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GREEN-LIOUVILLE APPROXIMATION AND CORRECT SOLVABILITY IN $L_p(\mathbb{R})$ OF THE GENERAL STURM-LIOUVILLE EQUATION

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Abstract. We consider the equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R},$$

where $f \in L_p(\mathbb{R})$, $p \in (1,\infty)$ and

$$r > 0, \quad \frac{1}{r} \in L_1^{\rm loc}(\mathbb{R}), \quad q \in L_1^{\rm loc}(\mathbb{R}).$$

For particular equations of this form, we suggest some methods for the study of the question on requirements to the functions r and q under which the above equation is correctly solvable in the space $L_p(\mathbb{R})$, $p \in (1,\infty)$.

Keywords: Green-Liouville approximation; correct solvability; general Sturm-Liouville equation

MSC 2020: 34B27, 34B24

1. Introduction

In the present paper, we consider the equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $f \in L_p(\mathbb{R})$, $p \in (1,\infty)$ and

$$r > 0, \quad \frac{1}{r} \in L_1^{\rm loc}(\mathbb{R}), \quad q \in L_1^{\rm loc}(\mathbb{R}). \tag{1.2}$$

We continue the study (see [3]–[5], [7]–[10]) of correct solvability of equation (1.1) in $L_p(\mathbb{R})$, $p \in (1,\infty)$. We now give precise definitions and conventions.

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Definition 1.1 ([5], [7])**.** We say that equation (1.1) is correctly solvable in the space $L_p(\mathbb{R})$, $p \in (1,\infty)$, if the following requirements hold:

(I) for any function $f \in L_p(\mathbb{R})$, equation (1.1) has a unique solution $y \in L_p(\mathbb{R})$;

(II) there is a constant $c(p) \in (0,\infty)$ such that regardless of the choice of the function $f \in L_p(\mathbb{R})$, the solution $y \in L_p(\mathbb{R})$ of (1.1) satisfies the inequality

$$\|y\|_{L_p(\mathbb{R})} \leqslant c(p)\|f\|_{L_p(\mathbb{R})}. \tag{1.3}$$

Here and in the sequel, a solution of (1.1) is understood to be an absolutely continuous function y together with ry' that satisfies equality (1.1) almost everywhere in $\mathbb{R}$. Note that below, for the sake of brevity, the problem on the validity of the requirement of Definition 1.1 is referred to as "problem (I)–(II)" or "the question on (I)–(II)". If requirements (I)–(II) are satisfied, we say that "problem (I)–(II) for (1.1) is solvable". By the letter c we denote positive constants, which are not essential for exposition and may differ even within a single chain of calculations. The symbols $\|\cdot\|_{L_p(\mathbb{R})}$ are shortened throughout to $\|\cdot\|_p$.

Let us now briefly describe our approach to problem (I)–(II). To this end, consider the homogeneous equation corresponding to (1.1):

$$(r(x)z'(x))' = q(x)z(x), \quad x \in \mathbb{R}. \tag{1.4}$$

Definition 1.2 ([3])**.** We say that a fundamental system of solutions (FSS) $\{u(x), v(x)\}$, $x \in \mathbb{R}$, of equation (1.4) is a principal fundamental system (PFSS) if the solutions u and v of (1.4) satisfy the relations

$$u(x) > 0, \quad v(x) > 0, \quad u'(x) \leqslant 0, \quad v'(x) \geqslant 0 \quad \text{for } x \in \mathbb{R}, \tag{1.5}$$

$$r(x)[v'(x)u(x) - u'(x)v(x)] = 1 \quad \text{for } x \in \mathbb{R}, \tag{1.6}$$

$$\lim_{x\to-\infty} \frac{v(x)}{u(x)} = \lim_{x\to\infty} \frac{u(x)}{v(x)} = 0. \tag{1.7}$$

In the sequel, a PFSS of (1.4) is denoted by $\{u, v\}$. In addition to (1.2), another standing assumption in this paper is the existence of a PFSS. (See Section 2 for the discussion on the validity of this assumption.) We emphasize that our standing restrictions are assumed to hold throughout the sequel, and we do not mention them in the statements.

Note that a PFSS is determined uniquely, up to constant mutually inverse factors at u and v, see Section 2. This implies that the function

$$\varrho(x) \overset{\text{def}}{=} u(x)v(x), \quad x \in \mathbb{R}, \tag{1.8}$$

is well-defined and is an implicit function in the coefficients of r and q of equation (1.1). One can show (see Section 2) that the converse is also true: a PFSS $\{u, v\}$

can be expressed via the functions r and ϱ (the Davies-Harrell formulas, see Section 2). Therefore, the function ϱ is called the *function generating* the PFSS $\{u, v\}$ (or just the function generating $\{u, v\}$).

Note the following important a priori equalities related to the function ϱ:

$$\int_{-\infty}^{0} \frac{\mathrm{d}t}{r(t)\varrho(t)} = \int_{0}^{\infty} \frac{\mathrm{d}t}{r(t)\varrho(t)} = \infty, \tag{1.9}$$

see [3], [10]. In particular, they allow one to introduce an auxiliary function $s(x)$, $x \in \mathbb{R}$, whose role will be clarified below, see [10]. To this end, fix $x \in \mathbb{R}$ and consider the equation in $s \geqslant 0$,

$$\int_{x-s}^{x+s} \frac{\mathrm{d}t}{r(t)\varrho(t)} = 1. \tag{1.10}$$

For $x \in \mathbb{R}$, (1.10) has a unique, finite, positive solution $s(x)$ in $s \geqslant 0$, see Section 2.

We now can formulate a criterion for the solvability of problem (I)–(II).

Theorem 1.3 ([10]). *Let* $p \in (1, \infty)$. *Equation* (1.1) *is correctly solvable in the space* L_p *if and only if* $\mathcal{D} < \infty$. *Here*

$$\mathcal{D} = \sup_{x\in\mathbb{R}} (\varrho(x)s(x)). \tag{1.11}$$

Remark 1.4. Note that the statement of Theorem 1.3 in [10] contains the condition $q \geqslant 0$. This condition originally appeared in [3] and was used only in the proof of existence of the PFSS of the equation (1.4). Since in [10] the existence of the PFSS of (1.4) is a priori required, then, in fact, already in [10] the condition $q \geqslant 0$ is redundant. The authors are grateful to the referee who drew their attention to this matter.

Now we comment on the applicability of this assertion. Clearly, the general criterion (1.11) is not applicable to particular equations of the form (1.1) because the functions ϱ and s are in general not computable. However, if for all $x \in \mathbb{R}$ one can find their two-sided sharp by order estimates then, by Theorem 1.3, solving problem (I)–(II) becomes an obvious task. This implies that when studying the question on (I)–(II) for the particular equation (1.1), the main problem is the search for estimates of the aforementioned type for the functions ϱ and s. To solve this problem, we first consider the function ϱ. The question concerning its estimates was studied in [3], [4], [8] under the condition

$$\lim_{|d|\to\infty} \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-d}^{x} q(t)\,\mathrm{d}t = \infty. \tag{1.12}$$

The assertions suggested in those papers are complemented here by a method of obtaining inequalities for ϱ, which is based on the classical Liouville-Green approximations for the FSS of equation (1.4) (see [14]), on the study of the asymptotics of a FSS of this equation (the Hartman-Wintner problem, see [2], [12] and Section 3 below), and on some results of [10]. Finally, the new inequalities for s,

$$c^{-1}r(x)\varrho(x) \leqslant s(x) \leqslant cr(x)\varrho(x), \quad x \in \mathbb{R} \tag{1.13}$$

(see Section 3), together with the estimates for ϱ guarantee the solution of problem (I)–(II).

We describe the structure of the paper. Section 2 contains preliminaries needed for the proofs. The results and comments are presented in Section 3. Section 4 is devoted to the proofs, and Section 5 contains examples of applications.

2. PRELIMINARIES

Below we present the definitions and facts used in the proofs.

2.1. PFSS and properties of the functions ϱ and s.

Theorem 2.1 ([3]). *Suppose that*

$$q \geqslant 0, \quad \int_{-\infty}^{0} q(t)\,\mathrm{d}t > 0, \quad \int_{0}^{\infty} q(t)\,\mathrm{d}t > 0. \tag{2.1}$$

Then equation (1.4) *has a PFSS.*

Theorem 2.2 ([3], [10]). *Suppose that equation* (1.4) *has a PFSS* $\{u, v\}$. *Then any other PFSS* $\{\tilde{u}, \tilde{v}\}$ *of this equation is of the form*

$$\tilde{u}(x) = \alpha u(x), \quad \tilde{v}(x) = \alpha^{-1}v(x), \quad x \in \mathbb{R}. \tag{2.2}$$

Here α *is an arbitrary fixed constant,* $\alpha \in (0, \infty)$.

Corollary 2.3 ([10]). *If equation* (1.4) *has a PFSS* $\{u, v\}$, *then the function*

$$\varrho(x) = u(x)v(x), \quad x \in \mathbb{R} \tag{2.3}$$

is well-defined (i.e., does not depend on the choice of a PFSS of equation (1.4)*).*

We have the following converse assertion to Corollary 2.3.

Corollary 2.4 ([11], [12], and [16], Section 19.53). *A PFSS of equation* (1.4) *admits the Davies-Harrell representation (see* (2.3)*):*

$$u(x) = \sqrt{\varrho(x)}\exp\biggl(-\frac{1}{2}\int_{x_0}^{x}\frac{\mathrm{d}\xi}{r(\xi)\varrho(\xi)}\biggr), \quad v(x) = \sqrt{\varrho(x)}\exp\biggl(\frac{1}{2}\int_{x_0}^{x}\frac{\mathrm{d}\xi}{r(\xi)\varrho(\xi)}\biggr). \tag{2.4}$$

Here $x \in \mathbb{R}, x_0$ *is the unique root of the equation* $u(x) = v(x)$, $x \in \mathbb{R}$.

Remark 2.5. Formulas (2.4) were obtained (in a special case) by Abel (see [16], Section 19.53); under conditions (1.2) and $r \equiv 1$, they were proven in [11]; Corollary 2.4 was obtained in [3].

Corollary 2.6 ([4]). *We have the following relations for a PFSS and the function* ϱ*:*

$$\frac{u'(x)}{u(x)} = -\frac{1 - r(x)\varrho'(x)}{2r(x)\varrho(x)}, \quad \frac{v'(x)}{v(x)} = \frac{1 + r(x)\varrho'(x)}{2r(x)\varrho(x)}, \quad x \in \mathbb{R}, \tag{2.5}$$

$$r(x)|\varrho'(x)| \leqslant 1, \quad x \in \mathbb{R}. \tag{2.6}$$

Corollary 2.7 ([10]). *We have the following relations for a PFSS* $\{u, v\}$ *and the function* ϱ *(see* (2.3)*):*

$$u(x) = v(x)\int_x^{\infty} \frac{\mathrm{d}t}{r(t)v^2(t)}, \quad v(x) = u(x)\int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)u^2(t)}, \quad x \in \mathbb{R}, \tag{2.7}$$

$$\varrho(x) = u(x)v(x) = v^2(x)\int_x^{\infty} \frac{\mathrm{d}t}{r(t)v^2(t)} = u^2(x)\int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)u^2(t)}, \quad x \in \mathbb{R}, \tag{2.8}$$

$$\int_{-\infty}^{0} \frac{\mathrm{d}t}{r(t)u^2(t)} < \infty, \quad \int_0^{\infty} \frac{\mathrm{d}t}{r(t)v^2(t)} < \infty, \tag{2.9}$$

$$\int_{-\infty}^{0} \frac{\mathrm{d}t}{r(t)v^2(t)} = \int_0^{\infty} \frac{\mathrm{d}t}{r(t)u^2(t)} = \infty. \tag{2.10}$$

Lemma 2.8 ([10]). *For any* $x \in \mathbb{R}$, *there is a unique solution for* $s \geqslant 0$ *of equation* (1.10), *denote it by* $s(x)$, $x \in \mathbb{R}$. *The function* s *is positive, continuous, and almost everywhere differentiable in* $\mathbb{R}$. *In addition, we have the relations*

$$|s(x+t) - s(x)| \leqslant |t| \quad \text{for } |t| \leqslant s(x), \ x \in \mathbb{R}, \tag{2.11}$$

$$|s'(x)| < 1 \quad \text{for almost all } x \in \mathbb{R}, \tag{2.12}$$

$$\lim_{x\to-\infty}(x + s(x)) = -\infty, \quad \lim_{x\to\infty}(x - s(x)) = \infty, \tag{2.13}$$

$$0 < s(x) \leqslant |x| \quad \text{for } |x| \gg 1. \tag{2.14}$$

Finally, there is a constant $c \in [1, \infty)$ *such that for all* $x \in \mathbb{R}$ *we have the inequality*

$$s(x) \leqslant c(1 + |x|). \tag{2.15}$$

Lemma 2.9 ([10]). *For* $t \in [x - s(x), x + s(x)]$, $x \in \mathbb{R}$, *we have the inequalities*

$$\mathrm{e}^{-1}\varrho(x) \leqslant \varrho(t) \leqslant \mathrm{e}\varrho(x), \quad \mathrm{e} = \exp(1), \tag{2.16}$$

$$\mathrm{e}^{-1}u(x) \leqslant u(t) \leqslant \mathrm{e}u(x), \quad \mathrm{e}^{-1}v(x) \leqslant v(t) \leqslant \mathrm{e}v(x). \tag{2.17}$$

2.2. The Hartman-Wintner problem. In this section, we consider the equations

$$(r(x)y'(x))' = q(x)y(x), \quad x \in \mathbb{R}, \tag{2.18}$$

$$(r(x)z(x))' = g(x)z(x), \quad x \in \mathbb{R}, \tag{2.19}$$

where the functions $r(x)$, $q(x)$ and $g(x)$ are real and continuous for $x \in \mathbb{R}$ and, in addition, $r(x) > 0$ for $x \in \mathbb{R}$. We also assume that equation (2.19) has a PFSS $\{u_1, v_1\}$.

Definition 2.10 ([2], [12]). We say that for equations (2.18) and (2.19) the Hartman-Wintner problem is solvable as $x \to \infty$ or as $x \to -\infty$ if there exists a FSS $\{\hat{u}(x), \hat{v}(x)\}$, $x \in [0, \infty)$ or $\{\tilde{u}(x), \tilde{v}(x)\}$, $x \in (-\infty, 0]$), respectively, of equation (2.18) such that

$$\lim_{x\to\infty} \frac{\hat{u}(x)}{u_1(x)} = \lim_{x\to\infty} \frac{\hat{v}(x)}{v_1(x)} = 1 \quad \text{or} \quad \lim_{x\to-\infty} \frac{\tilde{u}(x)}{u_1(x)} = \lim_{x\to-\infty} \frac{\tilde{v}(x)}{v_1(x)} = 1, \tag{2.20}$$

$$\frac{\hat{u}'(x)}{\hat{u}(x)} - \frac{u_1'(x)}{u_1(x)} = o\Big(\frac{1}{r(x)\varrho_1(x)}\Big) \text{ as } x \to \infty \quad \text{or} \tag{2.21}$$
$$\frac{\tilde{u}'(x)}{\tilde{u}(x)} - \frac{u_1'(x)}{u_1(x)} = o\Big(\frac{1}{r(x)\varrho_1(x)}\Big) \text{ as } x \to -\infty,$$

$$\frac{\hat{v}'(x)}{\hat{v}(x)} - \frac{v_1'(x)}{v_1(x)} = o\Big(\frac{1}{r(x)\varrho_1(x)}\Big) \text{ as } x \to \infty \quad \text{or} \tag{2.22}$$
$$\frac{\tilde{v}'(x)}{\tilde{v}(x)} - \frac{v_1'(x)}{v_1(x)} = o\Big(\frac{1}{r(x)\varrho_1(x)}\Big) \text{ as } x \to -\infty, \text{ respectively.}$$

Remark 2.11. Note that taking into account the goals of this paper, we intentionally made the statement of the Hartman-Wintner problem narrower compared to the original statement, see [2], [12]. For the sake of brevity, in the sequel, we refer to this problem as "problem (2.20)–(2.22)" for equations (2.18) and (2.19).

Put

$$\varrho_1(x) = u_1(x)v_1(x), \quad x \in \mathbb{R}, \quad \text{see (2.19)}, \tag{2.23}$$

$$(\Delta q)(x) = q(x) - g(x), \quad x \in \mathbb{R}, \tag{2.24}$$

$$I^{(-)}(x) = \int_{-\infty}^{x} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t, \quad x \in (-\infty, 0], \tag{2.25}$$

$$I^{(+)}(x) = \int_{x}^{\infty} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t, \quad x \in [0, \infty). \tag{2.26}$$

If the integrals $I^{(-)}(x)$, $x \leqslant 0$, and $I^{(+)}(x)$, $x \leqslant 0$, converge, at least conditionally, then we put

$$K^{(-)} = \int_{-\infty}^{0} \frac{I^{(-)}(x)^2\,\mathrm{d}x}{r(x)\varrho_1(x)}, \quad K^{(+)} = \int_{0}^{\infty} \frac{I^{(+)}(x)^2\,\mathrm{d}x}{r(x)\varrho_1(x)}. \tag{2.27}$$

Theorem 2.12 ([12], see also [2]). *If the integral $I^{(+)}(0)$ or $I^{(-)}(0)$ absolutely converges, then problem* (2.20)–(2.22) *for equations* (2.18) *and* (2.19) *is solvable as $x \to \infty$ or as $x \to -\infty$, respectively.*

Theorem 2.13 ([2]). *If the integral $I^{(+)}(0)$ or $I^{(-)}(0)$ converges at least conditionally and $K^{(+)} < \infty$ or $K^{(-)} < \infty$, then problem* (2.20)–(2.22) *for equations* (2.18)–(2.19) *is solvable as $x \to \infty$ or as $x \to -\infty$, respectively.*

Remark 2.14. Here is a useful consequence of Definition 2.10 which, it seems, has never been mentioned previously.

Corollary 2.15. *Suppose that problem* (2.20)–(2.22) *for equations* (2.18)–(2.19) *is solvable as $|x| \to \infty$. Suppose that the following inequality holds (cf.* (2.6)*):*

$$\sup_{x\in\mathbb{R}} |r(x)\varrho_1'(x)| < 1, \quad \varrho_1(x) = u_1(x)v_1(x), \quad x \in \mathbb{R}. \tag{2.28}$$

Then, together with (2.20), *we have the equalities*

$$\lim_{x\to\infty}\frac{\hat{u}'(x)}{u_1'(x)} = \lim_{x\to\infty}\frac{\hat{v}'(x)}{v_1'(x)} = 1 \quad \Big(\lim_{x\to-\infty}\frac{\tilde{u}'(x)}{u_1'(x)} = \lim_{x\to\infty}\frac{\tilde{v}'(x)}{v_1'(x)} = 1\Big). \tag{2.29}$$

Proof. All equalities (2.29) are checked in a similar way; therefore, we only consider the first one. Below $\varepsilon_1(x)$, $x \in [0,\infty)$, is a function tending to 0 as $x \to \infty$, according to Landau's definition of the symbol $o(\cdot)$. Thus, from (2.21), (2.22), (2.5), we obtain the equalities:

$$\begin{aligned}\Big|\frac{\hat{u}'(x)}{u_1'(x)} - \frac{\hat{u}(x)}{u_1(x)}\Big| &= \Big|\frac{\hat{u}(x)}{u_1'(x)}\Big|\Big|\frac{\hat{u}'(x)}{\hat{u}(x)} - \frac{u_1'(x)}{u_1(x)}\Big| = \frac{\hat{u}(x)}{u_1(x)}\Big|\frac{u_1(x)}{u_1'(x)}\Big|\frac{\varepsilon(x)}{r(x)\varrho_1(x)} \\ &= \frac{\hat{u}(x)}{u_1(x)}\frac{2r(x)\varrho_1(x)}{|1-r(x)\varrho_1'(x)|}\frac{\varepsilon_1(x)}{r(x)\varrho_1(x)} = \frac{\hat{u}(x)}{u_1(x)}\frac{\varepsilon_1(x)}{|1-r(x)\varrho_1'(x)|} \Rightarrow (2.29).\end{aligned}$$

□

Remark 2.16. Note that a criterion for the validity of condition (2.28) was obtained in [4].

2.3. Otelbaev's coverings of the real axis.

Definition 2.17 ([3], [6], [13]). Given $x \in \mathbb{R}$, a positive continuous function $\varkappa(t)$, $t \in \mathbb{R}$, and a sequence $\{t_n\}_{n\in N'}$, $N' = \{\pm1, \pm2, \pm\ldots\}$; consider the segments

$$\Delta_n = [\Delta_n^{(-)}, \Delta_n^{(+)}], \quad \Delta_n^{(\pm)} = t_n \pm \varkappa(t_n), \quad n \in N'.$$

We say that the sequence of segments $\{\Delta_n\}_{n=1}^{\infty}$ or $\{\Delta_n\}_{n=-\infty}^{-1}$ forms an $\mathbb{R}(x,\varkappa)$-covering of the semi-axis $[x,\infty)$ or $(-\infty,x]$, respectively, if the following conditions hold:

(1) $\Delta_n^{(+)} = \Delta_{n+1}^{(-)}$ for $n \geqslant 1$ or $\Delta_{n-1}^{(+)} = \Delta_n^{(-)}$ for $n \leqslant -1$;
(2) $\Delta_1^{(-)} = x$, $(\Delta_{-1}^{(+)} = x)$;
(3) $\bigcup\limits_{n\geqslant 1} \Delta_n = [x,\infty)$, $\Big(\bigcup\limits_{n\leqslant -1} \Delta_n = (-\infty,x]\Big)$.

Lemma 2.18 ([6], [13]). *Suppose that a positive continuous function $\varkappa(t)$, $t \in \mathbb{R}$, satisfies the conditions*

$$\lim_{t\to\infty}(t-\varkappa(t)) = \infty \quad \textit{or} \quad \lim_{t\to-\infty}(t+\varkappa(t)) = -\infty, \ \textit{respectively.} \tag{2.30}$$

Then for any $x \in \mathbb{R}$, there exists an $\mathbb{R}(x,\varkappa)$-covering of $[x,\infty)$ or $(-\infty,x]$, respectively.

2.4. Equivalent problems (I)–(II).

Definition 2.19 ([10]). Let functions $f_1(x)$ and $f_2(x)$ be defined, continuous and positive in $x \in (a,b)$, $-\infty \leqslant a < b < \infty$. We say that they are weakly equivalent for $x \in (a,b)$ (and write $f_1(x) \asymp f_1(x)$, $x \in (a,b)$) if there is a constant $c \in [1,\infty)$ such that the inequalities

$$c^{-1} f_1(x) \leqslant f_2(x) \leqslant c f_1(x) \tag{2.31}$$

hold for $x \in (a,b)$.

Below, together with equation (1.1), we consider the equation

$$-(r(x)y'(x))' + g(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{2.32}$$

where $f \in L_p$, $p \in (1,\infty)$, and

$$r > 0, \quad \frac{1}{r} \in L_1^{\mathrm{loc}}(\mathbb{R}), \quad g \in L_1^{\mathrm{loc}}(\mathbb{R}). \tag{2.33}$$

We also assume (without further mention in the statements) that the homogeneous equation

$$(r(x)z'(x))' = g(x)z(x), \quad x \in \mathbb{R}, \tag{2.34}$$

corresponding to equation (2.32) has a PFSS $\{u_1, v_1\}$. We emphasize that this is our standing assumption in this section.

Definition 2.20 ([10]). We say that problems (I)–(II) for equations (1.1) and (2.32) are (or are not) equivalent if they are (or are not) solvable together.

Theorem 2.21 ([10]). *Let ϱ and ϱ_1 be functions generating PFSS of equations* (1.4) *and* (2.34)*, respectively. If*

$$\varrho(x) \asymp \varrho_1(x), \quad x \in \mathbb{R}, \tag{2.35}$$

then problems (I)–(II) *for equations* (1.1) *and* (2.32) *are equivalent.*

Theorem 2.22 ([10]). *Let $r(x)$, $q(x)$ and $g(x)$ be continuous functions for $x \in \mathbb{R}$. Assume that $r(x) > 0$ for $x \in \mathbb{R}$. If the conditions*

(1) *equations* (1.4) *and* (2.34) *have PFSS $\{u, v\}$ and $\{u_1, v_1\}$, respectively;*
(2) *problems* (2.20)–(2.22) *for equations* (1.4) *and* (2.34) *are solvable as $x \to -\infty$ and $x \to \infty$;*

hold then the functions ϱ and ϱ_1, generating PFSS of equations (1.4) *and* (2.34)*, are weakly equivalent for $x \in \mathbb{R}$, see Definition* 2.19.

3. RESULTS AND COMMENTS

Recall that our standing requirements (the validity of conditions (1.2) and (2.33)), the existence of PFSS $\{u, v\}$ and $\{u_1, v_1\}$ of equations (1.4) and (2.34), and the conditions underlying the study of the Hartman-Wintner problem (see Section 2.2) are assumed to be satisfied throughout the sequel and are not mentioned in the statements.

Note also that all assertions stated below (except for Theorems 3.3, 3.5 and Corollary 3.4) rely on Theorem 2.21 and applications of relation (2.35), see the proofs of Theorems 3.1 and 3.7 in Section 4.

Theorem 3.1. *Suppose that the integrals*

$$\int_{-\infty}^{0} (q(t) - g(t))\varrho_1(t)\,\mathrm{d}t, \quad \int_{0}^{\infty} (q(t) - g(t))\varrho_1(t)\,\mathrm{d}t \tag{3.1}$$

absolutely converge. Then problems (I)–(II) *for equations* (1.1) *and* (2.32) *are equivalent. Here ϱ_1 is the function that generates the PFSS of equation* (2.34).

Remark 3.2. A special feature of Theorem 3.1 is that it has a minimal amount of requirements (relative to the framework of the present paper) to equations (1.4) and (2.34). Unfortunately, absolute convergence of integrals (3.1) is a too strong restriction, preventing wide application of Theorem 3.1, see Section 5. Therefore, our next goal consists of replacing the condition of absolute convergence of integrals (3.1) by a weaker restriction, i.e., conditional convergence, see Theorem 3.7 below.

Towards this end, we need two auxiliary theorems, on estimates (1.13) and on the relationship between the Liouville-Green approximations (see [14]) and the existence of a PFSS of equation (2.34). Note that each of these assertions is also interesting in its own right.

Theorem 3.3. *Let* ϱ *be a function generating a PFSS of equation* (1.4), *see* (1.8). *Suppose that there exist* $a \in [1, \infty)$ *and a continuously differentiable function* $\mu(x)$, $x \in \mathbb{R}$, *such that*

$$a^{-1}\mu(x) \leqslant r(x)\varrho(x) \leqslant a\mu(x), \quad x \in \mathbb{R}, \tag{3.2}$$

$$\lim_{|x|\to\infty} \mu'(x) = 0. \tag{3.3}$$

Then there exists a constant $c(a) \in [1, \infty)$ *such that the inequalities*

$$c(a)^{-1} r(x)\varrho(x) \leqslant s(x) \leqslant c(a) r(x)\varrho(x), \quad x \in \mathbb{R}, \tag{3.4}$$

hold, see Lemma 2.8.

Corollary 3.4. *Let* ϱ *be a function generating a PFSS of equation* (1.4) *(see* (1.8)*), and let* r *(see* (1.4)*) be continuously differentiable for all* $x \in \mathbb{R}$. *If*

$$(r(x)\varrho(x))' \to 0 \quad \text{as } |x| \to \infty, \tag{3.5}$$

then $s(x) \asymp r(x)\varrho(x)$, $x \in \mathbb{R}$, *see Definition* 2.19. *In particular, there exists* $\nu_0 \gg 1$ *such that*

$$4^{-1} r(x)\varrho(x) \leqslant s(x) \leqslant \frac{5}{8} r(x)\varrho(x) \quad \text{for } |x| \geqslant \nu_0. \tag{3.6}$$

Theorem 3.5. *Let* $r(x)$ *and* $q_1(x)$ *be twice continuously differentiable positive functions for* $x \in \mathbb{R}$, *and let* x_0 *be a point in* $\mathbb{R}$. *Put*

$$\alpha(x) = (4r(x)q_1(x))^{-1/4}, \qquad \beta(x) = \int_{x_0}^{x} \sqrt{\frac{q_1(t)}{r(t)}}\,\mathrm{d}t, \quad x \in \mathbb{R}, \tag{3.7}$$

$$u_1(x) = \alpha(x)\exp(-\beta(x)), \quad v_1(x) = \alpha(x)\exp(\beta(x)), \quad x \in \mathbb{R}. \tag{3.8}$$

Functions (3.8) *form FSS of the equation*

$$(r(x)z'(x))' = g(x)z(x), \quad \text{where } g(x) = q_1(x) + \frac{(r(x)\alpha'(x))'}{\alpha(x)}, \ x \in \mathbb{R}, \tag{3.9}$$

and we have the equality

$$r(x)(v_1'(x)u_1(x) - u_1'(x)v_1(x)) = 1, \quad x \in \mathbb{R}. \tag{3.10}$$

If, in addition, we have the relations

$$|(r(x)q_1(x))'| \leqslant 4\sqrt{r(x)q_1(x)^3}, \quad x \in \mathbb{R}, \tag{3.11}$$

$$\int_{-\infty}^{x_0} \sqrt{\frac{q_1(t)}{r(t)}}\,\mathrm{d}t = \int_{x_0}^{\infty} \sqrt{\frac{q_1(t)}{r(t)}}\,\mathrm{d}t = \infty, \tag{3.12}$$

then the FSS $\{u_1, v_1\}$ *of equation* (3.9) *is a PFSS with generating function*

$$\varrho_1(x) = \alpha^2(x) = \frac{1}{2\sqrt{r(x)q_1(x)}}, \quad x \in \mathbb{R}. \tag{3.13}$$

Corollary 3.6. *Suppose that all conditions of Theorem* 3.5 *hold. Then for any* $p \in [1, \infty)$, *problem* (I)–(II) *for the equation (see* (3.9)*)*

$$-(r(x)y'(x))' + g(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{3.14}$$

is solvable if $\mathcal{D}_1 > 0$. *Here*

$$\mathcal{D}_1 = \inf_{x\in\mathbb{R}} (q_1(x)). \tag{3.15}$$

To study the question posed in Remark 3.2, consider equations (3.14) and (3.16):

$$-(r(x)y'(x))' + (q_1(x) + \theta(x))y(x) = f(x), \quad x \in \mathbb{R}, \tag{3.16}$$

under the following assumptions:

(1) the functions r and q_1 satisfy all the conditions of Theorem 3.5;

(2) we have

$$\theta(x) \text{ is continuous for } x \in \mathbb{R}; \tag{3.17}$$

(3) the equation

$$(r(x)z'(x))' = (q_1(x) + \theta(x))z(x), \quad x \in \mathbb{R} \tag{3.18}$$

has a PFSS;

(4) we have the equality (see (3.13))

$$\lim_{|x|\to\infty} \mu_1'(x) = 0, \quad \text{where } \mu_1(x) = r(x)\varrho_1(x) = \frac{1}{2}\sqrt{\frac{r(x)}{q_1(x)}}, \ x \in \mathbb{R}. \tag{3.19}$$

To formulate the fifth assumption, we need some more notation. Namely, $s_1(x)$, $x \in \mathbb{R}$, is the function similar to s constructed from the functions r and ϱ_1 (see (3.13));

$$\widehat{\Delta}(x) = [\widehat{\Delta}^{(-)}(x), \widehat{\Delta}^{(+)}(x)], \quad \widehat{\Delta}^{(\pm)} = x \pm s_1(x), \quad x \in \mathbb{R}, \tag{3.20}$$

$$\omega(x) = [\omega^{(-)}(x), \omega^{(+)}(x)], \quad \omega^{(\pm)} = x \pm \frac{5}{8}\mu_1(x), \quad x \in \mathbb{R} \tag{3.21}$$

(see (3.19)); $\tau(x)$, $x \in \mathbb{R}$, is any positive continuously differentiable function which for all $|x| \geqslant \nu_0$ (ν_0 is the point from Corollary 3.4) satisfies the inequality (see (3.7), (3.17))

$$\frac{1}{r(x)} \sup_{\alpha,\beta \in \omega(x)} \left| \int_\alpha^\beta \Big(\theta(t) - \frac{(r(t)\alpha'(t))'}{\alpha(t)}\Big)\,\mathrm{d}t \right| \leqslant \tau(x), \quad |x| \geqslant \nu_0. \tag{3.22}$$

We can now formulate assumption (5):

(5) one can choose $\tau(x)$, $x \in \mathbb{R}$, such that together with (3.22) we have the equality

$$\lim_{|x|\to\infty} \delta(x) = 0, \quad \text{where } \delta(x) = \frac{|\tau'(x)|}{\tau(x)}\mu_1(x). \tag{3.23}$$

Theorem 3.7. *Suppose that* (1)–(5) *hold and that the integrals*

$$P_1^{(-)} = \int_{-\infty}^0 \tau(x)\,\mathrm{d}x, \quad P_1^{(+)} = \int_0^\infty \tau(x)\,\mathrm{d}x,$$

$$P_2^{(-)} = \int_{-\infty}^0 \frac{1}{\mu_1(x)}\Big(\int_{-\infty}^x \tau(t)\,\mathrm{d}t\Big)^2\mathrm{d}x, \quad P_2^{(+)} = \int_0^\infty \frac{1}{\mu_1(x)}\Big(\int_x^\infty \tau(t)\,\mathrm{d}t\Big)^2\mathrm{d}x$$

converge. Then problem (I)–(II) *for equations* (3.14) *and* (3.16) *is equivalent.*

Remark 3.8. Note that although the assumptions (1)–(5) are cumbersome, usually one has no difficulties when applying Theorem 3.7. Indeed requirements (1), (2) and (3) only fix the studied set-up; condition (4) makes the choice of a model equation (3.9) more precise by shrinking it, and the whole checking boils down to the test (3.19). The only essential requirement is condition (3.22), i.e., the a priori choice of a function $\tau(x)$, $x \in \mathbb{R}$, but this can be done using elementary estimates, see Section 5.

Checking (3.23), which also shrinks the choice of the function $\tau(x)$, $x \in \mathbb{R}$, is not problematic either. Finally, the convergence (divergence) of the integrals $P_1^{(-)}$ and $P_2^{(+)}$ is established by standard tools of calculus.

4. Proofs

Proof of Theorem 3.1.

Lemma 4.1. *Let* $x_0 \in \mathbb{R}$ *be such that*

$$\int_{x_0}^{\infty} |q(t) - g(t)|\varrho_1(t)\,\mathrm{d}t < \frac{1}{2}. \tag{4.1}$$

Denote by $C[x_0, \infty)$ *the Banach space of continuous and bounded functions* $\beta(x)$, $x \in [x_0, \infty)$, *with the norm*

$$\|\beta\|_{C[x_0,\infty)} = \sup_{x \geqslant x_0} |\beta(x)|. \tag{4.2}$$

In this space, the integral equation (see (2.8)*)*

$$\beta(x) = 1 + \int_x^{\infty} (q(t) - g(t))\left[v_1^2(t)\int_t^{\infty} \frac{\beta(\xi)\,\mathrm{d}\xi}{r(\xi)v_1^2(\xi)}\right]\mathrm{d}t, \quad x \geqslant x_0 \tag{4.3}$$

has a unique solution $\beta \in C[x_0, \infty)$. *This solution is almost everywhere differentiable, and*

$$\beta'(x) = (g(x) - q(x))v_1^2(x)\int_x^{\infty} \frac{\beta(\xi)\,\mathrm{d}\xi}{r(\xi)v_1^2(\xi)}, \quad x \geqslant x_0. \tag{4.4}$$

In addition,

$$\lim_{x\to\infty} \beta(x) = 1 \tag{4.5}$$

and one has the asymptotic equality

$$\beta(x) = 1 + \varepsilon(x), \quad |\varepsilon(x)| \leqslant c\int_x^{\infty} |q(t) - g(t)|\varrho_1(t)\,\mathrm{d}t, \quad x \geqslant x_0. \tag{4.6}$$

Proof. Define a linear operator $\mathcal{A}\colon C[x_0, \infty) \to C[x_0, \infty)$,

$$(\mathcal{A}\beta)(x) = \int_x^{\infty} (q(t) - g(t))\left[v_1^2(t)\int_t^{\infty} \frac{\beta(\xi)\,\mathrm{d}\xi}{r(\xi)v_1^2(\xi)}\right]\mathrm{d}t, \quad x \geqslant x_0. \tag{4.7}$$

From (2.8) and (4.1), it follows that

$$\begin{aligned} \|\mathcal{A}\beta\|_{C[x_0,\infty)} &\leqslant \sup_{x\geqslant x_0}\left[\int_x^{\infty} |q(t) - g(t)|\left(v_1^2(t)\int_t^{\infty}\frac{\mathrm{d}\xi}{r(\xi)v_1^2(\xi)}\right)\mathrm{d}t\right]\cdot\|\beta\|_{C[x_0,\infty)} \\ &= \left(\int_{x_0}^{\infty} |q(t) - g(t)|\varrho_1(t)\,\mathrm{d}t\right)\cdot\|\beta\|_{C[x_0,\infty)} \leqslant \frac{1}{2}\|\beta\|_{C[x_0,\infty)}. \end{aligned} \tag{4.8}$$

By (4.8), the operator $\mathcal{A}\colon C[x_0, \infty) \to C[x_0, \infty)$ is compressing, and therefore equation (4.3) has a unique solution $\beta \in C[x_0, \infty)$. Equality (4.6) follows from (4.1), (4.3) and (2.8) and gives (4.5); the proof of (4.4) is obvious. □

Put

$$\hat{u}(x) = v_1(x)\int_x^\infty \frac{\beta(t)\,\mathrm{d}t}{r(t)v_1^2(t)}, \quad x \in [x_0,\infty). \tag{4.9}$$

Here β is the solution of (4.3) from $C[x_0,\infty)$. Clearly, the function $\hat{u}(x)$, $x \geqslant x_0$, is continuous on $[x_0,\infty)$ and is almost everywhere differentiable. In particular, we have

$$\begin{aligned}
&r(x)\hat{u}'(x) = r(x)v_1'(x)\int_x^\infty \frac{\beta(t)\,\mathrm{d}t}{r(t)v_1^2(t)} - \frac{\beta(x)}{v_1(x)}, \quad x \geqslant x_0 \\
&\quad \Rightarrow (r(x)\hat{u}'(x))' = (r(x)v_1'(x))'\int_x^\infty \frac{\beta(t)\,\mathrm{d}t}{r(t)v_1^2(t)} - \beta(x)\frac{v_1'(x)}{v_1^2(x)} + \beta(x)\frac{v_1'(x)}{v_1^2(x)} - \frac{\beta'(x)}{v_1(x)} \\
&\quad = g(x)v_1(x)\int_x^\infty \frac{\beta(t)\,\mathrm{d}t}{r(t)v_1^2(t)} - \frac{\beta'(x)}{v_1(x)} = g(x)\hat{u}(x) - \frac{\beta'(x)}{v_1(x)}, \quad x \geqslant x_0 \\
&\quad \Rightarrow \beta'(x) = g(x)\hat{u}(x)v_1(x) - (r(x)\hat{u}_1'(x))'v_1(x), \quad x \geqslant x_0.
\end{aligned} \tag{4.10}$$

On the other hand (see (4.4) and (4.9)),

$$\beta'(x) = (g(x) - q(x))\hat{u}(x)v_1(x), \quad x \geqslant x_0. \tag{4.11}$$

Combining (4.10) and (4.11) and using (1.5), we get

$$(r(x)\hat{u}'(x))' = q(x)\hat{u}(x), \quad x \geqslant x_0. \tag{4.12}$$

Therefore, $\hat{u}$ is a solution of (1.4) on $[x_0,\infty)$.

Remark 4.2. Substitution (4.3) was first used in [1].

Extend the solution $\hat{u}$ to the whole real line. Put

$$\delta(x) = \sup_{t\geqslant x}|\varepsilon(t)| \Rightarrow \ \delta(x) \to 0 \quad \text{as } x \to \infty \tag{4.13}$$

(here ε is defined in (4.6)). Then for $x \gg 1$ we have the obvious relations (see (4.6)):

$$\begin{aligned}
&\hat{u}(x) = v_1(x)\int_x^\infty \frac{1 + (\beta(t) - 1)}{r(t)v_1^2(t)}\,\mathrm{d}t \leqslant v_1(x)\int_x^\infty \frac{\mathrm{d}t}{r(t)v_1^2(t)} + v_1(x)\int_x^\infty \frac{|\varepsilon(t)|\,\mathrm{d}t}{r(t)v_1^2(t)} \\
&\qquad \leqslant (1 + \delta(x))v_1(x)\int_x^\infty \frac{\mathrm{d}t}{r(t)v_1^2(t)} = (1 + \delta(x))u_1(x), \quad x \gg 1; \\
&\hat{u}(x) = v_1(x)\int_x^\infty \frac{1 + (\beta(t) - 1)}{r(t)v_1^2(t)}\,\mathrm{d}t \geqslant v_1(x)\int_x^\infty \frac{\mathrm{d}t}{r(t)v_1^2(t)} - v_1(t)\int_x^\infty \frac{|\varepsilon(t)|\,\mathrm{d}t}{r(t)v_1^2(t)} \\
&\qquad \geqslant (1 - \delta(x))v_1(x)\int_x^\infty \frac{\mathrm{d}t}{r(t)v_1^2(t)} = (1 - \delta(x))u_1(x), \quad x \gg 1 \\
&\quad \Rightarrow 1 - \delta(x) \leqslant \frac{\hat{u}(x)}{u_1(x)} \leqslant 1 + \delta(x), \quad x \gg 1 \Rightarrow \lim_{x\to\infty}\frac{\hat{u}(x)}{u_1(x)} = 1.
\end{aligned} \tag{4.14}$$

Further, since $\{u, v\}$ is a PFSS of equation (1.4), there exist constants α and β_0 $(0 < |\alpha| + |\beta_0|)$ such that

$$\hat{u}(x) = \alpha u(x) + \beta_0 v(x), \quad x \geqslant x_0. \tag{4.15}$$

Suppose that $\beta_0 \neq 0$ in (4.15). Let $x_1 \gg x_0$ be such that (see (1.7) and (4.13))

$$\max\Big\{\delta(x); \Big|\frac{\alpha}{\beta_0}\Big|\frac{u(x)}{v(x)}\Big\} \leqslant \frac{1}{2} \quad \text{for } x \geqslant x_1. \tag{4.16}$$

Then by Corollary 2.7, (4.14), (4.15) and (4.16), we obtain

$$\begin{aligned}\infty &= \int_{x_1}^{\infty} \frac{\mathrm{d}t}{r(t)u_1^2(t)} \leqslant \int_{x_1}^{\infty} \frac{(1+\delta(t))^2\,\mathrm{d}t}{r(t)\hat{u}(t)^2} \leqslant \frac{9}{4}\int_{x_1}^{\infty} \frac{\mathrm{d}t}{r(t)(\alpha u(t) + \beta_0 v(t))^2} \\ &= \frac{9}{4}\cdot\frac{1}{\beta_0^2}\int_{x_1}^{\infty} \frac{\mathrm{d}t}{r(t)(1 + (\alpha/\beta_0)\cdot(u(t)/v(t)))^2 v(t)^2} \\ &\leqslant \frac{9}{4\beta_0^2}\int_{x_1}^{\infty} \frac{\mathrm{d}t}{r(t)(1 - |\alpha/\beta_0|(u(t)/v(t)))^2 v^2(t)} \leqslant \frac{9}{\beta_0^2}\int_{x_1}^{\infty} \frac{\mathrm{d}t}{r(t)v^2(t)} < \infty.\end{aligned}$$

This gives a contradiction. Hence, $\beta_0 = 0$ and therefore

$$\hat{u}(x) = \alpha u(x), \quad x \geqslant x_0 \Rightarrow \alpha > 0. \tag{4.17}$$

Now, by (4.14) and (4.17), we have

$$\lim_{x\to\infty} \frac{u(x)}{u_1(x)} = \lim_{x\to\infty} \frac{u(x)}{\hat{u}(x)}\cdot\frac{\hat{u}(x)}{u_1(x)} = \frac{1}{\alpha}\lim_{x\to\infty}\frac{\hat{u}(x)}{u_1(x)} = \frac{1}{\alpha}. \tag{4.18}$$

Consider the relation between the solutions v and v_1 from the PFSS of equations (1.4) and (2.34), respectively. Let $x \geqslant x_1$, see (4.16). By Definition 1.2, we have

$$\Big(\frac{v(x)}{u(x)}\Big)' = \frac{1}{r(x)u^2(x)} \Rightarrow \frac{v(x)}{u(x)} = \frac{v(x_1)}{u(x_1)} + \int_{x_1}^{x} \frac{\mathrm{d}t}{r(t)u^2(t)}, \quad x \geqslant x_1, \tag{4.19}$$

$$\Big(\frac{v_1(x)}{u_1(x)}\Big)' = \frac{1}{r(x)u_1^2(x)} \Rightarrow \frac{v_1(x)}{u_1(x)} = \frac{v_1(x_1)}{u_1(x_1)} + \int_{x_1}^{\infty} \frac{\mathrm{d}t}{r(t)u_1^2(t)}, \quad x \geqslant x_1. \tag{4.20}$$

Put

$$c_1 = c_1(x_1) = \frac{v_1(x_1)}{u_1(x_1)}, \quad c_2 = c_2(x_1) = \frac{v(x_1)}{u(x_1)}.$$

Using Corollary 2.7, (4.18), (4.19) and (4.20) and L'Hôpital's rule, we get

$$\lim_{x\to\infty}\frac{v(x)}{v_1(x)} = \lim_{x\to\infty}\frac{u(x)}{u_1(x)}\cdot\frac{c_2 + \int_{x_1}^{x} r^{-1}(t)u^{-2}(t)\,\mathrm{d}t}{c_1 + \int_{x_1}^{x} r^{-1}(t)u_1^{-2}(t)\,\mathrm{d}t} = \frac{1}{\alpha}\lim_{x\to\infty}\frac{r(x)u_1^2(x)}{r(x)u^2(x)} = \frac{\alpha^2}{\alpha} = \alpha. \tag{4.21}$$

Thus, from (4.18) and (4.21), we obtain

$$\lim_{x\to\infty}\frac{\varrho(x)}{\varrho_1(x)} = \lim_{x\to\infty}\frac{u(x)}{u_1(x)}\cdot\frac{v(x)}{v_1(x)} = 1. \tag{4.22}$$

Since the functions ϱ and ϱ_1 are continuous and positive in $[0,\infty)$, by (4.22) we have $\varrho(x) \asymp \varrho_1(x)$, $x \in [0,\infty)$. In a similar way, we establish weak equivalence of these functions on $(-\infty,0]$ which gives (2.35). It remains to refer to Theorem 2.21. $\square$

Proof of Theorem 3.3. From the definition of $s(x)$, $x \in \mathbb{R}$ (see (1.10), Lemma 2.8 and (3.2)), it follows that

$$\tag{4.23} \begin{gathered} \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}t}{\mu(t)} = \int_{x-s(x)}^{x+s(x)} \frac{r(\xi)\varrho(\xi)}{\mu(\xi)} \cdot \frac{\mathrm{d}\xi}{r(\xi)\varrho(\xi)} \leqslant a \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}\xi}{r(\xi)\varrho(\xi)} = a, \\ \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}t}{\mu(t)} = \int_{x-s(x)}^{x+s(x)} \frac{r(\xi)\varrho(\xi)}{\mu(\xi)} \cdot \frac{\mathrm{d}\xi}{r(\xi)\varrho(\xi)} \geqslant \frac{1}{a} \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}\xi}{r(\xi)\varrho(\xi)} = \frac{1}{a} \\ \Rightarrow \frac{1}{a} \leqslant \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}t}{\mu(t)} \leqslant a, \quad x \in \mathbb{R}. \end{gathered}$$

Put

$$\Delta(x) = [\Delta^{(-)}(x), \Delta^{(+)}(x)], \quad \Delta^{(\pm)}(x) = x \pm s(x), \quad x \in \mathbb{R},$$
$$\varepsilon(x) = \max_{\xi \in \Delta(x)} |\mu'(\xi)|, \quad x \in \mathbb{R}.$$

Note that from (2.13) and (3.3), it follows that

$$\tag{4.24} \lim_{|x|\to\infty} \varepsilon(x) = 0.$$

In the following relations, we use Schwarz's inequality and (4.23):

$$\tag{4.25} \begin{aligned} 2s(x) &= \int_{x-s(x)}^{x+s(x)} \frac{\sqrt{\mu(t)}}{\sqrt{\mu(t)}}\,\mathrm{d}t \leqslant \Bigl(\int_{x-s(x)}^{x+s(x)} \mu(t)\,\mathrm{d}t\Bigr)^{1/2} \cdot \Bigl(\int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}t}{\mu(t)}\Bigr)^{1/2} \\ &\leqslant \sqrt{a}\Bigl(\int_{x-s(x)}^{x+s(x)} \mu(t)\,\mathrm{d}t\Bigr)^{1/2}, \quad x \in \mathbb{R} \\ \Rightarrow 4s^2(x) &\leqslant a \int_{x-s(x)}^{x+s(x)} \mu(t)\,\mathrm{d}t = a\Bigl[2\mu(x)s(x) + \int_{x-s(x)}^{x+s(x)} (\mu(t)-\mu(x))\,\mathrm{d}t\Bigr] \\ &\leqslant a\Bigl[2\mu(x)s(x) + \int_{x-s(x)}^{x+s(x)} \Bigl|\int_x^t |\mu'(\xi)|\,\mathrm{d}\xi\Bigr|\,\mathrm{d}t\Bigr], \quad x \in \mathbb{R}. \end{aligned}$$

Let $x_0 \gg 1$ be such that

$$\tag{4.26} \varepsilon(x) \leqslant 4 \cdot (5a)^{-1} \quad \text{for } |x| \geqslant x_0.$$

Then, according to (4.25) and (4.26), for $|x| \geqslant x_0$, we have

$$\tag{4.27} \begin{gathered} 4s^2(x) \leqslant a\Bigl[2\mu(x)s(x) + \varepsilon(x)\int_{x-s(x)}^{x+s(x)} |t-x|\,\mathrm{d}t\Bigr] \leqslant a\Bigl[2\mu(x)s(x) + \frac{4}{5a}s^2(x)\Bigr] \\ \Rightarrow 4 \leqslant 2a\frac{\mu(x)}{s(x)} + \frac{4}{5} \quad \text{for } |x| \geqslant x_0 \Rightarrow s(x) \leqslant \frac{5}{8}a\mu(x) \quad \text{for } |x| \geqslant x_0. \end{gathered}$$

Further, let $t \in \Delta(x)$ and $|x| \geqslant x_0$. Then (see (4.26) and (4.27))

$$(4.28)\quad |\mu(t)-\mu(x)| \leqslant \left|\int_x^t |\mu'(\xi)|\,\mathrm{d}\xi\right| \leqslant \frac{4}{5a}s(x) \leqslant \frac{4}{5a}\cdot\frac{5}{8}a\mu(x) = \frac{\mu(x)}{2}, \quad |x| \geqslant x_0$$

$$\Rightarrow \frac{1}{2} \leqslant \frac{\mu(t)}{\mu(x)} \leqslant 2 \quad \text{for } t \in \Delta(x),\ |x| \geqslant x_0.$$

Now for $|x| \geqslant x_0$, by (4.23) and (4.28), we have

$$(4.29)\qquad \frac{1}{a} \leqslant \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}t}{\mu(t)} = \frac{1}{\mu(x)}\int_{x-s(x)}^{x+s(x)} \frac{\mu(x)}{\mu(t)}\,\mathrm{d}t \leqslant 4\frac{s(x)}{\mu(x)},$$

$$a \geqslant \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}t}{\mu(t)} = \frac{1}{\mu(x)}\int_{x-s(x)}^{x+s(x)} \frac{\mu(x)}{\mu(t)}\,\mathrm{d}t \geqslant \frac{s(x)}{\mu(x)}$$

$$\Rightarrow \frac{1}{4a}\mu(x) \leqslant s(x) \leqslant a\mu(x) \quad \text{for } |x| \geqslant x_0.$$

Since the functions $\mu(x)$, $x \in \mathbb{R}$, and $s(x)$, $x \in \mathbb{R}$, are continuous and positive (see Lemma 2.8), by (4.29) we obtain (3.4). □

Proof of Corollary 3.4. Put

$$(4.30)\qquad \varepsilon(x) = \max_{\xi\in\Delta(x)} |\mu'(\xi)|, \quad \mu(x) = r(x)\varrho(x), \quad x \in \mathbb{R},$$

where $\Delta(x) = [x-s(x), x+s(x)]$.

From (3.5) and (2.13), it follows that

$$(4.31)\qquad \lim_{|x|\to\infty} \varepsilon(x) = 0.$$

Below we use Schwarz's inequality and (1.10):

(4.32)

$$2s(x) = \int_{\Delta(x)} \frac{\sqrt{\mu(t)}}{\sqrt{\mu(t)}}\,\mathrm{d}t \leqslant \left(\int_{\Delta(x)} \mu(t)\,\mathrm{d}t\right)^{1/2}\cdot\left(\int_{\Delta(x)} \frac{\mathrm{d}t}{\mu(t)}\right)^{1/2}$$

$$= \left(\int_{\Delta(x)} \mu(t)\,\mathrm{d}t\right)^{1/2}$$

$$\Rightarrow 4s^2(x) \leqslant \int_{\Delta(x)} \mu(t)\,\mathrm{d}t = 2\mu(x)s(x) + \int_{\Delta(x)} (\mu(t)-\mu(x))\,\mathrm{d}t$$

$$\leqslant 2\mu(x)s(x) + \int_{\Delta(x)} \left|\int_x^t |\mu'(\xi)|\,\mathrm{d}\xi\right| dt \leqslant 2\mu(x)s(x) + \varepsilon(x)\int_{\Delta(x)} |t-x|\,\mathrm{d}t.$$

Let ν_0 be such that $\varepsilon(x) \leqslant \frac{4}{5}$ for $|x| \geqslant \nu_0$, see (4.31). Then, continuing (4.32), we get

(4.33)

$$4s^2(x) \leqslant 2\mu(x)s(x) + \frac{4}{5}s^2(x) \Rightarrow 4 \leqslant 2\frac{\mu(x)}{s(x)} + \frac{4}{5} \Rightarrow s(x) \leqslant \frac{5}{8}\mu(x) \quad \text{for } |x| \geqslant \nu_0.$$

Further, for $|x| \geqslant \nu_0$, we have (see (4.33))

$$|\mu(t) - \mu(x)| \leqslant \left| \int_x^t |\mu'(\xi)| \, \mathrm{d}\xi \right| \leqslant \frac{4}{5} s(x) \leqslant \frac{4}{5} \cdot \frac{5}{8} \mu(x) = \frac{\mu(x)}{2} \tag{4.34}$$
$$\Rightarrow \frac{1}{2} \leqslant \frac{\mu(t)}{\mu(x)} \leqslant 2 \quad \text{for } t \in \Delta(x),\ x \geqslant \nu_0.$$

Now, from (4.34) and (1.10), it follows that

$$1 = \int_{x-s(x)}^{x+s(x)} \frac{\mathrm{d}t}{\mu(t)} = \int_{x-s(x)}^{x+s(x)} \frac{\mu(x)}{\mu(t)} \cdot \frac{\mathrm{d}t}{\mu(x)} \leqslant 2 \cdot 2 \frac{s(x)}{\mu(x)} \quad \text{for } |x| \geqslant \nu_0$$
$$\Rightarrow \mu(x) \leqslant 4s(x) \quad \text{for } |x| \geqslant \nu_0.$$

Thus, we have inequalities (3.6), see (4.33).

Since s and μ are continuous and positive in $\mathbb{R}$, from (3.6) it follows that $s(x) \asymp \mu(x)$, $x \in \mathbb{R}$, see Definition 2.19. □

Proof of Theorem 3.5. The following equalities are consequences of (3.7) and (3.8):

$$\begin{aligned} r(x)v_1'(x) &= \Big(\frac{r(x)\alpha'(x)}{\alpha(x)} + r(x)\beta'(x)\Big) v_1(x) \\ &= \Big(\frac{r(x)\alpha'(x)}{\alpha(x)} + \frac{1}{2\alpha^2(x)}\Big) v_1(x), \quad x \in \mathbb{R}, \end{aligned} \tag{4.35}$$

$$\begin{aligned} r(x)u_1'(x) &= \Big(\frac{r(x)\alpha'(x)}{\alpha(x)} - r(x)\beta'(x)\Big) u_1(x) \\ &= \Big(\frac{r(x)\alpha'(x)}{\alpha(x)} - \frac{1}{2\alpha^2(x)}\Big) u_1(x), \quad x \in \mathbb{R}. \end{aligned} \tag{4.36}$$

Consequently, (3.7), (3.8), (4.35) and (4.36) immediately imply (3.10). Further, differentiating (4.35), we get

$$\begin{aligned} (r(x)v_1'(x))' &= \Big(\frac{r(x)\alpha'(x)}{\alpha(x)} + \frac{1}{2\alpha^2(x)}\Big)' v_1(x) \\ &\quad + \Big(\frac{r(x)\alpha'(x)}{\alpha(x)} + \frac{1}{2\alpha^2(x)}\Big)\Big(\frac{\alpha'(x)}{\alpha(x)} + \frac{1}{2r(x)\alpha^2(x)}\Big) v_1(x) \\ &= \Big(\frac{1}{4r(x)\alpha^4(x)} + \frac{(r(x)\alpha'(x))'}{\alpha(x)}\Big) v_1(x) \\ &= \Big(q_1(x) + \frac{(r(x)\alpha'(x))'}{\alpha(x)}\Big) v_1(x), \quad x \in \mathbb{R}. \end{aligned}$$

In a similar way, we conclude that

$$(r(x)u_1'(x))' = \Big(q_1(x) + \frac{(r(x)\alpha'(x))'}{\alpha(x)}\Big) u_1(x), \quad x \in \mathbb{R}.$$

Hence, v_1 and u_1 form an FSS (see (3.10)) of equation (3.9). Further, from (4.35) and (4.36), it follows that the inequalities $v_1'(x) \geqslant 0$, $x \in \mathbb{R}$, and $u_1'(x) \leqslant 0$, $x \in \mathbb{R}$, (see (1.5)) hold provided one has the estimate

$$|\alpha'(x)| \leqslant \frac{1}{2r(x)\alpha(x)}, \quad x \in \mathbb{R}, \tag{4.37}$$

and by (3.7), inequalities (3.11) and (4.37) coincide. Finally, (3.8) and (3.12) imply the equalities (1.7) (written for $u_1(x)$, $x \in \mathbb{R}$ and $v_1(x)$, $x \in \mathbb{R}$). □

Proof of Corollary 3.6. Let $\mathcal{D}_1 > 0$. Then from (3.13), we get

$$\sup_{x\in\mathbb{R}}(r(x)\varrho_1^2(x)) = \sup_{x\in\mathbb{R}}\Big(r(x)\frac{1}{4r(x)q_1(x)}\Big) \leqslant \frac{1}{4}\big(\inf_{x\in\mathbb{R}} q_1(x)\big)^{-1} = \frac{1}{4\mathcal{D}_1} < \infty.$$

Since $\{u_1, v_1\}$ is a PFSS of equation (3.9), the function $s_1(x)$, $x \in \mathbb{R}$ (see Lemma 2.8) is well-defined, and we obtain (see (2.16))

$$\begin{aligned} 1 &= \int_{x-s_1(x)}^{x+s_1(x)} \frac{\mathrm{d}t}{r(t)\varrho_1(t)} = \int_{x-s_1(x)}^{x+s_1(x)} \frac{\varrho_1(t)\,\mathrm{d}t}{r(t)\varrho_1^2(t)} \geqslant 4\mathcal{D}_1 \int_{x-s_1(x)}^{x+s_1(x)} \varrho_1(t)\,\mathrm{d}t \\ &\geqslant \frac{8\mathcal{D}_1}{\mathrm{e}}\varrho_1(x)s_1(x), \quad x \in \mathbb{R} \Rightarrow \sup_{x\in\mathbb{R}}(\varrho_1(x)s_1(x)) \leqslant \frac{\mathrm{e}}{8\mathcal{D}_1} < \infty \quad (\text{here } \mathrm{e} = \exp(1)).\end{aligned}$$

To prove the corollary, it remains to refer to Theorem 1.3. □

Proof of Theorem 3.7. We begin by giving an outline of the proof. Below, using Theorems 2.13 and 2.22, we show that under the hypothesis of the theorem, the functions ϱ and ϱ_1 generating the PFSS of equations (3.9) and (3.18), are weakly equivalent in $\mathbb{R}$. By Theorem 2.21, this implies that problems (I)–(II) for equations (3.14) and (3.16) are equivalent, as required, see Corollary 3.6.

Let us now go into the details of the proof. Put

$$(\Delta q)(x) = \theta(x) - \frac{(r(x)\alpha'(x))'}{\alpha(x)}, \quad x \in \mathbb{R}, \tag{4.38}$$

see also notation (3.20), (3.21) and (3.19). We show that under the hypothesis of the theorem, the integral $I^{(+)}(0)$ (see (2.26)) converges at least conditionally. Let $[\alpha, \beta] \subseteq \widehat{\Delta}(x)$, $x \in \mathbb{R}$ (see (3.20)), and let ξ_1 and ξ_2 be points of the segment $[\alpha, \beta]$ that can be used (see below) in the second mean value theorem, see [15]. Then we have the relations (see (1.5))

$$\begin{aligned}\int_\alpha^\beta (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t &= \int_\alpha^\beta (\Delta q)(t)u_1(t)v_1(t)\,\mathrm{d}t = v_1(\beta)\int_{\xi_1}^\beta (\Delta q)(t)u_1(t)\,\mathrm{d}t \\ &= v_1(\beta)u_1(\xi_1)\int_{\xi_1}^{\xi_2} (\Delta q)(t)\,\mathrm{d}t.\end{aligned} \tag{4.39}$$

Now, from (2.17), (3.19), (3.22) and Theorem 3.3 for $x \geqslant x_0$ (x_0 is a point from Corollary 3.4), we obtain the inequality

(4.40)
$$\begin{aligned}\sup_{\alpha,\beta\in\widehat{\Delta}(x)}\Bigl|\int_\alpha^\beta(\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\Bigr| &= \sup_{\alpha,\beta\in\widehat{\Delta}(x)}\Bigl\{\Bigl[\frac{v_1(\beta)}{v_1(x)}\cdot\frac{u_1(\xi)}{u_1(x)}\Bigr]\cdot\varrho_1(x)\Bigl|\int_{\xi_1}^{\xi_2}(\Delta q)(t)\,\mathrm{d}t\Bigr|\Bigr\}\\ &\leqslant \mathrm{e}^2\varrho_1(x)\sup_{\alpha,\beta\in\widehat{\Delta}(x)}\Bigl|\int_\alpha^\beta(\Delta q)(t)\,\mathrm{d}t\Bigr|\\ &= \mathrm{e}^2 r(x)\varrho_1(x)\Bigl\{\frac{1}{r(x)}\sup_{\alpha,\beta\in\widehat{\Delta}(x)}\Bigl|\int_\alpha^\beta(\Delta q)(t)\,\mathrm{d}t\Bigr|\Bigr\}\\ &\leqslant \mathrm{e}^2 r(x)\varrho_1(x)\Bigl[\frac{1}{r(x)}\sup_{\alpha,\beta\in\omega(x)}\Bigl|\int_\alpha^\beta(\Delta q)(t)\,\mathrm{d}t\Bigr|\Bigr]\leqslant cs_1(x)\tau(x),\quad x\in\mathbb{R}.\end{aligned}$$

Suppose that on the segment $\widehat{\Delta}(x)$ the function $\tau(t)$ attains its maximal value at the point $t = x_0$. Then for $t \in \hat{\Delta}(x)$ and $x \gg 1$, we obtain the inequalities (see (3.2), (3.19), (3.23), (4.28), and (4.29)):

$$\begin{aligned}|\tau(t)-\tau(x_0)| &= \Bigl|\int_{x_0}^t|\tau'(\xi)|\,\mathrm{d}\xi\Bigr| = \Bigl|\int_{x_0}^t\frac{|\tau'(\xi)|}{\tau(\xi)}\mu_1(\xi)\cdot\frac{\tau(\xi)}{\mu_1(\xi)}\,\mathrm{d}\xi\Bigr|\\ &\leqslant c\Bigl(\max_{\xi\in\widehat{\Delta}(x)}\delta(\xi)\Bigr)\frac{1}{\mu_1(x)}\int_{\widehat{\Delta}(x)}\tau(\xi)\,\mathrm{d}\xi\\ &\leqslant c\Bigl(\max_{\xi\in\widehat{\Delta}(x)}\delta(\xi)\Bigr)\frac{s_1(x)}{\mu_1(x)}\tau(x_0)\leqslant c\Bigl(\max_{\xi\in\widehat{\Delta}(x)}\delta(\xi)\Bigr)\tau(x_0).\end{aligned}\tag{4.41}$$

From (2.13) and (3.23), it follows that there is a point $x_1 \gg 1$ such that the estimate

$$c\max_{\xi\in\widehat{\Delta}(x)}\delta(\xi)\leqslant\frac{1}{2}\quad\text{for}\quad x\geqslant x_1\tag{4.42}$$

holds, see (4.41). Then, by (4.41) and (4.42), we have

$$\tau(x_0)\leqslant 2\tau(t)\quad\text{for } t\in\widehat{\Delta}(x),\ x\geqslant x_1.\tag{4.43}$$

Finally, combining (4.40), (4.41) and (4.43), we obtain

$$\begin{aligned}\sup_{\alpha,\beta\in\widehat{\Delta}(x)}\Bigl|\int_\alpha^\beta(\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\Bigr| &\leqslant cs_1(x)\tau(x)\leqslant cs_1(x)\tau(x_0)\\ &\leqslant c\int_{\widehat{\Delta}(x)}\tau(t)\,\mathrm{d}t\quad\text{for } x\geqslant x_1.\end{aligned}\tag{4.44}$$

Let us consider the integral $I^{(+)}(0)$ (see (2.26)) and show that it converges at least conditionally. To this end, fix an arbitrary $\varepsilon \in (0,1)$ and choose $\mathcal{A}_0(\varepsilon) \gg 1$ in order to satisfy the estimate

$$\int_{\mathcal{A}_0(\varepsilon)}^{\infty} \tau(t)\,\mathrm{d}t < \varepsilon. \tag{4.45}$$

Further, since the function s_1 is continuous in $\mathbb{R}$ and one has (2.13), by Lemma 2.18 for any $\mathcal{A}_1 \in \mathbb{R}$, there exists an $\mathbb{R}(\mathcal{A}_1, s_1)$-covering of the semi-axis $[\mathcal{A}_1, \infty)$ by the segments $\{\widehat{\Delta}_n\}_{n=1}^{\infty}$, see Definition 2.17. Let $\mathcal{A}_2 \geqslant \mathcal{A}_1 \geqslant \mathcal{A}_0(\varepsilon)$ and $\mathcal{A}_2 \in \widehat{\Delta}_{n_0}$, $n_0 \geqslant 1$. Then we have the relations (see (4.45))

$$\begin{aligned} \left|\int_{\mathcal{A}_1}^{\mathcal{A}_2} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\right| &= \left|\sum_{n=1}^{n_0} \int_{\widehat{\Delta}_n} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t - \int_{\mathcal{A}_2}^{\widehat{\Delta}_{n_0}^{(+)}} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\right| \\ &\leqslant \sum_{n=1}^{n_0}\left|\int_{\widehat{\Delta}_n} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\right| + \sup_{\alpha,\beta\in\widehat{\Delta}_{n_0}}\left|\int_{\alpha}^{\beta} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\right| \\ &\leqslant \sum_{n=1}^{n_0} c\int_{\widehat{\Delta}_n} \tau(t)\,\mathrm{d}t + c\int_{\widehat{\Delta}_{n_0}} \tau(t)\,\mathrm{d}t \leqslant 2c\int_{\mathcal{A}_1}^{\widehat{\Delta}_{n_0}} \tau(t)\,\mathrm{d}t \\ &\leqslant 2c\int_{\mathcal{A}_0(\varepsilon)}^{\infty} \tau(t)\,\mathrm{d}t \leqslant 2c\varepsilon. \end{aligned} \tag{4.46}$$

Since in (4.46) c is an absolute constant, we conclude that the integral $I^{(+)}(0)$ converges at least conditionally.

The following inequalities are proved in the same way as estimate (4.46), and therefore, we do not make any additional comments on the following relations:

$$\begin{aligned} |I^{(+)}(x)| &= \left|\int_x^{\infty} (\Delta q)(t)\varrho_1(t)dt\right| = \left|\sum_{n=1}^{\infty}\int_{\widehat{\Delta}_n} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\right| \\ &\leqslant \sum_{n=1}^{\infty}\left|\int_{\widehat{\Delta}_n} (\Delta q)(t)\varrho_1(t)\,\mathrm{d}t\right| \leqslant \sum_{n=1}^{\infty} c\int_{\widehat{\Delta}_n} \tau(t)\,\mathrm{d}t = c\int_x^{\infty} \tau(t)\,\mathrm{d}t, \quad x \geqslant x_1. \end{aligned} \tag{4.47}$$

From (4.47), (3.25) and (3.19), we obtain the inequality that is needed to apply Theorem 2.13,

$$\int_{x_1}^{\infty} \frac{1}{r(t)\varrho_1(t)}(I^{(+)}(t))^2\,\mathrm{d}t \leqslant c\int_{x_1}^{\infty} \frac{1}{\mu_1(t)}\left(\int_t^{\infty} \tau(\xi)\,\mathrm{d}\xi\right)^2\mathrm{d}t < \infty. \tag{4.48}$$

This easily implies that $K^{(+)} < \infty$, see Theorem 2.13. In a similar way, we check that under the hypotheses of the theorem, the integral $I^{(-)}(0)$ converges (at least conditionally) and $K^{(-)} < \infty$, see (2.27). Thus, we conclude (see Theorem 2.13) that the Hartman-Wintner problems for equations (3.9) and (3.18) are solvable for $x \to \pm\infty$. By Theorem 2.22, this implies that for $x \in \mathbb{R}$ the functions ϱ and ϱ_1 generating the PFSS of equations (3.9) and (3.18) are weakly equivalent for $x \in \mathbb{R}$, see Definition 2.19. It remains to refer to Theorem 2.21. $\square$

5. EXAMPLES

In this section, we study problems (I)–(II) for equation (1.1) in the following situations:

$$r(x) = \sqrt{1+x^2}, \quad q(x) = (\mathrm{e}^x + \mathrm{e}^{-x}) + (1+x^2)^n \cos \mathrm{e}^{\gamma|x|}, \quad x \in \mathbb{R},\ n \geqslant 1,\ \gamma > 0, \tag{5.1}$$

$$r(x) = \sqrt{1+x^2}, \quad q(x) = (\mathrm{e}^x + \mathrm{e}^{-x}) + (\mathrm{e}^x + \mathrm{e}^{-x}) \cos \mathrm{e}^{\gamma|x|}, \quad x \in \mathbb{R},\ \gamma \geqslant \frac{5}{4} \tag{5.2}$$

Our study of these equations is only an illustration of the assertions of Section 3, and therefore, we do not use the methods of [8], [10]. Note that for the sake of brevity we do not present routine and cumbersome calculations in full, but all the omitted details can be easily recovered by the interested reader.

5.1. Study of the model homogeneous equation. In this section, we fully follow Theorem 3.5, including the notation. Set

$$r(x) = \sqrt{1+x^2}, \quad q_1(x) = \mathrm{e}^x + \mathrm{e}^{-x}, \quad x \in \mathbb{R}, \tag{5.3}$$

$$\alpha(x) = (4r(x)q_1(x))^{-1/4}, \quad x \in \mathbb{R}, \tag{5.4}$$

$$\beta(x) = \int_{x_0}^{x} \sqrt{\frac{q_1(t)}{r(t)}}, \quad x \in \mathbb{R},\ x_0 \in \mathbb{R}. \tag{5.5}$$

In (5.5), the point x_0 is fixed but can be chosen arbitrarily. Consider the equation (see (5.3), (5.4))

$$(r(x)z'(x))' = \Big(q_1(x) + \frac{(r(x)\alpha'(x))'}{\alpha(x)}\Big) z(x), \quad x \in \mathbb{R}. \tag{5.6}$$

According to Theorem 3.5, its FSS is given by the formulas

$$u_1(x) = \alpha(x)\mathrm{e}^{-\beta(x)}, \quad v_1(x) = \alpha(x)\mathrm{e}^{\beta(x)}, \quad x \in \mathbb{R} \tag{5.7}$$

(see (3.8)); and the relations (3.10), (3.11) and (3.12) hold, i.e., $\{u_1, v_1\}$ is a PFSS of equation (5.6). Note also the relations

$$\varrho_1(x) = u_1(x)v_1(x) = \alpha^2(x) = \frac{1}{2(1+x^2)^{1/4}(\mathrm{e}^x + \mathrm{e}^{-x})^{1/2}}, \quad x \in \mathbb{R} \text{ (see (3.13))}, \tag{5.8}$$

$$\mu_1(x) = r(x)\varrho_1(x) = \frac{1}{2}\cdot\frac{\sqrt[4]{1+x^2}}{\sqrt{\mathrm{e}^x+\mathrm{e}^{-x}}}, \quad x\in\mathbb{R} \text{ (see (3.13) and (3.19))}, \tag{5.9}$$

$$\frac{|(r(x)\alpha'(x))'|}{\alpha(x)} \leqslant c(1+x^2)^{1/2}, \quad x\in\mathbb{R}. \tag{5.10}$$

5.2. Study of problem (I)–(II) for non-homogeneous model equation (3.14)**.**

$$-\big(\sqrt{1+x^2}y'(x)\big)' + \Big(q_1(x) + \frac{(r(x)\alpha'(x))'}{\alpha(x)}\Big)y(x) = f(x), \quad x\in\mathbb{R}. \tag{5.11}$$

Since in case (5.3) the hypotheses of Theorem 3.5 are satisfied, by Corollary 3.6 problem (I)–(II) for equation (5.11) is solvable because $\mathcal{D}_1 > 0$ (see (3.15)),

$$\mathcal{D}_1 = \inf_{x\in\mathbb{R}} q_1(x) = \inf_{x\in\mathbb{R}}(\mathrm{e}^x+\mathrm{e}^{-x}) = 2 > 0.$$

5.3. Study of problem (I)–(II) for equation (1.1) **in case** (5.1)**.**

$$-\big(\sqrt{1+x^2}y'(x)\big)' + [(\mathrm{e}^x+\mathrm{e}^{-x}) + (1+x^2)^n\cos\mathrm{e}^{\gamma|x|}]y(x) = f(x), \qquad x\in\mathbb{R},\ n\geqslant 1,\ \gamma>0. \tag{5.12}$$

To apply Theorem 3.1, let us choose equation (5.11) as the second (model) equation. According to this theorem, we have to establish the inequalities (see (3.1) and (5.8))

$$\int_{-\infty}^0 \Big|\frac{(r(x)\alpha'(x))'}{\alpha(x)} - (1+x^2)^n\cos\mathrm{e}^{-\gamma x}\Big|\varrho_1(x)\,\mathrm{d}x < \infty, \tag{5.13}$$

$$\int_0^\infty \Big|\frac{(r(x)\alpha'(x))'}{\alpha(x)} - (1+x^2)^n\cos\mathrm{e}^{\gamma x}\Big|\varrho_1(x)\,\mathrm{d}x < \infty. \tag{5.14}$$

Since these inequalities are studied in the same way, below we consider only (5.14). Using (5.8) and (5.10), we get the obvious relations

$$\begin{aligned}\int_0^\infty \frac{|(r(x)\alpha'(x))'|}{\alpha(x)}\varrho_1(x)\,\mathrm{d}x &\leqslant c\int_0^\infty \frac{(1+x^2)^{1/2}\,\mathrm{d}x}{(1+x^2)^{1/4}(\mathrm{e}^x+\mathrm{e}^{-x})^{1/2}}\\ &\leqslant c\int_0^\infty \frac{x^{1/2}}{\mathrm{e}^{x/2}}\,\mathrm{d}x < \infty,\\ \int_0^\infty (1+x^2)^n|\cos\mathrm{e}^{\gamma x}|\varrho_1(x)\,\mathrm{d}x &\leqslant c\int_0^\infty \frac{(1+x^2)^n\,\mathrm{d}x}{(1+x^2)^{1/4}(\mathrm{e}^x+\mathrm{e}^{-x})^{1/2}}\\ &\leqslant c\int_0^\infty \frac{x^{2n-1/2}}{\mathrm{e}^{x/2}}\,\mathrm{d}x < \infty.\end{aligned}$$

Thus, the hypotheses of Theorem 3.1 are satisfied, and therefore problem (I)–(II) for equation (5.12) is solvable due to the assertions obtained in Section 5.2 and in Theorem 3.1.

5.4. Study of problem (I)–(II) for equation (1.1) in case (5.2).

$$-\big(\sqrt{1+x^2}y'(x)\big)' + [(\mathrm{e}^x+\mathrm{e}^{-x}) + (\mathrm{e}^x+\mathrm{e}^{-x})\cos \mathrm{e}^{\gamma|x|}]y(x) = f(x), \quad x\in\mathbb{R},\ \gamma>\frac{5}{4}. \tag{5.15}$$

Here we rely on Theorem 3.7 but as in Section 5.3, we choose (5.11) as the second (model) equation. Throughout below we assume that $x \gg 1$ (the case $x \ll -1$ is completely analogous).

Thus, we have to find a function $\tau(x)$, $x\in\mathbb{R}$, satisfying relations (3.22) and (3.23), see Theorem 3.7. The following inequality holds for $x \geqslant \nu_0$ (see Corollary 3.4 and (3.21))

$$\begin{aligned}\sup_{\xi_1,\xi_2\in\omega(x)}\bigg|\int_{\xi_1}^{\xi_2}(\Delta q)(t)\,\mathrm{d}t\bigg| &\leqslant \sup_{\xi_1,\xi_2\in\omega(x)}\bigg|\int_{\xi_1}^{\xi_2}\frac{|(r(t)\alpha'(t))'|}{\alpha(t)}\,\mathrm{d}t\bigg|\\ &\quad+\sup_{\xi_1,\xi_2\in\omega(x)}\bigg|\int_{\xi_1}^{\xi_2}(\mathrm{e}^t+\mathrm{e}^{-t})\cos \mathrm{e}^{\gamma t}\,\mathrm{d}t\bigg|, \quad x\geqslant\nu_0.\end{aligned} \tag{5.16}$$

Let us estimate the first integral in (5.16) (see Corollary 3.4, (5.9) and (5.10))

$$\begin{aligned}\sup_{\xi_1,\xi_2\in\omega(x)}\bigg|\int_{\xi_1}^{\xi_2}\frac{|(r(t)\alpha'(t))'|}{\alpha(t)}\,\mathrm{d}t\bigg| &\leqslant \sup_{t\in\omega(x)}\frac{|(r(t)\alpha'(t))'|}{\alpha(t)}\cdot\frac{5}{8}\mu_1(x)\\ &\leqslant c\Big(\sup_{t\in\omega(x)}\sqrt{1+t^2}\Big)\cdot\frac{\sqrt[4]{1+x^2}}{\sqrt{\mathrm{e}^x+\mathrm{e}^{-x}}}\\ &\leqslant c\frac{(1+x^2)^{3/4}}{\sqrt{\mathrm{e}^x+\mathrm{e}^{-x}}}\leqslant c\frac{x^{3/2}}{\mathrm{e}^{x/2}}, \quad x\geqslant\nu_0.\end{aligned} \tag{5.17}$$

To estimate the second integral in (5.16), first note that for $\gamma>\frac{5}{4}$ the function

$$\eta(t) = (\mathrm{e}^t+\mathrm{e}^{-t})\mathrm{e}^{-\gamma t}, \quad t\gg\nu_0,$$

is monotone decreasing. Therefore, by the second mean value theorem (see [15]), we have (below, ξ_1 is a point in the interval (ξ_1,ξ_2) appearing in the mean value theorem)

$$\begin{aligned}\bigg|\int_{\xi_1}^{\xi_2}(\mathrm{e}^t+\mathrm{e}^{-t})\cos \mathrm{e}^{\gamma t}\,\mathrm{d}t\bigg| &= \bigg|\int_{\xi_1}^{\xi_2}\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{\gamma \mathrm{e}^t}[\sin \mathrm{e}^{\gamma t}]'\,\mathrm{d}t\bigg| = \frac{\mathrm{e}^{\xi_1}+\mathrm{e}^{-\xi_1}}{\gamma \mathrm{e}^{\gamma\xi_1}}\bigg|\int_{\xi_3}^{\xi_2}(\sin \mathrm{e}^{\gamma t})'\,\mathrm{d}t\bigg|\\ &= \frac{\mathrm{e}^{\xi_1}+\mathrm{e}^{-\xi_1}}{\mathrm{e}^{\gamma\xi_1}}|\sin \mathrm{e}^{\gamma\xi_2}-\sin \mathrm{e}^{\gamma\xi_3}|\\ &\leqslant 2\cdot\frac{\mathrm{e}^{\xi_1}+\mathrm{e}^{-\xi_1}}{\mathrm{e}^{\gamma\xi_1}}\leqslant c\mathrm{e}^{(1-\gamma)x}, \quad x\gg\nu_0.\end{aligned} \tag{5.18}$$

Finally, we get (see (5.17) and (5.18)):

$$\begin{aligned}(5.19)\qquad \frac{1}{r(x)}\sup_{\xi_1,\xi_2\in\omega(x)}\Big|\int_{\xi_1}^{\xi_2}(\Delta q)(t)\,\mathrm{d}t\Big| &\leqslant c\frac{1}{\sqrt{1+x^2}}\Big(\frac{x^{3/2}}{\mathrm{e}^{x/2}}+\mathrm{e}^{(1-\gamma)x}\Big)\\ &\leqslant c\Big(\frac{x^{1/2}}{\mathrm{e}^{x/2}}+\frac{1}{x\mathrm{e}^{(\gamma-1)x}}\Big),\quad x\geqslant\tau_0,\ \gamma\geqslant\frac{5}{4}.\end{aligned}$$

Taking into account (5.19), set

$$(5.20)\qquad \tau(x)=c\Big(\frac{x^{1/2}}{\mathrm{e}^{x/2}}+\frac{1}{x\mathrm{e}^{(\gamma-1)x}}\Big),\quad x\geqslant\tau_0,\ \gamma\geqslant\frac{5}{4}.$$

The choice of τ given in (5.20) is kept as long as this function satisfies condition (3.23) (see (5.9))

$$\begin{aligned}(5.21)\quad \frac{|\tau'(x)|}{\tau(x)}\mu_1(x) &\leqslant c\frac{x^{1/2}\mathrm{e}^{-x}+(\gamma-1)x^{-1}\mathrm{e}^{(\gamma-1)x}}{x^{1/2}\mathrm{e}^{-x/2}+x^{-1}\mathrm{e}^{(\gamma-1)x}}\mu_1(x)\\ &\leqslant c(\mathrm{e}^{-x/2}+1)\mu_1(x)\leqslant c\mu_1(x)=c\frac{\sqrt[4]{1+x^2}}{\sqrt{\mathrm{e}^x+\mathrm{e}^{-x}}}\to 0\quad\text{as } x\to\infty.\end{aligned}$$

From (5.21) it follows that the choice of $\tau(x)$, $x\geqslant\tau_0$, $\gamma\geqslant\frac{5}{4}$ given in (5.20) is admissible, and we can proceed to checking conditions (3.24) and (3.25).

First note that by L'Hôpital's rule, we have the equalities (see (5.20))

$$(5.22)\qquad \int_x^\infty\frac{t^{1/2}}{\mathrm{e}^{t/2}}\,\mathrm{d}t=\frac{x^{1/2}}{\mathrm{e}^{x/2}}(1+\varepsilon_1(x)),\qquad \lim_{x\to\infty}\varepsilon_1(x)=0,$$

$$(5.23)\qquad \int_x^\infty\frac{\mathrm{d}t}{t\mathrm{e}^{(\gamma-1)t}}=\frac{1}{x\mathrm{e}^{(\gamma-1)x}}(1+\varepsilon_2(x)),\quad \lim_{x\to\infty}\varepsilon_2(x)=0,\ \gamma\geqslant\frac{5}{4}.$$

From (5.22) and (5.23) we obtain

$$\begin{gathered}(5.24)\quad \int_x^\infty\tau(t)\,\mathrm{d}t=c\Big[\frac{x^{1/2}}{\mathrm{e}^{x/2}}(1+\varepsilon_1(x))+\frac{1}{x\mathrm{e}^{(\gamma-1)x}}(1+\varepsilon_2(x))\Big]=\tau(x)(1+\varepsilon_3(x)),\\ \lim_{x\to\infty}\varepsilon_3(x)=0.\end{gathered}$$

From (5.24) it follows that the integral $P_1^{(+)}$ converges, see (3.24). In addition, by (5.9) and (5.24) we have the relations

$$\begin{aligned}&\int_x^\infty\frac{1}{\mu_1(t)}\Big(\int_t^\infty\tau(\xi)\,\mathrm{d}\xi\Big)^2\mathrm{d}t\\ &\quad\leqslant c\int_x^\infty\frac{\tau^2(t)}{\mu_1(t)}\,\mathrm{d}t=c\int_x^\infty\frac{(\mathrm{e}^t+\mathrm{e}^{-t})^{1/2}}{(1+t^2)^{1/4}}\Big[\frac{t}{\mathrm{e}^t}+2\frac{1}{t^{1/2}\mathrm{e}^{(\gamma-1/2)t}}+\frac{1}{t^2\mathrm{e}^{2(\gamma-1)t}}\Big]\,\mathrm{d}t<\infty.\end{aligned}$$

which hold for $x\gg 1$ and $\gamma\geqslant\frac{5}{4}$.

Hence, the integral $P_2^{(+)}$ converges. Since all the hypotheses of Theorem 3.7 are satisfied, problem (I)–(II) for equation (5.15) is solvable because this problem is solvable for equation (5.11).

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