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CONDITION NUMBERS OF HESSENBERG COMPANION MATRICES

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Abstract. The Fiedler matrices are a large class of companion matrices that include the well-known Frobenius companion matrix. The Fiedler matrices are part of a larger class of companion matrices that can be characterized by a Hessenberg form. We demonstrate that the Hessenberg form of the Fiedler companion matrices provides a straight-forward way to compare the condition numbers of these matrices. We also show that there are other companion matrices which can provide a much smaller condition number than any Fiedler companion matrix. We finish by exploring the condition number of a class of matrices obtained from perturbing a Frobenius companion matrix while preserving the characteristic polynomial.

Keywords: companion matrix; Fiedler companion matrix; condition number; generalized companion matrix

MSC 2020: 15A12, 15B99

1. Introduction

The Frobenius companion matrix is a template that provides a matrix with a prescribed characteristic polynomial. More recently, it was discovered that the Frobenius companion matrix belongs to a larger class of Fiedler companion matrices (see [5]), which in turn is a subset of the intercyclic companion matrices, see [4]. Other recent templates include nonsparse companion matrices (see [2]) and generalized companion matrices, see [6].

The Frobenius companion matrix is employed in algorithms that use matrix methods to determine roots of polynomials, but this matrix is not always well-conditioned, see [3]. Recent work (see [3]) has explored under what circumstances

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other Fiedler companion matrices can have a better condition number than the Frobenius matrix, with respect to the Frobenius norm. After covering background details in Section 2, we use a Hessenberg characterization of the Fiedler companion matrices in Section 3 to provide a concise argument for the condition number of a Fiedler companion matrix. The characterization allows us to avoid dealing with the particular permutation in Fiedler's construction of companion matrices (see [5]), as well as associated concepts around consecutions and inversions developed in [3]. In Section 4, we provide some examples of non-Fiedler companion matrices that demonstrate that there are intercyclic companion matrices that have a smaller condition number than any Fiedler companion matrix for some specific polynomials. In Section 5, we provide a method for constructing a generalized companion matrix that, in some cases, can improve on the condition number of any Fiedler companion matrix.

2. Technical definitions and background

In this section we recall the relevant background on companion matrices and condition numbers that will be required throughout the paper.

Let $n \geqslant 2$ be an integer and $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_0$. A *companion matrix* to $p(x)$ is an $n \times n$ matrix A over $\mathbb{R}[c_0, \ldots, c_{n-1}]$ such that the characteristic polynomial of A is $p(x)$. A *unit sparse companion matrix* to $p(x)$ is a companion matrix A that has $n-1$ entries equal to one, n variable entries $-c_0, \ldots, -c_{n-1}$, and the remaining $n^2 - 2n + 1$ entries equal to zero. The unit sparse companion matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \ldots & -c_{n-2} & -c_{n-1} \end{bmatrix}$$

is called the *Frobenius companion matrix* of $p(x)$. Sparse companion matrices have also been called intercyclic companion matrices due to the structure of the digraph associated with the matrix; see [7] and [4] for details.

The matrices in Figure 1 are examples of unit sparse companion matrices to $p(x) = x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$. The first matrix in Figure 1 is a Frobenius companion matrix. The matrices in Figure 2 are also companion matrices to $p(x)$, but they are not unit sparse since not every nonzero variable entry is the negative of a single coefficient of $p(x)$. Note that in the last matrix, the value of a can be any real number; when $a = 0$, then this matrix becomes a unit sparse companion matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -c_3 & 1 & 0 \\ 0 & -c_2 & 0 & 1 \\ -c_0 & -c_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -c_2 & -c_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & 0 & 0 \end{bmatrix}.$$

Figure 1. Some 4×4 unit sparse companion matrices.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -c_2 & 0 & 1 & 0 \\ -c_1 + c_3c_2 & 0 & -c_3 & 1 \\ -c_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -c_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -c_1 + c_3c_2 & -c_2 & 0 & 1 \\ -c_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -c_3 & 1 & 0 & 0 \\ -c_2 + a & 0 & 1 & 0 \\ -c_1 + ac_3 & -a & 0 & 1 \\ -c_0 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 2. Some 4×4 companion matrices.

$$\begin{bmatrix} -c_3 & 1 & 0 & 0 \\ -c_2 & 0 & 1 & 0 \\ -c_1 & 0 & 0 & 1 \\ -c_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -c_3 & -c_2 & -c_1 & -c_0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -c_0 \\ 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \end{bmatrix}.$$

Figure 3. Some companion matrices equivalent to the 4×4 Frobenius companion matrix.

Since the matrix transposition and permutation similarity affect neither the characteristic polynomial, nor the set of nonzero entries in a matrix, we call two companion matrices *equivalent* if one can be obtained from the other via the transposition and/or permutation similarity. The matrices in Figure 3 are equivalent to the 4×4 Frobenius companion matrix. Note that if A and B are equivalent matrices, then the multiset of entries in any row of A is exactly the multiset of entries of some row or column of B. No two matrices from Figures 1 and 2 are equivalent (assuming $a \neq 0$).

Fiedler in [5] introduced a class of companion matrices that are constructed as a product of certain block diagonal matrices. In particular, let F_0 be a diagonal matrix with diagonal entries $(1, \dots, 1, -c_0)$ and for $k = 1, \dots, n-1$, let

$$F_k = \begin{bmatrix} I_{n-k-1} & O & O \\ O & T_k & O \\ O & O & I_{k-1} \end{bmatrix} \quad \text{with } T_k = \begin{bmatrix} -c_k & 1 \\ 1 & 0 \end{bmatrix}.$$

Fiedler showed (see [5], Theorem 2.3) that the product of these n matrices, in any order, will produce a companion matrix of $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_0$. Consequently, given any permutation $\sigma = (\sigma_0, \sigma_2, \dots, \sigma_{n-1})$ of $\{0, 1, 2, \dots, n-1\}$, we say that $F_\sigma = F_{\sigma_0} F_{\sigma_1} \dots F_{\sigma_{n-1}}$ is a *Fiedler companion matrix.* The Frobenius companion matrix is a Fiedler companion matrix since the Frobenius companion matrix is equivalent to $F_0 F_1 \dots F_{n-1}$, as noted in [5].

In [4] it was demonstrated that every unit sparse companion matrix is equivalent to a unit lower Hessenberg matrix, as summarized in Theorem 2.1. Note that, for $0 \leqslant k \leqslant n-1$, the kth *subdiagonal* of a matrix $A = [a_{ij}]$ consists of the entries $\{a_{k+1,1}, a_{k+2,2}, \dots, a_{n,n-k}\}$. The 0th subdiagonal is usually called the *main diagonal* of a matrix.

Theorem 2.1 ([4], Corollary 4.3)**.** *Let $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_1 x + c_0$ be a polynomial over $\mathbb{R}$ with $n \geqslant 2$. Then A is an $n \times n$ unit sparse companion matrix to $p(x)$ if and only if A is equivalent to a unit lower Hessenberg matrix*

$$C = \left[\begin{array}{cc|c} \mathbf{0} & I_m & O \\ \hline \multicolumn{2}{c|}{R} & \begin{array}{c} I_{n-m-1} \\ \mathbf{0}^\top \end{array} \end{array}\right] \tag{2.1}$$

for some $(n-m) \times (m+1)$ matrix R with $m(n-1-m)$ zero entries, such that C has $-c_{n-1-k}$ on its kth subdiagonal for $0 \leqslant k \leqslant n-1$.

Note that in (2.1), the unit lower Hessenberg matrix C always has $C_{n,1} = -c_0$ and $R_{1,m+1} = -c_{n-1}$. Given this Hessenberg characterization of the unit sparse companion matrices, one can deduce the corresponding inverse matrix if $c_0 \neq 0$.

Lemma 2.2 ([7], Section 7)**.** *Let $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_1 x + c_0$ be a polynomial over $\mathbb{R}$ with $n \geqslant 2$. Suppose that C is a unit lower Hessenberg companion matrix to $p(x)$ as in* (2.1)*. Assuming $c_0 \neq 0$, if*

$$C = \left[\begin{array}{c|c|c} \mathbf{0} & I_m & O \\ \hline \mathbf{u} & H & I_{n-m-1} \\ \hline -c_0 & \mathbf{y}^\top & \mathbf{0}^\top \end{array}\right]$$

for some $\mathbf{u}$, $\mathbf{y}$, H, then

$$C^{-1} = \left[\begin{array}{c|c|c} \frac{1}{c_0}\mathbf{y}^\top & \mathbf{0}^\top & -\frac{1}{c_0} \\ \hline I_m & O & \mathbf{0} \\ \hline -\frac{1}{c_0}\mathbf{u}\mathbf{y}^\top - H & I_{n-m-1} & \frac{1}{c_0}\mathbf{u} \end{array}\right].$$

Throughout this paper, we use the *Frobenius norm* of an $n \times n$ matrix $A = [a_{ij}]$ given by

$$\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}.$$

Remark 2.3. If both A and B are unit sparse companion matrices to the same polynomial $p(x)$, then it follows that $\|A\| = \|B\|$ since A and B have exactly the same entries. Furthermore, if $A = PBP^\top$ for some permutation matrix P, then A^{-1} and B^{-1} also have the same entries, and hence $\|A^{-1}\| = \|B^{-1}\|$.

The *condition number* of A, denoted $\kappa(A)$, is defined to be

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|.$$

Remark 2.3 implies the following lemma.

Lemma 2.4. *If A and B are equivalent companion matrices, then $\kappa(A) = \kappa(B)$.*

3. Condition numbers of Fiedler matrices via the Hessenberg characterization

The condition numbers of Fiedler companion matrices were first calculated by de Terán, Dopico, and Pérez; see [3], Theorem 4.1. In this section we demonstrate how a characterization of Fiedler companion matrices via unit lower Hessenberg matrices, as given by Eastman et al. (see [4]), provides an efficient way to obtain the condition numbers for Fiedler companion matrices. Our approach avoids the use of the consecution-inversion structure sequence, described in Definition 2.3, of [3], which was used in the original computation of these numbers.

The following theorem gives a characterization of the Fiedler companion matrices in terms of unit lower Hesenberg matrices.

Theorem 3.1 ([4], Corollary 4.4). *If $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$ is a polynomial over $\mathbb{R}$ with $n \geqslant 2$, then F is an $n \times n$ Fiedler companion matrix to $p(x)$ if and only if F is equivalent to a unit lower Hessenberg matrix as in* (2.1) *with the additional property that if $-c_k$ is in position (i, j) then $-c_{k+1}$ is in position $(i-1, j)$ or $(i, j+1)$ for $1 \leqslant k \leqslant n-1$.*

An alternative way to describe the unit lower Hesenberg matrix in Theorem 3.1 is to say that the variable entries of R in (2.1) form a lattice-path from the bottom-left corner to the top-right corner of R. The first two matrices in Figure 1 are examples of Fiedler companion matrices since the variable entries of R form a lattice-path. The last matrix in Figure 1 is not a Fiedler companion matrix.

If F is a Fiedler companion matrix, the *initial step size* of F is the number of coefficients other than c_0 in the row or column containing both c_0 and c_1. The first matrix in Figure 1 has initial step size three and the second matrix in Figure 1 has initial step size one.

Remark 3.2. Note that equivalent matrices have the same initial step size since transpositions and permutation equivalence does not change the number of coefficients in the row or column containing c_0 and c_1.

Using Theorem 3.1 and Lemma 2.2, one can describe the nonzero entries of the inverse of a Fiedler companion matrix:

Lemma 3.3 ([3], Theorem 3.2). *Let* $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1x + c_0$ *be a polynomial over* $\mathbb{R}$ *with* $n \geqslant 2$ *and* $c_0 \neq 0$. *Let* F *be a Fiedler companion matrix to* $p(x)$ *with an initial step size* t. *Then*

(1) F^{-1} *has* $t+1$ *entries equal to* $-1/c_0, -c_1/c_0, \ldots, -c_t/c_0$,
(2) F^{-1} *has* $n-1-t$ *entries equal to* $c_{t+1}, c_{t+2}, \ldots, c_{n-1}$,
(3) F^{-1} *has* $n-1$ *entries equal to* 1, *and*
(4) *the remaining entries of* F^{-1} *are* 0.

Proof. Since F is a companion matrix to $p(x)$, by Theorem 2.1, the matrix F is equivalent to a lower Hessenberg matrix C of the form (2.1). Since F and C are equivalent, it follows that the matrices F^{-1} and C^{-1} are equivalent, so it suffices to show that the matrix C^{-1} satisfies conditions (1)–(4).

Since F is a Fiedler companion matrix, Theorem 3.1 implies that c_1 is either directly above c_0 in C or directly to the right of c_1. If c_1 is to the right of c_0 in C, then all other entries in the column containing c_0 are zero. Alternatively, if c_1 is above c_0, all entries to the right of c_0 in C are zero.

Lemma 2.2, which gives us the inverse of a unit lower Hessenberg matrix, applies to the matrix C. By our above observation, the vector $\mathbf{u}$ or the vector $\mathbf{y}$ must be the zero vector. Without loss of generality, let $\mathbf{y}^\top$ be zero, which means that ${-c_0}^{-1}\mathbf{u}\mathbf{y}^\top - H = -H$. If the initial step size of A is t, then there are t nonzero elements in $\mathbf{u}$, and it has the form

$$\mathbf{u} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -c_t \\ \vdots \\ -c_1 \end{bmatrix}.$$

By Lemma 2.2 the inverse of the matrix C then has the form

$$C^{-1} = \left[\begin{array}{c|c|c} \mathbf{0}^\top & \mathbf{0}^\top & -\dfrac{1}{c_0} \\ \hline I_m & O & \mathbf{0} \\ \hline -H & I_{n-m-1} & \dfrac{1}{c_0}\mathbf{u} \end{array}\right]. \tag{3.1}$$

From (3.1), we can describe the entries of C^{-1}: $m+n-m-1=n-1$ entries are 1 (coming from the submatrices I_m and I_{n-m-1}); $c_{t+1},\dots,c_{n-1}$, which all belong to the submatrix $-H$; the entry $-1/c_0$ from the top-right corner; and the entries $-c_1/c_0,\dots,-c_t/c_0$ from the term ${c_0}^{-1}\mathbf{u}$. Moreover, the rest of the entries of C^{-1} are zero. We have now shown that C^{-1}, and hence F^{-1}, has the desired properties. □

Remark 3.4. Lemma 3.3 mimics Theorem 3.2 of [3]. As observed in [7], the initial step size of a Fiedler companion matrix is equal to the number of initial consecutions or inversions of the permutation associated with the Fiedler companion matrix, as defined in [3].

We can now compute the condition number for any Fiedler companion matrix. This result first appeared in [3], but we can avoid the formal analysis of the permutation that was used to construct the Fiedler companion matrix, as well as the associated concepts of consecution and inversion of a permutation.

Theorem 3.5 ([3], Theorem 4.1). *Let $p(x)=x^n+c_{n-1}x^{n-1}+c_{n-2}x^{n-2}+\dots+c_1x+c_0$ be a polynomial over $\mathbb{R}$ with $n\geqslant 2$ and $c_0\neq 0$. Let F be a Fiedler companion matrix to $p(x)$ with an initial step size t. Then*

$$\kappa(F)^2=\|F\|^2\cdot\Big((n-1)+\frac{1+|c_1|^2+\dots+|c_t|^2}{|c_0|^2}+|c_{t+1}|^2+\dots+|c_{n-1}|^2\Big)$$

with

$$\|F\|^2=(n-1)+|c_0|^2+|c_1|^2+\dots+|c_{n-1}|^2.$$

Proof. This result follows from the fact that F is a unit sparse companion matrix (so it contains $n-1$ entries equal to 1 and the entries $-c_0,\dots,-c_{n-1}$), and Lemma 3.3, which describes the entries of F^{-1}. □

Because the condition number $\kappa(F)$ of a Fiedler companion matrix F depends only upon the initial step size and not the permutation σ, we can derive the following corollary.

Corollary 3.6 ([3], Corollary 4.3). *Let $p(x)=x^n+c_{n-1}x^{n-1}+c_{n-2}x^{n-2}+\dots+c_1x+c_0$ be a polynomial over $\mathbb{R}$ with $n\geqslant 2$ and $c_0\neq 0$. Let A and B be Fiedler companion matrices to the polynomial $p(x)$. If the initial step size of both A and B is t, then $\kappa(A)=\kappa(B)$.*

Since condition numbers of Fiedler companion matrices depend on the initial step size, let

$$S_t = \{F\colon\ F \text{ is a Fiedler companion matrix to } p(x) \text{ with the initial step size } t\},$$

and define $\kappa(t) = \kappa(F)$ for $F \in S_t$. We can now recover a result of [3] that allows us to compare the condition numbers of Fiedler matrices while again avoiding any reference to the permutation σ used to define a Fiedler matrix.

Corollary 3.7 ([3], Corollary 4.5). *Let $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1x + c_0$ be a polynomial over $\mathbb{R}$ with $n \geqslant 2$ and $c_0 \neq 0$. Then*
(1) *if $|c_0| < 1$, then $\kappa(1) \leqslant \kappa(2) \leqslant \ldots \leqslant \kappa(n-1)$;*
(2) *if $|c_0| = 1$, then $\kappa(1) = \kappa(2) = \ldots = \kappa(n-1)$; and*
(3) *if $|c_0| > 1$, then $\kappa(1) \geqslant \kappa(2) \geqslant \ldots \geqslant \kappa(n-1)$.*

Proof. Note that by Corollary 3.6, $\kappa(A)$ is the same for all $A \in S_t$, so $\kappa(t)$ is well-defined. The conclusions follow from Theorem 3.5. □

One of our new results is to compare the condition number of a Fiedler companion matrix of $p(x)$ to the condition number of other companion matrices of $p(x)$. In particular, if a Fiedler companion matrix F has a smaller condition number than another companion matrix C to the same polynomial $p(x)$, then the ratio $\kappa(C)/\kappa(F)$ can be bounded. This result is similar in spirit to [3], Theorem 4.12.

Theorem 3.8. *Let $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$ be a polynomial over $\mathbb{R}$ with $n \geqslant 2$, and $c_0 \neq 0$. Let F be a Fiedler companion matrix to $p(x)$. Further, suppose C is any companion matrix to $p(x)$ whose lower Hessenberg form is*

$$C = \left[\begin{array}{c|c|c} \mathbf{0} & I_m & O \\ \hline \mathbf{u}_C & H_C & I_{n-m-1} \\ \hline -c_0 & \mathbf{y}_C^\top & \mathbf{0}^\top \end{array}\right]$$

such that either $\mathbf{u}_C$ or $\mathbf{y}_C^\top$ is the zero vector. If $\kappa(F) \leqslant \kappa(C)$, then

$$1 \leqslant \frac{\kappa(C)}{\kappa(F)} \leqslant \kappa(F).$$

Proof. The conclusion that $1 \leqslant \kappa(C)/\kappa(F)$ is immediate from the hypothesis that $\kappa(F) \leqslant \kappa(C)$.

By Theorem 3.1 and Lemma 2.4, we can assume that F is in unit lower Hessenberg form. As such, let

$$F = \left[\begin{array}{c|c|c} \mathbf{0} & I_l & O \\ \hline \mathbf{u}_F & H_F & I_{n-l-1} \\ \hline -c_0 & \mathbf{y}_F^\top & \mathbf{0}^\top \end{array}\right]$$

and let t be the initial step size of F. We want to show that

$$\frac{\|C\| \cdot \|C^{-1}\|}{\|F\| \cdot \|F^{-1}\|} \leqslant \|F\| \cdot \|F^{-1}\|.$$

Since C and F are unit sparse companion matrices, $\|C\| = \|F\|$. It suffices to show that

$$\|C^{-1}\| \leqslant \|F\| \cdot \|F^{-1}\|^2.$$

Using equivalence, we may assume without loss of generality that $\mathbf{u}_C = \mathbf{0}$. By Lemma 2.2,

$$C^{-1} = \left[\begin{array}{c|c|c} \frac{1}{c_0}\mathbf{y}_C^\top & \mathbf{0}^\top & -\frac{1}{c_0} \\ \hline I_m & O & \mathbf{0} \\ \hline -H_C & I_{n-m-1} & \mathbf{0} \end{array}\right]$$

since $\mathbf{u}_C = \mathbf{0}$. Then

$$\|C^{-1}\|^2 = (n-1) + \left(\frac{1}{c_0}\right)^2 + \sum_{c_i \in \mathbf{y}_C^\top} \left|\frac{c_i}{c_0}\right|^2 + \sum_{c_k \in H_C} |c_k|^2, \tag{3.2}$$

where $c \in H$ (or $c \in \mathbf{y}$) means $-c$ is an entry in H (or $\mathbf{y}$, respectively). On the other hand, using Lemma 3.3,

$$\|F\|^2 \cdot \|F^{-1}\|^4 = \left[(n-1) + \sum_{i=0}^{n-1} |c_i|^2\right]\left[(n-1) + \left(\frac{1}{c_0}\right)^2 + \sum_{i=1}^{t}\left|\frac{c_i}{c_0}\right|^2 + \sum_{j=t+1}^{n-1} |c_j|^2\right]^2. \tag{3.3}$$

We want to show that $\|C^{-1}\| \leqslant \|F\| \cdot \|F^{-1}\|^2$ which is equivalent to showing that $\|C^{-1}\|^2 \leqslant \|F\|^2 \cdot \|F^{-1}\|^4$. To do this, for each of the four different summands in (3.2), we show that there exist distinct terms in $\|F\|^2 \cdot \|F^{-1}\|^4$ that are greater than or equal to the summand. Here we rely on the fact that there are no negative summands in (3.3).

Partially expanding out (3.3), we have

$$\|F\|^2 \cdot \|F^{-1}\|^4 = (n-1)^3 + (n-1)^2\left(\frac{1}{c_0}\right)^2 + (n-1)\left(\sum_{i=0}^{n-1} |c_i|^2\right)\left(\frac{1}{c_0}\right)^2$$
$$+ (n-1)^2 \sum_{j=0}^{n-1} |c_j|^2 + \text{other nonnegative terms.}$$

Consequently,

$$\begin{aligned}\|C^{-1}\|^2 &= (n-1) + \left(\frac{1}{c_0}\right)^2 + \sum_{c_i \in \mathbf{y}_C^\top} \left|\frac{c_i}{c_0}\right|^2 + \sum_{c_k \in H_C} |c_k|^2 \\ &\leqslant (n-1)^3 + (n-1)^2\left(\frac{1}{c_0}\right)^2 + (n-1)\left(\sum_{i=0}^{n-1} |c_i|^2\right)\left(\frac{1}{c_0}\right)^2 + (n-1)^2 \sum_{j=0}^{n-1} |c_j|^2 \\ &\leqslant \|F\|^2 \cdot \|F^{-1}\|^4.\end{aligned}$$

It now follows that $1 \leqslant \kappa(C)/\kappa(F) \leqslant \kappa(F)$. □

4. Striped companion matrices

In this section we explore a particular class of companion matrices known as striped companion matrices, which were introduced in [4]. A striped companion matrix to a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ has the property that the coefficients $-c_0, -c_1, \ldots, -c_{n-1}$ form horizontal stripes in the matrix. In particular, if $\mathbf{t} = (t_1, t_2, \ldots, t_r)$ is an ordered r-tuple of positive integers with $t_1 + t_2 + \ldots + t_r = n$, and $t_1 \geqslant t_i$ for $2 \leqslant i \leqslant n$, then we define the *striped companion matrix* $S_n(\mathbf{t})$ to be the companion matrix of unit Hessenberg form

$$S_n(\mathbf{t}) = \left[\begin{array}{c|c} \mathbf{0} \quad I_{t_1-1} & O \\ \hline R & \begin{array}{c} I_{n-t_1} \\ \mathbf{0}^\top \end{array} \end{array}\right] \tag{4.1}$$

with the $(n - t_1 + 1) \times t_1$ matrix R having r nonzero rows and with the ith nonzero row of R having t_i variables in the first t_i positions and $t_i - 1$ zero rows immediately above it in R, for $1 < i \leqslant r$. Note that this implies the first row of R is a nonzero row with t_1 leading nonzero entries. For example,

$$S_7(3,2,2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -c_2 & -c_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$S_8(3,3,2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -c_5 & -c_6 & -c_7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -c_2 & -c_3 & -c_4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As the next theorem shows, in some cases the striped companion matrices can have a better condition number than a Fiedler companion matrix.

Theorem 4.1. *Suppose $n = k(m+1)$ for some positive $k, m \in \mathbb{Z}$ and $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ with $c_0 = 1$, $c_1, \ldots, c_{n-1} \in \mathbb{R}$. There exists a striped companion matrix $S = S_n(k, \ldots, k)$ for $p(x)$ such that $\kappa(S) \leqslant \kappa(F)$ for every Fiedler companion matrix F if and only if*

$$\sum_{j=1}^{m}\biggl(\sum_{i=1}^{k-1} |c_i c_{jk} - c_{jk+i}|^2\biggr) \leqslant \sum_{j=1}^{m}\biggl(\sum_{i=1}^{k-1} |c_{jk+i}|^2\biggr). \tag{4.2}$$

P r o o f. Let $S = S_{k(m+1)}(k, \ldots, k)$, and let F be a Fiedler companion matrix. Since $\|S\| = \|F\|$ as noted in Remark 2.3, it suffices to show that $\|S^{-1}\| \leqslant \|F^{-1}\|$ if and only if the equation (4.2) holds. By Lemma 2.2,

$$S^{-1} = \left[\begin{array}{cccc|c|c} -c_1 & -c_2 & \ldots & -c_{k-1} & \mathbf{0}^\top & -1 \\ \hline & & I_{k-1} & & O & \mathbf{0} \\ \hline -c_1c_{mk} + c_{mk+1} & -c_2c_{mk} + c_{mk+2} & \ldots & -c_{k-1}c_{mk} + c_{(m+1)k-1} & & -c_{mk} \\ 0 & 0 & \ldots & 0 & & 0 \\ \vdots & \vdots & \ldots & \vdots & & \vdots \\ \vdots & \vdots & \ldots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & & 0 \\ -c_1c_{2k} + c_{2k+1} & -c_2c_{2k} + c_{2k+2} & \ldots & -c_{k-1}c_{2k} + c_{3k-1} & & -c_{2k} \\ 0 & 0 & \ldots & 0 & I_{mk} & 0 \\ \vdots & \vdots & \ldots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & & 0 \\ -c_1c_k + c_{k+1} & -c_2c_k + c_{k+2} & \ldots & -c_{k-1}c_k + c_{2k-1} & & -c_k \\ 0 & 0 & \ldots & 0 & & 0 \\ \vdots & \vdots & \ldots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & & 0 \end{array}\right].$$

Thus,

$$\|S^{-1}\|^2 = n + \sum_{j=1}^{k-1} |c_j|^2 + \sum_{j=1}^{m} |c_{jk}|^2 + \sum_{j=1}^{m}\biggl(\sum_{i=1}^{k-1} |c_i c_{jk} - c_{jk+i}|^2\biggr).$$

By Theorem 3.5,

$$\|F^{-1}\|^2 = n + \sum_{j=1}^{k-1} |c_j|^2 + \sum_{j=1}^{m} |c_{jk}|^2 + \sum_{j=1}^{m}\Big(\sum_{i=1}^{k-1} |c_{jk+i}|^2\Big).$$

Therefore $\kappa(S) \leqslant \kappa(F)$ if and only if

$$\sum_{j=1}^{m}\Big(\sum_{i=1}^{k-1} |c_i c_{jk} - c_{jk+i}|^2\Big) \leqslant \sum_{j=1}^{m}\Big(\sum_{i=1}^{k-1} |c_{jk+i}|^2\Big).$$

□

We can deduce the following corollary.

Corollary 4.2. *Suppose* $n = k(m+1)$ *for some* $m, k \in \mathbb{Z}$ *and* $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ *with* $c_0 = 1$, $c_1, \ldots, c_{n-1} \in \mathbb{R}$. *Suppose* F *is any Fiedler companion matrix for* $p(x)$. *If*

$$|c_i c_{jk} - c_{jk+i}| \leqslant |c_{jk+i}| \quad \text{for } 1 \leqslant j \leqslant m \text{ and } 1 \leqslant i \leqslant k-1,$$

then there exists a striped companion matrix $S = S_n(k, \ldots, k)$, *such that* $\kappa(S) \leqslant \kappa(F)$.

Example 4.3. Let

$$p(x) = x^9 + 8x^8 + 6x^7 + 2x^6 + 5x^5 + 8x^4 + 3x^3 + 3x^2 + 2x + 1.$$

Note that the inequalities in Corollary 4.2 hold. Let F be any Fiedler companion to $p(x)$ and consider the striped companion matrix $S = S_9(3,3,3)$, i.e.,

$$S_9(3,3,3) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -6 & -8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & -8 & -5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\|S\| = \|F\| = \sqrt{224}$, but $\kappa(S) = \sqrt{224}\sqrt{63} < \kappa(F) = \sqrt{224}\sqrt{224}$.

One extreme example of how the inequalities in Corollary 4.2 can be met is if $c_0 = 1$ and the striped companion matrix in line (4.1) has $\operatorname{rank}(R) = 1$. In this case, the inequalities are trivially met as described in the following corollary. A more general result can be developed for striped companion matrices with differing stripe lengths; see, e.g., [1], Section 4.2.

Corollary 4.4. *Given* $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1x + c_0$ *with* $c_0 = 1$ *and* $c_1, \ldots, c_{n-1} \in \mathbb{R}$*, let* S *be a striped companion matrix to the polynomial* $p(x)$*. If*

$$S = \left[\begin{array}{c|c} \mathbf{0} \quad I_m & O \\ \hline R & \begin{matrix} I_{n-m-1} \\ \mathbf{0}^\top \end{matrix} \end{array}\right]$$

with $\operatorname{rank}(R) = 1$*, then* $\kappa(S) \leqslant \kappa(F)$ *for any Fiedler companion matrix* F*.*

Proof. This result follows from Corollary 4.2 by observing that $|c_i c_{jk} - c_{jk+i}| = 0$ for all $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant k-1$, if and only if $\operatorname{rank}(R) = 1$. In particular, $\operatorname{rank}(R) = 1$ if and only if every 2×2 submatrix of R has zero determinant, which is true if and only if $|c_i c_{jk} - c_{jk+i}| = 0$ for $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant k-1$. Note that we use the fact that

$$\begin{bmatrix} -c_{jk} & -c_{jk+i} \\ -c_0 & -c_i \end{bmatrix}$$

is a 2×2 submatrix of R and $c_0 = 1$. □

Example 4.5. Let $b, k \in \mathbb{R}$ and consider the polynomial $p(x) = x^6 + (bk^3)x^5 + (bk^2)x^4 + (bk^2)x^3 + (bk)x^2 + kx + 1$. If

$$S = S_6(2,2,2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -bk^2 & -bk^3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -bk & -bk^2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -k & 0 & 0 & 0 & 0 \end{bmatrix}$$

and F is any Fiedler companion matrix for $p(x)$, then

$$\left(\frac{\kappa(F)}{\kappa(S)}\right)^2 = \frac{b^2k^6 + b^2k^4 + b^2k^4 + b^2k^2 + k^2 + 6}{b^2k^4 + b^2k^2 + k^2 + 6}.$$

In this case, for sufficiently large k,

$$\frac{\kappa(F)}{\kappa(S)} \approx k,$$

demonstrating a significantly better condition number for S compared to any Fiedler companion matrix.

As shown in Corollary 4.4, if the submatrix R in the striped companion matrix S has $\operatorname{rank}(R) = 1$, then the inequality $\kappa(S) \leqslant \kappa(F)$ holds for any Fiedler companion matrix F. Note that in the striped companion matrix given in Example 4.5, the corresponding submatrix R has rank one. Observe also that we can write $p(x)$ as

$$p(x) = q(x) + (bk)x^2q(x) + (bk^2)x^4q(x) + x^6 \quad \text{with } q(x) = 1 + kx.$$

This generalizes: if the matrix S in Corollary 4.4 has $\operatorname{rank}(R) = 1$, then $p(x) = x^n + q(x)f(x)$ for some polynomial $q(x)$ with $\deg(q(x)) = m$ and $\deg(f(x)) = n-m-1$. Moreover, Corollary 4.4 can be improved by giving an estimate on $\kappa(F)/\kappa(S)$ for any Fiedler companion matrix F.

Theorem 4.6. *Suppose $n = k(m+1)$ and $p(x) = q(x)+b_1x^kq(x)+b_2x^{2k}q(x)+\ldots+b_mx^{mk}q(x) + x^{(m+1)k}$ with*

$$q(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \ldots + a_1x + 1.$$

Suppose that $S = S_n(k, k, \ldots, k)$ and F is any Fiedler companion matrix to $p(x)$. If $(b_1^2 + \ldots + b_m^2)$ is sufficiently large, then

$$\left(\frac{\kappa(F)}{\kappa(S)}\right)^2 \approx (a_1^2 + \ldots + a_{k-1}^2 + 1),$$

or if $(a_1^2 + \ldots + a_{k-1}^2)$ is sufficiently large, then

$$\left(\frac{\kappa(F)}{\kappa(S)}\right)^2 \approx (1 + b_1^2 + \ldots + b_m^2).$$

Proof. By Remark 2.3, $\kappa(F)/\kappa(S) = \|F^{-1}\|/\|S^{-1}\|$. By Lemma 2.2,

$$\|S^{-1}\|^2 = a_1^2 + \ldots + a_{k-1}^2 + b_1^2 + \ldots + b_m^2 + n.$$

By Theorem 3.5 we can determine that

$$\|F^{-1}\|^2 = (1 + b_1^2 + \ldots + b_m^2)(a_1^2 + \ldots + a_{k-1}^2) + (b_1^2 + \ldots + b_m^2) + n.$$

Therefore,

$$\left[\frac{\kappa(F)}{\kappa(S)}\right]^2 = \frac{(1 + b_1^2 + \ldots + b_m^2)(a_1^2 + \ldots + a_{k-1}^2) + (b_1^2 + \ldots + b_m^2) + n}{(a_1^2 + \ldots + a_{k-1}^2) + (b_1^2 + \ldots + b_m^2) + n}$$

and the result follows. □

5. Generalized companion matrices: a case study

In the previous sections, we focused on the condition numbers of unit sparse companion matrices. In this section, we initiate an investigation into the condition numbers of a family of matrices that are not companion matrices, but have properties similar to companion matrices. To date, there appears to be little work done on this approach, so the work in this section can be seen as providing a proof-of-concept for future projects. These results can also be viewed in the broader context of developing the properties of generalized companion matrices; see, e.g., [4], [6]. Roughly speaking, given a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x_1 + c_0$, a generalized companion matrix A is a matrix whose entries are polynomials in $c_0, \ldots, c_n$ and whose characteristic polynomial is $p(x)$; see [6] for more explicit detail.

Instead of considering the general case, we focus on a particular family of matrices and their condition numbers. This case study shows that the condition numbers can improve on those of Frobenius (or Fiedler) companion matrices under some extra hypotheses.

We now define our special family. Let $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$ be a polynomial over $\mathbb{R}$ with $n \geqslant 2$ and let $a \in \mathbb{R}$ be any real number. Fix an integer $l \in \{3, \ldots, n-2\}$ and let

$$\mathbf{a}^\top = (-c_{n-1}, -c_{n-2}, \ldots, -c_{l+1}) \quad \text{and} \quad \mathbf{b}^\top = (-c_{l-2}, -c_{l-3}, \ldots, -c_1).$$

Then let

$$M_n(a,l) = \left[\begin{array}{c|c|c|c} \mathbf{a} & I_{n-l-1} & O & O \\ \hline \begin{array}{c} -c_l + a \\ -c_{l-1} + ac_{n-1} \end{array} & W & I_2 & O \\ \hline \mathbf{b} & O & O & I_{l-2} \\ \hline -c_0 & O & O & O \end{array}\right] \tag{5.1}$$

where W is a $2 \times (n-l-1)$ matrix having $W_{2,1} = -a$ and zeroes in every other entry.

Informally, the matrix $M_n(a,l)$ is constructed by starting with the Frobenius companion matrix which has all the coefficients of $p(x)$ in the first column. Then we fix a row that is neither the top row nor one of the bottom two rows (this corresponds to picking the l), and then add a to c_l in the $(n-l)$th row, and $-a$ in the column to the right and one below. We then also add ac_{n-1} to the first entry in the $(n-l+1)$th row. Note that when $a = 0$, $M_n(0,l)$ is equivalent to the Frobenius companion matrix. We can thus, view $M_n(a,l)$ as a perturbation of the Frobenius companion matrix when $a \neq 0$. As an example, the matrix $M_7(a,4)$ is given in Figure 4.

$$\begin{bmatrix} -c_6 & 1 & 0 & 0 & 0 & 0 & 0 \\ -c_5 & 0 & 1 & 0 & 0 & 0 & 0 \\ -c_4 + a & 0 & 0 & 1 & 0 & 0 & 0 \\ -c_3 + ac_6 & -a & 0 & 0 & 1 & 0 & 0 \\ -c_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -c_1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -c_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 4. The matrix $M_7(a,4)$.

We wish to compare the condition number of $M_n(a,l)$ with the Frobenius (and Fiedler) companion matrices. In some cases our new matrix $M_n(a,l)$ can provide us with a smaller condition number. The next lemma gives the inverse of $M_n(a,l)$ and shows that the characteristic polynomial of $M_n(a,l)$ is $p(x)$.

Lemma 5.1. *Let $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1 x + c_0$ be a polynomial over $\mathbb{R}$ with $n \geqslant 2$ and $c_0 \neq 0$. Let $a \in \mathbb{R}$ and $l \in \{3, \ldots, n-2\}$, and let $M = M_n(a,l)$ be constructed from $p(x)$ as above. Then*

(i) *the characteristic polynomial of M is $p(x)$, and*

(ii) *if $c_0 \neq 0$, then*

$$M^{-1} = \frac{1}{c_0}\left[\begin{array}{c|c|c|c} \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & -1 \\ \hline c_0 I_{n-l} & O & O & \mathbf{a} \\ \hline -c_0 W & c_0 I_2 & O & \begin{array}{c} -c_l + a \\ -c_{l-1} \end{array} \\ \hline O & O & c_0 I_{l-2} & \mathbf{b} \end{array}\right].$$

P r o o f. (i) We employ the fact that the determinant of a matrix is a linear function of its rows. In particular, if $M = M_n(a,l)$, we observe that row $n-l$ of $xI_n - M$ can be written as $\mathbf{u} + a\mathbf{v}$ for some vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$ is not a function of a. Row $n-l+1$ of $xI_n - M$ can also be written in a similar manner. Let $k = n-l$. Thus, applying linearity to the row k gives

$$\det(xI_n - M) = \det\left(xI_n - \left[\begin{array}{c|c|c|c} \mathbf{a} & I_{n-l-1} & O & O \\ \hline \begin{array}{c} -c_l \\ -c_{l-1} + ac_{n-1} \end{array} & W & I_2 & O \\ \hline \mathbf{b} & O & O & I_{l-2} \\ \hline -c_0 & O & O & O \end{array}\right]\right) + a(-1)x^l. \tag{5.2}$$

Note that the term $a(-1)x^l$ in (5.2) comes from computing the determinant of the matrix A' formed by replacing the kth row of the matrix $xI_n - M$ with the row $[-a \quad 0 \quad \ldots \quad 0]$. Doing a row expansion along the kth row of A', the determinant of A' is $(-1)^{k+1}(-a)\det(A'')$, where A'' is a block lower diagonal matrix with diagonal blocks D_1 and D_2. Furthermore, D_1 is a $(k-1)\times(k-1)$ lower triangular matrix with -1 on all the diagonal positions and D_2 is a $l \times l$ upper triangular matrix with x on all the diagonal positions. So $\det(A'') = (-1)^{k-1}x^l$, and hence $\det(A') = (-1)^{k+1}(-a)(-1)^{k-1}x^l = (-a)x^l$, as desired.

We now apply linearity to the row $k+1$ in the matrix that appears in (5.2); in particular, a similar argument shows that (5.2) is equal to

$$\det\left(xI - \left[\begin{array}{c|c|c|c} \mathbf{a} & I_{k-1} & O & O \\ \hline -c_l & & & \\ & O & I_2 & O \\ -c_{l-1} & & & \\ \hline \mathbf{b} & O & O & I_{l-2} \\ \hline -c_0 & O & O & O \end{array}\right]\right) + a(-1)x^l + ac_{n-1}(-1)x^{l-1} + a(x+c_{n-1})x^{l-1}. \tag{5.3}$$

Note that the first summand in (5.3) is the characteristic polynomial of a Frobenius companion matrix of $p(x)$, and hence is $p(x)$. Thus, (5.3) reduces to

$$p(x) + a(-1)x^l + ac_{n-1}(-1)x^{l-1} + a(x+c_{n-1})x^{l-1} = p(x).$$

(ii) A direct multiplication shows that the given matrix is the inverse M. □

Because both $M_n(a,l)$ and its inverse are known, we are able to compute its condition number. In the next lemma, instead of providing the general formula, we compute the condition number under the extra assumption that $c_0 = 1$ in the polynomial $p(x)$.

Lemma 5.2. *Let $p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1 x + c_0$ be a polynomial over $\mathbb{R}$ with $n \geqslant 2$, and suppose that $c_0 = 1$. Let $a \in \mathbb{R}$ and $l \in \{3, \ldots, n-2\}$, and let $M = M_n(a,l)$. Then*

$$\kappa(M)^2 = (v + a^2 + (a-c_l)^2 + (ac_{n-1} - c_{l-1})^2)(v + a^2 + (a-c_l)^2 + c_{l-1}^2 + 1)$$

with

$$v = n - c_{l-1}^2 - c_l^2 + \sum_{i=1}^{n-1} c_i^2.$$

The next result illustrates the desired proof-of-concept. In particular, the result shows that in special cases, the condition number of the matrix $M_n(a,l)$, which has properties similar to a companion matrix, has a condition number smaller than *any* Fiedler companion matrix. Although the scope of this result is limited, it does suggest that generalized companion matrices, and in particular perturbations of the Frobenius companion matrix, can provide better condition numbers in some cases.

Theorem 5.3. *Let $n \geqslant 2$, and fix $l \in \{3, \ldots, n-2\}$ and $t \in \mathbb{R}$. Set*

$$p(x) = x^n + tx^{n-1} + tx^l + t^2 x^{l-1} + 1.$$

Let $M = M_n(t, l)$. Then, for any Fiedler companion matrix F of $p(x)$,

$$\frac{\kappa(F)^2}{\kappa(M)^2} = \frac{(n + 2t^2 + t^4)^2}{(n + 2t^2)(n + 1 + 2t^2 + t^4)}.$$

In particular, for t for sufficiently large, $\kappa(F)/\kappa(M) \approx \frac{1}{\sqrt{2}} t$.

P r o o f. By Lemma 5.2, $\kappa(M)^2 = (1+t^2+(n-1)+a^2+(a-t)^2+(at-t^2)^2)(1+t^2+(n-1)+a^2+(a-t)^2+t^4+1)$. Setting $a = t$ gives $\kappa(M)^2 = (n+2t^2)(n+1+2t^2+t^4)$. We use Theorem 3.5 to compute $\kappa(F)^2$. Note that since $c_0 = 1$, $\kappa(F)$ is independent of the initial step size of F. Hence,

$$\kappa(F) = ((n-1) + 1 + t^4 + t^2 + t^2) = (n + 2t^2 + t^4).$$

Thus we have

$$\frac{\kappa(F)^2}{\kappa(M)^2} = \frac{(n + 2t^2 + t^4)^2}{(n + 2t^2)(n + 1 + 2t^2 + t^4)}.$$

The limit of the right hand side is $\frac{1}{2} t^2$ as $t \to \infty$, which implies the final statement. □

The following result gives another case, where we can make a matrix with smaller condition number than any other Fiedler companion matrix, providing additional evidence that generalized companion matrices may be of interest.

Theorem 5.4. *Let $n \geqslant 2$, and fix $l \in \{3, \ldots, n-2\}$. Let $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ with $c_0 = 1$, and $(c_l c_{n-1})^2 < 2c_{l-1}c_l c_{n-1} - 1$. Let $M = M_n(c_l, l)$. Then $\kappa(M) < \kappa(F)$ for every Fiedler companion matrix F of $p(x)$.*

P r o o f. Let $v = n - c_l^2 - c_{l-1}^2 + \sum\limits_{i=1}^{n-1} c_i^2$. Because $c_0 = 1$, by Theorem 3.5 all Fiedler companion matrices F have the condition number

$$\kappa(F) = (v + c_l^2 + c_{l-1}^2).$$

By Lemma 5.2, with $a = c_l$,

$$\begin{aligned}\kappa(M)^2 &= (v + c_l^2 + (c_l c_{n-1} - c_{l-1})^2)(v + c_l^2 + c_{l-1}^2 + 1)\\ &= (v + c_l^2 + c_{l-1}^2 + ((c_l c_{n-1})^2 - 2c_{l-1}c_l c_{n-1}))(v + c_l^2 + c_{l-1}^2 + 1).\end{aligned}$$

If we set $w = (v + c_l^2 + c_{l-1}^2)$, then $\kappa(M)^2 = (w - y)(w + 1)$ with $y = 2c_{l-1}c_l c_{n-1} - (c_l c_{n-1})^2$. But $y > 1$ by hypothesis, thus $\kappa(M)^2 < w^2 = \kappa(F)^2$. □

References

[1] *M. Cox*: On Conditions Numbers of Companion Matrices: M.Sc. Thesis. McMaster University, Hamilton, 2018.

[2] *L. Deaett, J. Fischer, C. Garnett, K. N. Vander Meulen*: Non-sparse companion matrices. Electron. J. Linear Algebra *35* (2019), 223–247. zbl MR doi

[3] *F. de Terán, F. M. Dopico, J. Pérez*: Condition numbers for inversion of Fiedler companion matrices. Linear Algebra Appl. *439* (2013), 944–981. zbl MR doi

[4] *B. Eastman, I.-J. Kim, B. L. Shader, K. N. Vander Meulen*: Companion matrix patterns. Linear Algebra Appl. *463* (2014), 255–272. zbl MR doi

[5] *M. Fiedler*: A note on companion matrices. Linear Algebra Appl. *372* (2003), 325–331. zbl MR doi

[6] *C. Garnett, B. L. Shader, C. L. Shader, P. van den Driessche*: Characterization of a family of generalized companion matrices. Linear Algebra Appl. *498* (2016), 360–365. zbl MR doi

[7] *K. N. Vander Meulen, T. Vanderwoerd*: Bounds on polynomial roots using intercyclic companion matrices. Linear Algebra Appl. *539* (2018), 94–116. zbl MR doi

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