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NEW EQUIVALENT CONDITIONS FOR HARDY-TYPE INEQUALITIES

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Abstract. We consider a Hardy-type inequality with Oinarov's kernel in weighted Lebesgue spaces. We give new equivalent conditions for satisfying the inequality, and provide lower and upper estimates for its best constant. The findings are crucial in the study of oscillation and non-oscillation properties of differential equation solutions, as well as spectral properties.

Keywords: integral operator; norm; weight function; Lebesgue space; Hardy-type inequality; kernel

MSC 2020: 26D10, 26D15, 47B01, 47B34, 47B37, 47B93, 47G10

1. Introduction

Let $(a,b)\subset R$ and u, v be weight functions on (a,b), i.e., positive measurable functions defined a.e. on (a,b). Let $1<p\leqslant q<\infty$ and consider the Hardy-type inequality

$$\left(\int_a^b\left(\int_a^x k(x,t)f(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q}\leqslant C\left(\int_a^b f^p(x)v(x)\,\mathrm{d}x\right)^{1/p} \tag{1.1}$$

for functions $f\geqslant 0$ a.e. in (a,b), where $k(x,t)$ is called a kernel of the inequality, which is a nonnegative measurable function defined a.e. on $(a,b)\times(a,b)$.

If $k(x,t)\equiv 1$, then (1.1) takes the form

$$\left(\int_a^b\left(\int_a^x f(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q}\leqslant C\left(\int_a^b f^p(x)v(x)\,\mathrm{d}x\right)^{1/p}, \tag{1.2}$$

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which holds if and only if the condition

$$A_M := \sup_{t\in(a,b)} \left(\int_t^b u(\tau)\,\mathrm{d}\tau\right)^{1/q} \left(\int_a^t v^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1} \tag{1.3}$$

holds, and the best constant C (the least constant for which the inequality holds) in (1.2) satisfies the estimates

$$A_M \leqslant C \leqslant l(p,q)A_M, \tag{1.4}$$

where $l(p,q)$ are values of p and q discovered by scholars in different times—in the range $p = q > 1$ in 1969 by Tomaselli (see [23]) and Talenti (see [22]); in 1972 Muckenhoupt (see [16]) derived the expression. The case $1 < p \leqslant q < \infty$ was investigated independently in 1978 by Bradley (see [2]) and in 1979 by Kokilashvili (see [7]) and obtained also by Maz'ja (see [15]) etc. (see also [10], [11] and [18] for more details). Let us present Opic's expression

$$l(p,q) = \left(1 + \frac{q}{p'}\right)^{1/q} \left(1 + \frac{p'}{q}\right)^{1/p'}, \tag{1.5}$$

which will further be used in the proofs of our results.

In 2001, Persson and Stepanov (see [19]) also gave a new equivalent condition

$$A_{\mathrm{PS}} = \sup_{t\in(a,b)} \left(\int_a^t v^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{-1/p} \left(\int_a^t u(\tau)\left(\int_a^\tau v^{1-p'}(\xi)\,\mathrm{d}\xi\right)^q \mathrm{d}\tau\right)^{1/q} < \infty \tag{1.6}$$

and for the best constant the estimates

$$A_{\mathrm{PS}} \leqslant C \leqslant p' A_{\mathrm{PS}}.$$

Therefore, conditions (1.3) and (1.6) are mutually equivalent, that is, if one condition is satisfied, the other is also. Later, in 2003 (see [24]) Wedestig gave parameter-dependent conditions, where it was shown in examples that it is a better approximation for the best constant of (1.2). As a result, any new equivalent condition is extremely useful for investigating the Hardy inequalities. In [12] Kufner, Persson and Wedestig, in [5] Gogatishvili, Kufner, Persson and Wedestig and in [4] Gogatishvili, Kufner and Persson gave a family of conditions ensuring the fulfillment of inequality (1.2).

The problem becomes significantly more difficult if the kernel $k(x,t)$ is not a constant function. The satisfying of (1.1) began to be studied in the last decades of the twentieth century. Let us now give some scientific conclusions concerning inequalities of this type. For example, Martin-Reyes and Sawyer in [14], Prokhorov and Stepanov in [21], Prokhorov in [20] considered (1.1) with the kernel $k(x,t) = (x-t)^{\alpha-1}$, $\alpha > 0$. Bloom and Kerman in [1] and Oinarov in [17] gave equivalent conditions with kernels $k(x,t)$ being a continuous nonnegative function increasing in the first argument, decreasing in the second argument and satisfying the following condition: There exists a number $h \geqslant 1$ such that $k(x,s) \leqslant h(k(x,t)+k(t,s))$ for all $a < s \leqslant t \leqslant x < b$. Functions $k(x,t)$ satisfying the above conditions are also called Oinarov's kernel. It was proved that the conditions

$$A_1 := \sup_{x\in(a,b)} \left(\int_x^b k^q(t,x)u(t)\,\mathrm{d}t\right)^{1/q} \left(\int_a^x v^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'} < \infty$$

and

$$A_2 := \sup_{x\in(a,b)} \left(\int_x^b u(t)\,\mathrm{d}t\right)^{1/q} \left(\int_a^x k^{p'}(x,t)v^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'} < \infty$$

are equivalent to (1.1). Kufner, Kuliev and Oinarov in [8] and Kufner, Kuliev and Persson in [9] also considered (1.1) with the kernel $k(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t)$. In the foregoing works, the authors focused mainly on finding equivalent conditions for the fulfillment of the inequality. But, in the theory of differential equations and spectral theory it is very important to find the lower and upper estimates for the best constant. Let us note that in [3] Drábek, Kuliev, Marletta also gave an estimate for the best constant for a special choice of the kernel $k(x,t) = x - t$ which appears from the fourth order nonlinear Shturm-Liouville problem, where they significantly used these estimates. In 2021, Kalybay and Baiarystanov (see [6]) proved the estimates

$$A \leqslant C \leqslant (h+1)^3 p^{1/q}(p')^{1/p'} A, \tag{1.7}$$

where $A = \max\{A_1, A_2\} < \infty$. Recently, Kuliev, Kulieva and Eshimova in [13] also obtained the estimate

$$A \leqslant C \leqslant X,$$

where X is a positive solution of the equation

$$X^{q'} - hq^{1/(q-1)}(q')^{1/(q-1)}A^{1/(q-1)}X = hq^{1/(q-1)}(p')^{q'/p'}A^{q'}, \quad q' = \frac{q}{q-1}. \tag{1.8}$$

In this work, we present new equivalent conditions. What can also be seen is the convenience of these conditions in the estimates obtained for the best constant of inequality (1.1), and for the kernel chosen as a constant function, the lower values of the estimates are better from the previously obtained values. In the second section we present the main results of the work, the proofs of which are given in the third section.

2. Main part

We give some new couple conditions which are equivalent to Oinarov's couple condition A_1 and A_2. The upper estimates are given by solutions of some nonlinear algebraic equations. Let us denote

$$A_1(s) = \left(\int_s^b k^q(t,s)u(t)\,\mathrm{d}t\right)^{1/q}\left(\int_a^s v^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'};$$

$$A_2(s) = \left(\int_s^b u(t)\,\mathrm{d}t\right)^{1/q}\left(\int_a^s k^{p'}(s,t)v^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'};$$

$$B_1(s) = \left(\int_s^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\right)^{p'}\mathrm{d}t\right)^{1/p'q}\left(\int_a^s v^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'q'};$$

$$B_2(s) = \left(\int_s^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)\,\mathrm{d}x\right)^{p'}\mathrm{d}t\right)^{1/p'}\left(\int_s^b u(t)\,\mathrm{d}t\right)^{-1/q'};$$

$$B_3(s) = \left(\int_a^s u(t)\left(\int_a^t k(t,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{q}\mathrm{d}t\right)^{1/q}\left(\int_a^s v^{1-p'}(t)\,\mathrm{d}t\right)^{-1/p};$$

$$B_4(s) = \left(\int_a^s u(t)\left(\int_a^t k^{p'}(t,x)v^{1-p'}(x)\,\mathrm{d}x\right)^{q}\mathrm{d}t\right)^{1/qp'}\left(\int_s^b u(t)\,\mathrm{d}t\right)^{1/qp}.$$

Our first main result reads:

Theorem 2.1. *Let $1 < p \leqslant q < \infty$ and the kernel $k(x,t)$ satisfy Oinarov's condition. Then inequality* (1.1) *holds if and only if*

$$B_1 := \sup_{s\in(a,b)} B_1(s) < \infty \quad \text{and} \quad B_2 := \sup_{s\in(a,b)} B_2(s) < \infty.$$

The best constant of inequality (1.1) *satisfies*

$$\max\Big\{\sup_{s\in(a,b)}[A_1^{p'q}(s) + qB_1^{p'q}(s)]^{1/p'q},\ \sup_{s\in(a,b)}[A_2^{p'}(s) + B_2^{p'}(s)]^{1/p'}\Big\} \leqslant C \leqslant X, \tag{2.1}$$

where X is a positive solution of the nonlinear equation

$$X^{q'} - h(qB_2)^{1/(q-1)}X = hq^{p'+1/p'(q-1)}(q')^{1/p'}B_1^{q'}. \tag{2.2}$$

Remark 2.2. Equation (2.2) has a unique positive solution since the function

$$h(x) = \frac{x^{q'}}{q^{(p'+1)/(p'(q-1))}(q')^{1/p'}B_1^{q'} + (qB_2)^{1/(q-1)}x}$$

is a continuous and monotone increasing function of x in the half line $(0,\infty)$, $h(0)=0$ and $h(\infty)=\infty$.

E x a m p l e 2.3. Let $1 < p \leqslant q = 2$ and $h \geqslant 1$. Then equation (2.2) and its positive solution take the forms

$$X^2 - 2hB_2X = 2^{(p'+2)/p'}hB_1^2 \quad \text{and} \quad X = \big(h + \sqrt{h^2 + 2^{(p'+2)/p'}h}\big)B,$$

respectively. Then corresponding estimates take the form

$$\max\Big\{\sup_{s\in(a,b)}[A_1^{2p'}(s) + 2B_1^{2p'}(s)]^{1/2p'}, \sup_{s\in(a,b)}[A_2^{p'}(s) + B_2^{p'}(s)]^{1/p'}\Big\}$$
$$\leqslant C \leqslant \big(hB_2 + \sqrt{h^2B_2^2 + 2^{(p'+2)/p'}hB_1^2}\big).$$

Our second main result reads:

Theorem 2.4. *Let $1 < p \leqslant q < \infty$ and the kernel $k(x,t)$ satisfy Oinarov's condition. Then inequality (1.1) holds if and only if*

$$B_3 := \sup_{s\in(a,b)} B_3(s) < \infty \quad \textit{and} \quad B_4 := \sup_{s\in(a,b)} B_4(s) < \infty.$$

The best constant of the inequality satisfies

$$\max\Big\{\sup_{s\in(a,b)}[A_1^q(s) + B_3^q(s)]^{1/q}, \sup_{s\in(a,b)}[A_2^{p'q}(s) + p'B_4^{p'q}(s)]^{1/p'q}\Big\} \leqslant C \leqslant X, \tag{2.3}$$

where X is a positive solution of the nonlinear equation

$$X^p - h(p'B_3)^{p-1}X = hp^{1/q}(p')^{(q+1)/(p'-1)q}B_4^p.$$

Corollary 2.5. *For (1.2) the corresponding B_1, B_2 and B_3, B_4 conditions take the forms:*

$$B_1 = \sup_{s\in(a,b)}\bigg(\int_s^b v^{1-p'}(t)\Big(\int_t^b u(x)\,\mathrm{d}x\Big)^{p'}\mathrm{d}t\bigg)^{1/p'q}\bigg(\int_a^s v^{1-p'}(t)\,\mathrm{d}t\bigg)^{1/p'q'};$$

$$B_2 = \sup_{s\in(a,b)}\bigg(\int_s^b v^{1-p'}(t)\Big(\int_t^b u(x)\,\mathrm{d}x\Big)^{p'}\mathrm{d}t\bigg)^{1/p'}\bigg(\int_s^b u(x)\,\mathrm{d}x\bigg)^{-1/q'};$$

$$B_3 = \sup_{s\in(a,b)}\bigg(\int_a^s u(t)\Big(\int_a^t v^{1-p'}(x)\,\mathrm{d}x\Big)^{q}\mathrm{d}t\bigg)^{1/q}\bigg(\int_a^s v^{1-p'}(t)\,\mathrm{d}t\bigg)^{-1/p};$$

$$B_4 = \sup_{s\in(a,b)}\bigg(\int_a^s u(t)\Big(\int_a^t v^{1-p'}(x)\,\mathrm{d}x\Big)^{q}\mathrm{d}t\bigg)^{1/qp'}\bigg(\int_s^b u(t)\,\mathrm{d}t\bigg)^{1/qp}.$$

It is known from [4] that the conditions are mutually equivalent since $B_1 = A_9(1/p'q')$, $B_2 = A_4(1/q')$, $B_3 = A_3(1/p) = A_{\mathrm{PS}}$ and $B_4 = A(1/pq)$, which *means that each condition is equivalent to validity of* (1.2). *Therefore, the lower estimate for the best constant of* (1.2) *is*

$$\max\{A_{M,1}, A_{M,2}, A_{M,3}, A_{M,4}\} \leqslant C,$$
$$A_{M,1} := \sup_{s\in(a,b)} [A_M^{p'q}(s) + qB_1^{p'q}(s)]^{1/p'q}, \quad A_{M,2} := \sup_{s\in(a,b)} [A_M^{p'}(s) + B_2^{p'}(s)]^{1/p'},$$
$$A_{M,3} := \sup_{s\in(a,b)} [A_M^{q}(s) + B_3^{q}(s)]^{1/q}, \quad A_{M,4} := \sup_{s\in(a,b)} [A_M^{p'q}(s) + p'B_4^{p'q}(s)]^{1/p'q}.$$

As a special case we derive that

$$\sup_{s\in(a,b)} [A_M^{q}(s) + A_{\rm PS}^{q}(s)]^{1/q} \leqslant C,$$
$$A_{\rm PS}(s) = \biggl(\int_a^s u(t)\biggl(\int_a^t v^{1-p'}(x)\,{\rm d}x\biggr)^q {\rm d}t\biggr)^{1/qp'} \biggl(\int_s^b u(t)\,{\rm d}t\biggr)^{1/qp}.$$

It is seen that the lower estimate is better than the estimates given in [11], *Theorem* 1.2 *and in* [10], *Theorem* 5.

3. Proofs

In this section we present proofs of the main theorems. First we deal with the duality principle, which will further be used.

Duality principle. When it comes to creating new conditions and working with conjugate inequality, the concept of duality is critical.

Lemma 3.1. *Let* $1 < p, q < \infty$, $0 < C < \infty$ *and* $k(x,t)$ *be a nonnegative measurable function on* $(a,b)\times(a,b)$. *Then inequality* (1.1) *holds for all positive measurable functions* f *on* (a,b) *if and only if "its dual inequality"*

$$(3.1)\quad \biggl(\int_a^b\biggl(\int_x^b k(t,x)g(t)\,{\rm d}t\biggr)^{p'} v^{1-p'}(x)\,{\rm d}x\biggr)^{1/p'} \leqslant C\biggl(\int_a^b g^{q'}(x)u^{1-q'}(x)\,{\rm d}x\biggr)^{1/q'}$$

holds for all nonnegative functions g *with the same best constant* C.

P r o o f. For the proof see page 78 in [11]. □

P r o o f of Theorem 2.1. *Necessity.* Here we first assume that Hardy-type inequality (1.1) holds, where C is the best constant (the smallest constant in which the inequality holds). It is known from Lemma 3.1 that inequality (1.1) holds if and only if dual inequality (3.1) holds. Now by choosing the function g_τ in (3.1) as

$$g_\tau(x) = k^{q-1}(x,\tau)u(x)\chi_{[\tau,b)}(x)$$

we get for the right-hand side of (3.1)

$$\left(\int_a^b g_\tau^{q'}(x)u^{1-q'}(x)\,\mathrm{d}x\right)^{1/q'} = \left(\int_\tau^b k^q(x,\tau)u(x)\,\mathrm{d}x\right)^{1/q'}$$

and for the left-hand side

$$\begin{aligned}S &:= \left(\int_a^b \left(\int_x^b k(t,x)g_\tau(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p'}\\ &= \left(\int_a^b \left(\int_x^b k(t,x)k^{q-1}(t,\tau)u(t)\chi_{[\tau,b)}(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p'}\\ &= \left(\int_a^\tau \left(\int_\tau^b k(t,x)k^{q-1}(t,\tau)u(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\right.\\ &\qquad \left.+\int_\tau^b \left(\int_x^b k(t,x)k^{q-1}(t,\tau)u(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p'}.\end{aligned}$$

Using monotonicity of k in the first integral $k(t,x) \geqslant k(t,\tau)$ for $x \geqslant \tau$ and in the second integral as $k(t,\tau) \geqslant k(t,x)$ for $\tau \geqslant x$, we have

$$\begin{aligned}S \geqslant \left(\int_a^\tau \left(\int_\tau^b k^q(t,\tau)u(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\right.\\ \left.+\int_\tau^b \left(\int_x^b k^q(t,x)u(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p'}.\end{aligned}$$

From these and (3.1) we obtain

$$\begin{aligned}\left(\int_\tau^b k^q(t,\tau)u(t)\,\mathrm{d}t\right)^{p'}\left(\int_a^\tau v^{1-p'}(x)\,\mathrm{d}x\right) + \int_\tau^b \left(\int_x^b k^q(t,x)u(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\\ \leqslant C^{p'}\left(\int_\tau^b k^q(x,\tau)u(x)\,\mathrm{d}x\right)^{p'/q'}.\end{aligned}$$

This implies that

$$A_1^{p'}(\tau) + \int_\tau^b \left(\int_x^b k^q(t,x)u(t)\,\mathrm{d}t\right)^{p'} v^{1-p'}(x)\,\mathrm{d}x\left(\int_\tau^b k^q(x,\tau)u(x)\,\mathrm{d}x\right)^{-p'/q'} \leqslant C^{p'},$$

i.e.,

$$A_1^{p'q}(\tau) + B_1^{p'q}(\tau) \leqslant C^{p'} A_1^{p'(q-1)}(\tau).$$

Using Young's inequality to the right-hand side of the estimate we obtain

$$A_1^{p'q}(\tau) + B_1^{p'q}(\tau) \leqslant \frac{C^{p'q}}{q} + \frac{A_1^{p'q}(\tau)}{q'}$$

and then

$$\sup_{\tau\in(a,b)} \{A_1^{p'q}(\tau) + qB_1^{p'q}(\tau)\} \leqslant C^{p'q}.$$

This proves the first part of (2.1) . Now we prove the second part. Choosing the function g_τ in (3.1) as

$$g_\tau(x) = u(x)\chi_{[\tau,b)}(x)$$

we get

$$\Big(\int_a^b g_\tau^{q'}(x)u^{1-q'}(x)\,\mathrm{d}x\Big)^{1/q'} = \Big(\int_\tau^b u(x)\,\mathrm{d}x\Big)^{1/q'}$$

and the left-hand side is estimated as follows:

$$\begin{aligned}
&\Big(\int_a^b \Big(\int_x^b k(t,x)g_\tau(t)\,\mathrm{d}t\Big)^{p'} v^{1-p'}(x)\,\mathrm{d}x\Big)^{1/p'} \\
&\qquad= \Big(\int_a^b \Big(\int_x^b k(t,x)u(t)\chi_{[\tau,b)}(t)\,\mathrm{d}t\Big)^{p'} v^{1-p'}(x)\,\mathrm{d}x\Big)^{1/p'} \\
&\qquad= \Big(\int_a^\tau \Big(\int_\tau^b k(t,x)u(t)\,\mathrm{d}t\Big)^{p'} v^{1-p'}(x)\,\mathrm{d}x \\
&\qquad\qquad+ \int_\tau^b \Big(\int_x^b k(t,x)u(t)\,\mathrm{d}t\Big)^{p'} v^{1-p'}(x)\,\mathrm{d}x\Big)^{1/p'}.
\end{aligned}$$

From these and (3.1) we obtain

$$\begin{aligned}
&\int_\tau^b \Big(\int_x^b k(t,x)u(t)\,\mathrm{d}t\Big)^{p'} v^{1-p'}(x)\,\mathrm{d}x\Big(\int_\tau^b u(x)\,\mathrm{d}x\Big)^{-p'/q'} \\
&\quad+\Big(\int_a^\tau \Big(\int_\tau^b k(t,x)u(t)\,\mathrm{d}t\Big)^{p'} v^{1-p'}(x)\,\mathrm{d}x\Big)\Big(\int_\tau^b u(x)\,\mathrm{d}x\Big)^{-p'/q'} \leqslant C^{p'},
\end{aligned}$$

i.e.,

$$B_2^{p'}(\tau) + \int_a^\tau \Big(\int_\tau^b k(t,x)u(t)\,\mathrm{d}t\Big)^{p'} v^{1-p'}(x)\,\mathrm{d}x\Big(\int_\tau^b u(x)\,\mathrm{d}x\Big)^{-p'/q'} \leqslant C^{p'}.$$

Using monotonicity of k—$k(t,x) \geqslant k(\tau,x)$ for $t \geqslant \tau$—we obtain

$$C^{p'} \geqslant B_2^{p'}(\tau) + \int_a^\tau k^{p'}(\tau,x)v^{1-p'}(x)\,\mathrm{d}x\Big(\int_\tau^b u(x)\,\mathrm{d}x\Big)^{p'/q},$$

i.e.,

$$B_2^{p'}(\tau) + A_2^{p'}(\tau) \leqslant C^{p'}.$$

The second part of (2.1) is also proved.

Sufficiency. Let us denote

$$I = \int_a^b \Big(\int_a^x k(x,t)f(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x,$$

then consequently using Fubini's theorem and Hölder's inequality we get

$$(3.2)\quad \begin{aligned} I &= q\int_a^b \bigg(\int_a^x k(x,t)f(t)\bigg(\int_a^t k(x,s)f(s)\,\mathrm{d}s\bigg)^{q-1}\mathrm{d}t\bigg)u(x)\,\mathrm{d}x \\ &= q\int_a^b f(t)\bigg(\int_t^b k(x,t)u(x)\bigg(\int_a^t k(x,s)f(s)\,\mathrm{d}s\bigg)^{q-1}\mathrm{d}x\bigg)\,\mathrm{d}t \\ &\leqslant q\|f\|_{p,v}\bigg(\int_a^b v^{1-p'}(t)\bigg(\int_t^b k(x,t)u(x)\bigg(\int_a^t k(x,s)f(s)\,\mathrm{d}s\bigg)^{q-1}\mathrm{d}x\bigg)^{p'}\mathrm{d}t\bigg)^{1/p'} \\ &= q\|f\|_{p,v}J^{1/p'}. \end{aligned}$$

We now proceed to the proof by estimating J. To do this, we estimate its inner integral separately. Using Hölder's inequality with exponents $[q]/(q-1)$ and $[q]/(1-\{q\})$, we have

$$\begin{aligned} J_1 &:= \int_t^b k(x,t)u(x)\bigg(\int_a^t k(x,s)f(s)\,\mathrm{d}s\bigg)^{q-1}\mathrm{d}x \\ &= \int_t^b \bigg(k^{\{q\}}(x,t)u(x)\bigg(\int_a^t k(x,s)f(s)\,\mathrm{d}s\bigg)^{[q]}\bigg)^{(q-1)/[q]}\bigg(k^q(x,t)u(x)\bigg)^{(1-\{q\})/[q]}\mathrm{d}x \\ &\leqslant \bigg(\int_t^b k^{\{q\}}(x,t)u(x)\bigg(\int_a^t k(x,s)f(s)\,\mathrm{d}s\bigg)^{[q]}\mathrm{d}x\bigg)^{(q-1)/[q]} \\ &\quad\times\bigg(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\bigg)^{(1-\{q\})/[q]}. \end{aligned}$$

We now derive the following estimate from the condition of Oinarov's kernel and then Newton's binomial formulae

$$(3.3)\quad \begin{aligned} J_1 &\leqslant h^{q-1}\bigg(\int_t^b k^{\{q\}}(x,t)u(x)\bigg(k(x,t)\int_a^t f(s)\,\mathrm{d}s \\ &\qquad + \int_a^t k(t,s)f(s)\,\mathrm{d}s\bigg)^{[q]}\mathrm{d}x\bigg)^{(q-1)/[q]}\bigg(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\bigg)^{(1-\{q\})/[q]} \\ &= h^{q-1}\bigg(\int_t^b k^{\{q\}}(x,t)u(x)\bigg(\sum_{n=0}^{[q]} C_{[q]}^n k^n(x,t)\bigg(\int_a^t f(s)\,\mathrm{d}s\bigg)^n \\ &\qquad\times\bigg(\int_a^t k(t,s)f(s)\,\mathrm{d}s\bigg)^{[q]-n}\bigg)\,\mathrm{d}x\bigg)^{(q-1)/[q]}\bigg(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\bigg)^{(1-\{q\})/[q]} \\ &= h^{q-1}\bigg(\sum_{n=0}^{[q]} C_{[q]}^n\bigg(\int_t^b k^{\{q\}+n}(x,t)u(x)\,\mathrm{d}x\bigg)\bigg(\int_a^t f(s)\,\mathrm{d}s\bigg)^n \\ &\qquad\times\bigg(\int_a^t k(t,s)f(s)\,\mathrm{d}s\bigg)^{[q]-n}\bigg)^{(q-1)/[q]}\bigg(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\bigg)^{(1-\{q\})/[q]}. \end{aligned}$$

Using Hölder's inequality to the first integral of the sum for $0 < n < [q]$ we have

$$\begin{aligned}
&\int_t^b k^{\{q\}+n}(x,t)u(x)\,\mathrm{d}x \\
&\qquad = \int_t^b (k^q(x,t)u(x))^{(n-1+\{q\})/(q-1)}(k(x,t)u(x))^{([q]-n)/(q-1)}\,\mathrm{d}x \\
&\qquad \leqslant \Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{(n-1+\{q\})/(q-1)}\Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{([q]-n)/(q-1)}.
\end{aligned}$$

Then (3.3) is estimated as follows:

$$\begin{aligned}
J_1 &\leqslant h^{q-1}\Big(\sum_{n=0}^{[q]} C_{[q]}^n\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{(n-1+\{q\})/(q-1)} \\
&\quad \times \Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{([q]-n)/(q-1)}\Big(\int_a^t f(s)\,\mathrm{d}s\Big)^n \\
&\quad \times \Big(\int_a^t k(t,s)f(s)\,\mathrm{d}s\Big)^{[q]-n}\Big)^{(q-1)/[q]}\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{(1-\{q\})/[q]} \\
&= h^{q-1}\Big(\sum_{n=0}^{[q]} C_{[q]}^n\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{n/(q-1)}\Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{([q]-n)/(q-1)} \\
&\quad \times \Big(\int_a^t f(s)\,\mathrm{d}s\Big)^n\Big(\int_a^t k(t,s)f(s)\,\mathrm{d}s\Big)^{[q]-n}\Big)^{(q-1)/[q]} \\
&= h^{q-1}\Big(\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{1/(q-1)}\int_a^t f(s)\,\mathrm{d}s \\
&\quad + \Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{1/(q-1)}\int_a^t k(t,s)f(s)\,\mathrm{d}s\Big)^{q-1}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_t^b k(x,t)u(x)\Big(\int_a^t k(x,s)f(s)\,\mathrm{d}s\Big)^{q-1}\,\mathrm{d}x \\
&\qquad \leqslant h^{q-1}\Big(\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{1/(q-1)}\int_a^t f(s)\,\mathrm{d}s \\
&\qquad\quad + \Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{1/(q-1)}\int_a^t k(t,s)f(s)\,\mathrm{d}s\Big)^{q-1}.
\end{aligned}$$

From this we get

$$J = \int_a^b v^{1-p'}(t)\Big(\int_t^b k(x,t)u(x)\Big(\int_a^t k(x,s)f(s)\,\mathrm{d}s\Big)^{q-1}\,\mathrm{d}x\Big)^{p'}\,\mathrm{d}t$$

$$\leqslant h^{(q-1)p'}\int_a^b v^{1-p'}(t)\bigg(\bigg(\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{1/(q-1)}\int_a^t f(s)\,\mathrm{d}s + \Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{1/(q-1)}\int_a^t k(t,s)f(s)\,\mathrm{d}s\bigg)^{(q-1)p'}\mathrm{d}t.$$

Using Minkowski's integral inequality, we obtain

$$\begin{aligned} J &\leqslant h^{(q-1)p'}\bigg(\bigg(\int_a^b v^{1-p'}(t)\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{p'} \\ &\quad\times\Big(\int_a^t f(s)\,\mathrm{d}s\Big)^{(q-1)p'}\mathrm{d}t\bigg)^{1/(q-1)p'} \\ &\quad+\bigg(\int_a^b v^{1-p'}(t)\Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{p'} \\ &\quad\times\Big(\int_a^t k(t,s)f(s)\,\mathrm{d}s\Big)^{(q-1)p'}\mathrm{d}t\bigg)^{1/(q-1)p'}\bigg)^{(q-1)p'} \\ &= h^{(q-1)p'}\Big(I_1^{1/(q-1)p'}+I_2^{1/(q-1)p'}\Big)^{(q-1)p'}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_a^b v^{1-p'}(t)\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{p'}\Big(\int_a^t f(s)\,\mathrm{d}s\Big)^{(q-1)p'}\mathrm{d}t, \\ I_2 &= \int_a^b v^{1-p'}(t)\Big(\int_t^b k(x,t)u(x)\,\mathrm{d}x\Big)^{p'}\Big(\int_a^t k(t,s)f(s)\,\mathrm{d}s\Big)^{(q-1)p'}\mathrm{d}t. \end{aligned}$$

From this and (3.2) we have that

$$I \leqslant q\|f\|_{p,v}J^{1/p'} \leqslant q\|f\|_{p,v}h^{q-1}(I_1^{1/(q-1)p'}+I_2^{1/(q-1)p'})^{q-1},$$

then we raise both sides of this inequality to the power of $1/(q-1)$, i.e.,

$$I^{1/(q-1)} \leqslant hq^{1/(q-1)}\|f\|_{p,v}^{1/(q-1)}(I_1^{1/(q-1)p'}+I_2^{1/(q-1)p'}).$$

Further, estimate I_1 and I_2, separately. To estimate I_1, we use Hardy inequality (1.2) of the exponents $\bar{p} := p$, $\bar{q} := (q-1)p'$ and weight functions $\bar{u} := v^{1-p'}(t)\big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\big)^{p'}$ and $\overline{v}(t) := v(t)$, i.e.,

(3.4)

$$\begin{aligned} I_1^{1/(q-1)p'} &= \bigg(\int_a^b v^{1-p'}(t)\Big(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\Big)^{p'}\Big(\int_a^t f(s)\,\mathrm{d}s\Big)^{(q-1)p'}\mathrm{d}t\bigg)^{1/(q-1)p'} \\ &\leqslant C_{p,(q-1)p'}\Big(\int_a^b f^p(t)v(t)\,\mathrm{d}t\Big)^{1/p}, \end{aligned}$$

which holds since its characteristic condition (1.3) is satisfied for $\bar{p} \leqslant \bar{q}$:

$$B_1^{q'} = \sup_{s\in(a,b)} \left(\int_s^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)\,\mathrm{d}x\right)^{p'} \mathrm{d}t\right)^{1/(q-1)p'} \left(\int_a^s v^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'} < \infty.$$

Using (1.5) and (1.4) we get the following upper estimate for the best constant $C_{p,(q-1)p'}$ in (3.4):

$$C_{p,(q-1)p'} \leqslant q^{1/p'(q-1)}(q')^{1/p'} B_1^{q'}.$$

Then $I_1^{1/(q-1)p'} \leqslant q^{1/p'(q-1)}(q')^{1/p'} B_1^{q'} \|f\|_{p,v}$.

Let us estimate I_2:

$$\begin{aligned}
I_2 &= \int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)\,\mathrm{d}x\right)^{p'} \left(\int_a^t k(t,s)f(s)\,\mathrm{d}s\right)^{(q-1)p'} \mathrm{d}t \\
&= \int_a^b \left(\int_a^t k(t,s)f(s)\,\mathrm{d}s\right)^{(q-1)p'} \mathrm{d}\left(-\int_t^b \left(\int_s^b k(x,s)u(x)\,\mathrm{d}x\right)^{p'} v^{1-p'}(s)\,\mathrm{d}s\right) \\
&= \int_a^b \left(\int_t^b \left(\int_s^b k(x,s)u(x)\,\mathrm{d}x\right)^{p'} v^{1-p'}(s)\,\mathrm{d}s\right) \mathrm{d}\left(\int_a^t k(t,s)f(s)\,\mathrm{d}s\right)^{(q-1)p'} \\
&= \int_a^b B_2^{p'}(t)\left(\int_t^b u(x)\,\mathrm{d}x\right)^{p'/q'} \mathrm{d}\left(\int_a^t k(t,s)f(s)\,\mathrm{d}s\right)^{(q-1)p'} \\
&\leqslant B_2^{p'} \int_a^b \left(\int_t^b u(x)\,\mathrm{d}x\right)^{p'/q'} \mathrm{d}\left(\int_a^t k(t,s)f(s)\,\mathrm{d}s\right)^{(q-1)p'} \\
&= B_2^{p'} \left(\left(\int_a^b \left(\int_t^b u(x)\,\mathrm{d}x\right)^{p'/q'} \mathrm{d}\left(\int_a^t k(t,s)f(s)\,\mathrm{d}s\right)^{(q-1)p'}\right)^{q'/p'}\right)^{p'/q'} \\
&\leqslant B_2^{p'} \left(\int_a^b u(x)\left(\int_a^x k(x,s)f(s)\,\mathrm{d}s\right)^q \mathrm{d}x\right)^{p'/q'} = B_2^{p'} I^{p'/q'}.
\end{aligned}$$

To get the last estimate we used Minkowski's integral inequality. Therefore,

$$I_2^{1/(q-1)p'} \leqslant B_2^{1/q-1} I^{1/q}.$$

From the above estimates we have finally obtained

$$\begin{aligned}
(3.5) \qquad I^{1/(q-1)} &\leqslant hq^{1/(q-1)}\|f\|_{p,v}^{1/(q-1)}(q^{1/p'(q-1)}(q')^{1/p'} B_1^{q'}\|f\|_{p,v} + B_2^{1/(q-1)} I^{1/q}) \\
&= hq^{(p'+1)/p'(q-1)}(q')^{1/p'} B_1^{q'}\|f\|_{p,v}^{q'} + h(qB_2\|f\|_{p,v})^{1/(q-1)} I^{1/q}.
\end{aligned}$$

Using Young's inequality $ab \leqslant a^q/q + b^{q'}/q'$ to the second term of the last line as

$$h(qB_2\|f\|_{p,v})^{1/(q-1)} I^{1/q} \leqslant \frac{h^q(qB_2\|f\|_{p,v})^{q'}}{q} + \frac{I^{1/(q-1)}}{q'},$$

then inserting this estimate into (3.5), we find that

$$I^{1/(q-1)} \leqslant hq^{(p'+1)/p'(q-1)}(q')^{1/p'}B_1^{q'}\|f\|_{p,v}^{q'} + h^q q^{q'-1}B_2^{q'}\|f\|_{p,v}^{q'} + \frac{I^{1/(q-1)}}{q'}$$

and then

$$I^{1/(q-1)} \leqslant hq^{1/p'(q-1)+q'}(q')^{1/p'}B_1^{q'}\|f\|_{p,v}^{q'} + h^q q^{q'}B_2^{q'}\|f\|_{p,v}^{q'}.$$

From this we obtain that

$$I^{1/q} \leqslant q(hq^{1/p'(q-1)}(q')^{1/p'} + h^q)^{1/q'}B\|f\|_{p,v}, \quad B = \max\{B_1, B_2\}.$$

This implies that the conditions $B_1 < \infty$ and $B_2 < \infty$ are sufficient for the inequality to hold. Further, we obtain the upper estimate for the best constant C. Using $I^{1/q} \leqslant C\|f\|_{p,v}$ in (3.5) we get

$$I^{1/(q-1)} \leqslant hq^{1/(q-1)}\|f\|_{p,v}^{q/(q-1)}(q^{1/p'(q-1)}(q')^{1/p'}B_1^{q'} + B_2^{1/(q-1)}C),$$

i.e.,

$$\left(\frac{I^{1/q}}{\|f\|_{p,v}}\right)^{q'} \leqslant hq^{1/(q-1)}(q^{1/p'(q-1)}(q')^{1/p'}B_1^{q'} + B_2^{1/(q-1)}C).$$

Then we have the estimate for the best constant

$$C^{q'} \leqslant hq^{1/(q-1)}(q^{1/p'(q-1)}(q')^{1/p'}B_1^{q'} + B_2^{1/(q-1)}C).$$

Consequently, we obtain

$$\frac{C^{q'}}{q^{(p'+1)/p'(q-1)}(q')^{1/p'}B_1^{q'} + (qB_2)^{1/(q-1)}C} \leqslant h.$$

Let us consider the function

$$f(x) = \frac{x^{q'}}{q^{(p'+1)/p'(q-1)}(q')^{1/p'}B_1^{q'} + (qB_2)^{1/(q-1)}x}$$

corresponding to the left-hand side of the estimate. It is easy to see that this function is monotone increasing and continuous in $(0,\infty)$, $f(0) = 0$ and $f(\infty) = \infty$, which implies that the equation $f(x) = h$ has exactly one positive solution in $(0,\infty)$. If X is a solution of the equation, which means that

$$X^{q'} = h(q^{(p'+1)/p'(q-1)}(q')^{1/p'}B_1^{q'} + (qB_2)^{1/(q-1)}X),$$

then $C \leqslant X$. The proof is complete. $\square$

P r o o f of Theorem 2.4. *Necessity.* Let Hardy-type inequality (1.1) hold and C be the best constant. Then choosing the function f_τ in (1.1) as

$$f_\tau(x) = v^{1-p'}(x)\chi_{(a,\tau]}(x),$$

we get the right-hand side

$$\left(\int_a^b f_\tau^p(x)v(x)\,\mathrm{d}x\right)^{1/p} = \left(\int_a^\tau v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p}$$

and the left-hand side is estimated as

$$\begin{aligned}
&\left(\int_a^b \left(\int_a^x k(x,t)f_\tau(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q} \\
&\quad= \left(\int_a^b \left(\int_a^x k(x,t)v^{1-p'}(t)\chi_{(a,\tau]}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q} \\
&\quad= \left(\int_a^\tau \left(\int_a^x k(x,t)v^{1-p'}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x + \int_\tau^b \left(\int_a^\tau k(x,t)v^{1-p'}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q}.
\end{aligned}$$

From these and (1.1) we obtain

$$\begin{aligned}
&\int_a^\tau \left(\int_a^x k(x,t)v^{1-p'}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\left(\int_a^\tau v^{1-p'}(x)\,\mathrm{d}x\right)^{-q/p} \\
&\qquad+ \int_\tau^b \left(\int_a^\tau k(x,t)v^{1-p'}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\left(\int_a^\tau v^{1-p'}(x)\,\mathrm{d}x\right)^{-q/p} \leqslant C^q,
\end{aligned}$$

i.e.,

$$B_3^q(\tau) + \int_\tau^b \left(\int_a^\tau k(x,t)v^{1-p'}(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\left(\int_a^\tau v^{1-p'}(x)\,\mathrm{d}x\right)^{-q/p} \leqslant C^q.$$

Using the monotonicity of k—$k(x,t) \geqslant k(x,\tau)$ for all $t \geqslant \tau$—we have

$$C^q \geqslant B_3^q(\tau) + \int_\tau^b k^q(x,\tau)u(x)\,\mathrm{d}x\left(\int_a^\tau v^{1-p'}(x)\,\mathrm{d}x\right)^{q/p'},$$

which implies

$$\sup_{\tau\in(a,b)} (B_3^q(\tau) + A_1^q(\tau)) \leqslant C^q.$$

The first part of (2.3) is proved.

Now we prove the second part. By choosing the function f_τ in (1.1) as

$$f_\tau(x) = k^{p'-1}(\tau,x)v^{1-p'}(x)\chi_{(a,\tau]}(x)$$

we get

$$\Big(\int_a^b f_\tau^p(x)v(x)\,\mathrm{d}x\Big)^{1/p} = \Big(\int_a^\tau k^{p'}(\tau,x)v^{1-p'}(x)\,\mathrm{d}x\Big)^{1/p}$$

and

$$\begin{aligned}
&\Big(\int_a^b\Big(\int_a^x k(x,t)f_\tau(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\Big)^{1/q}\\
&\quad= \Big(\int_a^b\Big(\int_a^x k(x,t)k^{p'-1}(\tau,t)v^{1-p'}(t)\chi_{(a,\tau]}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\Big)^{1/q}\\
&\quad= \Big(\int_a^\tau\Big(\int_a^x k(x,t)k^{p'-1}(\tau,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\\
&\qquad+\int_\tau^b\Big(\int_a^\tau k(x,t)k^{p'-1}(\tau,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\Big)^{1/q}.
\end{aligned}$$

Using the monotonicity of k in the first integral $k(\tau,t)\geqslant k(x,t)$ for $\tau\geqslant x$ and in the second integral as $k(x,t)\geqslant k(\tau,t)$ for $x\geqslant\tau$ we get

$$\begin{aligned}
&\geqslant \Big(\int_a^\tau\Big(\int_a^x k^{p'}(x,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\\
&\qquad+\int_\tau^b\Big(\int_a^\tau k^{p'}(\tau,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\Big)^{1/q}.
\end{aligned}$$

From these and (1.1) we obtain

$$\begin{aligned}
&\Big(\int_a^\tau\Big(\int_a^x k^{p'}(x,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x+\int_\tau^b\Big(\int_a^\tau k^{p'}(\tau,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\Big)^{1/q}\\
&\qquad\leqslant C\Big(\int_a^\tau k^{p'}(\tau,x)v^{1-p'}(x)\,\mathrm{d}x\Big)^{1/p},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_a^\tau\Big(\int_a^x k^{p'}(x,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x+\int_\tau^b\Big(\int_a^\tau k^{p'}(\tau,t)v^{1-p'}(t)\,\mathrm{d}t\Big)^q u(x)\,\mathrm{d}x\\
&\qquad\leqslant C^q\Big(\int_a^\tau k^{p'}(\tau,x)v^{1-p'}(x)\,\mathrm{d}x\Big)^{q/p},
\end{aligned}$$

or

$$B_4^{p'q}(\tau)A_2^{q/(1-p)}(\tau)+A_2^q(\tau)\leqslant C^q,$$

and then

$$B_4^{p'q}(\tau)+A_2^{p'q}(\tau)\leqslant C^qA_2^{q/(p-1)}(\tau).$$

Using Young's inequality to the right-hand side of the estimate we obtain

$$A_2^{p'q}(\tau) + B_4^{p'q}(\tau) \leqslant \frac{C^{p'q}}{p'} + \frac{A_2^{p'q}(\tau)}{p}$$

and then

$$\sup_{\tau\in(a,b)} (A_2^{p'q}(\tau) + p'B_4^{p'q}(\tau)) \leqslant C^{p'q}.$$

The second part of (2.3) is also proved.

Sufficiency. The sufficiency of the conditions is analogously proved as Theorem 2.1. In this instance, dual inequality (3.1) is used instead of inequality (1.1). The proof is given step by step, with the exception of the difference occurring at integration's orientation. The proof is complete. □

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