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LIE PERFECT, LIE CENTRAL EXTENSION AND GENERALIZATION OF NILPOTENCY IN MULTIPLICATIVE LIE ALGEBRAS

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Abstract. This paper aims to introduce and explore the concept of Lie perfect multiplicative Lie algebras, with a particular focus on their connections to the central extension theory of multiplicative Lie algebras. The primary objective is to establish and provide proof for a range of results derived from Lie perfect multiplicative Lie algebras. Furthermore, the study extends the notion of Lie nilpotency by introducing and examining the concept of local nilpotency within multiplicative Lie algebras. The paper presents an innovative adaptation of the Hirsch-Plotkin theorem specifically tailored for multiplicative Lie algebras.

Keywords: multiplicative Lie algebra; commutator; nilpotent group; perfect group; central extensions

MSC 2020: 17A99, 19G24, 20A99, 20F19

1. Introduction

Ellis in [4] conjectured that commutator identities of weight n could be derived from a set of five universal commutator identities of weight 2. He proved this conjecture for $n=2$ and $n=3$ using homological techniques. Later, Donaze and Ladra in [3] proved this conjecture for $n=4$. During the investigation of commutator identities, Ellis introduced a novel algebraic structure known as multiplicative Lie algebra, which is a generalization of groups and Lie rings. It gives a comprehensive understanding of algebraic structures. Note that a multiplicative Lie algebra need not be a Lie algebra but every Lie algebra can be considered as multiplicative Lie algebra by defining a multiplicative Lie product as Lie bracket. Therefore, it is natural to extend concepts from groups and Lie algebras to multiplicative Lie algebras and explore whether analogous results are valid in this structure.

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Point and Wantiez in [10] introduced the concepts of solvability and nilpotency in multiplicative Lie algebras, extending existing results from groups and Lie algebras to this new algebraic structure. Further Bak et al. in [1] discussed some structural properties and the homology of multiplicative Lie algebras. Donadze et al. in [2] defined the notion of the nonabelian tensor product of a multiplicative Lie algebra and discussed the extension theory for this. Lal and Upadhyay in [6] introduced and constructed the free multiplicative Lie algebra based on a given set. This led to development of the Schur multiplier for multiplicative Lie algebras. Pandey and Upadhyay in [9] have given partial classifications of multiplicative Lie products for some finite groups and proved that every multiplicative Lie product $\star$ on a group K can be determined by a homomorphism from $K \wedge K$ to K under certain conditions.

In the context of a group $(K, \cdot)$, two apparent multiplicative Lie products exist: the trivial one defined as $x \star y = 1$, and another given by $x \star y = [x, y]$ for all $x, y \in K$. We call $(K, \cdot, \star)$ the trivial and improper multiplicative Lie algebra, respectively. A multiplicative Lie product $\star$ on a group $(K, \cdot)$ is said to be proper if it does not coincide with the trivial or improper multiplicative Lie product on K. Not all groups possess proper multiplicative Lie products, and those lacking are known as Lie simple groups. Pandey, Lal, and Upadhyay in [7] introduced the concepts of the Lie commutator and multiplicative Lie center, shedding new light on solvable and nilpotent multiplicative Lie algebras and their relation to improper multiplicative Lie algebras. In this article, we define the concept of local nilpotency in multiplicative Lie algebras in terms of the Lie commutator introduced in [7] and prove an analogue of the Hirsch-Plotkin theorem. Extension theory is an important tool in the classification of any algebraic structure. The main aim of extension theory of multiplicative Lie algebras is to classify all multiplicative Lie algebras G having an ideal H such that $G/H \cong K$ for a multiplicative Lie algebra K. The extension theory of multiplicative Lie algebras has been studied by many authors, see [2], [6], [8]. In this article, we defined Lie perfect multiplicative Lie algebras and developed the Lie central extension theory of multiplicative Lie algebras using the multiplicative Lie center introduced in [7].

This article is structured as follows: In Section 2, we provide a review of basic definitions and results concerning multiplicative Lie algebras. In Section 3, we introduce Lie perfect multiplicative Lie algebras, establish identities for Lie commutators, and demonstrate that the multiplicative Lie center is a subset of both the Lie center and the center of the multiplicative Lie algebra. In Section 4, we delve into the central extension theory for multiplicative Lie algebras and establish some results. In Section 5, we define the Lie idealizer, explore locally Lie nilpotent multiplicative Lie algebras, and prove that the product of two locally Lie nilpotent multiplicative Lie algebras is itself locally Lie nilpotent.

2. Multiplicative Lie algebra

In this section, we recall some definitions and results related to multiplicative Lie algebras, which are used in proving our results.

Definition 2.1 ([4]). A multiplicative Lie algebra is a triple $(K, \cdot, \star)$, where $(K, \cdot)$ is a group and $\star$ is a binary operation on K (called the *multiplicative Lie product*) satisfying the following identities:

(1) $x \star x = 1$,
(2) $x \star (y \cdot z) = (x \star y) \cdot {}^{y}(x \star z)$,
(3) $(x \cdot y) \star z = {}^{x}(y \star z) \cdot (x \star z)$,
(4) $((x \star y) \star {}^{y}z) \cdot ((y \star z) \star {}^{z}x) \cdot ((z \star x) \star {}^{x}y) = 1$,
(5) ${}^{z}(x \star y) = {}^{z}x \star {}^{z}y$ for all $x, y, z \in K$, where ${}^{z}x = zxz^{-1}$.

Note: If $(K, \cdot)$ is an abelian group, then $(K, \cdot, \star)$ is called a *Lie ring.*

Ellis in [4] established the following identities for the multiplicative Lie algebra.

Proposition 2.2 ([4], [6], [12]). *Let $(K, \cdot, \star)$ be a multiplicative Lie algebra. Then the following identities hold:*

(1) $1 \star x = 1 = x \star 1$,
(2) $(x \star y)(y \star x) = 1$,
(3) ${}^{(x \star y)}(u \star v) = {}^{[x,y]}(u \star v)$,
(4) $[x \star y, z] = [x, y] \star z$,
(5) $(x^{-1} \star y) = {}^{x^{-1}}((x \star y)^{-1})$ *and also* $(x \star y^{-1}) = {}^{y^{-1}}((x \star y)^{-1})$
for all $x, y, z, u, v \in K$.

Definition 2.3 ([1], [6]). Let $(K, \cdot, \star)$ be a multiplicative Lie algebra. Then a subgroup G of K is said to be a subalgebra of K if $x \star y \in G$ for all $x, y \in G$. Further G is said to be an ideal of K if it is a normal subgroup of K and $x \star y \in G$ for all $x \in K$ and for all $y \in G$.

Remark 2.4. Using Proposition 2.2, identity (4) above, it can be seen that the group center $Z(K)$ of the multiplicative Lie algebra K and the Lie center $LZ(K) = \{x \in K\colon\ x \star y = 1 \text{ for all } y \in K\}$ are ideals of K.

Remark 2.5. In any group $(K, \cdot)$, we have two apparent multiplicative Lie products, the trivial one given by $x \star_1 y = 1$ and another given by $x \star_2 y = [x, y]$ for all $x, y \in K$. Authors in [6] called $(K, \cdot, \star_i)$, $i = 1, 2$ the trivial and improper multiplicative Lie algebras, respectively.

Definition 2.6 ([7]). Let $(K, \cdot, \star)$ be a multiplicative Lie algebra and $a, b \in K$. Then the element $(a \star b)^{-1}[a, b]$ of K is called *Lie commutator* of a, b and is denoted by ${}^{L}[a, b]$, where $[a, b]$ is the commutator of a and b.

Note. The subgroup generated by the set $\{{}^L[a,b]\colon\ a\in K,\ b\in G\}$ will be denoted by ${}^L[K,G]$, where G is a subset of K.

Proposition 2.7 ([7]). *Let G be an ideal of K. Then ${}^L[K,G]$ is an ideal of K.*

Note. The ideal ${}^L[K,K]$ is called the *Lie commutator* of K.

Definition 2.8 ([7]). The set $MZ(K)=\{a\in K\colon\ {}^L[a,b]=1 \text{ for all } b\in K\}$ is an ideal of K and is termed as the multiplicative Lie center of K.

Proposition 2.9 ([7], [12]). *Let K be a multiplicative Lie algebra. Then*
(1) ${}^L[a,a]=1$,
(2) ${}^L[a,b]{}^L[b,a]=1$,
(3) ${}^L[ab,c]={}^a({}^L[b,c]){}^L[a,c]$,
(4) ${}^L[a,bc]={}^L[a,b]{}^b({}^L[a,c])$,
(5) ${}^a({}^L[b,c])={}^L[{}^ab,{}^ac]$,
(6) ${}^L[a^{-1},b]={}^{a^{-1}}({}^L[b,a])$ *and* ${}^L[a,b^{-1}]={}^{b^{-1}}({}^L[b,a])$,
(7) ${}^{{}^L[a,b]}(x\star y)=(x\star y)$, *in particular* $[{}^L[a,b],(x\star y)]=1$
for all $a,b,c,x,y\in K$.

3. Lie perfect multiplicative Lie algebra

Definition 3.1. A multiplicative Lie algebra K is said to be Lie perfect if

$$K={}^L[K,K].$$

Remark 3.2. In [1], a multiplicative Lie algebra $(K,\cdot,\star)$ is said to be perfect if $K=K\star K$, while in [6] K is perfect if $K=(K\star K)[K,K]$. In the case of the Lie rings, these three definitions coincide. We give two examples that show our definition of Lie perfect is different from the one given in [1] and [6].

Example 3.3. Let K be a perfect group with trivial multiplicative Lie product, i.e., $x\star y=1$ for all $x,y\in K$. Clearly, $K\star K=\{1\}$, i.e., $K\star K\neq K$. Thus, it is not perfect in the sense of [1]. But ${}^L[K,K]=\langle\{{}^L[a,b]=(a\star b)^{-1}[a,b]\colon\ a,b\in K\}\rangle=\langle\{[a,b]\colon\ a,b\in K\}\rangle=[K,K]=K$. Hence, K is Lie perfect.

Example 3.4. Let K be a perfect group with improper multiplicative Lie product defined by $x\star y=[x,y]$ for all $x,y\in K$. Clearly, $K\star K=[K,K]=K$ and $(K\star K)[K,K]=K$. Thus, it is perfect in the sense of [6]. But ${}^L[K,K]=\langle\{{}^L[a,b]=(a\star b)^{-1}[a,b]\colon\ a,b\in K\}\rangle=\langle\{[a,b]^{-1}[a,b]\colon\ a,b\in K\}\rangle=\{1\}$. Hence, K is not Lie perfect.

Proposition 3.5. *Let K_1 and K_2 be two multiplicative Lie algebras. Then* ${}^L[K_1 \times K_2, K_1 \times K_2] = ({}^L[K_1, K_1] \times {}^L[K_2, K_2])$ *and* $MZ(K_1 \times K_2) = MZ(K_1) \times MZ(K_2)$.

P r o o f. By the definition of the Lie commutator, ${}^L[K_1 \times K_2, K_1 \times K_2] = \langle {}^L[(a,b),(c,d)] \colon a, c \in K_1,\ b, d \in K_2 \rangle$. Now,

$$\begin{aligned} {}^L[(a,b),(c,d)] &= ((a,b) \star (c,d))^{-1}[(a,b),(c,d)] = ((a \star c),(b \star d))^{-1}([a,c],[b,d]) \\ &= ((a \star c)^{-1}[a,c],(b \star d)^{-1}[b,d]) = ({}^L[a,c], {}^L[b,d]). \end{aligned}$$

Therefore, ${}^L[K_1 \times K_2, K_1 \times K_2] = ({}^L[K_1, K_1] \times {}^L[K_2, K_2])$. Also, $MZ(K_1 \times K_2) = \{(a,b) \colon {}^L[(a,b),(c,d)] = 1 \text{ for all } (c,d) \in K_1 \times K_2\}$. Since ${}^L[(a,b),(c,d)] = ({}^L[a,c], {}^L[b,d])$, this implies ${}^L[(a,b),(c,d)] = 1$ if and only if ${}^L[a,c] = 1$ and $L[b,d] = 1$. Hence, $MZ(K_1 \times K_2) = MZ(K_1) \times MZ(K_2)$. □

Remark 3.6. If K is a Lie perfect multiplicative Lie algebra, then for any ideal H, the quotient K/H is also Lie perfect. Moreover, if K' is another Lie perfect multiplicative Lie algebra, then using Proposition 3.5 we can prove that $K \times K'$ is also a Lie perfect multiplicative Lie algebra.

Now we prove some Lie commutator identities which are needed to prove Proposition 3.8 and Lie-Hirsch-Plotkin Theorem 5.11.

Proposition 3.7. *Let K be a multiplicative Lie algebra. Then*
(1) ${}^L[a,b] \star [x,y] = {}^L[(a \star b),[x,y]]$,
(2) ${}^L[a,b] \star (x \star y) = {}^L[a \star b, x \star y]$,
(3) ${}^L[a,b] \star {}^L[x,y] = {}^L[a \star b, {}^L[x,y]]$,
(4) $x \star {}^L[a,b] = {}^L[x, a \star b]$,
(5) $[x, {}^L[a,b]] = {}^L[x,[a,b]]$
for all $a, b, x, y \in K$.

P r o o f. We give a proof of identities (1) and (4), the rest of them can be done similarly.

$$\begin{aligned} (1)\quad {}^L[a,b] \star [x,y] &= ((a \star b)^{-1}[a,b]) \star [x,y] = {}^{(a\star b)^{-1}}([a,b] \star [x,y])((a \star b)^{-1} \star [x,y]) \\ &= {}^{(a\star b)^{-1}}[(a \star b),[x,y]]^{(a\star b)^{-1}}((a \star b) \star [x,y])^{-1} \\ &\qquad \text{(by Proposition 2.2, identity (4))} \\ &= {}^{(a\star b)^{-1}}([(a \star b),[x,y]]((a \star b) \star [x,y])^{-1}) \\ &= {}^{(a\star b)^{-1}}({}^L[(a \star b),[x,y]]) \\ &= {}^L[(a \star b),[x,y]] \quad \text{(by Proposition 2.9, identity (7))}. \end{aligned}$$

$$
\begin{aligned}
(4)\qquad x\star {}^{L}[a,b] &= x\star(a\star b)^{-1}[a,b] = (x\star(a\star b)^{-1})^{(a\star b)^{-1}}(x\star[a,b])\\
&= {}^{(a\star b)^{-1}}(x\star(a\star b))^{-1\,(a\star b)^{-1}}[x,a\star b]\\
&\qquad\qquad \text{(by Proposition 2.2, identity (4))}\\
&= {}^{(a\star b)^{-1}}((x\star(a\star b))^{-1}[x,a\star b])\\
&\qquad\qquad \text{(by Definition 2.1, identity (5))}\\
&= {}^{(a\star b)^{-1}}({}^{L}[x,(a\star b)])\\
&= {}^{L}[x,a\star b] \quad \text{(by Proposition 2.9, identity (7)).}
\end{aligned}
$$

□

Proposition 3.8. *Let $(K,\cdot,\star)$ be a Lie perfect multiplicative Lie algebra. Then*
(a) $MZ(K)\subseteq LZ(K)$,
(b) $MZ(K)\subseteq Z(K)$.

P r o o f. Let $x\in MZ(K)$ and $y\in K$. Since K is Lie perfect, there exist $x_i,y_i\in K$, $i=1,2,\ldots,n$ such that $y=\prod\limits_{i=1}^{n}{}^{L}[x_i,y_i]$.

(a) To prove $x\in LZ(G)$, we need to prove that $x\star y=1$. Consider

$$
\begin{aligned}
(x\star y) &= \left(x\star\prod_{i=1}^{n}{}^{L}[x_i,y_i]\right)\\
&= (x\star{}^{L}[x_1,y_1])({}^{{}^{L}[x_1,y_1]}(x\star{}^{L}[x_2,y_2]))\ldots\left({}^{\prod_{i=1}^{n-1}{}^{L}[x_i,y_i]}(x\star{}^{L}[x_n,y_n])\right).
\end{aligned}
$$

We establish this identity by induction on n and using Definition 2.1, identity (2). Using Proposition 3.7, identity (4), we get $x\star{}^{L}[x_k,y_k]={}^{L}[x,x_k\star y_k]$ for $k=1,2,\ldots,n$. Since $x\in MZ(K)$, ${}^{L}[x,x_k\star y_k]=1$ for $k=1,2,\ldots,n$. Hence, we get $x\star y=1$, i.e., $x\in LZ(K)$. This implies that $MZ(K)\subseteq LZ(K)$.

(b) To prove $x\in Z(K)$, we show that $[x,y]=1$. Consider

$$
\begin{aligned}
[x,y] &= \left[x,\prod_{i=1}^{n}{}^{L}[x_i,y_i]\right]\\
&= [x,{}^{L}[x_1,y_1]]({}^{{}^{L}[x_1,y_1]}[x,{}^{L}[x_2,y_2]])\ldots\left({}^{\prod_{i=1}^{n-1}{}^{L}[x_i,y_i]}[x,{}^{L}[x_n,y_n]]\right).
\end{aligned}
$$

By Proposition 3.7, identity (5), we get $[x,{}^{L}[x_k,y_k]]={}^{L}[x,[x_k,y_k]]$ for $k=1,2,\ldots,n$. Since $x\in MZ(K)$, therefore, ${}^{L}[x,[x_k,y_k]]=1$ for $k=1,2,\ldots,n$. Hence, we get $[x,y]=1$, i.e., $x\in Z(K)$. This implies that $MZ(K)\subseteq Z(K)$. □

4. Lie central extension of a multiplicative Lie algebra

Definition 4.1. An extension

$$E(H,K) \equiv \{1\} \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to \{1\}$$

of multiplicative Lie algebras is called *Lie central extension* if $\alpha(H) \subseteq MZ(G)$.

Note. Such extensions are frequently encountered; for example, every Lie nilpotent, multiplicative Lie algebra can be constructed from improper multiplicative Lie algebras by means of central extension of sequences.

Remark 4.2. In [1] Bak, et al. called an extension central if $H \subseteq LZ(G)$, while Lal and Upadhyay in [6] called an extension central if $H \subseteq Z(G) \cap LZ(G)$. From Remark 3.3 of [7] it can be seen that $Z(G) \cap LZ(G) \subseteq MZ(G)$. Therefore, every central extension of [6] is a Lie central extension. In the next example we see that a Lie central extension need not be a central extension in the sense of [1] and [6].

Example 4.3. Consider the extension of multiplicative Lie algebra $\mathbb{Z}_2$ given by $E(A_3, \mathbb{Z}_2) \equiv \{1\} \to A_3 \xrightarrow{i} S_3 \xrightarrow{p} \mathbb{Z}_2 \to \{1\}$, where A_3 and S_3 have the multiplicative Lie product given by $a \star b = [a,b]$. By the definition of multiplicative Lie center, $MZ(S_3) = S_3$ and $i(A_3) = A_3 \subseteq S_3$. Thus, the extension $E(A_3, \mathbb{Z}_2)$ is a Lie central extension. Since the Lie center $LZ(S_3) = \{a \in S_3 \colon a \star b = 1 \text{ for all } b \in S_3\} = \{a \in S_3 \colon [a,b] = 1 \text{ for all } b \in S_3\} = \{1\}$ and $Z(S_3) \cap LZ(S_3) = \{1\}$, $A_3 \subsetneq LZ(S_3)$. Thus, the extension $E(A_3, \mathbb{Z}_2)$ is not a central extension in the sense of [1] and [6].

Remark 4.4. In the next example we see that central extensions of an improper multiplicative Lie algebra need not be an improper multiplicative Lie algebra.

Example 4.5. Consider the extension of multiplicative Lie algebra $\mathbb{Z}_2$ given by $E(\mathbb{Z}_2, \mathbb{Z}_2) \equiv \{1\} \to \mathbb{Z}_2 \xrightarrow{i} V_4 \xrightarrow{p} \mathbb{Z}_2 \to \{1\}$, where $V_4 = \langle a, b \colon a^2 = b^2 = 1,\ ab = ba \rangle$ is Klein's four group and $\star$ on V_4 is given by $a \star b = a$. As $a \star b \neq [a,b]$, this implies that V_4 is not an improper multiplicative Lie algebra.

Definition 4.6. A Lie central extension

$$E(H,K) \equiv \{1\} \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to \{1\}$$

of multiplicative Lie algebras is called a *free Lie central extension* if given any Lie central extension

$$E(H',K') \equiv \{1\} \to H' \xrightarrow{\alpha'} G \xrightarrow{\beta'} K' \to \{1\}$$

and a homomorphism $f\colon K \to K'$, there exists a morphism (g, h, f) from $E(H, K)$ to $E(H', K')$ such that the following diagram is commutes:

$$\begin{array}{ccccccccc}
E(H,K) \equiv \{1\} & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & \{1\} \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \\
E(H',K') \equiv \{1\} & \longrightarrow & H' & \xrightarrow{\alpha'} & G & \xrightarrow{\beta'} & K' & \longrightarrow & \{1\}.
\end{array}$$

Remark 4.7. Let K be a multiplicative Lie algebra with free presentation

$$E(R, K) \equiv \{1\} \to R \xrightarrow{i} F \xrightarrow{\nu} K \to \{1\}.$$

Consider the ideal ${}^L[R, F]$. Since R is an ideal of F, ${}^L[R, F] \subseteq R$, we get an extension

$$E'\Big(\frac{R}{{}^L[R,F]}, K\Big) \equiv \{1\} \to \frac{R}{{}^L[R,F]} \xrightarrow{i'} \frac{F}{{}^L[R,F]} \xrightarrow{\nu'} K \to \{1\}$$

induced by $E(R, K)$. Then $E'\big(\frac{R}{{}^L[R,F]}, K\big)$ is a Lie central extension, where ν' is defined by $\nu'(a + {}^L[R, F]) = \nu(a)$. Further, one can show that $E'\big(\frac{R}{{}^L[R,F]}, K\big)$ is a free Lie central extension similar to the case of groups, see [5].

Proposition 4.8. *Let K be a Lie perfect multiplicative Lie algebra and*

$$E(H, K) \equiv \{1\} \to H \xrightarrow{i} G \xrightarrow{\nu} K \to \{1\}$$

be a Lie central extension of K. Then the Lie commutator ideal ${}^L[G, G]$ is Lie perfect.

Proof. Since K is Lie perfect, $\nu({}^L[G, G]) = K = {}^L[K, K]$. Thus, for any $a \in G$, there exists $g \in {}^L[G, G]$ such that $\nu(a) = \nu(g)$, $\nu(ag^{-1}) = 1$, i.e., $ag^{-1} \in \mathrm{Ker}(\nu) = \mathrm{Im}\, i$. This implies that there exists $h \in H$ such that $h = ag^{-1}$, $a = hg$, where $h \in H$ and $g \in {}^L[G, G]$. Let $a' = h'g'$, where $h' \in H$ and $g' \in {}^L[G, G]$. Then

$$\begin{aligned}
{}^L[a, a'] &= {}^L[hg, h'g'] = {}^h({}^L[g, h'g'])^L[h, h'g'] = {}^h({}^L[g, h'g']) = {}^h({}^L[g, h'])^{hh'}({}^L[g, g']) \\
&= {}^{hh'}({}^L[g, g']) = L[{}^{hh'}g, {}^{hh'}g'].
\end{aligned}$$

We get ${}^L[a, a'] \in {}^L[{}^L[G, G], {}^L[G, G]]$. This implies that $L[G, G] \subseteq {}^L[{}^L[G, G], {}^L[G, G]]$. □

Proposition 4.9. *Let $E(H, K) \equiv \{1\} \to H \xrightarrow{i} G \xrightarrow{\varphi} K \to \{1\}$ be a Lie central extension of K such that $H \subseteq Z(G)$ and G be a Lie perfect multiplicative Lie algebra. Let f and g be homomorphisms from G to G' including the morphisms $(f|_H, f, I_K)$ and $(g|_H, g, I_K)$ from $E(H, K)$ to Lie central extension $E'(H, K) \equiv \{1\} \to H \xrightarrow{i} G' \xrightarrow{\chi} K \to \{1\}$. Then $f = g$.*

P r o o f. Since G is Lie perfect, it is sufficient to show that $f({}^L[x,y]) = g({}^L[x,y])$, for all $x, y \in G$. As $(f|_H, f, I_K)$ and $(g|_H, g, I_K)$ are morphisms between $E(H,K)$ and $E'(H,K)$, the diagram

$$\begin{array}{ccccccccc} E(H,K) \equiv \{1\} & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{\varphi} & K & \longrightarrow & \{1\} \\ & & {\scriptstyle f|_H}\downarrow{\scriptstyle g|_H} & & {\scriptstyle f}\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle I_K} & & \\ E'(H,K) \equiv \{1\} & \longrightarrow & H & \xrightarrow{i} & G' & \xrightarrow{\chi} & K & \longrightarrow & \{1\} \end{array}$$

is commutative. Thus, for any $x \in G$, $\chi(f(x)) = \varphi(x) = \chi(g(x))$, which implies $f(x)g(x)^{-1} \in \mathrm{Ker}(\chi)$, i.e., there exists unique $h \in H$ such that $f(x) = g(x)h$. Now $f({}^L[k,k']) = {}^L[f(k), f(k')] = {}^L[g(k)h, g(k')h'] = {}^L[g(k), g(k')] = g({}^L[k,k'])$ (for $h, h' \in MZ(G)$ and using Proposition 2.9, (3), (4)). This proves the result. □

Definition 4.10. A Lie central extension

$$E(H,K) \equiv \{1\} \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to \{1\}$$

is called a *universal Lie central extension* by K if given any Lie central extension

$$E'(L,K) \equiv \{1\} \to L \xrightarrow{\alpha'} U \xrightarrow{\beta'} K \to \{1\}$$

by K, there is a unique homomorphism φ from G to U inducing a morphism (χ, φ, I_K) from $E(H,K)$ to $E'(L,K)$.

Remark 4.11. A universal Lie central extension of K is unique up to equivalence.

Proposition 4.12. *If $E(H,K) \equiv \{1\} \to H \xrightarrow{i} U \xrightarrow{\nu} K \to \{1\}$ is a universal Lie central extension, then U is Lie perfect.*

P r o o f. Suppose that U is not Lie perfect. Then $\frac{U}{{}^L[U,U]} \neq {}^L[U,U]$. Consider the exact sequence

$$E\Big(\frac{U}{{}^L[U,U]}, K\Big) \equiv \{1\} \to \frac{U}{{}^L[U,U]} \xrightarrow{i_1} \frac{U}{{}^L[U,U]} \times K \xrightarrow{p_2} K \to \{1\},$$

where i_1 is the first inclusion and p_2 is projection homomorphism. We know $\frac{U}{{}^L[U,U]}$ has improper multiplicative Lie product (see [7], Proposition 3.4), therefore $MZ\big(\frac{U}{{}^L[U,U]}\big) = \frac{U}{{}^L[U,U]}$ and $\frac{U}{{}^L[U,U]} \subseteq MZ\big(\frac{U}{{}^L[U,U]} \times K\big) = MZ\big(\frac{U}{{}^L[U,U]}\big) \times MZ(K)$. Hence, $E\big(\frac{U}{{}^L[U,U]}, K\big)$ is a Lie central extension of K. Further, consider the maps $(f,\nu)(u) = (u{}^L[U,U], \nu(u))$ and $(0,\nu)(u) = ({}^L[U,U], \nu(u))$. Then (f,ν) and $(0,\nu)$

are two distinct multiplicative Lie algebra homomorphisms from U to $\frac{U}{{}^L[U,U]} \times K$, which induce two distinct morphisms from $E(H,K)$ to $E\big(\frac{U}{{}^L[U,U]}, K\big)$. This leads to a contradiction that $E(H,K)$ is a universal Lie central extension. Therefore, U is a Lie perfect multiplicative Lie algebra. □

Proposition 4.13. *Every Lie perfect multiplicative Lie ring K admits a universal Lie central extension.*

P r o o f. Suppose K is a perfect Lie ring. Let

$$E(R,K) \equiv \{1\} \to R \xrightarrow{i} F \xrightarrow{\nu} K \to \{1\}$$

be a free multiplicative Lie algebra presentation of K. Then we have a Lie central extension

$$E\Big(\frac{R}{{}^L[R,F]}, K\Big) \equiv \{1\} \to \frac{R}{{}^L[R,F]} \xrightarrow{\bar{i}} \frac{F}{{}^L[R,F]} \xrightarrow{\bar{\nu}} K \to \{1\}.$$

Since K is perfect, we have another Lie central extension

$$E\Big(\frac{R \cap {}^L[F,F]}{{}^L[R,F]}, K\Big) \equiv \{1\} \to \frac{R \cap {}^L[F,F]}{{}^L[R,F]} \xrightarrow{\bar{i}} \frac{{}^L[F,F]}{{}^L[R,F]} \xrightarrow{\bar{\nu}} K \to \{1\}$$

of K. Now we prove that $E\big(\frac{R}{{}^L[R,F]}, K\big)$ is a universal Lie central extension of K. Let

$$E'(H,K) \equiv \{1\} \to H \xrightarrow{i} G \xrightarrow{\nu} K \to \{1\}$$

be a Lie central extension of K. Since F is a free multiplicative Lie algebra, there exists a homomorphism $\varphi\colon F \to G$ inducing a morphism $(\varphi|_R, \varphi, I_K)$ from $E(R,K)$ to $E'(H,K)$, which gives a morphism from $E\big(\frac{R}{{}^L[R,F]}, K\big)$ to $E'(H,K)$, which in turn gives a morphism from $E\big(\frac{R\cap {}^L[F,F]}{{}^L[R,F]}, K\big)$ to $E'(H,K)$. Since K is a Lie ring and $\frac{{}^L[F,F]}{{}^L[R,F]}$ is Lie perfect, by Proposition 4.9 such morphism will be unique. This proves our result. □

5. Generalization of Lie nilpotency in multiplicative Lie algebras

In [7], authors have developed the notion of solvability and nilpotency in multiplicative Lie algebras. The lower central series $\gamma_n(K)$ of a multiplicative Lie algebra K is defined in the following way: $\gamma_0(K) = K$, $\gamma_1(K) = {}^L[K, \gamma_0(K)]$ with this assumption suppose $\gamma_n(K)$ has been already defined, now define $\gamma_{n+1}(K) = {}^L[K, \gamma_n(K)]$. This gives us a descending chain

$$K = \gamma_0(K) \trianglerighteq \gamma_1(K) \trianglerighteq \gamma_2(K) \ldots \gamma_k(K) \trianglerighteq \gamma_{k+1}(K) \ldots$$

of ideals which we call the lower central series of K. The upper central series of ideals $Z_n(K)$ of K is defined inductively as $Z_0(K) = \{1\}$, $Z_1(K) = MZ(K)$. Clearly $\frac{Z_1(K)}{(Z_0(K))} = MZ\big(\frac{K}{(Z_0(K))}\big)$. Suppose $Z_n(K)$ has been already defined. Then define $Z_{n+1}(K)$ as follows:

$$\frac{Z_{n+1}(K)}{Z_n(K)} = MZ\Big(\frac{K}{Z_n(K)}\Big).$$

This gives us an ascending chain

$$\{1\} = Z_0(K) \trianglelefteq Z_1(K) \trianglelefteq Z_2(K) \trianglelefteq \ldots Z_n(K) \trianglelefteq Z_{n+1}(K) \ldots$$

of ideals, which is called the *upper central series* of K. Further, it can be seen that

$$Z_{n+1}(K) = \{k \in K\colon\ {}^L[k,h] \in Z_n(K)\ \forall\, h \in K\}.$$

Definition 5.1. A multiplicative Lie algebra $(K, \cdot, *)$ is said to be Lie nilpotent if $\gamma_n(K) = \{1\}$ or equivalently, $Z_n(K) = K$ for some $n \in \mathbb{N}$. A multiplicative Lie algebra K is said to be Lie nilpotent of class n if $\gamma_n(K) = \{1\}$ but $\gamma_{n-1}(K) \neq \{1\}$ or equivalently, $Z_n(K) = K$ but $Z_{n-1}(K) \neq K$.

Proposition 5.2. *Let K be a Lie nilpotent multiplicative Lie algebra and $G \neq \{1\}$ be an ideal in K. Then $G \cap MZ(K)$ is a nontrivial ideal of both K and G.*

Proof. Since K is Lie nilpotent, the upper central series

$$\{1\} = Z_0(K) \trianglelefteq Z_1(K) \trianglelefteq \ldots \trianglelefteq Z_n(K) = K$$

terminates at K for some $n \in \mathbb{N}$. Then there exists least positive integer k such that $G \cap Z_k(K) \neq \{1\}$. Since ${}^L[G \cap Z_k(K), K]$ is a subset of $G \cap Z_{k-1}(K)$ and k is the least integer, $G \cap Z_{k-1}(K) = \{1\}$, i.e., ${}^L[G \cap Z_k(K), K] = \{1\}$. By the definition of the multiplicative Lie center $MZ(K)$ we get $G \cap Z_k(K) \subseteq G \cap MZ(K)$, i.e., $G \cap MZ(K) \neq \{1\}$. Since G and $MZ(K)$ are ideals of K, $G \cap MZ(K)$ is also an ideal of both K and G. □

Remark 5.3. From Proposition 5.2, we can say that in a Lie nilpotent multiplicative Lie algebra K, every minimal ideal is contained in the multiplicative Lie center $MZ(K)$ of K, for if G is a minimal ideal, then $G \cap MZ(K) = G$ implies $G \subseteq MZ(K)$.

Now we provide the statement of Fitting's theorem for multiplicative Lie algebras, which was proven in [7] and for a group can be seen in [11]. We will utilize this theorem in the proof of our main theorem, see Theorem 5.11.

Theorem 5.4 ([7]). *Let G and H be two Lie nilpotent ideals of K of classes n and m, respectively, such that $MZ(G)$ and $MZ(H)$ are also ideals of K. Then GH is a Lie nilpotent ideal of K of class at most $n + m$.*

Definition 5.5. A multiplicative Lie algebra is said to be locally Lie nilpotent if all of its finitely generated subalgebras are Lie nilpotent.

Definition 5.6. A multiplicative Lie algebra K is said to satisfy the L-max condition if for every finitely generated Lie nilpotent subalgebra L of K, every subalgebra of L is finitely generated.

Proposition 5.7. *Homomorphic images and subalgebras of a locally Lie nilpotent multiplicative Lie algebra are locally Lie nilpotent.*

Next, we define the Lie idealizer and prove two propositions which give us conditions for a subalgebra to be the ideal of a multiplicative Lie algebra, which is used in the proof of Theorem 5.11.

Definition 5.8. Let K be a multiplicative Lie algebra and G be a subalgebra of K. An element $k \in K$ is called a *Lie idealizer* of G if
(a) $kgk^{-1} \in G$ for all $g \in G$ and
(b) ${}^L[k, g] \in G$ for all $g \in G$.

Proposition 5.9. *Let G be a subalgebra of the multiplicative Lie algebra K generated by the subset $X = \{a_i \in K\colon 1 \leqslant i \leqslant n\}$. Also, let $k \in K$ satisfy $ka_ik^{-1} \in G$ and ${}^L[k, a_i] \in G$ for each $a_i \in X$. Then k is a Lie idealizer of G.*

Proof. Let $L = \{g \in G\colon kgk^{-1} \in G, {}^L[k, g] \in G\}$. To prove that k is a Lie idealizer of G, it is enough to show that L is a subalgebra of G containing X, i.e., $xy^{-1} \in L$ and $x \star y \in L$ for all $x, y \in L$. From the definition of L, $X \subseteq L$. Let $x, y \in L$. Then kxk^{-1}, kyk^{-1}, ${}^L[k, x]$ and ${}^L[k, y] \in G$. To prove that $xy^{-1} \in L$, consider

$$kxy^{-1}k^{-1} = kxk^{-1}ky^{-1}k^{-1} = (kxk^{-1})(kyk^{-1})^{-1} \in G \tag{5.1}$$

and

(5.2)
$$\begin{aligned} {}^L[k, xy^{-1}] &= ({}^L[k, x])^x({}^L[k, y^{-1}]) \text{ (by Proposition 2.9, identity (4))} \\ &= ({}^L[k, x])^{xy^{-1}}({}^L[k, y])^{-1} \in G \text{ (by Proposition 2.9, identities (6) and (2))}. \end{aligned}$$

From, (5.1) and (5.2) we get $xy^{-1} \in L$. Now, we prove that $x \star y \in L$. Consider $k(x \star y)k^{-1} = {}^k(x \star y) = {}^kx \star {}^ky \in G$ (as G is a subalgebra of K) and ${}^L[x \star y, k] = ((x \star y) \star k)^{-1}[x \star y, k]$. We can write $(x \star y) \star k = ((x \star y) \star {}^{yy^{-1}}k)$. By Jacobi identity, we have

$$\begin{aligned} ((x \star y) \star {}^{yy^{-1}}k)^{-1} &= ((y \star {}^{y^{-1}}k) \star {}^{{}^{y^{-1}}k}x)(({}^{y^{-1}}k \star x) \star {}^xy) \\ &= ({}^{y-1}(y \star k) \star {}^{{}^{y^{-1}}k}x)({}^{y^{-1}}(k \star {}^yx) \star {}^xy). \end{aligned}$$

First consider

$$({}^{y^{-1}}(y \star k) \star {}^{y^{-1}k}x) = ({}^{y^{-1}}(y \star k) \star (y^{-1}kyxyk^{-1}y^{-1})).$$

We have $y \star k = [y,k]({}^{L}[k,y])^{-1} \in G$ and $y^{-1}kyxyk^{-1}y^{-1} \in L$. We know that G is a subalgebra and $L \subseteq G$, thus we have $({}^{y^{-1}}(y \star k) \star {}^{y^{-1}k}x) \in G$. Now consider

$$\begin{aligned}({}^{y^{-1}}(k \star {}^{y}x) \star {}^{x}y) &= ({}^{y^{-1}}([k,{}^{y}x]({}^{L}[{}^{y}x,k])^{-1} \star {}^{x}y) \quad \text{(by Definition 2.6)}\\ &= ({}^{y^{-1}}([k,{}^{y}x]^{L}[k,{}^{y}x] \star {}^{x}y) \quad \text{(by Proposition 2.9, identity (2))}.\end{aligned}$$

Since $[k,{}^{y}x]^{L}[k,{}^{y}x] \in G$ and ${}^{x}y \in L$, this implies $({}^{y^{-1}}(k \star {}^{y}x) \star {}^{x}y) \in G$. Hence, we get $((x \star y) \star {}^{yy^{-1}}k)^{-1} = ((x \star y) \star k)^{-1} \in G$. Next, consider

$$\begin{aligned}[x \star y, k] &= [x,y] \star k \quad \text{(by Proposition 2.2, identity (4))}\\ &= {}^{L}[k,[x,y]][k,[x,y]]^{-1} \quad \text{(by Definition 2.6)}.\end{aligned}$$

Since L is a subgroup of G and $k[x,y]k^{-1} \in G$, this implies $[k,[x,y]]^{-1} \in G$ and ${}^{L}[k,[x,y]] \in G$ (because $[x,y] \in L$). Thus, we get $[x \star y, k] \in G$. Now, we have $((x \star y) \star k)^{-1}$ and $[x \star y, k] \in G$. Therefore, we get ${}^{L}[x \star y, k] \in L$. Hence, L becomes a subalgebra of K, i.e., $x \star y \in L$. Since $X \subseteq L$ and L is a subalgebra of G, we conclude that k is a Lie idealizer of G. □

Proposition 5.10. *Let G be a subalgebra of K generated by a subset $X = \{a_i \in K\colon 1 \leqslant i \leqslant n\}$ and K be generated by a subset $Y = \{b_j \in K\colon 1 \leqslant j \leqslant m\}$. If $b_j a_i b_j{}^{-1} \in G$ and ${}^{L}[b_j, a_i] \in G$ for each $a_i \in X$, $b_j \in Y$, then G is an ideal of K.*

Proof. Let $Z = \{x \in K\colon xGx^{-1} \subseteq G \,\&\, {}^{L}[x,G] \subseteq G\}$. Clearly, by Proposition 5.9, $Y \subseteq Z$. To show that Z is an ideal of K, we prove that $g \star x$ and $xy^{-1} \in Z$ for all $x, y \in Z$ and $g \in K$. Let $x, y \in Z$. Then $xgx^{-1}, ygy^{-1}, {}^{L}[x,g], {}^{L}[y,g] \in G$ for each $g \in G$. To show $xy^{-1} \in Z$, consider

$$xy^{-1}g(xy^{-1})^{-1} = xy^{-1}gyx^{-1} = x(ygy^{-1})^{-1}x^{-1} \in G \tag{5.3}$$

and

(5.4)

$$\begin{aligned}{}^{L}[xy^{-1},g] &= {}^{x}({}^{L}[y^{-1},g])({}^{L}[x,g]) \quad \text{(by Proposition 2.9, identity (3))}\\ &= {}^{xy^{-1}}({}^{L}[y,g])^{-1}({}^{L}[x,g]) \in G \text{ (by Proposition 2.9, identities (6) and (2))}.\end{aligned}$$

From (5.3) and (5.4) we get $xy^{-1} \in Z$, i.e., Z is a subgroup of K. Now we show that $x \star y \in Z$. Let $g \in G$. Consider

$$\begin{aligned}(x \star y)g(x \star y)^{-1} &= (x \star y)g(x \star y)^{-1}g^{-1}g = [(x \star y), g]g = ([x,y] \star g)g\\ &= [[x,y],g]({}^{L}[[x,y],g])^{-1}g \quad \text{(by Definition 2.6)}.\end{aligned}$$

Since Z is a group $[x,y] \in Z$, this implies $[x,y]g[x,y]^{-1}$ and ${}^{L}[[x,y],g] \in G$. Hence, we have $(x \star y)g(x \star y)^{-1} \in G$. Now we prove that ${}^{L}[x \star y, g] \in G$. Consider ${}^{L}[x \star y, g] = ((x \star y) \star g)^{-1}[x \star y, g]$. By Jacobi identity, we have

$$\begin{aligned}((x \star y) \star {}^{yy^{-1}}g)^{-1} &= ((y \star {}^{y^{-1}}g) \star {}^{{}^{y^{-1}}g}x)(({}^{y^{-1}}g \star x) \star {}^{x}y) \\ &= ({}^{y-1}(y \star g) \star {}^{{}^{y^{-1}}g}x)({}^{y^{-1}}(g \star {}^{y}x) \star {}^{x}y).\end{aligned}$$

First consider

$$\begin{aligned}({}^{y-1}(y \star g) \star {}^{{}^{y^{-1}}g}x) &= z \star {}^{{}^{y^{-1}}g}x, \text{ where } z = ({}^{y-1}(y \star g)) \\ &= {}^{y^{-1}g}({}^{({}^{y^{-1}}g)^{-1}}z \star x) \\ &= ({}^{y^{-1}}g \cdot {}^{({}^{y^{-1}}g)^{-1}} z \star x)({}^{y^{-1}}g \star x)^{-1} \text{ (by Definition 2.1, identity (3))} \\ &= ((z({}^{y^{-1}}g)) \star x)({}^{y^{-1}}g \star x)^{-1}.\end{aligned}$$

Since $z \in G$ (as $y \star g = [y,g]({}^{L}[y,g])^{-1} \in G$) and $({}^{y^{-1}}g) \in G$, consequently we get $({}^{y^{-1}}g \star x) = [{}^{y^{-1}}g, x]({}^{L}[{}^{y^{-1}}g, x]) \in G$ and $(z({}^{y^{-1}}g)) \star x \in G$, (by Definition 2.1, identity (3)). So, $({}^{y-1}(y \star g) \star {}^{{}^{y^{-1}}g}x) \in G$. Now, consider

$$({}^{y^{-1}}(g \star {}^{y}x) \star {}^{x}y) = ({}^{y^{-1}}([g, {}^{y}x]({}^{L}[{}^{y}x, g])^{-1}) \star {}^{x}y) \quad \text{(by Definition 2.6)}.$$

Since Z is a subgroup of $G, {}^{y}x \in Z$ and $g \star {}^{y}x = ({}^{y}x \star g)^{-1}[{}^{y}x, g]([{}^{y}x, g])^{-1} = {}^{L}[{}^{y}x, g]({}^{y}xg({}^{y}x)^{-1}g^{-1})^{-1} \in G$, this implies $({}^{y^{-1}}(g \star {}^{y}x) \star {}^{x}y) \in G$. Hence, we get $((x \star y) \star g)^{-1} \in G$. Now, we consider

$$[x \star y, g] = [x,y] \star g = {}^{L}[g,[x,y]][g,[x,y]]^{-1} \quad \text{(by Definition 2.6)}.$$

Since $[x,y] \in Z$, we get that ${}^{L}[g,[x,y]][g,[x,y]]^{-1} \in G$, i.e., $[x \star y, g] \in G$. So, ${}^{L}[x \star y, g] = ((x \star y) \star g)^{-1}[x \star y, g] \in G$. Hence, $x \star y \in Z$, i.e., Z is a subalgebra of G. Since $Y \subseteq Z$, this implies $Z = K$. So, for each $x \in K$, $xgx^{-1} \in G$ and ${}^{L}[x,g] \in G$. To show that G is an ideal of K, let $k \in K$ and $g \in G$. Then $k \star g = (g \star k)^{-1}[g,k][g,k]^{-1} = {}^{L}[g,k]kgk^{-1}g^{-1} \in G$ (for $kgk^{-1} \in G$ and ${}^{L}[g,k] \in G$). Therefore, G is an ideal of K. □

Now we prove an analogue of Hirsch-Plotkin theorem for multiplicative Lie algebras.

Theorem 5.11. *Let K be a multiplicative Lie algebra with the locally Lie nilpotent ideals G and H such that $MZ(G)$ and $MZ(H)$ are ideals of K. Then the product ideal $J = GH$ is also locally Lie nilpotent.*

P r o o f. Let $Y = \langle\{g_1h_1, g_2h_2, \ldots, g_nh_n\colon\ g_i \in G,\ h_i \in H\}\rangle$ be a finitely generated subalgebra of J and M be a subalgebra of K generated by the set $\{g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n\}$. Since Y is a subalgebra of M, to prove that Y is Lie nilpotent it is sufficient to prove that M is Lie nilpotent. Let Z be the subalgebra of M generated by the set $S = \{g_1, g_2, \ldots, g_n, \alpha_j\}$ and α_j is one of the following forms:

$$[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], \quad 1 \leqslant i \leqslant n,\ l \geqslant 1,$$
$$^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], \quad 1 \leqslant i \leqslant n,\ l \geqslant 1,$$
$$(^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, h_{j_{l+2}}, \ldots, h_{j_k}]), \quad 1 \leqslant i \leqslant n,\ l \geqslant 1,\ k > l.$$

Now, by Propositions 5.9, 5.10 and 3.7, we show that Z is an ideal of M, i.e., we need to prove that $h\alpha_jh^{-1}, {}^L[h, \alpha_j] \in Z$ for all α_j and $h \in \{h_1, h_2, \ldots, h_n\}$.

Case (1): Take $\alpha_j = [g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}]$ and $h \in \{h_1, h_2, \ldots, h_n\}$. Then

$$\begin{aligned} h[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}]h^{-1} &= [h, [g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}]][g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}] \\ &= [g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}, h]^{-1}[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}] \in Z, \end{aligned}$$

and $^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h] \in Z$.

Case (2): Take $\alpha_j = {}^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}]$, then

$$\begin{aligned} h^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}]h^{-1} &= [h, {}^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}]]^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}] \\ &= {}^L[h, [g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}]]^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}] \in Z, \end{aligned}$$

and $^L[{}^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h] = {}^L[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}, h] \in Z$.

Case (3): Take $\alpha_j = {}^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, h_{j_{l+2}}, \ldots, h_{j_k}]$, then

$$\begin{aligned} &h^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, h_{j_{l+2}}, \ldots, h_{j_k}]h^{-1} \\ &\quad = [h, {}^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, \ldots, h_{j_k}]] \\ &\qquad \times {}^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, h_{j_{l+2}}, \ldots, h_{j_k}] \\ &\quad = (^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}, h_{j_{l+1}}], h_{j_{l+2}}, \ldots, h_{j_k}, h])^{-1} \\ &\qquad \times (^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, h_{j_{l+2}}, \ldots, h_{j_k}]) \in Z, \end{aligned}$$

and

$$\begin{aligned} {}^L[h, \alpha_j] &= {}^L[h, {}^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, h_{j_{l+2}}, \ldots, h_{j_k}]] \\ &= (^L[{}^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, h_{j_{l+2}}, \ldots, h_{j_k}], h])^{-1} \\ &= (^L[[g_i, h_{j_1}, h_{j_2}, \ldots, h_{j_l}], h_{j_{l+1}}, \ldots, h_{j_k}, h])^{-1} \in Z. \end{aligned}$$

Hence, by all the three cases Z becomes an ideal of M. Now, it is easily seen that α_j's are in the subalgebra (say I) of H generated by the elements h_j, $[g_i, h_j]$ and $^L[g_i, h_j]$.

Since H is locally Lie nilpotent and I is finitely generated, so I is Lie nilpotent and satisfies the L-max condition on its subalgebras. Thus, the subalgebra generated by the α_j's is finitely generated. Hence, it is worth noticing that the subalgebra Z is finitely generated and contained in G, so it is Lie nilpotent. Hence, we have embedded the subalgebra generated by $\{g_1, g_2, \ldots, g_n\}$ into a Lie nilpotent ideal Z of M. Similarly, we can embed the subalgebra generated by $\{h_1, h_2, \ldots, h_m\}$ into another Lie nilpotent ideal X of M. Also we deduce that $M \subseteq ZX$ is Lie nilpotent by Fitting Theorem 5.4. Since Y is a subalgebra of M, therefore Y is Lie nilpotent. □

Example 5.12. Let $K = D_4 = \langle a, b \colon b^4 = a^2 = 1, ab = b^{-1}a \rangle$ be the multiplicative Lie algebra with $a \star_1 b = [a, b] = b^2$ and $a \star_2 b = b$. Take $a \star_1 b = b^2$. Then D_4 is locally nilpotent, because $\gamma_1(K) = {}^L[K, K] = \langle \{(a \star_1 b)^{-1}[a, b] \colon a, b \in D_4\} \rangle = 1$. Now, take $a \star_2 b = b$ and the subgroup $V_4 = \{1, b^2, a, b^2a\}$ of D_4. Then $\gamma_0(V_4) = V_4, \gamma_1(V_4) = {}^L[V_4, V_4] = \langle \{(r \star_2 s)^{-1}[r, s] \colon r, s \in V_4\} \rangle = \langle \{(r \star_2 s)^{-1} \colon r, s \in V_4\} \rangle = \{1, b^2\}$ and $\gamma_n(V_4) = \{1, b^2\}$ for all $n \geqslant 2$, this implies that V_4 is not Lie nilpotent. Hence, D_4 is not locally Lie nilpotent.

Remark 5.13. We can also form a locally Lie nilpotent, multiplicative Lie algebra by forming the direct sum $K = \bigoplus_{n=3}^{\infty} D_{2^n}$.

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