}x, \Big(\frac{r}{\mu}\Big)^{1/(k-1)} |\Omega|\Big\} < m_0^*$$

for all $\varepsilon \in (0,1)$. Consequently, we obtain from these estimates with problems (3.31) and (3.32) that

$$F_\varepsilon(t) = \int_\Omega u_\varepsilon^p \,\mathrm{d}x \leqslant s_1, \quad t > 0$$

for all $\varepsilon \in (0,1)$ by an ODE comparison principle. The proof is complete. $\square$

Proof of Theorem 1.2. Based on the estimate $\int_\Omega u_\varepsilon^p \,\mathrm{d}x \leqslant s_1$ for $p \in (\frac{1}{2}n(n+2)/(n+1), n]$ in Lemma 3.4 and uniformly in time lower-bound estimate of v_ε in Lemma 3.2, we obtain the global boundedness of solutions to the regularized problem (2.4) via a similar argument as that in [20], Lemma 2.3, i.e., $\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant \widetilde{C}$ with some $\widetilde{C} > 0$ for all $t > 0$ and $\varepsilon \in (0,1)$. Consequently, we conclude that the generalized solution (u, v) is globally bounded as well by taking $\varepsilon = \varepsilon_j \searrow 0$. $\square$

4. GLOBAL BOUNDEDNESS OF THE CLASSICAL SOLUTION FOR $n = 1$

Firstly, we introduce the global existence of the classical solution to (1.5) for $n = 1$.

Lemma 4.1. *Let $n = 1$ and (u_0, v_0) satisfy (1.6). Then system (1.5) admits a global classical solution.*

Proof. The standard Banach fixed theorem entails that system (1.5) possesses a local classical solution (u,v) in $\Omega\times(0,T_{\max})$ with some $T_{\max}\leqslant\infty$. Then, based on the a priori point-in-wise lower bound of v and the restriction on dimension, it will be shown for $n=1$ that the local solution (u,v) is in fact global via some fundamental energy estimates, i.e., $T_{\max}=\infty$. $\square$

Here, we give some fundamental facts to the classical solution.

Lemma 4.2. *Let (u,v) be the global classical solution to (1.5) for $n=1$. Then*

$$\int_\Omega u\,\mathrm{d}x\leqslant m^*,\quad t>0 \tag{4.1}$$

with $m^*>0$ determined in Lemma 2.5, and

$$\int_\Omega \frac{v_x^2}{v^2}\,\mathrm{d}x\leqslant|\Omega|,\quad t>0. \tag{4.2}$$

Now, we establish the crucial estimate on u in L^p-norm with $p>1$.

Lemma 4.3. *For $p>1$, it holds with some $M_9>0$ that*

$$\|u\|_{L^p(\Omega)}\leqslant M_9,\quad t>0. \tag{4.3}$$

Proof. Denote $C_{12}:=\max\{1,\max\limits_{s>0}(r+1)s-\mu s^k\}$ for $s>0$. According to the representation formula for u, it is known that

$$\begin{aligned}u={}&\mathrm{e}^{t(\partial^2/\partial x^2-1)}u_0-\chi\int_0^t\mathrm{e}^{(t-s)(\partial^2/\partial x^2-1)}\Big(\frac{u}{v}v_x\Big)_x\,\mathrm{d}x\\&+\int_0^t\mathrm{e}^{(t-s)(\partial^2/\partial x^2-1)}((r+1)u-\mu u^k)\,\mathrm{d}x,\end{aligned} \tag{4.4}$$

and by Lemma 1.3 of [21] with $p>2$ that

$$\begin{aligned}\|u\|_{L^p(\Omega)}\leqslant{}&(\|u_0\|_{L^\infty(\Omega)}+C_{12})|\Omega|^{1/p}\\&+\chi K_1\int_0^t(1+(t-s)^{-1+1/(2p)})\mathrm{e}^{-\lambda_1(t-s)}\Big\|\frac{u}{v}v_x\Big\|_{L^1(\Omega)}\\\leqslant{}&C_{13}+C_{14}\sup_{s\in(0,t)}\Big\|\frac{u}{v}v_x\Big\|_{L^1(\Omega)},\quad t>0\end{aligned} \tag{4.5}$$

with $C_{13} = (\|u_0\|_{L^\infty(\Omega)} + C_{12})|\Omega|^{1/p}$ and $C_{14} = \chi K_1 \int_0^t (1+(t-s)^{-1+1/(2p)}) \mathrm{e}^{-\lambda_1(t-s)}\,\mathrm{d}s$ for some $K_1, \lambda_1 > 0$. In addition, we get by (4.1) with (4.2) and the interpolation inequality that

$$\begin{aligned}\int_\Omega \frac{u}{v}|v_x|\,\mathrm{d}x &\leqslant \left(\int_\Omega \frac{v_x^2}{v^2}\,\mathrm{d}x\right)^{1/2} \|u\|_{L^2(\Omega)} \\ &\leqslant |\Omega|^{1/2} \|u\|_{L^p(\Omega)}^{p/(2(p-1))} \|u\|_{L^1(\Omega}^{(p-2)/(2(p-1))} \\ &\leqslant |\Omega|^{1/2} m^{*(p-2)/(2(p-1))} \|u\|_{L^p(\Omega)}^{p/(2(p-1))}.\end{aligned}$$

This together with (4.5) and Young's inequality entails that

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\leqslant C_{13} + C_{14}|\Omega|^{1/2} m^{*(p-2)/(2(p-1))} \sup_{s\in(0,t)} \|u\|_{L^p(\Omega)}^{p/(2(p-1))} \\ &\leqslant C_{13} + \frac{1}{2}\sup_{s\in(0,t)} \|u\|_{L^p(\Omega)} + m^*(2\sqrt{|\Omega|}C_{14})^{2(p-1)/(p-2)}, \quad t > 0.\end{aligned} \tag{4.6}$$

Consequently, the proof of (4.3) is complete by (4.6) with

$$M_9 = 2C_{13} + 2m^\star(2\sqrt{|\Omega|}C_{14})^{2(p-1)/(p-2)}.$$

□

Proof of Theorem 1.3. Using Lemma 1.3 of [21] with (4.4), we get from (4.2) and (4.3) for $p \in (1,2)$ that

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} &\leqslant C_{12} + \|u_0\|_{L^\infty(\Omega)} \\ &\quad + \chi K_1 \int_0^t (1 + (t-s)^{-1/2-1/(2p)}) \mathrm{e}^{-\lambda_1(t-s)} \left\|\frac{u}{v}v_x\right\|_{L^p(\Omega)} \mathrm{d}s \\ &\leqslant C_{12} + \|u_0\|_{L^\infty(\Omega)} \\ &\quad + \chi K_1 |\Omega|^{1/2} \int_0^t (1 + (t-s)^{-1/2-1/(2p)}) \mathrm{e}^{-\lambda_1(t-s)}\,\mathrm{d}s \sup_{s\in(0,t)} \|u\|_{L^{2p/(2-p)}(\Omega)} \\ &\leqslant C_{12} + \|u_0\|_{L^\infty(\Omega)} + \chi K_1 M_9 |\Omega|^{1/2} \int_0^t (1 + (t-s)^{-1/2-1/(2p)}) \mathrm{e}^{-\lambda_1(t-s)}\,\mathrm{d}s \\ &=: \overline{M}, \quad t > 0,\end{aligned}$$

which concludes the global boundedness of classical solution in dimension one due to the constant $\overline{M}$ independent of t. □

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