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THE GENERALIZED TOEPLITZ OPERATORS ON THE FOCK SPACE F_α^2

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Abstract. Let μ be a positive Borel measure on the complex plane $\mathbb{C}^n$ and let $j = (j_1, \ldots, j_n)$ with $j_i \in \mathbb{N}$. We study the generalized Toeplitz operators $T_\mu^{(j)}$ on the Fock space F_α^2. We prove that $T_\mu^{(j)}$ is bounded (or compact) on F_α^2 if and only if μ is a Fock-Carleson measure (or vanishing Fock-Carleson measure). Furthermore, we give a necessary and sufficient condition for $T_\mu^{(j)}$ to be in the Schatten p-class for $1 \leqslant p < \infty$.

Keywords: generalized Toeplitz operator; boundedness; compactness; Schatten class; Fock space

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1. Introduction

Let $\mathbb{C}^n$ be the complex n-space. For points $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$z\overline{w} = \sum_{i=1}^{n} z_i \overline{w_i}, \quad |z|^2 = z\overline{z}.$$

For any positive parameter α, we consider the Gaussian measure

$$\mathrm{d}\lambda_\alpha(z) = \Big(\frac{\alpha}{\pi}\Big)^n \mathrm{e}^{-\alpha|z|^2} \, \mathrm{d}v(z),$$

where $\mathrm{d}v$ is an ordinary volume measure on $\mathbb{C}^n$. It is easy to check that $\mathrm{d}\lambda_\alpha$ is a probability measure on $\mathbb{C}^n$. For $1 \leqslant p < \infty$, the Lebesgue space $L_\alpha^p(\mathbb{C}^n)$ consists of all measurable functions f for which

$$\|f\|_{p,\alpha} = \left\{ \int_{\mathbb{C}^n} |f(z)\mathrm{e}^{-\alpha|z|^2/2}|^p \, \mathrm{d}v(z) \right\}^{1/p} < \infty.$$

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The Fock space F^p_α consists of all entire functions on $\mathbb{C}^n$ that are also in the space $L^p_\alpha(\mathbb{C}^n)$. The space F^p_α is a closed subspace of $L^p_\alpha(\mathbb{C}^n)$. So we get that F^2_α is a Hilbert space with inner product inherited from $L^2(\mathbb{C}^n)$:

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}\,\mathrm{d}\lambda_\alpha(z).$$

Let

$$e_j(z) = \sqrt{\frac{\alpha^{|j|}}{j!}}\,z^j,$$

where $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $j! = j_1! \ldots j_n!$ and $|j| = j_1 + \ldots + j_n$. Then $\{e_j\}$ forms an orthonormal basis for F^2_α. For any fixed $z \in \mathbb{C}^n$, the reproducing kernel function $K_z(w)$ for the Fock space F^2_α is

$$K_z(w) = K(w, z) = \sum_l \overline{e_l(z)}e_l(w) = \mathrm{e}^{\alpha\bar{z}w}.$$

In the following discussion, let $k_z = K_z/\|K_z\|$ denote the normalized reproducing kernel function for F^2_α, where $\|\cdot\|$ denotes the norm in F^2_α.

For any fixed $z \in \mathbb{C}^n$, define the operator U_z on F^2_α by

$$U_z f = (f \circ \varphi_z)k_z,$$

where $\varphi_z(w) = z - w$ for $w \in \mathbb{C}^n$. Then it is easy to check that U_z is unitary and self-adjoint. For any $f, g \in F^2_\alpha$, let $f \otimes g$ be the rank-one operator defined on F^2_α, that is,

$$(f \otimes g)h = \langle h, g\rangle f \quad \forall\, h \in F^2_\alpha.$$

In particular, the rank one operator $E_j := e_j \otimes e_j$ is in fact the orthogonal projection onto the subspace generated by e_j. Let $L^\infty(\mathbb{C}^n)$ be the space of measurable functions f on $\mathbb{C}^n$ such that

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)|\colon\ z \in \mathbb{C}^n\} < \infty.$$

Note that

$$\langle U_z E_0 U_z f, g\rangle = f(z)\overline{g(z)}\mathrm{e}^{-\alpha|z|^2} \quad \forall\, f, g \in F^2_\alpha \quad \forall\, z \in \mathbb{C}^n, \tag{1.1}$$

then the traditional Toeplitz operator T_a on F^2_α with the symbol $a \in L^\infty(\mathbb{C}^n)$ is defined by

$$T_a = \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} U_z E_0 U_z a(z)\,\mathrm{d}v(z),$$

where the integral converges in the weak operator topology.

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and let $\mathrm{d}A$ denote the normalized Lebesgue area measure on $\mathbb{D}$, and let $\mathrm{d}m_\beta(z) = (\beta+1)(1-|z|^2)^\beta\,\mathrm{d}A(z)$, $\beta > -1$. For any fixed $\beta > -1$, the weighted Bergman space $L_a^2(\mathrm{d}m_\beta)$ is a Hilbert space consisting of the analytic functions on $\mathbb{D}$, that are also in the space $L^2(\mathbb{D}, \mathrm{d}m_\beta)$ of square integrable functions on $\mathbb{D}$. If $\beta = 0$, for convenience, the Bergman space is denoted by L_a^2. For $z \in \mathbb{D}$, let W_z on L_a^2 by $W_z f = (f \circ \varphi)\varphi'$, where $\varphi_z(w) = (z-w)/(1-\overline{z}w)$. Then W_z is unitary and self-adjoint on L_a^2. Let $L^\infty(\mathbb{D}, \mathrm{d}A)$ be the space of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)|\colon\ z \in \mathbb{D}\} < \infty.$$

Let R be an operator defined on L_a^2 such that the matrix of R under the orthonormal basis $\{\zeta_k\}_{k\geqslant 0} = \{\sqrt{k+1}z^k\}_{k\geqslant 0}$ of L_a^2 is diagonal, and let $a \in L^\infty(\mathbb{D}, \mathrm{d}A)$. Engliš in [5] considered the following operator defined as

$$R_a := \int_{\mathbb{D}} W_z R W_z a(z)\,\mathrm{d}\widetilde{A}(z), \tag{1.2}$$

where $\mathrm{d}\widetilde{A}(z) = \mathrm{d}A(z)/(1-|z|^2)^2$, and showed that if R is in the trace class, then $\|R_a\| \leqslant \|R\|_{\mathrm{tr}}\|a\|_\infty$. Since the operator R is an l^1 linear combination of the projections $H_k = \zeta_k \otimes \zeta_k$ with the trace norm of R given by the correspondent l^1-norm of its eigenvalues, the above result is equivalent to $\|T_a^{(k)}\| \leqslant \|a\|_\infty$ for all integers $k \geqslant 0$, where the operator $T_a^{(k)}$ is defined by

$$T_a^{(k)} := \int_{\mathbb{D}} W_z H_k W_z a(z)\,\mathrm{d}\widetilde{A}(z). \tag{1.3}$$

Let $k \geqslant 0$. Suárez in [14] defined the following generalized Toeplitz operators on the Bergman space L_a^2:

$$T_\mu^{(k)} := \int_{\mathbb{D}} W_z H_k W_z (1-|z|^2)^{-2}\,\mathrm{d}\mu(z). \tag{1.4}$$

He, using Carleson measure conditions, characterized the boundedness and compactness of the operator $T_\mu^{(k)}$ on the Bergman space. The authors in [16] gave a necessary and sufficient condition for $T_\mu^{(k)} \in S_p$ $(1 \leqslant p < \infty)$ and a sufficient condition for $T_\mu^{(k)} \in S_p$ $(0 < p < 1)$.

Based on the research of the above scholars, we considered similar operator on the Fock space F_α^2, and later found that the operator we defined is essentially a localized operator. Let μ be a Borel measure on $\mathbb{C}^n$. We now define the following generalized Toeplitz operators on the Fock space F_α^2:

$$T_\mu^{(j)} := \Big(\frac{\alpha}{\pi}\Big)^n \int_{\mathbb{C}^n} U_z E_j U_z\,\mathrm{d}\mu(z) \tag{1.5}$$

for all $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$, where the integral converges in the weak operator topology. That is,

$$\langle T_\mu^{(j)} f, g\rangle = \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} \langle U_z f, e_j\rangle \langle e_j, U_z g\rangle \,\mathrm{d}\mu(z), \quad f, g \in F_\alpha^2 \tag{1.6}$$

for all $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$. In particular, the traditional Toeplitz operator with symbol μ is

$$T_\mu = \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} U_z E_0 U_z \,\mathrm{d}\mu(z). \tag{1.7}$$

If $a \in L^\infty(\mathbb{C}^n)$, let $\mathrm{d}\mu = a\,\mathrm{d}v$. We get

$$T_a^{(j)} = \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} U_z E_j U_z a(z) \,\mathrm{d}v(z). \tag{1.8}$$

Toeplitz operators have been widely studied in the contexts of Hardy and Bergman spaces on various domains, and a large number of techniques and methods have been developed over the past twenty years or so, see [11], [17], [18]. In [15], the authors gave a sufficient conditions for a densely-defined operator on Fock space to be bounded or compact, under the boundedness condition, they characterize the compactness of the operator in terms of its Berezin transform.

The generalized Toeplitz operator can be used as a generalization of Toeplitz operator. When dealing with the boundedness, compactness and Schatten-p class of generalized Toeplitz operators, the previous methods for solving Toeplitz operators may no longer be applicable. So, in this paper, we solve the boundedness, compactness and Schatten p-class of generalized Toeplitz operators induced by positive Borel measure on F_α^2 by using new methods. We know that the Schatten-p $(0 < p < 1)$ class of Toeplitz operator has been solved, see [19]. Unfortunately, when dealing with the Schatten-p $(0 < p < 1)$ class of the generalized Toeplitz operator, we did not find a good method.

2. BOUNDEDNESS OF GENERALIZED TOEPLITZ OPERATORS

For a fixed $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$, suppose μ is a positive Borel measure that satisfies the condition

$$\int_{\mathbb{C}^n} |(z-w)^j|^2 |K(z,w)| \mathrm{e}^{-\alpha|w|^2} \,\mathrm{d}\mu(w) < \infty \tag{2.1}$$

for $z \in \mathbb{C}^n$. Because of the exponential form of the reproducing kernel, it is clear that the above is equivalent to

$$\int_{\mathbb{C}^n} |(z-w)^j|^2 |K(z,w)|^2 \mathrm{e}^{-\alpha|w|^2} \,\mathrm{d}\mu(w) < \infty \quad \text{for } z \in \mathbb{C}^n. \tag{2.2}$$

If μ satisfies condition (2.1), then the Toeplitz operator $T_\mu^{(j)}$ is well-defined on a dense subset of F_α^2. In fact, if

$$f(w) = \sum_{k=1}^{N} c_k K(w, w_k)$$

is any finite linear combination of kernel functions in F_α^2, then it follows from (2.2) and the Cauchy-Schwarz inequality that $T_\mu^{(j)}(f)$ is well defined. It is checked that the set of all finite linear combinations of kernel functions is dense in F_α^2.

All measures in this paper will be assumed to satisfy inequality (2.1). In partular, we can define a function $\widetilde{\mu}$ on $\mathbb{C}^n$ as follows:

$$\widetilde{\mu}(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 \mathrm{e}^{-\alpha|w|^2} \, \mathrm{d}\mu(w), \quad z \in \mathbb{C}^n.$$

Inequality (2.2) implies that $\widetilde{\mu}$ is well-defined. We called $\widetilde{\mu}$ the Berezin transform of μ.

Given a point $z \in \mathbb{C}^n$ and $r > 0$, write

$$B(z, r) = \{w : |w - z| < r\}$$

for the Euclidean disk centered at z with radius r. For a Borel measure μ on $\mathbb{C}^n$, the average of μ on $B(z, r)$ is simply writen as

$$\widehat{\mu}_{r,j}(z) = \int_{B(z,r)} |\varphi_z(w)^j|^2 \, \mathrm{d}\mu(w),$$

since the Lebesgue volume $v(B(z, r)) = \int_{B(z,r)} \mathrm{d}v$ is a constant when z varies in $\mathbb{C}^n$. For more information about the Berezin transform of μ and the average of μ one can refer to [8], [9].

The Weyl operator

$$W_z w(\xi) = \mathrm{e}^{\alpha\xi\cdot\overline{z} - \alpha|z|^2/2} w(\xi - z)$$

acts unitarily on $L_\alpha^2(\mathbb{C}^n)$. Let $w \in F_\alpha^2$, and $f \in L^\infty(\mathbb{C}^n)$, the Gabor-Daubechies localization operator $L_f^{(w)}$ is the operator acting on F_α^2 and defined, in the weak sence, by

$$\langle L_f^{(w)} u, \xi\rangle = \int_{\mathbb{C}^n} f(z) \langle u, W_z w\rangle \langle W_z w, \xi\rangle \, \mathrm{d}v(z) \quad \forall\, u,\ \xi \in F_\alpha^2.$$

The operator $T_\mu^{(j)}$ defined in (1.5) is essentially a localization operator. Let $BC^\infty(\mathbb{C})$ be the space of all $C^\infty(\mathbb{C})$ functions whose partial derivatives are bounded.

Let $w \in F^2_\pi$ and $f \in BC^\infty(\mathbb{C})$, the authors in [1] showed that a localization operator as a Toeplitz operator, their relationship can be expressed as

$$L_f^{(w)} = T_{D(w)f}.$$

That is, $L_f^{(w)}$ is equivalent to a Toeplitz operator $T_{D(w)f}$, whose symbol is a differential operator $D(w)f$, where the coefficients are constants explicitly determined by w. For more information about the localization operator one refer to [2], [3], [4], [6], [7], [10].

Theorem 2.1. *If $a \in L^\infty(\mathbb{C}^n)$, then $T_a^{(j)}$ is bounded on F^2_α, that is, there exists a positive constant M such that*

$$\|T_a^{(j)}\| \leqslant M\|a\|_\infty. \tag{2.3}$$

The proof can easily be adapted from the one of Proposition 2 in [6]. The routine details are omitted here.

In order to prove the boundedness and compactness of generalized Toeplitz operators on F^2_α, we need the following lemma.

Lemma 2.2. *For $w, z \in \mathbb{C}^n$, let $t = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\alpha z\overline{w})}$. Then $U_zU_w = U_{\varphi_z(w)}V_t$, where $V_tf(u) = tf(-u)$ for $f \in F^2_\alpha$.*

Proof. Since $\varphi_w \circ \varphi_z \circ \varphi_{\varphi_z(w)} = -I$, where I is identity, for any $f \in F^2_\alpha$ we have

$$\begin{aligned}U_wU_zf(\xi) &= f \circ \varphi_z \circ \varphi_w(\xi)k_z(\varphi_w(\xi))k_w(\xi)\\ &= f \circ \varphi_{\varphi_z(w)}(-\xi)\mathrm{e}^{\alpha\overline{z}\varphi_w(\xi) - \alpha|z|^2/2 + \alpha\overline{w}\xi - \alpha|w|^2/2}\\ &= f \circ \varphi_{\varphi_z(w)}(-\xi)k_{\varphi_z(w)}(-\xi)\mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(\alpha z\overline{w})} = V_{\overline{t}}U_{\varphi_z(w)}f(\xi).\end{aligned}$$

It is easy to show that $V_t^* = V_{\overline{t}}$, therefore $U_zU_w = U_{\varphi_z(w)}V_t$, where $t = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\alpha z\overline{w})}$. □

Each operator S on F^2_α induces a function $\widetilde{S}$ on $\mathbb{C}^n$, namely,

$$\widetilde{S}(z) = \langle Sk_z, k_z\rangle, \quad z \in \mathbb{C}^n.$$

We called $\widetilde{S}$ the Berezin transform of S.

Let μ be a positive Borel measure on $\mathbb{C}^n$. We say that μ is a Fock-Carleson measure on F^2_α if there exists a constant $M > 0$ such that

$$\int_{\mathbb{C}^n} |f(z)\mathrm{e}^{-\alpha|z|^2/2}|^2 \,\mathrm{d}\mu(z) \leqslant M \int_{\mathbb{C}^n} |f(z)\mathrm{e}^{-\alpha|z|^2/2}|^2 \,\mathrm{d}v(z)$$

for all entire functions f. Next we give the main theorem of this section.

Theorem 2.3. *Let μ be a positive Borel measure on $\mathbb{C}^n$. Then $T_\mu^{(j)}$ is bounded on F_α^2 if and only if μ is a Fock-Carleson measure, in which case, there exists a positive constant M such that*

$$M^{-1}\|\widetilde{\mu}\|_\infty \leqslant \|T_\mu^{(j)}\| \leqslant M\|\widetilde{\mu}\|_\infty.$$

Proof. The case of $j = 0$ is trivial. We only need to consider the case of $j \neq 0$. Suppose that μ is a Fock-Carleson measure. For any $F(\xi) = \sum a_m e_m(\xi) \in F_\alpha^2$ and $t = (t_1, \ldots, t_n)$ with $0 \leqslant t_i \leqslant 2\pi$, $s = (s_1, \ldots, s_n)$ with $0 \leqslant s_i < \infty$, it is easy to check that

$$\begin{aligned}(2.4)\qquad |\langle F, U_{se^{it}}e_j\rangle|^2 &= |\langle F(\xi), e_j\circ\varphi_{se^{it}}(\xi)k_{se^{it}}(\xi)\rangle|^2\\ &= |\langle F(\xi), e^{-ijt}e_j\circ\varphi_{se^{it}}(\xi)k_{se^{it}}(\xi)\rangle|^2\\ &= |\langle F(\xi), (U_se_j)(e^{-it}\xi)\rangle|^2 = |\langle F(e^{it}\xi), (U_se_j)(\xi)\rangle|^2\\ &= \sum_{m,l} a_m\overline{a_l}\langle e_m(e^{it}\xi), (U_se_j)(\xi)\rangle\overline{\langle e_l(e^{it}\xi), (U_se_j)(\xi)\rangle}\\ &= \sum_{m,l} a_m\overline{a_l}e^{i(m-l)t}\langle e_m, U_se_j\rangle\overline{\langle e_l, U_se_j\rangle}.\end{aligned}$$

Then

$$\begin{aligned}(2.5)\qquad \underbrace{\int_0^{2\pi}\cdots\int_0^{2\pi}}_{n}|\langle F, U_{se^{it}}e_j\rangle|^2\frac{dt_1}{2\pi}\cdots\frac{dt_n}{2\pi} &= \sum_m |a_m|^2|\langle e_m, U_se_j\rangle|^2\\ &\geqslant |a_j|^2|\langle e_j, U_se_j\rangle|^2 = |\langle F, e_j\rangle|^2|\langle e_j, U_se_j\rangle|^2\\ &= |\langle F, e_j\rangle|^2\underbrace{\int_0^{2\pi}\cdots\int_0^{2\pi}}_{n}|\langle e_j, U_{se^{it}}e_j\rangle|^2\frac{dt_1}{2\pi}\cdots\frac{dt_n}{2\pi}.\end{aligned}$$

Multiplying by $2^n\alpha^n s_1\ldots s_n e^{-\alpha s^2}$ both sides of (2.5) and integrating yields

$$\Big(\frac{\alpha}{\pi}\Big)^n\int_{\mathbb{C}^n}|\langle F, U_ze_j\rangle|^2 e^{-\alpha|z|^2}\,dv(z) \geqslant |\langle F, e_j\rangle|^2\Big(\frac{\alpha}{\pi}\Big)^n\int_{\mathbb{C}^n}|\langle e_j, U_ze_j\rangle|^2 e^{-\alpha|z|^2}\,dv(z),$$

that is,

$$(2.6)\qquad \int_{\mathbb{C}^n}|\langle F, U_ze_j\rangle|^2\,d\lambda_\alpha(z) \geqslant |\langle F, e_j\rangle|^2\int_{\mathbb{C}^n}|\langle e_j, U_ze_j\rangle|^2\,d\lambda_\alpha(z).$$

In particular, let $F = U_wf$, by (2.6), we have

$$(2.7)\qquad \int_{\mathbb{C}^n}|\langle U_wf, U_ze_j\rangle|^2\,d\lambda_\alpha(z) \geqslant |\langle U_wf, e_j\rangle|^2\int_{\mathbb{C}^n}|\langle e_j, U_ze_j\rangle|^2\,d\lambda_\alpha(z).$$

By Lemma 2.2, writing $t = \mathrm{e}^{\mathrm{i}\operatorname{Im}(\alpha w\overline{z})}$, we have

$$\begin{aligned}|\langle U_w f, U_z e_j\rangle| &= |\langle f, U_w U_z e_j\rangle| = |\langle f, U_{\varphi_w(z)} V_t e_j\rangle| \\ &= |\langle f, (-1)^{|j|} t U_{\varphi_w(z)} e_j\rangle| = |\langle f, U_{\varphi_w(z)} e_j\rangle|,\end{aligned}$$

where $V_t h(u) = t h(-u)$ for $h \in F^2_\alpha$. Taking the change of variables $\varsigma = \varphi_w(z)$ in the left of (2.7) yields

$$\int_{\mathbb{C}^n} |\langle f, U_\varsigma e_j\rangle|^2 |k_w(\varsigma)|^2 \,\mathrm{d}\lambda_\alpha(\varsigma) \geqslant |\langle U_w f, e_j\rangle|^2 \int_{\mathbb{C}^n} |\langle e_j, U_z e_j\rangle|^2 \,\mathrm{d}\lambda_\alpha(z). \tag{2.8}$$

Integrating with respect to $(\alpha/\pi)^n \,\mathrm{d}\mu(w)$ both sides of (2.8), we get

$$\begin{aligned}&\Big(\frac{\alpha}{\pi}\Big)^n \int_{\mathbb{C}^n} \widetilde{\mu}(\varsigma) |\langle f, U_\varsigma e_j\rangle|^2 \,\mathrm{d}v(\varsigma) \\ &\qquad \geqslant \Big(\frac{\alpha}{\pi}\Big)^n \int_{\mathbb{C}^n} |\langle U_w f, e_j\rangle|^2 \,\mathrm{d}\mu(w) \int_{\mathbb{C}^n} |\langle e_j, U_z e_j\rangle|^2 \,\mathrm{d}\lambda_\alpha(z),\end{aligned} \tag{2.9}$$

that is,

$$\langle T^{(j)}_{\widetilde{\mu}} f, f\rangle \geqslant \int_{\mathbb{C}^n} |\langle e_j, U_z e_j\rangle|^2 \,\mathrm{d}\lambda_\alpha(z) \langle T^{(j)}_\mu f, f\rangle. \tag{2.10}$$

It is easy to check that $\int_{\mathbb{C}^n} |\langle e_j, U_z e_j\rangle|^2 \,\mathrm{d}\lambda_\alpha(z)$ is a positive constant. By Theorem 2.1 and (2.10), we get that $T^{(j)}_\mu$ is bounded on F^2_α and there exists a constant $M > 0$ such that

$$\|T^{(j)}_\mu\| \leqslant M \|T^{(j)}_{\widetilde{\mu}}\| \leqslant M \|\widetilde{\mu}\|_\infty. \tag{2.11}$$

We now assume that $T^{(j)}_\mu$ is a bounded operator. Let $f(x) = x^j/\mathrm{e}^{\alpha(x_1+\ldots+x_n)}$, where $x = (x_1, \ldots, x_n)$ with $x_i \in [0,\infty)$ and let $f_i(x_i) = x_i^{j_i}\mathrm{e}^{-\alpha x_i}$. We have

$$f_{i\max}(x_i) = f\Big(\frac{j_i}{\alpha}\Big) = \frac{j_i^{j_i}}{\alpha^{j_i}}\mathrm{e}^{-j_i}.$$

Hence,

$$f_{\max}(x) = \prod_{i=1}^n f_{i\max}(x_i) = \frac{j^j}{\alpha^{|j|}}\mathrm{e}^{-|j|}.$$

Let $(j_i - \frac{1}{2})/\alpha \leqslant x_i \leqslant (j_i+1)/\alpha$, that is, $x_i = (j_i + y_i)/\alpha$ with $-\frac{1}{2} \leqslant y_i \leqslant 1$, then

$$f(x) = f\Big(\frac{j+y}{\alpha}\Big) = \frac{(j+y)^j}{\alpha^{|j|}\mathrm{e}^{|j|+(y_1+\ldots+y_n)}} \geqslant \frac{(j_1-1/2)^{j_1}\ldots(j_n-1/2)^{j_n}}{\alpha^{|j|}\mathrm{e}^{|j|+n}} \geqslant \frac{1}{2^n\alpha^{|j|}\mathrm{e}^{|j|+n}}.$$

If $(j_i - \frac{1}{2})/\alpha \leqslant |z_i|^2 \leqslant (j_i+1)/\alpha$, then

$$|z|^{2j}\mathrm{e}^{-\alpha|z|^2} \geqslant \frac{1}{2^n\alpha^{|j|}\mathrm{e}^{|j|+n}}. \tag{2.12}$$

Now, let $z_j := \bigl(\sqrt{j_1/\alpha}, \ldots, \sqrt{j_n/\alpha}\bigr)$, and $0 < r \leqslant |z_j| = \sqrt{|j|/\alpha}$, we know that $B(z_j, r)$ is contained in $B(0, |z_j| + r) \setminus B(0, |z_j| - r)$. Thus, we choose $r \leqslant |z_j|$ small enough such that

$$\frac{\sum_{i=1}^{n}(j_i - 1/2)}{\alpha} \leqslant (|z_j| - r)^2 \quad \text{and} \quad |z_j| + r \leqslant \sqrt{\frac{\sum_{i=1}^{n}(j_i + 1)}{\alpha}} \tag{2.13}$$

for all $j \neq 0$. We denote $\sqrt{x} = 0$ if $x < 0$. Let $\xi = \Bigl(\sqrt{(j_1 - \frac{1}{2})/\alpha}, \ldots, \sqrt{(j_n - \frac{1}{2})/\alpha}\Bigr)$ and $\eta = \bigl(\sqrt{(j_1 + 1)/\alpha}, \ldots, \sqrt{(j_n + 1)/\alpha}\bigr)$, then $B(z_j, r)$ is contained in $B(0, |\eta|) \setminus B(0, |\xi|)$ implying that the inequalities in (2.12) hold for $z \in B(z_j, r)$. We next see that there exists $r \leqslant \frac{1}{4}/\sqrt{\alpha(|j| + n)}$ such that (2.13) holds.

Making a change in (2.13), we have

$$r \leqslant \min\biggl\{\frac{n}{2\alpha(\sqrt{|j|/\alpha} + \sqrt{(|j| - n/2)/\alpha})}, \frac{n}{\alpha(\sqrt{|j|/\alpha} + \sqrt{(|j| + n)/\alpha})}\biggr\}.$$

It is easy to show that

$$\min\biggl\{\frac{n}{2\alpha(\sqrt{|j|/\alpha} + \sqrt{(|j| - n/2)/\alpha})}, \frac{n}{\alpha(\sqrt{|j|/\alpha} + \sqrt{(|j| + n)/\alpha})}\biggr\} \geqslant \frac{1}{4\sqrt{\alpha(|j| + n)}}.$$

Thus, we get

$$\begin{aligned} \widetilde{T_\mu^{(j)}}(w) &= \langle T_\mu^{(j)} k_w, k_w\rangle = \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} |\langle U_z e_j, k_w\rangle|^2 \,\mathrm{d}\mu(z) \\ &= \frac{\alpha^{|j|+n}}{\pi^n j!} \int_{\mathbb{C}^n} |(U_z \xi^j)(w)|^2 \mathrm{e}^{-\alpha|w|^2} \,\mathrm{d}\mu(z) \\ &= \frac{\alpha^{|j|+n}}{\pi^n j!} \int_{\mathbb{C}^n} |\varphi_z(w)^j|^2 |k_z(w)|^2 \mathrm{e}^{-\alpha|w|^2} \,\mathrm{d}\mu(z) \\ &= \frac{\alpha^{|j|+n}}{\pi^n j!} \int_{\mathbb{C}^n} |\varphi_w(z)^j|^2 \mathrm{e}^{-\alpha|\varphi_w(z)|^2} \,\mathrm{d}\mu(z) \\ &\geqslant \frac{\alpha^{|j|+n}}{\pi^n j!} \int_{B(\varphi_w(z_j), r)} |\varphi_w(z)^j|^2 \mathrm{e}^{-\alpha|\varphi_w(z)|^2} \,\mathrm{d}\mu(z) \\ &\geqslant \frac{\alpha^n}{2^n \pi^n \mathrm{e}^{|j|+n} j!} \mu(B(\varphi_w(z_j), r)) \end{aligned} \tag{2.14}$$

for any $r \leqslant \frac{1}{4} n/\sqrt{\alpha(|j| + n)}$.

Taking the supremum for $w \in \mathbb{C}^n$ and using $\{\varphi_w(z_j) \colon w \in \mathbb{C}^n\} = \mathbb{C}^n$ for any fix $z_j \in \mathbb{C}^n$, we get

$$\|T_\mu^{(j)}\| \geqslant \|\widetilde{T_\mu^{(j)}}\|_\infty \geqslant \frac{\alpha^n}{2^n \pi^n \mathrm{e}^{|j|+n} j!} \sup_\nu \mu(B(\nu, r)) \tag{2.15}$$

for any $r \leqslant \frac{1}{4}/\sqrt{\alpha(|j|+n)}$. Hence, let $r = \frac{1}{4}/\sqrt{\alpha(|j|+n)}$, make appropriate adjustments in Lemma 2.3 and Theorem 4.2 in [8], we get that μ is a Fock-Carleson measure. Furthermore, there exists a constant $M > 0$ such that

$$\|\widetilde{\mu}\|_\infty \leqslant M \sup_\nu \mu\Big(B\Big(\nu, \frac{1}{4\sqrt{\alpha(|j|+n)}}\Big)\Big) \leqslant M\|T_\mu^{(j)}\|. \tag{2.16}$$

This completes the proof. □

3. COMPACTNESS OF GENERALIZED TOEPLITZ OPERATORS

We say that a positive Borel measure μ on $\mathbb{C}^n$ is a vanishing Fock-Carleson measure if

$$\lim_{n\to\infty} \int_{\mathbb{C}^n} |f_n(z)\mathrm{e}^{-\alpha|z|^2/2}|^2 \, \mathrm{d}\mu(z) = 0,$$

where $\{f_n\}$ is a bounded sequence in F_α^2 that converges to 0 uniformly on compact subsets of the complex plane. A positive measure μ is a vanishing Fock-Carleson measure if and only if $\mu(B(z,r)) \to 0$ as $|z| \to \infty$, see [8], [19].

Lemma 3.1. *Let $f \in F_\alpha^2$ and $\varepsilon > 0$, and let there exist a $\delta = \delta(f,\varepsilon) > 0$. If $|z_1 - z_2| < \delta$, then $\|U_{z_1} f - U_{z_2} f\| < \varepsilon$.*

Proof. The proof is similar to the proof of Lemma 4.3 in [13]. Here we give a brief proof. Since the polynomials are dense in F_α^2, it is enough to assume that f is a polynomial. If $|z_1 - z_2| < \delta$, then $z_2 = \varphi_{z_1}(\eta)$ with $|\eta| < \delta$. By Lemma 2.2, we get

$$\begin{aligned}(I - U_{\varphi_{z_1}(\eta)} U_{z_1}) f(w) &= f(w) - V_t U_\eta f(w) = f(w) - U_\eta f(-w)t \\ &= f(w) - f(\eta + w) k_\eta(-w) t,\end{aligned}$$

where $t = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\alpha z_1 \overline{\eta})}$.

When $\eta \to 0$, we get that $t \to 1$ uniformly in z_1. So the above expression tends to 0 uniformly in z_1, w. Therefore,

$$\|U_{z_1} f - U_{z_2} f\| = \|(I - U_{\varphi_{z_1}(\eta)} U_{z_1}) f\| < \varepsilon$$

if $\eta \to 0$, that is, if $\delta \to 0$. □

Lemma 3.2. *If $a \in L^\infty(\mathbb{C}^n)$ has compact support on $\mathbb{C}^n$, then $T_a^{(j)}$ is compact on F_α^2.*

Proof. Let $f_n \to 0$ weakly in F^2_α, that is, f_n converges to 0 uniformly on each compact set of $\mathbb{C}^n$ and $\sup\limits_n \|f_n\| \leqslant M_0$. For any $l \in F^2_\alpha$, using Theorem 2.1, we have

$$\begin{aligned}
|\langle T^{(j)}_a f_n, l\rangle| &\leqslant \Big(\frac{\alpha}{\pi}\Big)^n \int_{\mathbb{C}^n} |\langle U_z f_n, e_j\rangle| |\langle U_z l, e_j\rangle| |a(z)| \,\mathrm{d}v(z) \\
&\leqslant \Big(\frac{\alpha}{\pi}\Big)^n \Big(\int_{\mathbb{C}^n} |\langle U_z f_n, e_j\rangle|^2 |a(z)| \,\mathrm{d}v(z)\Big)^{1/2} \Big(\int_{\mathbb{C}^n} |\langle U_z l, e_j\rangle|^2 |a(z)| \,\mathrm{d}v(z)\Big)^{1/2} \\
&\leqslant M\|a\|_\infty \|l\| \Big(\int_{\operatorname{supp} a} |\langle U_z f_n, e_j\rangle|^2 \,\mathrm{d}v(z)\Big)^{1/2} \\
&\leqslant M\|a\|_\infty \|l\| [v(\operatorname{supp} a)]^{1/2} \sup_{z\in \operatorname{supp} a} |\langle U_z f_n, e_j\rangle|.
\end{aligned}$$

Furthermore,

$$\|T^{(j)}_a f_n\| = \sup_{\|l\|=1} |\langle T^{(j)}_a f_n, l\rangle| \leqslant M\|a\|_\infty [v(\operatorname{supp} a)]^{1/2} \sup_{z\in \operatorname{supp} a} |\langle U_z f_n, e_j\rangle|. \tag{3.1}$$

Now, we just need to prove that $H_n(z) = \langle f_n, U_z e_j\rangle \to 0$ uniformly on any compact sets of $\mathbb{C}^n$. Lemma 3.1 tells us that the function $z \mapsto U_z e_j$ is uniformly continuous on compact sets of $\mathbb{C}^n$. Thus, by Cauchy-Schwarz inequality, we show that $H_n(z)$ are equicontinuous on compact sets. Using $f_n \to 0$ weakly in F^2_α again, then $H_n(z) \to 0$ pointwise. Arzela-Ascoli Theorem (see [12]) implies that $H_n(z) \to 0$ uniformly on compact sets. $\square$

We are now ready to prove the main result of this section.

Theorem 3.3. *Let μ be a positive Borel measure on $\mathbb{C}^n$. Then $T^{(j)}_\mu$ is compact on F^2_α if and only if μ is a vanishing Fock-Carleson measure.*

Proof. Suppose that $T^{(j)}_\mu$ is compact. Since $k_w \to 0$ weakly in F^2_α as $w \to \infty$ and $\|k_w\| = 1$, then

$$|\widetilde{T^{(j)}_\mu}(w)| = |\langle T^{(j)}_\mu k_w, k_w\rangle| \leqslant \|T^{(j)}_\mu k_w\| \to 0, \quad w \to \infty.$$

By (2.14), there are $z_j \in \mathbb{C}^n$ and $0 < r < \infty$ such that

$$\mu(B(\varphi_w(z_j), r)) \to 0, \quad w \to \infty.$$

For any fixed $z_j \in \mathbb{C}^n$, $\{\varphi_w(z_j)\colon w \in \mathbb{C}^n\} = \mathbb{C}^n$. Hence,

$$\mu(B(v, r)) \to 0, \quad v \to \infty.$$

Conversely, suppose that μ is a vanishing Fock-Carleson measure and let $0 < K < \infty$. By (2.10), we get

$$0 \leqslant T^{(j)}_\mu \leqslant M T^{(j)}_{\widetilde{\mu}} = M T^{(j)}_{\chi_K \widetilde{\mu}} + M T^{(j)}_{\chi_{(\mathbb{C}^n \setminus K)} \widetilde{\mu}}.$$

By Lemma 4.2, the first operator in the sum is compact, we get that $T^{(j)}_{\chi_K\widetilde{\mu}}$ is compact. There exists a constant $M_1 > 0$ such that $T^{(j)}_{\chi_{(\mathbb{C}^n\setminus K)}\widetilde{\mu}} \leqslant M_1\|\chi_{(\mathbb{C}^n\setminus K)}\widetilde{\mu}\|_\infty \to 0$ when $K \to \infty$. Hence, $T^{(j)}_\mu$ is compact. This completes the proof. □

4. Schatten class generalized Toeplitz operators

We are going to determine when a generalized Toeplitz operator $T^{(j)}_\mu$ on F^2_α belongs to the Schatten class S_p. This section is devoted to the situation $1 \leqslant p < \infty$. Background information about the Schatten classes S_p can be found in [18] for example.

If T is positive on F^2_α, then mimicking the proof of Theorem 6.4 in [18] we can show that

$$\operatorname{tr}(T) = \int_{\mathbb{C}^n} \langle TK_z, K_z\rangle \,\mathrm{d}\lambda_\alpha(z) = \Big(\frac{\alpha}{\pi}\Big)^n \int_{\mathbb{C}^n} \widetilde{T}(z)\,\mathrm{d}v(z). \tag{4.1}$$

In particular, S is in the trace class S_1 if and only if the integral above converges. We next give the main theorem of this section.

Theorem 4.1. *Suppose μ is a positive Borel measure on $\mathbb{C}^n$, $1 \leqslant p < \infty$. Then the following conditions are equivalent:*

(1) $T^{(j)}_\mu \in S_p$ *on* F^2_α;

(2) $\widetilde{T^{(j)}_\mu}(z) \in L^p(\mathbb{C}^n, \mathrm{d}v(z))$;

(3) *there exists some* $r > 0$ *such that* $\widehat{\mu}_{r,j}(z) \in L^p(\mathbb{C}^n, \mathrm{d}v(z))$.

Proof. (1) ⇒ (2). Suppose $T^{(j)}_\mu \in S_p$ on F^2_α. Since $T^{(j)}_\mu \geqslant 0$, by (4.1), we get

$$\|T^{(j)}_\mu\|^p_{S_p} = \operatorname{tr}((T^{(j)}_\mu)^p) = \int_{\mathbb{C}^n} \langle (T^{(j)}_\mu)^p K_z, K_z\rangle \,\mathrm{d}\lambda_\alpha(z) = \Big(\frac{\alpha}{\pi}\Big)^n \int_{\mathbb{C}^n} \langle (T^{(j)}_\mu)^p k_z, k_z\rangle \,\mathrm{d}v(z).$$

Since $1 \leqslant p < \infty$ and k_z is a unit vector in F^2_α, by Proposition 1.31 of [18], we get

$$\|T^{(j)}_\mu\|^p_{S_p} \geqslant \Big(\frac{\alpha}{\pi}\Big)^n \int_{\mathbb{C}^n} \langle T^{(j)}_\mu k_z, k_z\rangle^p \,\mathrm{d}v(z),$$

and then $\widetilde{T^{(j)}_\mu}(z) \in L^p(\mathbb{C}^n, \mathrm{d}v(z))$.

(2) ⇒ (3). By (2.14),

$$\widetilde{T^{(j)}_\mu}(z) = \frac{\alpha^{|j|+n}}{\pi^n j!} \int_{\mathbb{C}^n} |\varphi_z(w)^j|^2 \mathrm{e}^{-\alpha|\varphi_z(w)|^2} \,\mathrm{d}\mu(w) \geqslant \frac{\alpha^{|j|+n}}{\mathrm{e}^{\alpha r^2}\pi^n j!} \int_{B(z,r)} |\varphi_z(w)^j|^2 \,\mathrm{d}\mu(w)$$

and then we get

$$\widehat{\mu}_{r,j}(z) \in L^p(\mathbb{C}^n, \mathrm{d}v(z)).$$

In order to prove that (3) $\Rightarrow$ (1), we need some preparations.

Let $1 \leqslant p < \infty$, $\varphi \in L^p(\mathbb{C}^n, \mathrm{d}v)$. The generalized Toeplitz operator $T_\varphi^{(j)}$ on F_α^2 is defined as

$$T_\varphi^{(j)} = \int_{\mathbb{C}^n} U_z E_j U_z \varphi(z) \, \mathrm{d}v(z),$$

where the integral converges in the weak operator topology. $\square$

Lemma 4.2. *If $1 \leqslant p < \infty$, and if $\varphi \in L^p(\mathbb{C}^n, \mathrm{d}v)$, then $T_\varphi^{(j)} \in S_p$ on F_α^2.*

Proof. Note that this result is a similar case of Theorem 1 (d) in [5]. Here we omit the details of the proof. $\square$

Now we prove that (3) $\Rightarrow$ (1) in Theorem 4.1. Let $r > 0$ such that

$$\widehat{\mu}_{r,j}(z) \in L^p(\mathbb{C}^n, \mathrm{d}v(z)),$$

then by Lemma 4.2, $T_{\widehat{\mu}_{r,j}}^{(j)} \in S_p$. It is sufficient to show that there exists a positive constant M such that $T_\mu^{(j)} \leqslant M T_{\widehat{\mu}_{r,j}}^{(j)}$. In fact, for any $f \in F_\alpha^2$, by Fubini's theorem,

$$\begin{aligned}
\langle T_{\widehat{\mu}_{r,j}}^{(j)} f, f\rangle &= \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} |\langle U_z f, e_j\rangle|^2 \widehat{\mu}_{r,j}(z) \, \mathrm{d}v(z) \\
&= \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} |\langle U_z f, e_j\rangle|^2 \int_{\mathbb{C}^n} |\varphi_w(z)^j|^2 \chi_{B(w,r)}(z) \, \mathrm{d}\mu(w) \, \mathrm{d}v(z) \\
&= \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} \Bigl(\int_{B(w,r)} |\varphi_w(z)^j|^2 |\langle U_z f, e_j\rangle|^2 \, \mathrm{d}v(z)\Bigr) \, \mathrm{d}\mu(w).
\end{aligned}$$

For $w \in \mathbb{C}^n$ we denote $S_w := \{z \in \mathbb{C}^n \colon \frac{1}{2} r/\sqrt{n} < |z_i - w_i| < r/\sqrt{n},\ i = 1, \ldots, n\}$. Hence,

$$\langle T_{\widehat{\mu}_{r,j}}^{(j)} f, f\rangle \geqslant \Bigl(\frac{r^2}{4n}\Bigr)^{|j|} \Bigl(\frac{\alpha}{\pi}\Bigr)^n \int_{\mathbb{C}^n} \Bigl(\int_{S_w} |\langle U_z f, e_j\rangle|^2 \, \mathrm{d}v(z)\Bigr) \, \mathrm{d}\mu(w).$$

Next we just prove that the inequality

$$\int_{S_w} |\langle U_z f, e_j\rangle|^2 \, \mathrm{d}v(z) \geqslant M_{r,j} |\langle U_w f, e_j\rangle|^2 \tag{4.2}$$

holds for a positive constant $M_{r,j}$ depending only on r, j.

By (2.4) and (2.5), after simple calculations, we get

$$\int_{S_0} |\langle F, U_z e_j\rangle|^2 \, \mathrm{d}v(z) \geqslant |\langle F, e_j\rangle|^2 \int_{S_0} |\langle e_j, U_z e_j\rangle|^2 \, \mathrm{d}v(z). \tag{4.3}$$

Let $F(\xi) = (U_w f)(\xi)$, by (4.3), we get

$$\int_{S_0} |\langle U_w f, U_z e_j\rangle|^2 \,\mathrm{d}v(z) \geqslant |\langle U_w f, e_j\rangle|^2 \int_{S_0} |\langle e_j, U_z e_j\rangle|^2 \,\mathrm{d}v(z). \tag{4.4}$$

Using Lemma 3.1, the function $z \mapsto \langle e_j, U_z e_j\rangle$ is uniformly continuous and compact on $\mathbb{C}^n$. Note that $\langle e_j, U_0 e_j\rangle = 1$, then

$$\int_{S_0} |\langle e_j, U_z e_j\rangle|^2 \,\mathrm{d}v(z)$$

is a finite positive constant depending on r and j. On the other hand, note that $U_w U_z = U_{\varphi_w(z)} V_t$, where $t = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\alpha z\overline{w})}$, $V_t f(u) = t f(-u)$ for $f \in F^2_\alpha$. Consequently, $|\langle U_w f, U_z e_j\rangle| = |\langle f, U_{\varphi_w(z)} e_j\rangle|$, and the change of variable $\nu = \varphi_w(z)$ in the left of (4.4) yields

$$\int_{S_w} |\langle f, U_\nu e_j\rangle|^2 \,\mathrm{d}v(\nu) \geqslant |\langle U_w f, e_j\rangle|^2 \int_{S_0} |\langle e_j, U_z e_j\rangle|^2 \,\mathrm{d}v(z). \tag{4.5}$$

Hence, (4.2) holds, we complete the proof.

Let $1 < p < \infty$. For any fixed $z \in \mathbb{C}^n$, define the operator $U_z\colon\ F^p_\alpha \to F^p_\alpha$ such that

$$U_z f = (f \circ \varphi_z)\varphi_z' \quad \forall\, f \in F^p_\alpha.$$

Then U_z is bounded. It is easy to check that

$$U_z^* g = (g \circ \varphi_z)\varphi_z' \quad \forall\, g \in F^q_\alpha,$$

where $p^{-1} + q^{-1} = 1$.

Let S be a bounded operator on F^p_α, and let $S_z = U_z S U_z$. The Berezin transform of S is the function $\widetilde{S}$ defined on $\mathbb{C}^n$ such that

$$\widetilde{S}(z) = \langle S k_z, k_z\rangle,$$

where

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f\overline{g} \,\mathrm{d}\lambda_\alpha.$$

Let $E_j := e_j \otimes e_j$ be the rank one operator defined on F^p_α such that

$$E_j f = \langle f, e_j\rangle e_j, \quad f \in F^p_\alpha.$$

Let $a \in L^\infty(\mathbb{C}^n)$, and let $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$. The generalized Toeplitz operator $T^{(j)}_a$ on F^p_α is defined as

$$T^{(j)}_a := \int_{\mathbb{C}^n} U_z E_j U_z a(z) \,\mathrm{d}v(z),$$

where the integral converges in the weak operator topology.

Let $w \in F^2_\alpha$ and $f \in L^\infty(\mathbb{C}^n)$. Engliš in [6] characterized the boundedness of the localization operator $L_f^{(w)}$ on Fock space F^2_α by Bargmann transform. When $1 \leqslant p < \infty$, we cannot characterize the boundedness of operators $T_a^{(j)}$ induced by $a \in L^\infty(\mathbb{C}^n)$ by using Bargmann transform. So, we raise the following conjecture.

Conjecture 4.3. *Let $1 \leqslant p < \infty$, suppose $a \in L^\infty(\mathbb{C}^n)$. Then $T_a^{(j)}$ is bounded on F^p_α.*

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