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GLOBAL CLASSICAL SOLUTIONS IN A SELF-CONSISTENT CHEMOTAXIS(-NAVIER)-STOKES SYSTEM

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Abstract. The self-consistent chemotaxis-fluid system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c) + \nabla\cdot(n\nabla\phi), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u - n\nabla\phi + n\nabla c, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases}$$

is considered under no-flux boundary conditions for n, c and the Dirichlet boundary condition for u on a bounded smooth domain $\Omega\subset\mathbb{R}^N$ $(N=2,3)$, $\kappa\in\{0,1\}$. The existence of global bounded classical solutions is proved under a smallness assumption on $\|c_0\|_{L^\infty(\Omega)}$.

Both the effect of gravity (potential force) on cells and the effect of the chemotactic force on fluid are considered here, and thus the coupling is stronger than the most studied chemotaxis-fluid systems. The literature on self-consistent chemotaxis-fluid systems of this type so far concentrates on the nonlinear cell diffusion as an additional dissipative mechanism. To the best of our knowledge, this is the first result on the boundedness of a self-consistent chemotaxis-fluid system with linear cell diffusion.

Keywords: chemotaxis; Navier-Stokes system; self-consistent; global existence; boundedness

MSC 2020: 35K55, 35Q92, 35Q35, 92C17

1. Introduction

Chemotaxis-fluid system. Since Patlak in [21] introduced the first model of chemotaxis in 1953, Keller and Segel in [16] derived a similar model under different

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assumptions in 1970. The theory of chemotaxis experienced an extreme development in the past decades, because it combines several interests: it is strongly related to applications and it involves a rich class of mathematically interesting problems. Many details of this phenomenon and appropriate mathematical models are given in a vast literature. Here we refer to the surveys [1], [2], [12] and the references therein.

To model chemotaxis of cell populations, like *Bacillus subtilis*, in a viscous fluid, the authors in [26] proposed the chemotaxis-fluid system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega,\ t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u - n \nabla \phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

as a model for the unknown (n, c, u, P) in the physical domain $\Omega \subset \mathbb{R}^N$, where the coefficient κ is related to the strength of nonlinear fluid convection and ϕ is a gravitational potential function. In particular, in (1.1) it is assumed that the presence of bacteria with density denoted by $n = n(x,t)$ affects the fluid motion, and $u = u(x,t)$ and $P = P(x,t)$ represent the velocity field of the incompressible fluid and an associated pressure, respectively. Moreover, it is assumed that both cells and oxygen, the latter with concentration $c = c(x,t)$, are transported by the fluid and diffuse randomly.

In the past decade, there was a tremendous amount of mathematical results on well-posedness and asymptotic behavior for the system (1.1) posed in bounded or unbounded domains. Particularly, when Ω is a bounded domain, questions have meanwhile been answered to quite a comprehensive extent in the case when the homogeneous boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0, \quad x \in \partial\Omega,$$

are imposed: Indeed, Lorz showed local existence of weak solutions to system (1.1) with $\kappa = 0$ in [20]. The well-posed problem in the two-dimensional case has been quite completely solved. For instance, Winkler proved that the system (1.1) admits a unique global classical solution when $\kappa = 1$, and such a classical solution will stabilize to a spatially uniform equilibrium in [33] and [34]. Furthermore, Zhang and Li showed an exponential convergence rate in [41]. Recently, the small-convection limit problem was investigated in [29]. For the three-dimensional case with Stokes-governed fluid, global weak solutions have been constructed in [33]. Then Winkler showed that for a full chemotaxis-Navier-Stokes system there exist global bounded weak solutions, which become eventually smooth and classical after some relaxation time, see [36], [37]. In the case of the whole space, there exist some works addressing the problem of well-posedness and large time behavior obtained in [5], [6], [8], [9], [19], [23], [43].

Apart from the system (1.1) itself, the case of nonlinear diffusion is often considered in the study of the chemotaxis-fluid model of material consumption with signals. The authors of [7] extended the system (1.1) to one with a porous medium-type diffusion as

$$
(1.2)\qquad \begin{cases} n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u - n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0. \end{cases}
$$

In comparison to the linear diffusion case, the nonlinear diffusion mechanisms of this type may suppress the occurrence of blow-up phenomena. Under the hypothesis of $\kappa = 0$, some suitable large m, say $m > \frac{7}{6}$, $m > \frac{8}{7}$, $m > \frac{9}{8}$ and $m > \frac{11}{4} - \sqrt{3}$, have been successively proved to be sufficient for the global existence and boundedness of weak solutions to an associated Neumann-Neumann-Dirichlet initial-boundary value problem for all reasonably regular initial data in three-dimensional bounded domains, see [15], [35], [38], [44]. Recently, in a three-dimensional bounded domain, the global boundedness of solution has been proved for the case $m > 1$, see [14]. Results on global existence and boundedness in two-dimensional chemotaxis-fluid systems with nonlinear cell diffusion can be found in [7], [13] and [25]. On the other hand, under $\kappa \neq 0$, the existence of global weak solutions is proved when $m \geqslant \frac{2}{3}$ in the three-dimensional case, see [42]. For more recent results, we refer to previous studies [18], [22] and references therein.

Self-consistent chemotaxis-fluid system. Although the model (1.1) has been used for numerical computations in the biophysical literature (see [26]), Lorz pointed out that it could be more realistic to include both the impact of gravity on cells and the effect of the chemotactic force on fluid, see [20]. To model such a biological phenomenon, Di Francesco et al. in [7], and Lorz in [20] proposed the following self-consistent chemotaxis-Navier-Stokes system

$$
(1.3)\qquad \begin{cases} n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(n\nabla c) + \nabla\cdot(n\nabla\phi), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u - n\nabla\phi + n\nabla c, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases}
$$

where $m \geqslant 1$. As is mentioned in [4], the reasoning behind the coupling $n\nabla c$ in the third equation of (1.3) is that the fluid exerts frictional force on the moving cells to make the cells move without acceleration and thus, that the reaction forces act on the fluid, which also matches the nonlinear cross-diffusion term of the cell density in the first equation of (1.3). In contrast to the systems (1.1) and (1.2), a stronger coupling here brings essential difficulty in mathematics. The appearance of the coupling term $n\nabla c$ leads to stronger nonlinearity, making it difficult to close the entropy estimate. When $m \in (\frac{3}{2}, 2)$, Di Francesco et al. obtained the global existence of weak solutions to the Neumann-Neumann-Dirichlet boundary problem for the system (1.3) with $\kappa = 0$ in two-dimensional bounded domains, see [7]. Furthermore, global weak solutions have been constructed for any $m > 1$ later for this chemotaxis-Stokes system and even the fully chemotaxis-Navier-Stokes system in [27] and [40], respectively. Recently, under the hypothesis of $m > \frac{4}{3}$, Wang and Zhao proved the existence of global weak solutions to the initial-boundary value problem of system (1.3) in a three-dimensional setting with $\kappa = 0$, see [30].

However, to the best of our knowledge, few authors considered the situation of linear cell diffusion, that is, the case of $m = 1$ in (1.3). An exception is the very recent work, see [4]. When $m = 1$, the authors studied the global well-posedness and decay property of solutions to the Cauchy problem for (1.3) under a suitable smallness assumption of the initial data. Our objective in this paper is to study the self-consistent chemotaxis(-Navier)-Stokes system with linear cell diffusion in bounded domains.

Main results. In this paper, we consider the initial-boundary value problem

$$(1.4)\qquad \begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\nabla c) + \nabla \cdot (n\nabla\phi), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla)u + \nabla P = \Delta u - n\nabla\phi + n\nabla c, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ \nabla n \cdot \nu = \nabla c \cdot \nu = 0, \quad u = 0, & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x), c(x,0) = c_0(x), u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ with smooth boundary $\partial\Omega$, where $\kappa \in \{0, 1\}$ and ν denotes the unit outward normal vector field on $\partial\Omega$. The gravitational potential function ϕ is supposed to be a given parameter function.

We assume throughout this paper that the initial data (n_0, c_0, u_0) satisfy

$$(1.5)\qquad \begin{cases} n_0 \in C^0(\overline{\Omega}) & \text{with } n_0 \geqslant 0 \text{ and } n_0 \not\equiv 0 \text{ in } \overline{\Omega}, \\ c_0 \in W^{1,\infty}(\Omega) & \text{with } c_0 \geqslant 0 \text{ and } c_0 \not\equiv 0 \text{ in } \overline{\Omega}, \\ u_0 \in D(\mathcal{A}^\alpha) & \text{for some } \alpha \in \left(\dfrac{N}{4}, 1\right) \end{cases}$$

where $\mathcal{A} := -\mathcal{P}\Delta$ denotes the Stokes operator with the domain $D(\mathcal{A}) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega)$ and $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0\}$. For more details of Stokes operator, we can refer to [10], [17].

As for the gravitational potential ϕ in (1.4), we require that it is independent of time and satisfies

$$\phi \in W^{2,\infty}(\Omega) \tag{1.6}$$

and

$$\frac{\partial \phi}{\partial \nu} = 0, \quad x \in \partial\Omega. \tag{1.7}$$

Under these assumptions, we can establish boundedness of the global classical solutions to the system (1.4). Precisely, we have the following uniform boundedness result.

Theorem 1.1. *Let $N \in \{2,3\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that ϕ fulfills* (1.6)–(1.7). *Let either of the following conditions hold.*

(i) $N = 2$, $\kappa = 1$;

(ii) $N = 3$, $\kappa = 0$.

There is $\delta_0 > 0$ with the following property: If the initial data fulfill (1.5) *and*

$$\|c_0\|_{L^\infty(\Omega)} < \delta_0, \tag{1.8}$$

then (1.4) *admits a global classical solution (n, c, u, P), which is bounded in the sense that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C \quad \forall\, t > 0 \tag{1.9}$$

with some positive constant C.

The remainder part of this paper is organized as follows. In Section 2, we state the local solvability of system (1.4) and some lemmas to be used in subsequent proofs. In order to overcome the difficulties caused by stronger nonlinearity from the force term $n\nabla c$, we need higher regularity for n (see Lemma 3.1) to control this "new" term when we establish an entropy inequality. Specifically, we first derive some L^p $(p > 1)$ estimates for n under suitable smallness assumption on $\|c_0\|_{L^\infty(\Omega)}$ before some further regularity of c and u. In addition, an iteration technique is used to establish the necessary estimates of u and ∇c in Section 3. In Section 4, the boundedness of the global classical solution in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with $\kappa = 1$ is established. Finally, we establish the boundedness of the global classical solution in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ with $\kappa = 0$ in Section 5.

2. Preliminaries

Without essential difficulty, the above system (1.4) is locally solvable in the classical sense by an adaptation of the well-established fixed point argument, which is similar to Lemma 2.1 in [33]. We give the following lemma without proof.

Lemma 2.1. *Let $N \in \{2,3\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and $\kappa \in \mathbb{R}$. Assume the initial data (n_0, c_0, u_0) satisfy (1.5), and ϕ fulfills (1.6)–(1.7). Then there exist $T_{\max} \in (0,\infty]$ and a unique classical solution (n,c,u,P) to the system (1.4) satisfying*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ c \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap L^\infty([0,T_{\max}); W^{1,\infty}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap L^\infty([0,T_{\max}); D(\mathcal{A}^\alpha)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ P \in C^{1,0}(\overline{\Omega} \times [0,T_{\max})). \end{cases} \tag{2.1}$$

Moreover, we have $n > 0$ and $c > 0$ in $\overline{\Omega} \times [0,T_{\max})$, and if $T_{\max} < \infty$, then

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad \text{as } t \to T_{\max}, \tag{2.2}$$

where α is taken from (1.5).

In the following lemmas of this paper, we put $\tau := \min\{1, \frac{1}{2}T_{\max}\}$.

As a preparation for the reasoning below, let us introduce the following elementary ODI lemma, the proof of which can be found in Lemma 3.4 of [39], see also Lemma 2.3 of [28].

Lemma 2.2. *Let $T > 0$, $a > 0$ and $y \in C^0([0,T)) \cap C^1((0,T))$ be such that*

$$y'(t) + ay(t) \leqslant g(t) \quad \forall\, t \in (0,T),$$

where $g \in L^1_{\mathrm{loc}}([0,T))$ has the property that

$$\frac{1}{\tau}\int_t^{t+\tau} g(s)\,\mathrm{d}s \leqslant b \quad \forall\, t \in (0,T)$$

with some $\tau > 0$ and $b > 0$. Then

$$y(t) \leqslant y(0) + \frac{b\tau}{1 - \mathrm{e}^{-a\tau}} \quad \forall\, t \in (0,T).$$

It is easy to obtain the L^1-norm estimate of n and L^∞-norm estimate of c by a direct integration of the first equation of (1.4) and using the maximum principle.

Lemma 2.3. *Let* (n, c, u, P) *be a classical solution of* (1.4). *It follows that*

$$\int_\Omega n(\cdot, t) = \int_\Omega n_0 \quad \forall t \in (0, T_{\max}) \tag{2.3}$$

and

$$\|c(\cdot, t)\|_{L^\infty(\Omega)} \leqslant \|c_0\|_{L^\infty(\Omega)} \quad \forall t \in (0, T_{\max}). \tag{2.4}$$

Let $Y_\varepsilon v := (1 + \varepsilon \mathcal{A})^{-1} v$ for $v \in L^2_\sigma(\Omega)$. For the approximate Navier-Stokes system

$$\begin{cases} \partial_t v + (Y_\varepsilon v \cdot \nabla) v = \Delta v + \nabla P + f, & x \in \Omega,\ t > 0, \\ \nabla \cdot v = 0, & x \in \Omega,\ t > 0, \\ v = 0, & x \in \partial\Omega,\ t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.5}$$

our next lemma shows that under an assumption involving a spatio-temporal L^2-bound for the force f, solutions remain bounded in $W^{1,2}(\Omega)$. A proof of the following lemma can be found in Lemma 3.3 of [28].

Lemma 2.4 (Lemma 3.3 in [28]). *Let* $T \in (0, \infty]$ *and* $\tau \in (0, T)$, *and let* $v_0 \in C^0(\overline{\Omega}; \mathbb{R}^2) \cap W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega)$. *Then for all* $b > 0$ *there exists* $C(b, \tau) > 0$ *such that if* $f \in C^0(\overline{\Omega} \times [0, T); \mathbb{R}^2)$, $v \in C^0(\overline{\Omega} \times [0, T); \mathbb{R}^2) \cap C^0([0, T); W^{1,2}_0(\Omega; \mathbb{R}^2)) \cap C^{2,1}(\overline{\Omega} \times (0, T); \mathbb{R}^2)$, *and* $P \in C^{1,0}(\overline{\Omega} \times (0, T))$ *solve* (2.5) *for some* $\varepsilon \in [0, 1)$ *and they satisfy*

$$\frac{1}{\tau} \int_t^{t+\tau} \int_\Omega |f(x, s)|^2 \leqslant b \quad \forall t \in (0, T - \tau),$$

then

$$\int_\Omega |\nabla v(\cdot, t)|^2 \leqslant C(b, \tau) \quad \forall t \in (0, T).$$

3. Some basic regularity estimates for n, u and ∇c

In this section, we obtain boundedness of n in $L^p(\Omega)$ under the assumption that $\|c_0\|_{L^\infty(\Omega)}$ is suitably small. The approach is based on the weighted estimate of $\int_\Omega n^p \varphi(c)$ with an appropriate choice of φ which has been developed in [31] and has been applied in several publications such as [3], [11], [24].

Lemma 3.1. *Let $p > 1$. There is $\delta_0 := \delta_0(p) > 0$ with the property: If the initial data satisfy* (1.5)–(1.7) *and*

$$\|c_0\|_{L^\infty(\Omega)} < \delta_0, \tag{3.1}$$

then there exists $C > 0$ such that the first component of the solution in (1.4) *satisfies*

$$\|n(\cdot,t)\|_{L^p(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}) \tag{3.2}$$

and

$$\int_t^{t+\tau} \int_\Omega n^{p-2} |\nabla n|^2 \leqslant C \quad \forall\, t \in (0, T_{\max} - \tau). \tag{3.3}$$

Proof. Let $p > 1$, $0 < h < (p-1)/(32p)$. We can find $\delta_0 > 0$ satisfying

$$8p(p-1)\delta_0^2 \leqslant h(h+1), \tag{3.4}$$

$$4p\delta_0 \leqslant h+1. \tag{3.5}$$

Under the assumption (3.1), we can put $\varphi(c) = (\delta_0 - c)^{-h}$ according to (2.4), thus $\varphi(c) > 0$. Elementary calculus shows that

$$\varphi'(c) = h(\delta_0 - c)^{-h-1} > 0, \tag{3.6}$$

$$\varphi''(c) = h(h+1)(\delta_0 - c)^{-h-2} > 0. \tag{3.7}$$

Using the first two equations in (1.4), upon integrating by part we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega n^p \varphi(c) &= \int_\Omega p n^{p-1} n_t \varphi(c) + \int_\Omega n^p \varphi'(c) c_t \\ &= \int_\Omega p n^{p-1} \varphi(c)(\Delta n - \nabla\cdot(n\nabla c) + \nabla\cdot(n\nabla\phi) - u\cdot\nabla n) \\ &\quad + \int_\Omega n^p \varphi'(c)(\Delta c - nc - u\cdot\nabla c) \\ &= \int_\Omega (-\nabla n + n\nabla c - n\nabla\phi)\cdot(p(p-1)n^{p-2}\varphi(c)\nabla n + p n^{p-1}\varphi'(c)\nabla c) \\ &\quad - \int_\Omega \nabla c\cdot(p n^{p-1}\varphi'(c)\nabla n + n^p\varphi''(c)\nabla c) - \int_\Omega n^{p+1}\varphi'(c)c \\ &\quad - \int_\Omega p n^{p-1}\varphi(c) u\cdot\nabla n - \int_\Omega n^p\varphi'(c) u\cdot\nabla c \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
= & -p(p-1)\int_\Omega n^{p-2}\varphi(c)|\nabla n|^2 - p\int_\Omega n^{p-1}\varphi'(c)\nabla n\cdot\nabla c \\
& + p(p-1)\int_\Omega n^{p-1}\varphi(c)\nabla n\cdot\nabla c + p\int_\Omega n^p\varphi'(c)|\nabla c|^2 \\
& - p(p-1)\int_\Omega n^{p-1}\varphi(c)\nabla n\cdot\nabla\phi - p\int_\Omega n^p\varphi'(c)\nabla c\cdot\nabla\phi \\
& - p\int_\Omega n^{p-1}\varphi'(c)\nabla n\cdot\nabla c - \int_\Omega n^p\varphi''(c)|\nabla c|^2 - \int_\Omega n^{p+1}\varphi'(c)c
\end{aligned}
$$

for all $t\in(0,T_{\max})$, where we used the identity

$$
\begin{aligned}
-p\int_\Omega n^{p-1}\varphi(c)u\cdot\nabla n - \int_\Omega n^p\varphi'(c)u\cdot\nabla c &= -\int_\Omega \varphi(c)u\cdot\nabla n^p - \int_\Omega n^p u\cdot\nabla\varphi(c) \\
&= \int_\Omega n^p\varphi(c)(\nabla\cdot u) = 0.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\varphi(c) &+ p(p-1)\int_\Omega n^{p-2}\varphi(c)|\nabla n|^2 + \int_\Omega n^p\varphi''(c)|\nabla c|^2 \\
&\leqslant 2p\int_\Omega n^{p-1}\varphi'(c)|\nabla n||\nabla c| + p(p-1)\int_\Omega n^{p-1}\varphi(c)|\nabla n||\nabla c| \\
&\quad + p\int_\Omega n^p\varphi'(c)|\nabla c|^2 + p(p-1)\int_\Omega n^{p-1}\varphi(c)|\nabla n||\nabla\phi| \\
&\quad + p\int_\Omega n^p\varphi'(c)|\nabla c||\nabla\phi|
\end{aligned}
\tag{3.9}
$$

for all $t\in(0,T_{\max})$. Here the Young inequality yields that

$$
\begin{aligned}
2p\int_\Omega n^{p-1}\varphi'(c)|\nabla n||\nabla c| \leqslant{} & \frac{p(p-1)}{8}\int_\Omega n^{p-2}\varphi(c)|\nabla n|^2 \\
& + \frac{8p}{p-1}\int_\Omega n^p\varphi'^2(c)\varphi^{-1}(c)|\nabla c|^2
\end{aligned}
\tag{3.10}
$$

and

$$
p(p-1)\int_\Omega n^{p-1}\varphi(c)|\nabla n||\nabla c| \leqslant \frac{p(p-1)}{8}\int_\Omega n^{p-2}\varphi(c)|\nabla n|^2 + 2p(p-1)\int_\Omega n^p\varphi(c)|\nabla c|^2
\tag{3.11}
$$

and

$$
\begin{aligned}
p(p-1)&\int_\Omega n^{p-1}\varphi(c)|\nabla n||\nabla\phi| \\
&\leqslant p(p-1)\|\nabla\phi\|_{L^\infty(\Omega)}\int_\Omega n^{p-1}\varphi(c)|\nabla n| \\
&\leqslant \frac{p(p-1)}{8}\int_\Omega n^{p-2}\varphi(c)|\nabla n|^2 + 2p(p-1)\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^p\varphi(c).
\end{aligned}
\tag{3.12}
$$

Since $\varphi(c)$, $\varphi'(c)$ and $\varphi''(c)$ are bounded from above and below, we derive that there exist $m > 0$ and $M > 0$ such that

$$m=\min\{\delta_0^{-h}, h\delta_0^{-h-1}, h(h+1)\delta_0^{-h-2}\},$$
$$M=\max\{(\delta_0 - \|c_0\|_{L^\infty(\Omega)})^{-h}, h(\delta_0 - \|c_0\|_{L^\infty(\Omega)})^{-h-1}, h(h+1)(\delta_0-\|c_0\|_{L^\infty(\Omega)})^{-h-2}\}.$$

Therefore, we obtain

$$\begin{aligned} p\int_\Omega n^p\varphi'(c)|\nabla c||\nabla\phi| &\leqslant p\|\nabla\phi\|_{L^\infty(\Omega)}\int_\Omega n^p\varphi'(c)|\nabla c| \\ &\leqslant \frac{m}{5}\int_\Omega n^p|\nabla c|^2 + \frac{5p^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4m}\int_\Omega n^p{\varphi'}^2(c) \end{aligned} \tag{3.13}$$

by using the Young inequality again. Substituting (3.10)–(3.13) into (3.9), we immediately obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\varphi(c) + \frac{5p(p-1)}{8}\int_\Omega n^{p-2}\varphi(c)|\nabla n|^2 \\ &\qquad + \int_\Omega\Big(\varphi''(c) - \frac{8p}{p-1}\cdot\frac{{\varphi'}^2(c)}{\varphi(c)} - 2p(p-1)\varphi(c) - p\varphi'(c) - \frac{m}{5}\Big)n^p|\nabla c|^2 \\ &\quad\leqslant \frac{5p^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4m}\int_\Omega n^p{\varphi'}^2(c) + 2p(p-1)\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^p\varphi(c) \end{aligned} \tag{3.14}$$

for all $t \in (0, T_{\max})$. Now using (3.4)–(3.7), we see that

$$\frac{8p}{p-1}\cdot\frac{{\varphi'}^2}{\varphi(c)} = \frac{8p}{p-1}h^2(\delta_0-c)^{-h-2} \leqslant \frac{8p}{p-1}\cdot\frac{p-1}{32p}h(\delta_0-c)^{-h-2} \leqslant \frac{1}{4}\varphi''(c),$$
$$2p(p-1)\varphi(c) = 2p(p-1)(\delta_0-c)^{-h} \leqslant \frac{1}{4}\varphi''(c),$$
$$p\varphi'(c) = ph(\delta_0-c)^{-h-1} \leqslant \frac{1}{4}\varphi''(c), \quad \frac{m}{5} \leqslant \frac{1}{4}\varphi''(c).$$

Thus, the term

$$\int_\Omega\Big(\varphi''(c) - \frac{8p}{p-1}\cdot\frac{{\varphi'}^2(c)}{\varphi(c)} - 2p(p-1)\varphi(c) - p\varphi'(c) - \frac{m}{5}\Big)n^p|\nabla c|^2$$

on the left hand side of (3.14) is nonnegative and we immediately deduce that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\varphi(c) + \frac{5p(p-1)}{8}\int_\Omega n^{p-2}\varphi(c)|\nabla n|^2 \\ &\quad\leqslant \frac{5p^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4m}\int_\Omega n^p{\varphi'}^2(c) + 2p(p-1)\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^p\varphi(c) \end{aligned} \tag{3.15}$$

for all $t \in (0, T_{\max})$. Combining the value of M and m, we conclude from (3.15) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\varphi(c) + \frac{5mp(p-1)}{8}\int_\Omega n^{p-2}|\nabla n|^2 \leqslant \frac{5M^2p^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4m}\int_\Omega n^p + 2Mp(p-1)\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^p \tag{3.16}$$

for all $t \in (0, T_{\max})$.

By using the Gagliardo-Nirenberg inequality and (2.2), we can find that there exists $C_1 > 0$ such that for all $t \in (0, T_{\max})$,

$$\|n\|^p_{L^p(\Omega)} = \|n^{p/2}\|^2_{L^2(\Omega)} \leqslant C_1(\|\nabla n^{p/2}\|^{2\alpha}_{L^2(\Omega)} \cdot \|n^{p/2}\|^{2(1-\alpha)}_{L^{2/p}(\Omega)} + \|n^{p/2}\|^2_{L^{2/p}(\Omega)}), \tag{3.17}$$

where $\alpha \in (0,1)$ satisfies

$$\begin{cases} \alpha = \dfrac{p-1}{p}, & \text{if } N = 2,\\[2mm] \alpha = \dfrac{\frac{1}{2}p - \frac{1}{2}}{\frac{1}{2}p - \frac{1}{6}}, & \text{if } N = 3. \end{cases}$$

Therefore, by the Young inequality and the fact $2\alpha < 2$ $(N = 2, 3)$, we can find the existence of $C_2 > 0$ such that

$$\Big(\frac{5M^2p^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4m} + 2Mp(p-1)\|\nabla\phi\|^2_{L^\infty(\Omega)}\Big)\int_\Omega n^p \leqslant \frac{mp(p-1)}{8}\int_\Omega n^{p-2}|\nabla n|^2 + C_2 \tag{3.18}$$

for all $t \in (0, T_{\max})$.

Substituting (3.18) into (3.16), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\varphi(c) + \frac{1}{2}mp(p-1)\int_\Omega n^{p-2}|\nabla n|^2 \leqslant C_2 \quad \forall\, t \in (0, T_{\max}). \tag{3.19}$$

Since $\varphi(c)$ is bounded from above and below, (3.3) results from the above inequality upon integrating on $(t, t+\tau)$ for all $t \in (0, T_{\max} - \tau)$. By using (3.18) and the boundedness of $\varphi(c)$, there exists $C_3 > 0$ such that for all $t \in (0, T_{\max})$, we have

$$\int_\Omega n^p\varphi(c) \leqslant M\int_\Omega n^p \leqslant C_3\int_\Omega n^{p-2}|\nabla n|^2 + C_3. \tag{3.20}$$

Thus, combining (3.19) with (3.20), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\varphi(c) + C_4\int_\Omega n^p\varphi(c) \leqslant C_5 \quad \forall\, t \in (0, T_{\max}) \tag{3.21}$$

with some $C_4, C_5 > 0$. We can infer from (3.21) and Lemma 2.2 that

$$\int_\Omega n^p \varphi(c) \leqslant C_6$$

for some $C_6 > 0$. Recalling the boundedness of $\varphi(c)$, we conclude that (3.2) holds. □

We next try to obtain some information on the time evolution also for $\int_\Omega |\nabla c|^2/c$. In fact, employing Lemma 3.2 in [27], we can obtain

Lemma 3.2 (Lemma 3.2 in [27]). *Suppose that* (1.5)–(1.7) *hold. There exist positive constants* k, K, C *such that for all* $t \in (0, T_{\max})$ *we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla c|^2}{c} + k\int_\Omega \frac{|\nabla c|^4}{c^3} + k\int_\Omega |\nabla c|^4 \leqslant -\int_\Omega \nabla n \cdot \nabla c + K\int_\Omega |\nabla u|^2 + C. \tag{3.22}$$

To counteract the second integral on the right hand side of (3.22), we need to analyse the time evolution for $\int_\Omega |u|^2$.

Lemma 3.3. *Suppose that* (1.5)–(1.7) *hold. Let* k, K *be the constants in* (3.22). *Then there exists a constant* $C > 0$ *such that for all* $t \in (0, T_{\max})$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \leqslant \frac{k}{4K}\|\nabla c\|^4_{L^4(\Omega)} + C. \tag{3.23}$$

P r o o f. Multiplying u in the third equation of (1.4), then integrating it over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = -\int_\Omega nu\cdot\nabla\phi + \int_\Omega nu\cdot\nabla c \tag{3.24}$$

for all $t \in (0, T_{\max})$. Recalling the Poincaré inequality we can find a constant $K_p > 0$ fulfilling

$$\|\varphi\|^2_{L^2(\Omega)} \leqslant K_p\|\nabla\varphi\|^2_{L^2(\Omega)} \quad \forall\, \varphi \in W^{1,2}_0(\Omega). \tag{3.25}$$

Making use of the Young inequality and the boundness of $\nabla\phi$ we can estimate the first term in the right hand side of (3.24) as

$$\begin{aligned} -\int_\Omega nu\cdot\nabla\phi &\leqslant \frac{1}{4K_p}\int_\Omega |u|^2 + K_p\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2 \\ &\leqslant \frac{1}{4}\int_\Omega |\nabla u|^2 + K_p\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2 \end{aligned} \tag{3.26}$$

for all $t \in (0, T_{\max})$. Noticing (3.2), we can easily see that there exists $C_1 > 0$ satisfying

$$-\int_\Omega nu\cdot\nabla\phi \leqslant \frac{1}{4}\int_\Omega |\nabla u|^2 + C_1 \tag{3.27}$$

for all $t \in (0, T_{\max})$. Now we estimate the second term on the right hand side of (3.24). The Young inequality, the Hölder inequality, (3.25) and (3.2) tell that for all $t \in (0, T_{\max})$ there exists a constant $C_2 > 0$ satisfying

$$\begin{aligned} \int_\Omega nu \cdot \nabla c &\leqslant \frac{1}{4K_p}\|u\|^2_{L^2(\Omega)} + K_p\|n\nabla c\|^2_{L^2(\Omega)} \\ &\leqslant \frac{1}{4}\int_\Omega |\nabla u|^2 + K_p\Big(\frac{2KK_p}{k}\|n\|^4_{L^4(\Omega)} + \frac{k}{8KK_p}\|\nabla c\|^4_{L^4(\Omega)}\Big) \\ &\leqslant \frac{1}{4}\int_\Omega |\nabla u|^2 + \frac{k}{8K}\|\nabla c\|^4_{L^4(\Omega)} + C_2. \end{aligned} \tag{3.28}$$

By substituting (3.27)–(3.28) into (3.24), we see that there exists a constant $C_3 > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \leqslant \frac{k}{8K}\|\nabla c\|^4_{L^4(\Omega)} + C_3 \quad \forall\, t \in (0, T_{\max}). \tag{3.29}$$

One can easily obtain the inequality (3.23) from (3.29). □

Combining the result of Lemma 3.2 with Lemma 3.3, we thus establish the following energy-type inequality.

Lemma 3.4. *Suppose that* (1.5)–(1.7) *hold. There exists* $C > 0$ *such that*

$$\int_\Omega |\nabla c|^2 + \int_\Omega |u|^2 \leqslant C \quad \forall\, t \in (0, T_{\max}), \tag{3.30}$$

$$\int_t^{t+\tau}\int_\Omega |\nabla c|^4 + \int_t^{t+\tau}\int_\Omega |\nabla u|^2 \leqslant C \quad \forall\, t \in (0, T_{\max} - \tau). \tag{3.31}$$

Proof. Let k, K be the constants in (3.22). Firstly, we estimate the first term on the right hand of (3.22). The Hölder inequality and the Young inequality imply the existence of $C_k > 0$ and $C_1 > 0$ such that

$$\begin{aligned} -\int_\Omega \nabla n \cdot \nabla c \leqslant \int_\Omega |\nabla n||\nabla c| &\leqslant C_k\|\nabla n\|^{4/3}_{L^{4/3}(\Omega)} + \frac{k}{4}\|\nabla c\|^4_{L^4(\Omega)} \\ &= C_k\|n^{(p-2)/2} \cdot \nabla n \cdot n^{(2-p)/2}\|^{4/3}_{L^{4/3}(\Omega)} + \frac{k}{4}\|\nabla c\|^4_{L^4(\Omega)} \\ &\leqslant C_1\|n^{(p-2)/2}|\nabla n|\|^{4/3}_{L^2(\Omega)} \cdot \|n^{(2-p)/2}\|^{4/3}_{L^4(\Omega)} + \frac{k}{4}\|\nabla c\|^4_{L^4(\Omega)} \end{aligned} \tag{3.32}$$

for all $t \in (0, T_{\max})$. Due to the fact that (3.2) holds as $1 < p < 2$, we can deduce that there exist $C_2 > 0$ satisfying

$$\|n^{(p-2)/2}|\nabla n|\|^{4/3}_{L^2(\Omega)} \cdot \|n^{(2-p)/2}\|^{4/3}_{L^4(\Omega)} \leqslant C_2\|n^{(p-2)/2}|\nabla n|\|^{4/3}_{L^2(\Omega)}. \tag{3.33}$$

Next, substituting (3.33) into (3.32), and using the Young inequality, we immediately obtain that there exist $C_3 > 0$ and $C_4 > 0$ such that

$$(3.34) \qquad -\int_\Omega \nabla n \cdot \nabla c \leqslant C_3 \int_\Omega n^{p-2}|\nabla n|^2 + \frac{k}{4}\int_\Omega |\nabla c|^4 + C_4 \quad \forall t \in (0, T_{\max}).$$

On the other hand, multiplying (3.23) by $2K$ and combining the resulting inequality with (3.22), (3.34), we derive that for all $t \in (0, T_{\max})$, there exists a constant $C_5 > 0$ such that

$$(3.35) \quad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} + 2K\int_\Omega |u|^2\Big) + k\int_\Omega |\nabla c|^4 + 2K\int_\Omega |\nabla u|^2 + k\int_\Omega \frac{|\nabla c|^4}{c^3} \\ &\qquad \leqslant \Big(\frac{k}{4} + \frac{k}{4K}\cdot 2K\Big)\int_\Omega |\nabla c|^4 + K\int_\Omega |\nabla u|^2 + C_3\int_\Omega n^{p-2}|\nabla n|^2 + C_5, \end{aligned}$$

that is

$$(3.36) \quad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} + 2K\int_\Omega |u|^2\Big) + \frac{k}{4}\int_\Omega |\nabla c|^4 + K\int_\Omega |\nabla u|^2 + k\int_\Omega \frac{|\nabla c|^4}{c^3} \\ &\qquad \leqslant C_3\int_\Omega n^{p-2}|\nabla n|^2 + C_5. \end{aligned}$$

It follows from the Poincaré inequality that

$$(3.37) \qquad \int_\Omega |u|^2 \leqslant K_p \int_\Omega |\nabla u|^2 \quad \forall t \in (0, T_{\max})$$

with some $K_p > 0$. We use the Young inequality again to find that for all $t \in (0, T_{\max})$, there exists $C_6 > 0$ such that

$$(3.38) \qquad \int_\Omega \frac{|\nabla c|^2}{c} \leqslant 4kK_p\int_\Omega \frac{|\nabla c|^4}{c^3} + \frac{1}{16kK_p}\int_\Omega c \leqslant 4kK_p\int_\Omega \frac{|\nabla c|^4}{c^3} + C_6.$$

From (3.36) to (3.38), there exists a constant $C_7 > 0$ such that

$$(3.39) \quad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} + 2K\int_\Omega |u|^2\Big) + \frac{1}{4K_p}\Big(\frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} + 2K\int_\Omega |u|^2\Big) \\ &\qquad + \frac{k}{4}\int_\Omega |\nabla c|^4 + \frac{k}{2}\int_\Omega \frac{|\nabla c|^4}{c^3} + \frac{K}{2}\int_\Omega |\nabla u|^2 \leqslant C_3\int_\Omega n^{p-2}|\nabla n|^2 + C_7 \end{aligned}$$

for all $t \in (0, T_{\max})$. Now, if we define

$$y(t) := \Big(\frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} + 2K\int_\Omega |u|^2\Big)(\cdot, t)$$

and

$$h(t) := \Big(\frac{k}{4}\int_\Omega |\nabla c|^4 + \frac{k}{2}\int_\Omega \frac{|\nabla c|^4}{c^3} + \frac{K}{2}\int_\Omega |\nabla u|^2\Big)(\cdot, t),$$

then (3.39) implies that y satisfies the ODI

$$y'(t) + \frac{1}{4K_p} y(t) + h(t) \leqslant C_3 \int_\Omega n^{p-2} |\nabla n|^2 + C_7 \quad \forall\, t \in (0, T_{\max}).$$

We, together with (3.3) and Lemma 2.2, can find $C_8 > 0$ satisfying

$$\sup_{t \in (0, T_{\max})} \left(\int_\Omega \frac{|\nabla c|^2}{c} + \int_\Omega |u|^2 \right) \leqslant C_8.$$

Another integration of (3.39) thereupon shows that

$$\sup_{t \in (0, T_{\max} - \tau)} \left(\int_t^{t+\tau} \int_\Omega \frac{|\nabla c|^4}{c^3} + \int_t^{t+\tau} \int_\Omega |\nabla u|^2 + \int_t^{t+\tau} \int_\Omega |\nabla c|^4 \right) \leqslant C_9$$

with some positive constant C_9. As

$$\int_\Omega |\nabla c|^2 = \int_\Omega \frac{|\nabla c|^2}{c} \cdot c \leqslant \|c_0\|_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla c|^2}{c},$$
$$\int_t^{t+\tau} \int_\Omega |\nabla c|^4 = \int_t^{t+\tau} \int_\Omega \frac{|\nabla c|^4}{c^3} \cdot c^3 \leqslant \|c_0\|_{L^\infty(\Omega)}^3 \int_t^{t+\tau} \int_\Omega \frac{|\nabla c|^4}{c^3},$$

we immediately obtain (3.30) and (3.31). □

Remark 3.1. The arguments of Lemmas 3.1–3.4 do not depend on dimension N or the value of κ.

4. Boundedness in the two-dimensional case ($N = 2, \kappa = 1$)

The conclusion of Lemma 3.1 together with the boundedness of $\int_t^{t+\tau} \int_\Omega |\nabla c|^4$ obtained in Lemma 3.4 imply the following regularity estimate for u. Firstly, we use Lemma 2.4 to obtain the boundedness of $\|u(\cdot, t)\|_{L^p(\Omega)}$ for $p > 1$.

Lemma 4.1. *Let $N = 2$, $\kappa = 1$. Suppose that* (1.5)–(1.7) *hold. Then for all $p > 1$, there exists $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}). \tag{4.1}$$

Proof. Due to $\tau = \min\{1, \frac{1}{2} T_{\max}\}$, we can first obtain from Lemma 3.1, Lemma 3.4 and assumption on ϕ that there exist $C_1, C_2 > 0$ such that

$$\frac{1}{\tau} \int_t^{t+\tau} \int_\Omega |n \nabla \phi|^2 \leqslant \frac{1}{\tau} \int_t^{t+\tau} \|n(\cdot, t)\|_{L^2(\Omega)}^2 \|\nabla \phi\|_{L^\infty(\Omega)}^2 \leqslant C_1 \tag{4.2}$$

and

$$\frac{1}{\tau} \int_t^{t+\tau} \int_\Omega |n \nabla c|^2 \leqslant \frac{1}{\tau} \int_t^{t+\tau} \|n(\cdot, t)\|_{L^4(\Omega)}^2 \|\nabla c\|_{L^4(\Omega)}^2 \leqslant C_2 \tag{4.3}$$

for all $t \in (0, T_{\max} - \tau)$. Then we infer from Lemma 2.4 that

$$\int_\Omega |\nabla u|^2 \leqslant C_3 \quad \forall\, t \in (0, T_{\max}) \tag{4.4}$$

with $C_3 > 0$. Since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, the $L^p(\Omega)$ estimate of u is an immediate consequence of the Poincaré inequality and (4.4). □

Lemma 4.2. *Let $N = 2$. Suppose that* (1.5)–(1.7) *hold. Then for all $p > 2$, if $2 < q < p$, there exists $C > 0$ such that*

$$\|\nabla c(\cdot, t)\|_{L^q(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}). \tag{4.5}$$

P r o o f. Let $p_0 = 2p/(p+2)$. Thanks to (3.2), (2.4), (4.1), (3.30), the variation-of-constants formula, well-known smoothing estimates for the Neumann heat semi-group (see [32]) and the Hölder inequality, there exist $k_i > 0$ $(i = 1, 2, 3)$ such that for all $t \in (0, T_{\max})$, we have

$$\begin{aligned} \|\nabla c(\cdot, t)\|_{L^q(\Omega)} &\leqslant k_1 \|\nabla c_0\|_{L^q(\Omega)} + \int_0^t (t-s)^{-1/2-(1/p_0-1/q)} \mathrm{e}^{-\lambda(t-s)} \|(nc + u \cdot \nabla c)(\cdot, s)\|_{L^{p_0}(\Omega)}\, \mathrm{d}s \\ &\leqslant k_1 \|\nabla c_0\|_{L^q(\Omega)} + \int_0^t (t-s)^{-1+(p-q)/(pq)} \mathrm{e}^{-\lambda(t-s)} (\|n(\cdot, s)\|_{L^{p_0}(\Omega)} \|c(\cdot, s)\|_{L^\infty(\Omega)} \\ &\quad + \|u(\cdot, s)\|_{L^p(\Omega)} \|\nabla c(\cdot, s)\|_{L^2(\Omega)})\, \mathrm{d}s \\ &\leqslant k_1 \|\nabla c_0\|_{L^q(\Omega)} + k_2 \int_0^t (t-s)^{-1+(p-q)/(pq)} \mathrm{e}^{-\lambda(t-s)}\, \mathrm{d}s \leqslant k_1 \|\nabla c_0\|_{L^q(\Omega)} + k_2 k_3, \end{aligned} \tag{4.6}$$

where

$$\int_0^t (t-s)^{-1+(p-q)/(pq)} \mathrm{e}^{-\lambda(t-s)}\, \mathrm{d}s \leqslant k_3$$

due to $-1 + (p-q)/(pq) > -1$. We thus arrive at (4.5). □

Now we prove the uniform boundedness of $\|A^\alpha u(\cdot, t)\|_{L^2(\Omega)}$, $\|c(\cdot, t)\|_{W^{1,\infty}(\Omega)}$ and $\|n(\cdot, t)\|_{L^\infty(\Omega)}$.

Lemma 4.3. *Let $N = 2$, $\kappa = 1$. Suppose that* (1.5)–(1.7) *hold. Then there are $\alpha \in (\frac{1}{2}, 1)$ and $C > 0$ such that*

$$\|\mathcal{A}^\alpha u(\cdot, t)\|_{L^2(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}), \tag{4.7}$$

$$\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}) \tag{4.8}$$

as well as

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}). \tag{4.9}$$

P r o o f. From the variation-of-constants formula for u and the contractivity of the Stokes semigroup in $L^2(\Omega)$, we know that

(4.10)

$$\|\mathcal{A}^\alpha u(\cdot,t)\|_{L^2(\Omega)} \leqslant \|\mathcal{A}^\alpha u_0\|_{L^2(\Omega)} + \int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot,s)\nabla\phi)\|_{L^2(\Omega)}\,\mathrm{d}s$$
$$+ \int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot,s)\nabla c(\cdot,s))\|_{L^2(\Omega)}\,\mathrm{d}s$$
$$+ \int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}((u(\cdot,s)\cdot\nabla)u(\cdot,s))\|_{L^2(\Omega)}\,\mathrm{d}s \ \forall\, t\in(0,T_{\max}).$$

Thanks to Lemma 3.1, Lemma 4.2, properties of the Stokes operator, the Hölder inequality and (1.6), there exist $C_i > 0$ $(i = 1,2,3,4,5)$ such that for all $0 < s < t < T_{\max}$, we have

(4.11)

$$\int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot,s)\nabla\phi)\|_{L^2(\Omega)}\,\mathrm{d}s \leqslant C_1\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\mu(t-s)}\|n(\cdot,s)\nabla\phi\|_{L^2(\Omega)}\,\mathrm{d}s$$
$$\leqslant C_1C_2\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\mu(t-s)}\,\mathrm{d}s \leqslant C_1C_2C_3$$

and

$$\text{(4.12)}\qquad \int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot,s)\nabla c(\cdot,s))\|_{L^2(\Omega)}\,\mathrm{d}s$$
$$\leqslant C_4\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\mu(t-s)}\|n(\cdot,s)\|_{L^{2q/(q-2)}(\Omega)}\|\nabla c(\cdot,s)\|_{L^q(\Omega)}\,\mathrm{d}s$$
$$\leqslant C_4C_5\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\mu(t-s)}\,\mathrm{d}s \leqslant C_3C_4C_5,$$

where

$$\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\mu(t-s)}\,\mathrm{d}s \leqslant C_3$$

due to $\alpha < 1$. Moreover, since $\frac{1}{2} < \alpha < 1$, we obtain

$$\text{(4.13)}\qquad \int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}((u(\cdot,s)\cdot\nabla)u(\cdot,s))\|_{L^2(\Omega)}\,\mathrm{d}s$$
$$\leqslant C_6\int_0^t (t-s)^{-\alpha-(1/r-1/2)}\mathrm{e}^{-\lambda(t-s)}\|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\|_{L^r(\Omega)}\,\mathrm{d}s$$

for all $t\in(0,T_{\max})$ with some $C_6 > 0$, where $r\in(1,2)$ satisfies that

$$r > \frac{2}{3-2\alpha}.$$

Recalling (4.1) and (4.4), we can use the Hölder inequality to find $C_7 > 0$ fulfilling

$$\text{(4.14)}\qquad \|(u\cdot\nabla)u\|_{L^r(\Omega)} \leqslant \|u\|_{L^{2r/(2-r)}(\Omega)}\|\nabla u\|_{L^2(\Omega)} \leqslant C_7.$$

Thereupon, collecting (4.10)–(4.14) we can obtain (4.7). As a consequence, we have

$$\|u\|_{L^\infty(\Omega)} \leqslant C_8 \quad \forall\, t \in (0, T_{\max}) \tag{4.15}$$

with some $C_8 > 0$, since $D(\mathcal{A}^\alpha) \hookrightarrow L^\infty(\Omega)$ due to the fact that $\alpha \in (\frac{1}{2}, 1)$.

By an application of the variation-of-constants formula and the smoothing estimates for the Neumann heat semigroup, we can estimate

$$\begin{aligned} \|\nabla c(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant C_9 \|\nabla c_0\|_{L^\infty(\Omega)} + \int_0^t (t-s)^{-1/2-1/q} \mathrm{e}^{-\lambda(t-s)} \\ &\quad\times \|(nc + u\cdot\nabla c)(\cdot,s)\|_{L^q(\Omega)}\,\mathrm{d}s \\ &\leqslant C_{10} + C_{10} \int_0^t (t-s)^{-1/2-1/q} \mathrm{e}^{-\lambda(t-s)} \\ &\quad\times (\|n(\cdot,s)\|_{L^q(\Omega)} + \|\nabla c(\cdot,s)\|_{L^q(\Omega)})\,\mathrm{d}s \\ &\leqslant C_{11} \quad \forall\, t \in (0, T_{\max}) \end{aligned} \tag{4.16}$$

with some positive constants C_9, C_{10} and C_{11}, where we used the boundedness of $\|c\|_{L^\infty(\Omega)}$, (4.5), (3.2) and (4.15). Finally, by a straightforward iteration procedure of Moser-type as in [25] or [35], we arrive at the L^∞ estimate for n, that is,

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C_{12} \quad \forall\, t \in (0, T_{\max}) \tag{4.17}$$

with some positive constant C_{12}. The proof of Lemma 4.3 is completed. □

5. Boundedness in the three-dimensional case ($N = 3, \kappa = 0$)

In this section, we deal with the chemotaxis-Stokes system in the three-dimensional setting ($N = 3, \kappa = 0$) and only assume $\|c_0\|_{L^\infty(\Omega)}$ to be small. We first improve the regularity for u and ∇c in a suitable way.

Lemma 5.1. *Let $N = 3$, $\kappa = 0$. Suppose that* (1.5)–(1.7) *hold. Then there are $\alpha \in (\frac{3}{4}, 1)$ and $C > 0$ such that*

$$\|\mathcal{A}^\alpha u(\cdot,t)\|_{L^2(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}), \tag{5.1}$$

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}). \tag{5.2}$$

P r o o f. The proof is very similar to that of Lemma 4.3, we only need to deal with the term with less regularity of ∇c. From the variation-of-constants formula for u and the contractivity of the Stokes semigroup in $L^2(\Omega)$, we obtain that

(5.3)
$$\begin{aligned}\|\mathcal{A}^\alpha u(\cdot,t)\|_{L^2(\Omega)} \leqslant \|\mathcal{A}^\alpha u_0\|_{L^2(\Omega)} &+ \int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot,s)\nabla\phi)\|_{L^2(\Omega)}\,\mathrm{d}s \\ &+ \int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot,s)\nabla c(\cdot,s))\|_{L^2(\Omega)}\,\mathrm{d}s \quad \forall\, t\in(0,T_{\max}).\end{aligned}$$

The estimate of the second term on the right hand side of (5.3) is equal to (4.11). Now, we proceed to estimate the third term on the right hand side of (5.3). By using properties of the Stokes operator and the Hölder inequality, there exist $C_i>0$ $(i=1,2,3)$ such that for all $0<s<t<T_{\max}$, we have

$$\begin{aligned}(5.4)\qquad &\int_0^t \|\mathcal{A}^\alpha \mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{P}(n(\cdot,s)\nabla c(\cdot,s))\|_{L^2(\Omega)}\,\mathrm{d}s \\ &\leqslant C_1\int_0^t (t-s)^{-\alpha-3(1/w-1/2)/2}\mathrm{e}^{-\lambda(t-s)}\|n(\cdot,s)\nabla c(\cdot,s)\|_{L^w(\Omega)}\,\mathrm{d}s \\ &\leqslant C_2\|n(\cdot,s)\|_{L^{2w/(2-w)}(\Omega)}\|\nabla c(\cdot,s)\|_{L^2(\Omega)} \leqslant C_3,\end{aligned}$$

where $w\in(\frac{3}{2},2)$ satisfies the inequality

$$w>\frac{6}{7-4\alpha}.$$

Combining (5.3), (4.11) with (5.4), we see that $\|\mathcal{A}^\alpha u(\cdot,t)\|_{L^2(\Omega)}$ is bounded for all $t\in(0,T_{\max})$. Since $\alpha\in(\frac{3}{4},1)$, the embedding theorem implies the boundedness of $\|u\|_{L^\infty(\Omega)}$. Thus the proof is complete. □

Lemma 5.2. *Let $N=3$. Suppose that (1.5)–(1.7) hold. For all $2<q^*<6$, there exists $C>0$ such that*

$$(5.5)\qquad \|\nabla c(\cdot,t)\|_{L^{q^*}(\Omega)} \leqslant C \quad \forall\, t\in(0,T_{\max}).$$

P r o o f. Let $2<q^*<6$. Thanks to the variation-of-constants formula and the L^p-L^q estimates, Lemma 3.1, (2.4), (3.30) and (5.2), there exist $\gamma_i>0$ $(i=1,2,3)$

such that

$$\begin{aligned}(5.6)\qquad &\|\nabla c(\cdot,t)\|_{L^{q^*}(\Omega)}\\ &\leqslant \gamma_1\|\nabla c_0\|_{L^{q^*}(\Omega)}+\gamma_1\int_0^t (t-s)^{-1/2-3(1/2-1/q^*)/2}\mathrm{e}^{-\lambda(t-s)}\\ &\qquad\times\|(nc+u\cdot\nabla c)(\cdot,s)\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\leqslant \gamma_1\|\nabla c_0\|_{L^{q^*}(\Omega)}+\gamma_1\int_0^t (t-s)^{-5/4+3/(2q^*)}\mathrm{e}^{-\lambda(t-s)}\\ &\qquad\times(\|n(\cdot,s)\|_{L^2(\Omega)}\|c(\cdot,s)\|_{L^\infty(\Omega)}+\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,s)\|_{L^2(\Omega)})\,\mathrm{d}s\\ &\leqslant \gamma_1\|\nabla c_0\|_{L^{q^*}(\Omega)}+\gamma_2\int_0^t (t-s)^{-5/4+3/(2q^*)}\mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}s\\ &\leqslant \gamma_1\|\nabla c_0\|_{L^{q^*}(\Omega)}+\gamma_2\gamma_3\quad\forall\,t\in(0,T_{\max}),\end{aligned}$$

where

$$\int_0^t (t-s)^{-5/4+3/(2q^*)}\mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}s\leqslant\gamma_3$$

due to $-\frac{5}{4}+3/(2q^*)>-1$. We conclude that (5.5) holds. □

Having enough regularity for both u and ∇c, we are ready to prove the boundedness of $\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|n(\cdot,t)\|_{L^\infty(\Omega)}$.

Lemma 5.3. *Let* $N=3$. *Suppose that* (1.5)–(1.7) *hold. For all* $3<q^*<6$, *there exists* $C>0$ *such that*

$$(5.7)\qquad \|\nabla c(\cdot,t)\|_{L^\infty(\Omega)}\leqslant C\quad\forall\,t\in(0,T_{\max}),$$

$$(5.8)\qquad \|n(\cdot,t)\|_{L^\infty(\Omega)}\leqslant C\quad\forall\,t\in(0,T_{\max}).$$

P r o o f. Let $3<q^*<6$. By an application of the variation-of-constants formula and the smoothing estimates for the Neumann heat semigroup, we can estimate

$$\begin{aligned}(5.9)\qquad &\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)}\\ &\leqslant C_1\|\nabla c_0\|_{L^\infty(\Omega)}+\int_0^t (t-s)^{-1/2-3/(2q^*)}\mathrm{e}^{-\lambda(t-s)}\|(nc+u\cdot\nabla c)(\cdot,s)\|_{L^{q^*}(\Omega)}\,\mathrm{d}s\\ &\leqslant C_2+C_2\int_0^t (t-s)^{-1/2-3/(2q^*)}\mathrm{e}^{-\lambda(t-s)}(\|n(\cdot,s)\|_{L^{q^*}(\Omega)}+\|\nabla c(\cdot,s)\|_{L^{q^*}(\Omega)})\,\mathrm{d}s\\ &\leqslant C_3\quad\forall\,t\in(0,T_{\max})\end{aligned}$$

with some positive constants C_1, C_2 and C_3, where we used the boundness of $\|c\|_{L^\infty(\Omega)}$, (5.2), (3.2) and (5.5). Finally, by a straightforward iteration procedure of Moser-type as in [25] or [35], we arrive at the L^∞ estimate for n, that is,

$$(5.10)\qquad \|n(\cdot,t)\|_{L^\infty(\Omega)}\leqslant C_4\quad\forall\,t\in(0,T_{\max})$$

with some positive constant C_4. Thus the proof is complete. □

Proof of Theorem 1.1. Collecting Lemmas 4.3, 5.1 and 5.3, we conclude that there exists a positive constant C that does not depend on t such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C \quad \forall\, t \in (0, T_{\max}),$$

which together with (2.2) implies that $T_{\max} = \infty$. Moreover, (n, c, u, P) is uniformly bounded. Hence, we have proved Theorem 1.1. $\square$

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